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A shifted analogue to ribbon tableaux

Ezgi Kantarci Oğuz

We introduce a shifted analogue of the ribbon tableaux defined by James and Kerber [3]. For any positive integer k, we give a bijection between the k-ribbon fillings of a shifted shape and regular fillings of a $\lfloor k/2 \rfloor$-tuple of shapes called its k-quotient. We define the corresponding generating functions, and prove that they are symmetric, Schur positive and Schur Q-positive. Then we introduce a Schur Q-positive q-refinement.

1. Introduction

The study of ribbon tableaux on shifted shapes combines two existing areas of work: the theory of ribbon tableaux and Schur's Q-functions. Ribbon tableaux introduced by James and Kerber [3] have applications to the representations of the symmetric group over a field of finite characteristic. Schur's Q-functions come up as the symmetric functions that correspond to the shifted diagrams. They have a connection to the irreducible spin characters of the symmetric group, analogous to that of Schur functions and irreducible characters of linear representations [6]. Since their introduction in [9], applications to diverse mathematical fields have been discovered, including the cohomology of isotropic Grassmannians [4] and polynomial solutions to the BKP equation in hydrodynamics.

In this work, we are merging these two ideas to initiate a combinatorial theory of ribbon tableaux for shifted shapes. The k-quotients and k-cores for shifted shapes were previously studied by Morris and Yaseen in 1986 [8] in the context of modular representations. We expand upon their work, reformulating it in a more explicit way that is analogous to the ribbon tilings of unshifted shapes due to James and Kerber [3]. We also look at standard and semi-standard fillings of these shapes, and define shifted k-ribbon functions. There is a bijective correspondence between standard and semi-standard fillings of the k-ribbon tableaux with fillings of its k-quotient, analogous to the unshifted case, which gives a positive expansion in terms of Schur's Q-functions.

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Our work was partly motivated by the possiblity of providing a shifted analogue for the LLT polynomials of Lascoux, Leclerc and Thibon which extend the theory of ribbon tableaux in the unshifted case, and have connections to the Fock space representation of the universal enveloping algebra of quantum affine $\mathfrak{sl}_n$ [5]. While we are able to provide a Schur Q-positive q-analogue derived from the unshifted LLT polynomials using the quotient, this analogue unfortunately does not lift the underlying combinatorial structure to ribbon shifted tableaux. In fact, this paper provides a negative result in that the spin statistic does not have a natural analogue for shifted ribbons, along with counterexamples that should prove valuable for further research.

The layout of this paper is as follows: In Section 2, we recall the notions of Schur functions, Schur's Q-functions and ribbon tableaux. In Section 3, we give a graphical description of k-ribbons on a shifted diagram, which differs from the standard case in that we have some 'double ribbons', which are allowed to contain 2×2 boxes. We define the shifted k-ribbon tableaux and the corresponding P- and Q-functions, as well as state our main theorem giving an expansion of a shifted ribbon Q-function in terms of Schur's Q-funtions. Sections 4 and 5 give the combinatorial constructions necessary to prove this, including a new type of object that comes up in shifted k-quotients which we call folded tableaux. We give bijections between ribbon fillings of a shifted diagram and its k-quotient, both in the standard and semi-standard case. In Section 5, we give a description of the shifted ribbon functions in terms of peak functions. Lastly, in Section 7, we define a q-refinement of the shifted ribbon function and prove its Schur Q-positivity. We further discuss the difficulties of defining a direct analogue of the spin statistic from the unshifted case, and provide some counterexamples.

2. Preliminaries

2.1. Schur functions

A *composition* of n is a list $C = (c_1, c_2, \ldots, c_k)$ of positive integers adding up to n called its *parts*. Here, n is called the *size* of the composition, denoted $|C|$, and the number of its parts is called its *height* denoted $\mathrm{ht}(C)$. Compositions of n are partially ordered by refinement, where $D = (d_1, d_2, \ldots, d_l)$ refines $C = (c_1, c_2, \ldots, c_k)$ if there exist integers $a_1, ...a_{k-1}$ such that $c_1 = d_1 + d_2 + \cdots + d_{a_1}$ and $c_i = d_{a_{i-1}+1} + d_{a_{i-1}+2} \cdots + d_{a_i}$ for all $i > 1$. A composition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ is a *partition* if it satisfies $\mu_i \geq \mu_{i+1}$ for all i. With every partition, we associate a *Young diagram*, an array with μ_i boxes on row i.

For a given partition μ, a *semi-standard Young tableau* of shape μ is a filling of the boxes of its Young diagram with positive integers such that each column will be increasing from bottom to top, and each row will be non-decreasing from left to right. A semi-standard tableau that contains each of the numbers from 1 to n exactly once is called *standard*. We will denote the set of semi-standard tableaux of shape μ by $SSYT(\mu)$, and the set of standard ones by $SYT(\mu)$.

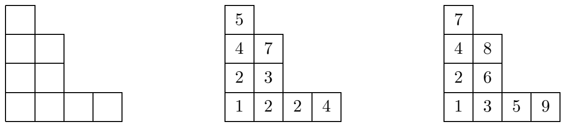


Figure 1: The diagram $\mu = (4, 2, 2, 1)$ with semi-standard and standard fillings.

For a partition μ we define its *Schur function* as follows:

$$s_\mu(X) = \sum_{T \in SSYT(\mu)} X^T \tag{1}$$

Here X^T denotes the monomial where the power of x_i is given by the number of times i occurs in T. The semi-standard filling in Figure 1 corresponds to the monomial $x_1x_2^3x_3x_4^2x_5x_7$.

The *reading word* of a tableau is a reading of all its labels from left to right, top to bottom. For example, the semi-standard tableau from Figure 1 has the reading word 547231224, where as the standard one has the reading word 748261359.

Note that the reading word of a standard tableau S of shape μ gives a permutation of numbers from 1 to $|\mu|$, so we can talk about its descent, peak and spike sets. The *descent set* of a standard tableau T is defined as follows:

$$\begin{aligned} \mathrm{Des}(T) &= \{i \mid i \text{ is to the right of } i+1 \text{ in the reading word of } T\} \\ &\subset [n-1] \end{aligned}$$

If the reading word of T is given by the permutation σ, its descents are exactly the coordinates i where $\sigma^{-1}(i) > \sigma^{-1}(i+1)$. We will also be interested in the peak points where $\sigma^{-1}(i)$ is higher than its neighbors, and spike points where $\sigma^{-1}(i)$ is higher or lower than both its neighbors. Note that peak and spike points only depend on the descent set of σ. In general, for any set $D \subset [n]$, the peak and spike sets of D are given by:

$$\begin{aligned} \mathrm{Peak}(D) &= \{i \mid i \neq 1 \text{ and } i \in \mathrm{D} \text{ and } i-1 \notin \mathrm{D}\} \\ \mathrm{Spike}(D) &= \{i \mid i \in \mathrm{D} \text{ and } i-1 \notin \mathrm{D} \text{ or } i \notin \mathrm{D} \text{ and } i-1 \in \mathrm{D}\}. \end{aligned}$$

Throughout this work, we will mainly be interested in the case when D is the descent set of the reading word for a tableau. For a tableau T, we will use to notations $\mathrm{Peak}(T)$ and $\mathrm{Spike}(T)$ to denote $\mathrm{Peak}(\mathrm{Des}(T))$ and $\mathrm{Spike}(\mathrm{Des}(T))$ respectively. The standard tableau from Figure 1 has descent, peak and spike sets $\{1,3,5,6\}$, $\{3,5\}$ and $\{2,3,4,5,7\}$ respectively. In 1984, Gessel [1] showed that the Schur function for a partition μ can be expressed in terms of descent sets:

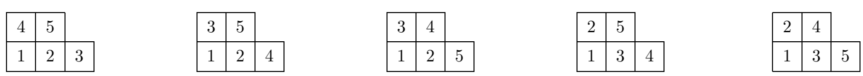
$$s_\mu(X) = \sum_{T \in SYT(\mu)} F_{\mathrm{Des}(T)}(X)$$

where $F_D(X)$, $D \subset [n-1]$ denotes Gessel's fundamental basis for quasisymmetric functions defined by:

$$F_D(X) = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_m \\ t \in D \Rightarrow i_t \ne i_{t+1}}} x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_m} \tag{2}$$

This formula allows us to calculate the Schur function of a partition using only its standard fillings. For example, the Schur function of (3,2), whose standard fillings are given in Figure 2 is:

$$s_{(3,2)}(X) = F_2(X) + F_3(X) + F_{1,4}(X) + F_{1,3}(X) + F_{2,4}(X)$$



45123 35124 34125 25134 24135

Figure 2: The standard tableaux of shape $(3,2)$ and their reading words.

2.2. Ribbon tableaux

Given two diagrams $\mu \subset \nu$, the *skew diagram* ν/μ is the diagram of ν minus the boxes that correspond to μ. A diagram is connected if any two boxes on it are connected by a path going through edges connecting two boxes on the diagram. A *k-ribbon* on an unshifted diagram is a connected skew-diagram of k boxes containing no 2×2 square. A k-ribbon R is removable from diagram μ if $\mu/\nu = R$ for some $\nu \subset \mu$. A diagram with no removable k-ribbon is called a k-core.

On a given k-ribbon R, the rightmost lowest box is called the *head* of the R. A set of disjoint ribbons form a *horizontal strip* if their disjoint union is a (not necessarily connected) skew-shape and their heads lie on different columns. A *semi-standard k-ribbon tableau* of shape μ is a sequence of shifted diagrams $\mu_0 \subset \mu_1 \subset \cdots \subset \mu_n = \mu$ where μ_0 is a k-core, and each μ_i/μ_{i-1} is a horizontal k-ribbon strip, the ribbon on which we label by i. A semi-standard k-ribbon tableau is called *standard* if all labels from 1 to n occur exactly once. The *reading word* of ribbon tableau is a reading of the labels on the heads of the ribbons, left to right, top to bottom, and its descent set is the descent set of its reading word. For example, the 3-ribbon tableau given in Figure 3 has the reading word 4123, and the corresponding descent set $\{3\}$. The generating function for the k-ribbon tableaux of shape μ is given by:

$$GF^{(k)}_{\mu/\mu_0}(X) = \sum_{T \in SSRT_k(\mu)} X^T = \sum_{S \in SRT_k(\mu)} F_{\mathrm{Des}(S)}(X),$$

where $SSRT_k(\mu)$ denotes the set of semi-standard k-ribbon tableaux of shape μ, and $SRT_k(\mu)$ denotes the set of standard ones.

Figure 3: Examples of Standard (left) and semi-standard (right) 3-ribbon tableaux of shape $(6, 3, 3, 2)$.

James and Kerber [3] showed that there is a weight-preserving bijection between semi-standard ribbon tableaux of shape μ, and semi-standard fillings of a k-tuple of unshifted shapes $(\mu^0, \mu^1, \ldots, \mu^{k-1})$ called the *k-quotient of μ*. This shows that

$$GF^{(k)}_{\mu/\mu_0}(X) = s_{\mu^0}(X) s_{\mu^1}(X) \cdots s_{\mu^{k-1}}(X). \tag{3}$$

The *spin* of a ribbon R, defined by Lascoux, Leclerc and Thibon [5] is $(|R| - \mathrm{ht}(R) - 1)/2$, where $|R|$ denotes the size of the ribbon (the number of boxes it contains) and $\mathrm{ht}(R)$ denotes height of a ribbon (the number of rows of it intersects). Note that the spin of a ribbon is not necessarily an integer. For a semi-standard k-ribbon tableaux T of shape μ, we define the *spin* of T to be the sum of the spins of all ribbons on T. The *cospin* of T is given by $spin(T*) - spin(T)$ where $T*$ is the semi-standard k-ribbon tableaux of shape μ with the maximum spin. The cospin is an integer for every tableau T.

Multiplying the fundamental quasisymmetric function for each tableau by a variable q raised to the tableau's cospin gives us the LLT-polynomial:

$$GF^{(k)}_{\mu/\mu_0}(X;q) = \sum_{T \in SRT_k(\mu)} q^{cospin(T)} F_{\mathrm{Des}(T)}(X). \tag{4}$$

The LLT-polynomials can be written as a sum of Schur polynomials with coefficients from $\mathbb{Z}^+[q]$ (See [2] for details).

2.3. Shifted tableaux

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is called *strict* if all its parts are distinct. With every strict partition, we associate a *shifted diagram*, which is an array with λ_i boxes on row i, where row i is shifted $k-i$ steps to the right, forming a staircase shape. For any box C on a shifted diagram, we define its *diagonal* value to be $diag(C) = col(C) - row(C) + 1$. Note that the smallest diagonal value is 1 and is attained only at the leftmost diagonal which is denoted the *main diagonal* of λ.

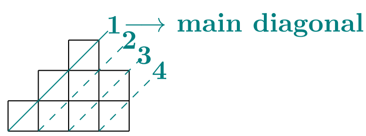


Figure 4: The shifted diagram for $\lambda = (4, 3, 1)$ with diagonals labelled with diagonal values.

A *semi-standard shifted tableau* of shape λ is a filling of its boxes with elements from the marked alphabet $1' < 1 < 2' < 2 < 3' < 3 < \cdots$ such that each row will be non-decreasing from left to right with no repeated marked numbers, and each column will be non-decreasing from bottom to top with no repeated unmarked numbers. A semi-standard shifted tableau of shape λ that contains each of the numbers $1, 2, \ldots, |\lambda|$ exactly once (possibly marked), it is called *marked standard*, and if they are all unmarked it is called *standard*. We will denote the set of semi-standard shifted tableaux of shape λ by $\mathrm{SsShT}(\lambda)$, the set of marked standard ones by $\mathrm{SShT}'(\lambda)$ and the set of the standard ones by $\mathrm{SShT}(\lambda)$ (See Figure 5).

$$\begin{array}{|c|c|c|c|} \cline{3-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c|}{} & 2' & 3' & 4' \\ \hline 1 & 1 & 2' & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|} \cline{3-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 7 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c|}{} & 4' & 6' & 8 \\ \hline 1 & 2 & 3' & 5 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|} \cline{3-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 7 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c|}{} & 4 & 6 & 8 \\ \hline 1 & 2 & 3 & 5 \\ \hline \end{array}$$

Figure 5: Examples of semi-standard (left), marked standard (middle) and standard (right) shifted tableaux for $\lambda = (4, 3, 1)$. The tableaux given here are related by a standardization algorithm, which will be introduced in Section 5.

Schur's Q- and P-functions for a strict partition λ are defined as follows:

$$Q_\lambda(X) = \sum_{S \in \mathrm{SsShT}(\lambda)} X^{|S|}, \tag{5}$$

$$P_\lambda(X) = 2^{-\mathrm{ht}(\lambda)} \sum_{S \in \mathrm{SsShT}(\lambda)} X^{|S|} = \sum_{S \in \mathrm{SsShT}^*(\lambda)} X^{|S|} \tag{6}$$

where $\mathrm{SsShT}^*(\lambda)$ denotes the set of semi-standard tableaux of shape λ with no marked entries on the main diagonal, and $X^{|S|}$ is the monomial where the power of x_i is equal to the number of times i or i' occurs in S. The semi-standard filling in Figure 5, for example, corresponds to the monomial $x_1^2 x_2^3 x_3^2 x_4$.

The *reading word* of a shifted tableau is, like in the unshifted case, a reading of all its labels from left to right, top to bottom. The definitions of descent, peak and spike sets can be extended to the reading words of marked standard tableaux by first moving all marked coordinates to the beginning and then reversing their order and working with the corresponding word. For instance, $\mathrm{Des}(74'6'8123'5) = \mathrm{Des}(36478125) = \{2, 5\}$.

Lemma 2.1. *Let T' be a marked standard shifted tableau and T be the standard shifted tableau obtained from T' by deleting the marks of the labels. If $i \in \mathrm{Des}(T)$, then $i \in \mathrm{Des}(T')$ if and only if i is unmarked in T'. If $i \notin \mathrm{Des}(T)$, then $i \in \mathrm{Des}(T')$ if and only if $i+1$ is marked in T.*

Proof. This follows directly from the definition of descents on marked tableaux. □

Like in the case of Schur functions, Schur's Q-functions can be expanded in terms of the fundemental quasisymmetric functions:

$$Q_\lambda(X) = \sum_{T' \in \mathrm{SShT}'(\lambda)} F_{\mathrm{Des}(T')}(X).$$

For this expansion, we only look at the marked standard tableaux of shape λ. An expansion that also eliminates the markings and only considers the standard fillings was given by Stembridge [11]:

$$Q_\lambda(X) = \sum_{T \in \mathrm{SShT}(\lambda)} 2^{|\mathrm{Peak}(T)|+1} G_{\mathrm{Peak}(T)}(X).$$

Here, the functions $G_{\mathrm{Peak}(T)}$ are the *peak functions* defined in [11]. For a subset P of $[2, 3, .., n-1]$ with no consecutive entries the corresponding peak function is given by:

$$G_P(X) = \sum_{\substack{D \in [n-1] \\ \mathrm{Spike}(D) \supset P}} F_D(X).$$

Given two strict partitions μ and λ with $\mu \subset \lambda$, the *skew-shifted diagram* $\lambda \backslash \mu$ is the diagram for λ with the boxes corresponding to the diagram of μ deleted.

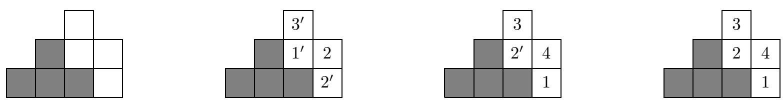


Figure 6: The skew-shifted diagram $(4,3,1)\backslash(3,1)$ with a corresponding semi-standard, marked standard and standard filling.

We can apply the above definitions to the skew-shifted diagrams to get skew-shifted tableaux. More precisely, the set of semi-standard shifted tableaux of shape $\lambda\backslash\mu$, denoted $(\lambda\backslash\mu)$ is given by all the fillings of $\lambda\backslash\mu$ from the marked alphabet with non-decreasing columns and rows such that we have no unmarked numbers repeated along columns and no marked numbers repeated along rows (See Figure 6). We will denote the marked standard fillings of $\lambda\backslash\mu$ (where we use each number from 1 to n once, possibly marked) by $\mathrm{SShT}'(\lambda\backslash\mu)$ and its standard fillings (where we use each number from 1 to n once, unmarked) by $\mathrm{SShT}(\lambda\backslash\mu)$. This gives rise to a skew analogue for Schur's Q-function:

$$\begin{aligned} Q_{\lambda\backslash\mu}(X) &= \sum_{S \in \mathrm{SsShT}(\lambda\backslash\mu)} X^{|S|} = \sum_{T' \in \mathrm{SShT}'(\lambda\backslash\mu)} F_{\mathrm{Des}(T')}(X) \\ &= \sum_{T \in \mathrm{SShT}(\lambda\backslash\mu)} G_{\mathrm{Peak}(T)}(X). \end{aligned}$$

It was shown by Stembridge that the skew-shifted Q-functions expand positively into Schur's Q-functions:

Theorem 2.2. (Stembridge [10]) *There exist coefficients $f^{\lambda}_{\mu,\nu} \in \mathbb{N}$ satisfying:*

$$Q_{\lambda\backslash\mu}(X) = \sum_{\nu} f^{\lambda}_{\mu,\nu} Q_{\nu}(X), \qquad P_{\mu}(X)P_{\nu}(X) = \sum_{\lambda} f^{\lambda}_{\mu,\nu} P_{\lambda}(X)$$

where $f^{\lambda}_{\mu,\nu} = 0$ unless $|\mu| + |\nu| = |\lambda|$.

3. Shifted ribbon tableaux

3.1. Ribbons on shifted diagrams

On a shifted diagram, we call the columns strictly to the left of the last row its *shifted region*, and the rest its *unshifted region*. Note that the unshifted region uniquely determines the diagram[1].

The definition of the hook of a box on a shifted diagram depends on whether the box falls into the shifted region. For any box C in the unshifted region, the *hook* of C is the union of C, with the boxes above it in its column, and the boxes to its right in the row. For a box in the shifted region, its hook additionally includes the row of boxes directly above the highest box in the column of C. The number of boxes in its hook is called the *hook length* of C (See Figure 7).

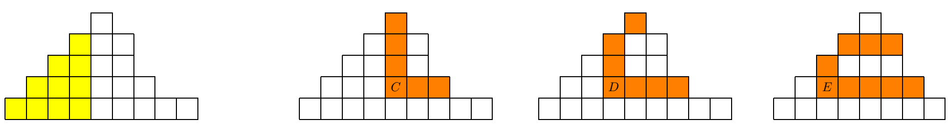


Figure 7: Shifted region of $(9, 6, 4, 3, 1)$ and hooks of the boxes C, D and E.

Alternatively, we can calculate hook lengths by using a ghost copy of λ reflected along the main diagonal (Figure 8, left). The hook length of any box C is given by the number of boxes above or to the right of C along with C (Figure 8, right). We can recover the hooks by reflecting the cells on the ghost copy back to the shape.

Definition 3.1. *We define a* single-ribbon *on λ to be a connected skew-shifted diagram with each box on a different diagonal.*

[1] We deviate from convention in defining the shifted region, to define hook lengths more easily.

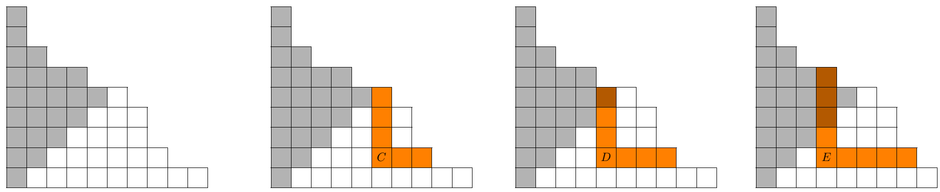


Figure 8: Ghost copy of $(9, 6, 4, 3, 1)$ and reflected hooks of the boxes C, D and E.

Some important notations we will use about ribbons throughout the paper are heads and tails of ribbons. We call the box with the highest diagonal value the *head* of R, denoted by $H(R)$, and the one with the lowest diagonal value the *tail* of R, denoted $T(R)$. Note that for a single-ribbon R, $|R| =\text{diag}(H(R))-\text{diag}(T(R)) + 1$.

Definition 3.2. *A* double-ribbon *of size k is a union of two disjoint single-ribbons R and S of sizes $r \geq s$ with $r + s = k$ such that the tails of both are on the main diagonal of $\lambda - R$, and their union forms a skew-shifted shape.*

This definition of double ribbons is motivated by the moves on the k-abacus, which will be discussed in Section 3.2. We define the head of a double ribbon to be the lowest of the boxes with the highest diagonal value, and its tail to be the highest of the boxes with the lowest diagonal value (See Figure 9). We use the notations $|R|$ for the size of a ribbon (the number of boxes it contains) and $\text{ht}(R)$ for the height of a ribbon (the number of rows of λ that it intersects) as in the unshifted case.

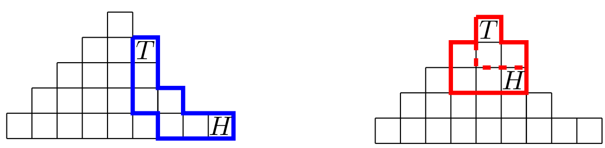


Figure 9: A single ribbon (left) and a double ribbon (right).

Definition 3.3. *A k-*ribbon *is a single or double ribbon of size k. A k-ribbon R is* removable *if $\lambda - R$ is a shifted diagram.*

Proposition 3.4. *For any removable k-ribbon R on λ, there is no box on R strictly to the right or strictly below $H(R)$.*

Proposition 3.5. *A shifted diagram λ has a removable single k-ribbon with diag$(H(R)) = m$ if and only if it has a part of size m, no part of size $m - k$*

and $m \geq k$. If it exists, it is the unique ribbon R where $\lambda - R$ has the part m replaced with $m-k$. Furthermore, λ has a removable double k-ribbon with $diag(H(R)) = a$ if and only if $a < k$ and λ has parts of sizes a and $k-a < a$. It is the unique ribbon R where $\lambda - R$ is λ with the parts of size a and $k-a$ removed.

Proof. If λ has a part of size m and no part of size $m-k$, then removing the outermost boxes from diagonals m to $m-k+1$ gives us a removable ribbon of size k, with $diag(H(R)) = m$. In particular, if $lambda_i$ is the smallest part greater than $m-k$ and $\lambda_j = m$, this ribbon decreases the length of λ_j to λ_{j-1}, λ_{j-1} to λ_{j-2} and so on, ending up with reducing the length of λ_i to $m-k$. Conversely, if R is a removable ribbon with $diag(H(R)) = m$, $\lambda \backslash R$ is itself a skew-shifted diagram, which means there is no box on λ above $T(R)$ or to the left of $H(R)$, and R contains the outermost box of each diagonal from m to $m-k+1$. As $H(R)$ is at the end of a row, and has diagonal value m, λ contains a part of size m. Similarly, as $T(R)$ is on a row with at least $m+1$ boxes, and as the row above has no box above $T(R)$ it has less than m boxes, so λ has no part of size m. The case for the double ribbon follows as the double ribbon is a union of two single ribbons, and a ribbon of size a with $diag(H(R)) = a$ will have its tail on the main diagonal. □

A corollary of this proposition is that no shape can have a removable double ribbon of size (t, t).

Theorem 3.6. *A shifted diagram λ admits no removable k-ribbon iff it has no boxes with hook length equal to k.*

In this case, we call λ a *k-core*.

Proof. We claim that there is a bijection between removable k-ribbons and boxes with hook length equal to k, where boxes in the unshifted part correspond to single ribbons and boxes in the shifted part correspond to double ribbons. Under this bijection it is clear that if a diagram admits no k ribbons, it can not have a box with hook length k and vice versa.

Let C have hook length equal to k. First let us look at the case when C is in the unshifted part. Let R be the single k-ribbon consisting of the outermost box on each diagonal that the hook of C passes. As the head and tail of R are the endpoints of the hook of C, R is removable. Conversely, if R is a single k-ribbon, the box on the row of $H(R)$ and the column of $T(R)$ has hook length k and is on the unshifted part.

Now let us assume C is a box in the shifted part, so that its hook includes the row above the column of C. Assume this row contains t boxes. This means C is on a row of size $k-t$, and by Proposition 3.5 the shape has a unique removable double ribbon of size $(t, k-t)$. Conversely, if R is a removable double ribbon of size $(t, k-t)$ with $t < k-t$, the diagram has rows i and j with sizes t and $k-t$. The box on row j and column right below row i falls on the shifted part and has hook length k. □

3.2. The k-abacus correspondence

The k-abaci come up in the modular representation theory of the affine symmetric group. The interested reader is advised to refer to [7] for more details. In this section, we will show that removing a k-ribbon on a shifted tableaux is equivalent to making a move on the k-abacus, tying our theory in with the existing one.

A k-abacus consists of k columns that are called *runners*. The runners are labeled by $1, 2, 3, \ldots, k$, and the positions on these runners are labeled as follows:

1	2	3	$\cdots$	k
1	2	3	$\cdots$	k
k+1	k+2	k+3	$\cdots$	2k
2k+1	2k+2	2k+3	$\cdots$	3k
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Two runners i and j are called k-*conjugate* if $i + j = k$. Each position on a runner is either empty or contains a *bead*. To any shifted diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ we will associate the k-abacus with beads on positions $\lambda_1, \lambda_2, \ldots, \lambda_n$. For example, for the diagram $\lambda = (16, 11, 10, 9, 8, 7, 4, 3, 1)$ we get the 5-abacus:

a_1	a_2	a_3	a_4	a_5
$\bullet$	$\circ$	$\bullet$	$\bullet$	$\circ$
$\circ$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
$\bullet$	$\circ$	$\circ$	$\circ$	$\circ$
$\bullet$	$\circ$	$\circ$	$\circ$	$\circ$
$\circ$	$\circ$	$\circ$	$\circ$	$\circ$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Given a strict partition λ, we identify each runner a_i in its abacus with a shifted shape $\alpha^{(i)}$, by treating the runners as the abacus of a shifted 1-core. More precisely, $\alpha^{(i)}$ will be the shifted shape $(\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_t^{(i)})$ where $k(\alpha_1^{(i)}-1)+i, \ldots, k(\alpha_t^{(i)}-1)+i$ are the parts of λ that are equal to i modulo k. We will call the k-tuple $[\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n)}]$ the abacus representation λ. The 5-abacus representation of the example $(16, 11, 10, 9, 8, 7, 4, 3, 1)$ with the abacus above is $[(4, 3, 1), (2), (2, 1), (2, 1), (2)]$.

There are three types of moves allowed on the k-abacus of λ [7]:

- *Type I Move:* Sliding one bead one position higher in its runner if that position is unoccupied.
- *Type II Move:* Removing a bead from the top row of column k.
- *Type III Move:* Removing two beads from the first row, if they are on two conjugate runners.

After a move on the k-abacus, we get a new shifted diagram $\lambda* \subset \lambda$. A *Type I* move corresponds to replacing a part of size $m+k$ with one of size m, whereas a *Type II* move corresponds to removing two parts of sizes adding up to k. By Proposition 3.5, we have the following correspondence:

Theorem 3.7. *Making a move on the k-abacus of λ is equivalent to removing a k-ribbon from λ. In particular,*

1. Single-Ribbon Correspondence: *Removing a single-ribbon A with $diag(H_A) = m+k$ and $diag(T_A) = m+1$ is equivalent to making a Type I move from position $m+k$ to position m for $m > 0$ and making a Type II move if $m = 0$.*
2. Double-Ribbon Correspondence: *Removing a double-ribbon of size $(t, k-t)$ is equivalent to making a Type III move removing top beads t and $k-t$ from conjugate runners a_t and a_{k-t}.*

If an abacus has no moves, it is called a k-core. For any abacus, there corresponds a unique k-core one can reach by applying moves that is independent of the choice of the moves [7]. Theorem 3.7 implies that the k-core of a shifted diagram corresponds to the k-core of the abacus, which gives us the following corollary.

Corollary 3.8. *The k-core of a shifted diagram is unique.*

3.3. Shifted ribbon tableaux

Definition 3.9. *A* standard k-ribbon tableau *of shape λ is a sequence of shifted diagrams $\lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(n)} = \lambda$, where each $A_i = \lambda^{(i)} \backslash \lambda^{(i-1)}$, $i = 1, 2, \ldots, n$ is a k-ribbon, and $\lambda^{(0)}$ is a k-core. For each i, we label ribbon A_i with i.*

Definition 3.10. *A skew-shifted diagram S on a shifted diagram λ is called a* horizontal k-ribbon strip *(resp.* vertical k-ribbon strip*) if there exists a sequence of shifted diagrams $\lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(t)} = \lambda$, where:*

1. *Each $R_i := \lambda^{(i)} \setminus \lambda^{(i-1)}$ is a k-ribbon.*

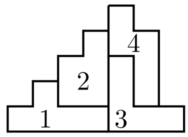


Figure 10: A 5-Ribbon Tableau with no 5-core of shape $\lambda = (7, 5, 4, 3, 1)$.

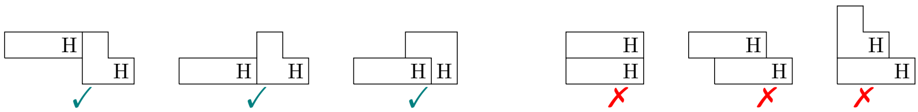


Figure 11: Horizontal strip examples and non-examples.

2. *$H(R_i)$ is strictly to the right of (resp. strictly above) $H(R_{i-1})$ for each i.*
3. *$S = \bigcup_{i=1}^{n} R_i = \lambda \setminus \lambda^{(0)}$.*

Definition 3.11. *A* semi-standard shifted k-ribbon tableau *is given by a sequence $\lambda^{(0)} \subset \lambda^{(1')} \subset \lambda^{(1)} \subset \lambda^{(2')} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(n)} = \lambda$, where:*

- *$\lambda^{(0)}$ is the k-core of λ.*
- *Each $\lambda^{(i)} \setminus \lambda^{(i')}$ is a (possibly empty) horizontal k-strip.*
- *Each $\lambda^{(i')} \setminus \lambda^{(i-1)}$ is a (possibly empty) vertical k-strip.*

We number the ribbons on the strip $\lambda^{(i)} \setminus \lambda^{(i')}$ by i and the ribbons forming the strip $\lambda^{(i')} \setminus \lambda^{(i-1)}$ by i' for each $i = 1, 2, \ldots, n$.

Definition 3.12. *For a shifted shape λ, we define its k-ribbon Q and P-functions as follows:*

$$RQ_{\lambda}^{(k)}(X) = \sum_{S \in \mathrm{SsShT}^{(k)}(\lambda)} X^{|S|}, \qquad RP_{\lambda}^{(k)}(X) = \sum_{S \in \mathrm{SsShT}^{*(k)}(\lambda)} X^{|S|}$$

where $\mathrm{SsShT}^{(k)}(\lambda)$ *is the set of semi-standard shifted k-ribbon tableaux of shape λ, and* $\mathrm{SsShT}^{*(k)}(\lambda)$ *is its subset consisting of tableaux with no marked entries on the ribbons that have boxes on the main diagonal.*

The example illustrated in Figure 12 gives the following ribbon Q- and P-functions restricted to two variables:

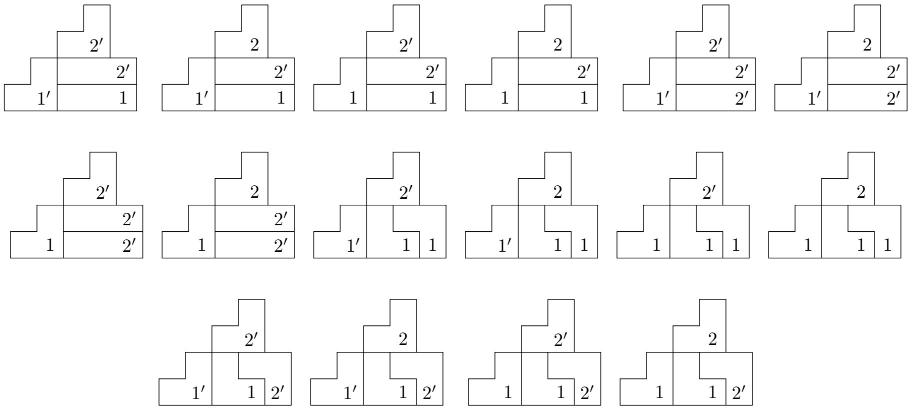


Figure 12: Semi-standard 3-ribbon fillings of $(5, 4, 2, 1)$ with numbers ≤ 2.

$$\begin{aligned} RQ^{(3)}_{(5,4,2,1)}(x_1, x_2) &= 4x_1^3x_2 + 8x_1^2x_2^2 + 4x_1x_2^3 = Q_{3,1}(x_1, x_2) \\ RP^{(3)}_{(5,4,2,1)}(x_1, x_2) &= x_1^3x_2 + 2x_1^2x_2^2 + x_1x_2^3 = P_{3,1}(x_1, x_2) \end{aligned}$$

Note that in this example, the ribbon Q and P functions are Schur Q and P positive respectively. One of our main results in this paper will be to show that the positivity holds true in general, and give a formula for the ribbon Q-functions in terms of Schur's Q-functions.

Proposition 3.13. *All the shifted k-ribbon tableaux of shape λ have the same number of ribbons that have a box on the main diagonal, called the k-*length *of λ, denoted $\ell^{(k)}(\lambda)$. Consequentally, the Q and P k-ribbon functions for λ are related by a scalar:*

$$RQ^{(k)}_\lambda(X) = 2^{\ell^{(k)}(\lambda)} RP^{(k)}_\lambda(X).$$

Proof. By Theorem 3.7, the ribbons that have a box on the main diagonal correspond to the moves on the abacus where beads are removed. The total number is independent of the order of the moves. In fact, if we denote the number of beads on runner i by $|a_i|$ then:

$$\ell^{(k)}(\lambda) = |a_k| + \sum_{i<(k/2)} max\{|a_i|, |a_{k-i}|\}.$$

This means that for every S in $\mathrm{SsShT}^{*(k)}(\lambda)$, we have $2^{\ell^{(k)}(\lambda)}$ ways of marking the $\ell^{(k)}(\lambda)$ ribbons with boxes on the main diagonal to get tableaux in $\mathrm{SsShT}^{(k)}(\lambda)$. $\square$

4. Folded tableaux

In this section, we will introduce an operation combining two shifted shapes to get an unshifted shape with a specialized diagonal, which we will call a folded diagram. We will later use the folded diagrams, along with their corresponding tableaux in describing the k-quotient of a shifted shape. We will use the notation δ_n to denote the staircase partition $(n, n-1, \ldots, 1)$ of size n.

Definition 4.1. *A* folded diagram $\Gamma = (\gamma, \mathfrak{d})$ *is an unshifted diagram* γ *called the* underlying shape *of* Γ *along with a specialized main diagonal* $\mathfrak{d}$ *which does not necessarily intersect* γ*.*

Definition 4.2. *Let* α *and* β *be shifted shapes (or equivalently strict partitions) . Denote* $n = min\{\mathrm{ht}(\alpha), \mathrm{ht}(\beta)\}$*. Their* combination*, which will be denoted by* $\alpha \diamond \beta$ *will be the folded diagram we obtain by:*

- *Step I: If one of the shapes has height* m *larger than* n*, delete its* $m-n$ *leftmost columns, so that both shapes have the same number of boxes in their main diagonals.*
- *Step II: Transpose* α*.*
- *Step III: Paste the two diagrams together along their main diagonals, and label this diagonal* $\mathfrak{d}$*.*

Example:

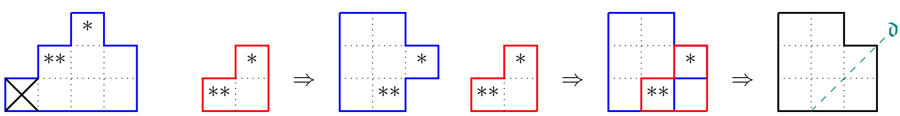


Figure 13: The combination of $\alpha = (4, 3, 1)$ and $\beta = (2, 1)$.

Proposition 4.3. *Any given folded diagram* $(\gamma, \mathfrak{d})$ *can be uniquely described as a combination of two shifted shapes.*

Proof. Let $\gamma^T = (\gamma_1^T, \ldots, \gamma_s^T)$ denote the transpose of γ. If $\mathfrak{d}$ is k units to the right of the main diagonal and going through $t > 0$ boxes, then $(\gamma, \mathfrak{d}) = \alpha \diamond \beta$ where $\alpha = \delta_{k+t} + (\gamma_{t+1}, \gamma_{t+2}, \ldots, \gamma_n)^T$ and $\beta = \delta_t + (\gamma_{k+t+1}^T, \gamma_{k+t+2}^T, \ldots, \gamma_s^T)^T$.

If $\mathfrak{d}$ is $k \geq 0$ units to the right of the bottom left corner and outside the shape (the case $k < 0$ is symmetrical) we have $(\gamma, \mathfrak{d}) = \alpha \diamond \beta$ where $\alpha = \delta_k + \gamma^T$ and β is empty.

The process can be inverted by transposing the cells weakly to the left of $\mathfrak{d}$ and completing them to a shifted shape to get α, and completing the cells weakly to the right of $\mathfrak{d}$ to a shifted shape get β (Follow Figure 13 right to left). $\square$

Definition 4.4. *A* standard folded tableau *of shape* $\Gamma = (\gamma, \mathfrak{d})$ *is a filling of the boxes of* γ *using numbers* $1, 2, \ldots, n$ *each exactly once, with numbers increasing left to right and bottom to top.*

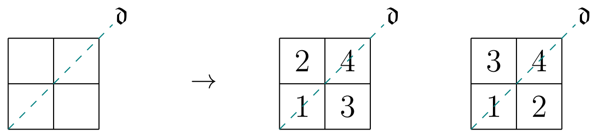


Figure 14: Standard Folded Tableaux of shape $\Gamma = ((2, 2), \mathfrak{d})$.

Definition 4.5. *A* semi-standard folded tableau *of shape* $\Gamma = (\gamma, \mathfrak{d})$ *is a semi-standard filling of the skew-shifted diagram* γ *with the rules inverted weakly above the specialized diagonal (above* $\mathfrak{d}$ *we can have no repeated unmarked numbers on the same row, and no two repeated marked numbers on the same column).*

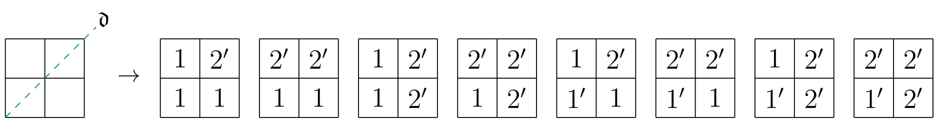


Figure 15: semi-standard folded tableaux of shape $\Gamma = ((2, 2), \mathfrak{d})$, using numbers $\leq 2'$.

Definition 4.6. *We define the folded P- and Q-functions as follows.*

$$Q^f_{(\gamma,\mathfrak{d})}(X) = \sum_{S \in \mathrm{SSFT}(\gamma,\mathfrak{d})} X^{|S|}$$

$$P^f_{(\gamma,\mathfrak{d})}(X) = 2^{-\mathrm{size}(\mathfrak{d})} Q^f_{(\gamma,\mathfrak{d})}(X)$$

where $SSFT(\gamma, \mathfrak{d})$ *denotes the set of all semi-standard folded tableaux of shape* $\Gamma = (\gamma, \mathfrak{d})$*, and* $\mathrm{size}(\mathfrak{d})$ *denotes the number of boxes on* $\mathfrak{d}$*.*

The folded shape $((2, 2), \mathfrak{d})$ illustrated in Figures 14 and 15 has the following folded P- and Q-functions:

$P^f_{(22,\mathfrak{d})}(X) = m_{31}(X) + 2m_{22}(X) + 4m_{211}(X) + 8m_{1111}(X) = s_{31}(X) + s_{22}(X) + s_{2111}(X) = P_{(31)}(X)$

$Q^f_{(22,\mathfrak{d})}(X) = 2^{\text{size}(\mathfrak{d})} P^f_{(22,\mathfrak{d})}(X) = 4(s_{31}(X) + s_{22}(X) + s_{2111}(X)) = Q_{(31)}(X)$

Theorem 4.7. *The function $Q^f_{(\gamma,\mathfrak{d})}(X)$ is independent of the choice of the main diagonal $\mathfrak{d}$.*

Proof. We will prove this by giving a weight preserving bijection between SSFT$(\gamma, \mathfrak{d})$ with semi-standard fillings of the skew-shifted shape γ.

Let $S \in SSFT(\gamma, \mathfrak{d})$. We start with inverting the markings of all boxes weakly above the main diagonal. Each connected piece weakly above the main diagonal containing only i or i's forms a ribbon, with at most one i' on each row (on the rightmost box), and at most one i on each column (on the lowest box). The next step is to go from bottom to the top in the ribbon, flipping i' with the leftmost i of each row (Figure 16, left) or flipping the i with the highest i' on each column (Figure 16, right). As the algorithm corrects the ordering when the process is repeated for all i, we get with a semi-standard skew-shifted filling of γ.

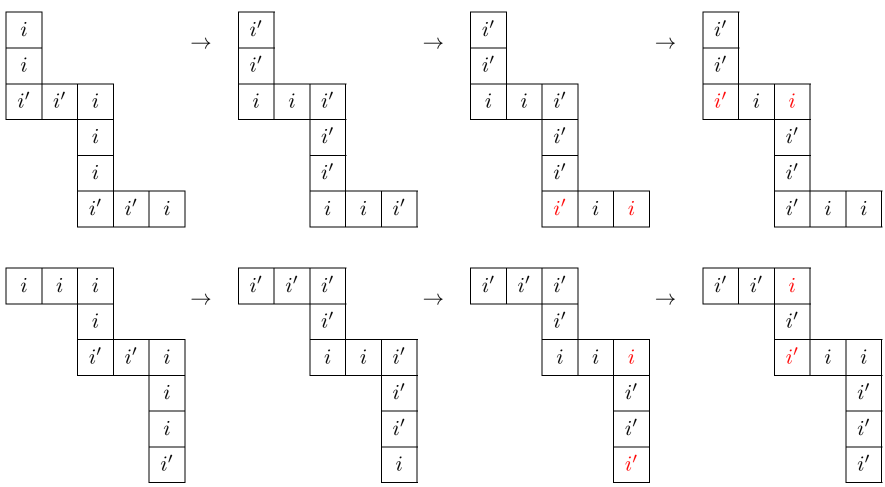


Figure 16: Examples of inverting markings on an i-ribbon.

The process can be inverted by inverting the markings weakly above the main diagonal again, and this time working our way from top to bottom on each ribbon, flipping any i' with the lowest i on columns and flipping any i with the rightmost i' in rows. □

Corollary 4.8. *The folded Q-functions are Schur Q-positive. In fact*

$$Q^f_{(\gamma,\mathfrak{d})}(X) = Q_{\lambda/\delta_n}(X) = \sum_{\varepsilon} f^{\lambda}_{\varepsilon,\delta_n} Q_{\varepsilon}(X),$$

where λ is a shifted shape with $\lambda/\delta_n = \gamma$ and $f^{\lambda}_{\varepsilon,\delta_n}$ are the non-negative integers defined by:

$$P_{\varepsilon} P_{\delta_n} = \sum_{\lambda} f^{\lambda}_{\varepsilon,\delta_n} P_{\lambda}.$$

Proof. By Theorem 4.7, the folded Q-function is independent of the main diagonal, so we can take $\mathfrak{d}$ to be completely above the shape. In this case the semi-standard fillings of $(\gamma, \mathfrak{d})$ are exactly the semi-standard fillings of the diagram γ seen as a skew-shifted shape. □

As the folded P function depends on the size of the main diagonal, it is not independent of $\mathfrak{d}$. Instead, it tells us that $Q^f_{\gamma}(X)$ is divisible by 2^d, where d is the size of the longest diagonal on γ.

Corollary 4.9. *An unshifted shape and its conjugate have the same folded Q-function. In particular, for two shifted shapes α and β, $Q^f_{\alpha\diamond\beta}(X) = Q^f_{\beta\diamond\alpha}(X)$.*

Proof. For an unshifted shape γ, the conjugation operation gives a bijection of folded tableaux $(\gamma, \mathfrak{d})$ with $\mathfrak{d}$ above the shape and folded tableaux $(\gamma^T, \mathfrak{d}^T)$ with $\mathfrak{d}^T$ below the shape. As the folded Q-function is independent of the placement of the specialized diagonal, we have: $Q^f_{\gamma}(X) = Q^f_{\gamma^T}(X)$. □

5. Quotients of ribbon tableaux

In this section, we will introduce the k-quotient for a shifted diagram, and we will give a bijection between the k-ribbon tableaux and the fillings of its k-quotient. Our definition of the k-quotient extends the one given by Morris and Yaseen in [8] by specialized diagonals which we will use for a direct bijection between semi-standard k-ribbon tableaux and semi-standard fillings of its k-quotient.

Definition 5.1. *The k-quotient of a shifted shape λ with k-abacus representation $[\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}]$ will consist of $\lfloor k/2 \rfloor$ folded shapes and one shifted shape, defined as follows:*

If k is odd,
$\overline{\Phi^k}(\lambda) = (\alpha^{(1)} \diamond \alpha^{(k-1)}, \alpha^{(2)} \diamond \alpha^{(k-2)}, \ldots, \alpha^{((k-1)/2)} \diamond \alpha^{((k+1)/2)}, \alpha^{(k)})$,

If k is even,
$\overline{\Phi^k}(\lambda) = (\alpha^{(1)} \diamond \alpha^{(k-1)}, \alpha^{(2)} \diamond \alpha^{(k-2)}, \ldots, \alpha^{(k/2-1)} \diamond \alpha^{(k/2+1)}, \alpha^{(k/2)} \diamond \varnothing, \alpha^{(k)})$.

The strict partition $\lambda = (16, 11, 10, 9, 8, 7, 5, 4, 3, 1)$ has the 5-quotient:

$$\overline{\Phi^5}(\lambda) = ((4,3,1) \diamond (2,1), (2) \diamond (2,1), (2,1)) = (((3,3,2), \mathfrak{d}_1), ((3), \mathfrak{d}_2), (2,1)).$$

where $\mathfrak{d}_1$ and $\mathfrak{d}_2$ are the specialized diagonals given by the combination operation.

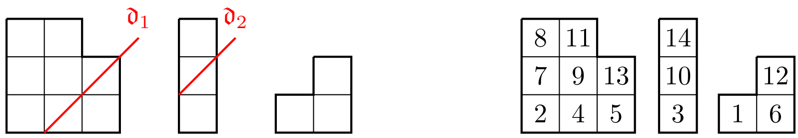


Figure 17: The 5-quotient of $\lambda = (16, 11, 10, 9, 8, 7, 5, 4, 3, 1)$ and a standard filling.

We call a simultaneous semi-standard filling of the $\lfloor k/2 \rfloor$ folded shapes and the shifted shape a *semi-standard filling* of the k-quotient. If this filling uses each number from 1 to n exactly once, unmarked, it will be called a *standard filling*. Next, we develop a bijection between the fillings of the ribbon tableaux with the fillings of its quotient.

Theorem 5.2. *There is a bijection Φ^k_λ between standard k-ribbon tableaux of shape λ and standard fillings of its k-quotient preserving diagonal values. That is, two ribbons that have the same diagonal value will be mapped to the same shape and diagonal in the quotient.*

An instance of this bijection can be seen in Figure 19. Notice that the ribbons 1 and 12 that are matched to the main diagonal of $\alpha^{(5)}$ are exactly those whose heads lie on diagonal 5, and the second diagonal of $\alpha^{(5)}$ is matched to ribbon 6 with diagonal value 10. To prove Theorem 5.2, first consider a k-ribbon tableau T of shape λ with abacus representation $[\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}]$. As each ribbon corresponds to a move in the abacus representation of λ, T uniquely corresponds to a sequence of moves from $[\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}]$ to the k-core of λ. As we can move independently on each runner pair (a_i, a_{k-i}) and on a_k, it will suffice to match the moves on a_k moves to shifted tableaux of shape $\alpha^{(k)}$, and the moves on runner pairs (a_i, a_{k-i}) to moves on $\alpha^{(i)} \diamond \alpha^{(k-i)}$for each i.

Lemma 5.3. *There is a bijection between sequences of moves from a_k to the empty runner and standard shifted tableaux of shape $\alpha^{(k)}$, where a move from row d to $d-1$ on the abacus corresponds to a box on diagonal d.*

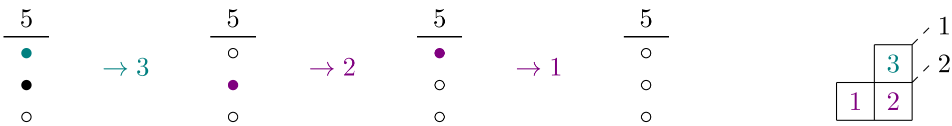


Figure 18: Moves on runner a_5 can be matched to a standard filling of the shifted diagram $\alpha^{(5)} = (2, 1)$.

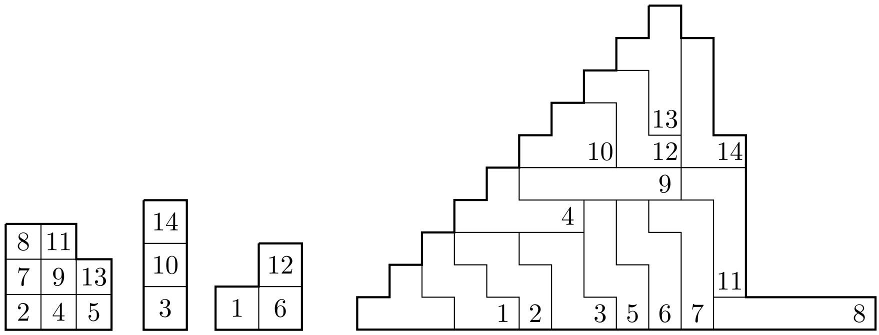


Figure 19: A 5-ribbon tableau with the corresponding filling of its 5-quotient.

Proof. A bead on the ith row of runner a_k will make a total of i moves, $i-1$ to one row higher, and one last move to be removed. We map these moves to a row of i boxes so that the move from position j to $j-1$ will correspond to a box on diagonal j, and the removal move will correspond a box on the main diagonal. For example, in Figure 18 the top bead can make only one move which is matched to a row of size one on $\alpha^{(5)}$. The lower bead has two moves to make which are matched to a row of size two.

In general, if a_k has beads on positions $i_1 > i_2 > \cdots > i_t$ we map the moves to the shifted diagram $\alpha^{(k)} = (i_1, i_2, \ldots, i_t)$. Let us number the moves in decreasing order with numbers from 1 to $|\alpha^{(k)}|$. This will give us a filling of $\alpha^{(k)}$, with the conditions that beads can only move to unoccupied positions (meaning rows need to increase left to right), and a bead can only move higher (meaning columns increase bottom to top). Note that these conditions exactly correspond to the tableaux conditions. $\square$

Now, we can turn our attention to runner pairs a_i, a_{k-i}.

Lemma 5.4. *There is a bijection between sequences of moves from the pair of runners (a_i, a_{k-i}) to the abacus core and standard unshifted tableau of shape $\alpha^{(i)} \diamond \alpha^{(k-i)}$, where moves removing beads are mapped to the specialized diagonal of $\alpha^{(i)} \diamond \alpha^{(k-i)}$, and moves on runner a_i (resp. a_{k-i}) from row r to $r-1$ are mapped to the diagonal d units to the left (resp. right).*

Proof. The move sequences on each runner can be matched to a shifted tableau of corresponding shape as in Claim 1, except now we have an additional constraint: To remove a bead from the first row one runner, we must simultaneously remove a bead from the first row of the second runner. This implies that the main diagonals of the two shapes must contain the same numbers, and they will be on the main diagonal, so it makes sense to transpose on of the shapes and paste them along main diagonals. An example of this can be seen in Figure 21. Note that the moves on a_1 are to the left of the special diagonal, and the moves on a_4 are to the right. Move 7 of removing two top row beads simultaneously falls on the special diagonal.

Also note that if one runner has q more beads than the other one, these can not be removed, and the moves which depend on the removal of these beads can not be made, which means the the subdiagram of shape δ_q inside the larger shape will be left empty. Deleting the corresponding boxes gives us a filling of $\alpha^{(i)} \diamond \alpha^{(k-i)}$. ☐

When k is even, we can move ribbons up on runner $a_{k/2}$ but not remove them, as if it has a conjugate runner with no beads. Theorem 5.2 now follows as any remark we made on $\alpha^{(i)} \diamond \alpha^{(k-i)}$ above automatically applies to $\alpha^{(k/2)} \diamond \varnothing$.

Remark 5.5. *The correspondence of diagonals gives us a way of labeling the diagonals of the quotient to match the values of the original shape. This way, the shifted shape $\alpha^{(k)}$ will have diagonals $0, k, 2k\dots$ and the folded shapes $\alpha^{(i)} \diamond \alpha^{(k-i)}$ will have diagonals: $\{\dots i+2k, i+k, k-i, 2k-i, 3k-i\dots\}$ where the specialized diagonal $\mathfrak{d}_i$ will have the diagonal value $k-i$.*

Note that the diagonal values $i < (k-1)/2$ do not appear. The reason of this is our convention of setting the head of double ribbon to be the head of the larger piece.

Corollary 5.6. *A k-ribbon R has a box on the main diagonal of λ if and only if $diag(R) \leq k$.*

Definition 5.7. *Consider a semi-standard k-ribbon tableaux T of shape λ, with $|\lambda| = n$. The* standardization *of T, denoted $St(T)$ is the standard k-ribbon tableaux of shape λ that we obtain by the following numbering:*

- *We number the boxes in the order $1' < 1 < 2' < 2 < \cdots$*
- *If there is more than one box of label i, we order them so that their diagonal values will be increasing.*
- *If there is more than one box of label i', we order them so that their diagonal values will be decreasing.*

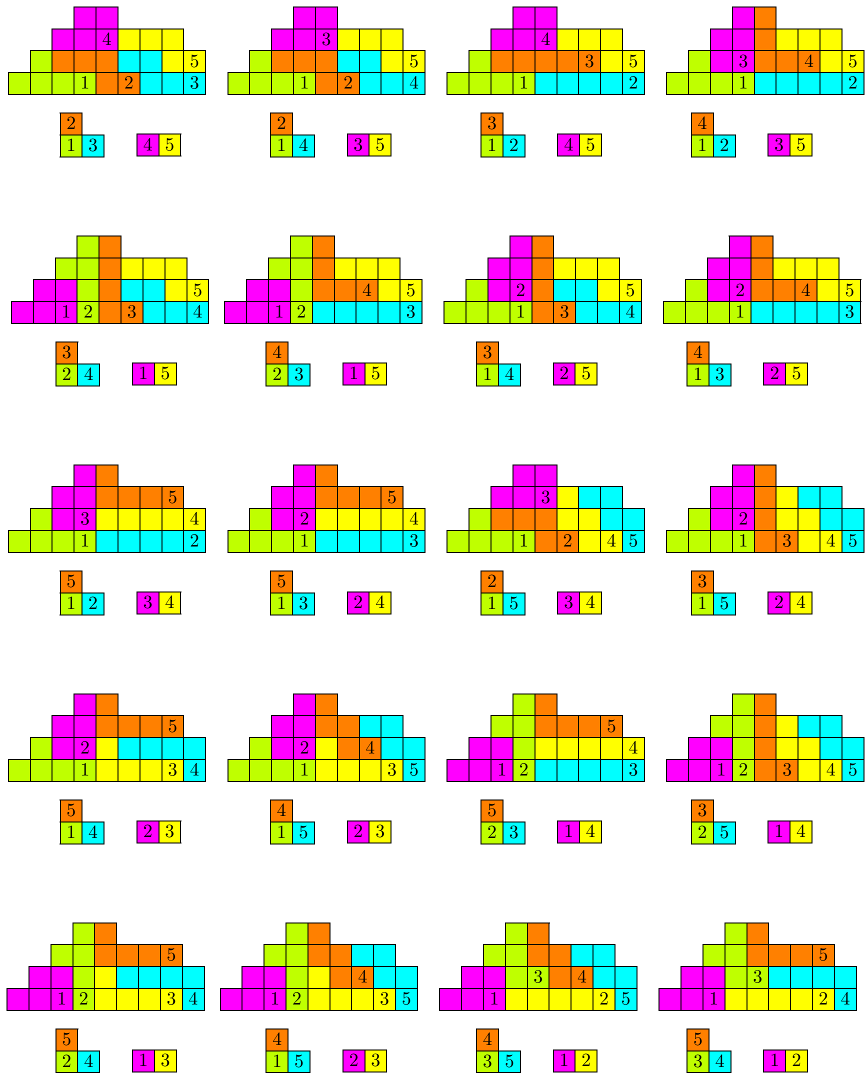


Figure 20: Correspondence of the standard 5-ribbon tableaux of shape $(9, 8, 6, 2)$ and the standard fillings of its 5- quotient $[(2, 1), (2), \varnothing]$.

Proposition 5.8. *$St(T)$ is well defined.*

Proof. As ribbons labeled i form a horizontal strip, they can be removed in the order diagonals are decreasing. Similarly, ribbons labeled i' form a vertical strip and can be removed in the order their diagonals are increasing. $\square$

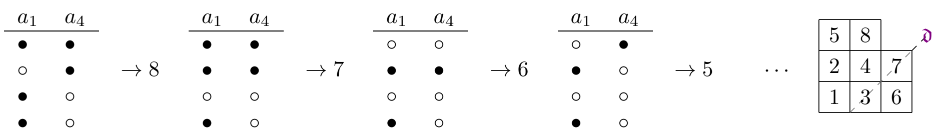


Figure 21: Moves on runners a_1 and a_4 give a standard filling of the folded diagram $\alpha^{(1)} \diamond \alpha^{(4)}$.

Using the labeling of the diagonals from Remark 5.5, we can also do the same standardization operation on the semi-standard fillings of the k-quotient.

Proposition 5.9. *We can extend Φ^k_λ to a weight preserving bijection between semi-standard k-ribbon tableaux of shape λ, and semi-standard fillings of its k-quotient.*

Proof. Let T be a semi-standard k-ribbon tableaux of shape λ, given by the sequence $\lambda_0 \subset \lambda_{1'} \subset \lambda_1 \subset \lambda_{2'} \subset \lambda_2 \subset \cdots \subset \lambda_t = \lambda$ of shifted diagrams. As our definition of standardization respects the inclusion order, $St(T)$ restricted to any λ_i gives a standardization of λ_i. The same is true for $\lambda_{i'}$. Let us apply the Φ^k_λ to the standardization of T. This gives us a bijection ϕ between ribbons of T and the boxes on the k-quotient. This also can be restricted to the subdiagrams λ_i and λ'_i, giving a sequence $\Phi^k(St(\lambda_0)) \subset \Phi^k(St(\lambda_{1'})) \subset \Phi^k(St(\lambda_1)) \subset \cdots \subset \Phi^k(St(\lambda_t)) = \Phi^k(\lambda)$. Here, the subset relation is defined pointwise in the $(k+1)$-tuples of quotient diagrams. We first claim that the filling of the k-quotient obtained by this is a semi-standard filling, and is equal to $\Phi^k_\lambda(T)$ if the filling T is standard. The second part of the claim follows from the definition of $\Phi^k_\lambda(T)$. For the first part, we need to show that the filling of each $a^{(i)} \diamond a^{(k-i)}$ gives a semi-standard folded shape and the filling of $a^{(k)}$ gives a semi-standard shifted shape. Let us look first at the case of $a^{(k)}$. Assume the ribbons R_1 and R_2 are both labeled j and the corresponding boxes B_1 and B_2 in the quotient are on the same column, with B_1 higher. Then we have $diag(B_1) < diag(B_2)$, implying $diag(R_1) < diag(R_2)$, which can not happen as the standardization algorithm would give B_1 a smaller number than B_2. they form a horizontal strip and have different diagonal values. Let us assume $diag(R_1) < diag(R_2)$. Then, if the boxes corresponding to R_1 and R_2 on the quotient are in the same column, the box for R_2 is higher. Symmetrically, no two boxes marked j' can be on the same row, so we indeed have a semi-standard filling of $a^{(k)}$. Now consider the boxes marked j on $a^{(i)} \diamond a^{(k-i)}$ in some i. As they come from the difference $\Phi^k(St(\lambda_j))\backslash\Phi^k(St(\lambda_{j'}))$, they form a skew shape. Also, the boxes that are labeled j to the right of the main diagonal form a horizontal strip by the same reasoning in the case of λ. The boxes labeled j to the left of the main diagonal form a vertical strip, as we have the inverted version of the same rules. The j' case is again symmetrical.

Now let us define the inverse of this operation. Given a semi-standard filling $\bar{T}$ of the k-quotient, as the k-quotient has the diagonal values induced by λ, we can apply the same standardization algorithm to the quotient, to get a standard filling $St(\bar{T})$ of the quotient. Applying Φ_λ^{k-1} to this filling gives a standard filling of λ. We can use this bijection between boxes of the quotient and ribbons of λ to carry the labels in $\bar{T}$ to the corresponding ribbons in λ. Note that, this inverts the above operation by definition. It remains to show that the inverse operation takes $\bar{T}$ to a semi-standard filling of λ. Let R and S be two ribbons marked j on λ. We will show that they form a horizontal strip. The case of j' is symmetrical. First note that we can not have $diag(R) = diag(S)$, as that would imply the corresponding boxes in the quotient are both in the same $a^{(i)} \diamond a^{(k-i)}$ (or both in $a^{(k)}$) on the same diagonal, which is not possible. Let us assume, without loss of generality, that $diag(R) > diag(S)$. Then, in the standardization, the label of R will be higher than the label of S, meaning R can be removed before S: $H(R)$ can not be below $H(S)$ in the same column. In this case $H(R)$ is strictly to the right of $H(S)$ implying they form a horizontal strip, as otherwise $H(R)$ would be strictly below $H(S)$ in a row strictly to the left, giving us no possible way to label the ribbon containing the box C in the same row as $H(R)$ and the same column as $H(S)$. □

This bijective relationship shows that the k-ribbon Q-function is equal to the product of the Q-functions of its quotient:

Theorem 5.10. *The k-ribbon Q-function of a shifted shape λ with k-abacus representation $[\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}]$ has the following expansion in terms of Schur's Q-functions:*

$$RQ_\lambda^{(k)}(X) = Q_{a^{(k)}}(X) \prod_{i \leq \lfloor k/2 \rfloor} Q_{\mu_i}(X) \tag{7}$$

where μ_i is the underlying skew-unshifted shape of $a^{(i)} \diamond a^{(k-i)}$ if $i < k/2$ and $a^{(k/2)} \diamond \varnothing$ if $i = k/2$.

Corollary 5.11. *Q-ribbon functions expand positively into Schur's Q-functions.*

Proof. This follows from the last theorem and the Schur Q-positivity of the skew-shifted Schur Q-functions (Theorem 2.2). □

Note that the Schur's Q-functions are themselves k-ribbon Q-functions for any k, as $k\lambda = (k\lambda_1, k\lambda_2, \dots, k\lambda_n)$ has $RQ^{(k)}_{k\lambda}(X) = Q_\lambda$.

6. Peak functions of ribbon tableaux

The *reading word* of a shifted k-ribbon tableau is a reading of the labels on the heads of the ribbons, left to right, top to bottom.

Definition 6.1. *A* marked standard shifted k-ribbon tableau T' *of shape λ is defined to be a standard shifted k-ribbon tableau T of shape λ together with a subset M of $[n]$ determining the marked coordinates.*

We identify $T' = (T, M)$ with the diagram of T where the label of R_i is replaced with i' for all i in M, as in Figure 22. We also let $\mathrm{Mark}(T) = \{(T, M) \mid M \subset [n]\}$ denote the set of the 2^n marked versions of T.

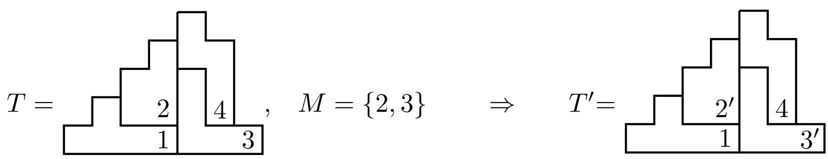


Figure 22: The marked 5-ribbon tableau T' has reading word $2'413'$.

Theorem 6.2. *The k-ribbon Q-function of a shifted shape λ we have can be written in terms of descent functions and peak functions as follows:*

$$
\begin{aligned}
RQ^{(k)}_\lambda(X) &= \sum_{T' \in \mathrm{SShT'}^{(k)}(\lambda)} F_{\mathrm{Des}(T')}, \\
RQ^{(k)}_\lambda(X) &= \sum_{T \in \mathrm{SShT}^{(k)}(\lambda)} 2^{|\mathrm{Peak}(\mathrm{Des}(T))|+1} G_{\mathrm{Peak}(\mathrm{Des}(T))}.
\end{aligned}
$$

where $\mathrm{SShT'}^{(k)}(\lambda)$ is the set of marked standard shifted k-ribbon tableaux of shape λ.

We postpone the proof of Theorem 6.2 to the end of this section to develop some necessary machinery. A *run* of a subset D of $[n]$ is a maximal subset of consecutive numbers. We will denote by $\mathrm{Run}(D)$ the set of the runs of D. Note that D is the disjoint union of its runs. For example, $D = \{2, 3, 5, 8, 9, 10\} = \{2, 3\} \cup \{5\} \cup \{8, 9, 10\}$.

Proposition 6.3. *We can calculate the number of the peaks of a tableau T from its descent set as follows:*

$$|\mathrm{Peak}(T)| = \begin{cases} |\mathrm{Run}(\mathrm{Des}(T))| & 1 \notin \mathrm{Des}(T) \\ |\mathrm{Run}(\mathrm{Des}(T))| - 1 & 1 \in \mathrm{Des}(T) \end{cases}.$$

Proof. This follows from the fact that the elements j of the peak set satisfy $j \in D$ and $j-1 \notin D$ for all $j > 1$, so that elements of the peak set are given by the smallest elements of each run, with the exception of the case if there is a run starting with 1. □

Lemma 6.4. *For any $T' \in \mathrm{Mark}(T)$, the descent set of T' is independent of whether a given $i \leq n$ is marked if and only if:*

- *$i > 1$ with $i-1 \in \mathrm{Des}(T)$, $i \notin \mathrm{Des}(T)$ or*
- *$i = 1 \notin \mathrm{Des}(T)$.*

The number of such i is given by $|\mathrm{Peak}(T)| + 1$.

Proof. The first part comes from Lemma 2.1. Note that as $\mathrm{Des}(T) \subset [n-1]$, the number 1 larger than the highest descent satisfies this condition. The other numbers $i > 1$ that satisfy these conditions are the i one lower than the lowest number of a run of $\mathrm{Des}(T)$. That means, if $1 \notin \mathrm{Des}(T)$, we have as many as the number of runs of those. If $1 \in \mathrm{Des}(T)$, we have $|\mathrm{Run}(\mathrm{Des}(T))| - 1$ of those and 1 is a descent. In both cases, there are $|\mathrm{Peak}(T)| + 1$ numbers in total that satisfy the conditions. □

Proposition 6.5. *For any $T' \in \mathrm{Mark}(T)$, we have $\mathrm{Spike}(T') \supset \mathrm{Peak}(T)$.*

Proof. Note that if $i \in \mathrm{Peak}(\mathrm{Des}(T))$, we have $i \in \mathrm{Des}(T)$ and $i-1 \notin \mathrm{Des}(T)$. For any given T' if i is unmarked on T', then $i \in \mathrm{Des}(T')$ and $i-1 \notin \mathrm{Des}(T')$ so $i \in \mathrm{Spike}(T')$. Otherwise i is marked, so that we have $i \notin \mathrm{Des}(T')$ and $i-1 \in \mathrm{Des}(T')$, implying again that $i \in \mathrm{Spike}(T')$. □

The proposition above shows that the descent map takes the elements of $\mathrm{Mark}(T)$ to subsets D of $[n-1]$ with $\mathrm{Spike}(D) \subset \mathrm{Peak}(T)$. Next, we will show that this map is surjective. In fact, we will prove the stronger statement that the preimage of every element is of the same size.

Lemma 6.6. *Assume D is a subset of $[n-1]$ satisfying $\mathrm{Spike}(D) \supset \mathrm{Peak}(T)$. Then, there is a marked version T' of T such that $\mathrm{Des}(T') = D$.*

Proof. Let us generate a marked version T' of T as follows: Starting with $i = 1$, at Step i we mark i if $i \in \mathrm{Des}(T)$ and $i \notin D$, and we mark $i+1$ if $i \notin \mathrm{Des}(T)$ and $i \in D$ (marking the same number a second time has no effect). Then we move on to the next number, until we have gone through all $i \leq n-1$.

Let us verify that the descent set of T' is indeed equal to D. For a fixed i assume $i \in \mathrm{Des}(T)$. Then by Lemma 2.1 i is a descent of T' iff i is unmarked. Therefore, it is sufficent to show i is unmarked iff $i \in D$. If $i \notin D$, then we marked i on Step i, so $i \in \mathrm{Des}(T')$. Otherwise, $i \in D$, and we can only have marked i at step $i-1$. This implies $i-1 \notin \mathrm{Des}(T)$ and $i-1 \in D$. This contradicts our assumption $\mathrm{Spike}(D) \supset \mathrm{Peak}(T)$ as i is a peak of T but not a spike of T. The case $i \notin \mathrm{Des}(T)$ is similar. ☐

Proposition 6.7. *The descent map taking elements of* $\mathrm{Mark}(T)$ *to subsets* D *of* $[n-1]$ *with* $\mathrm{Spike}(D) \supset \mathrm{Peak}(\mathrm{Des}(T))$ *is a* 2^m *to one cover, where* $m = |\mathrm{Peak}(\mathrm{Des}(T))| + 1$.

Proof. The number of subsets D of $[n-1]$ with $\mathrm{Spike}(D) \supset \mathrm{Peak}(\mathrm{Des}(T))$ is given by $2^{n-1-|\mathrm{Peak}(\mathrm{Des}(T))|}$. By Lemma 6.6, we know that the descent map is surjective. By Lemma 6.4, the preimage of each element under the descent map contains at least 2^m elements. $2^m \times 2^{n-1-|\mathrm{Peak}(\mathrm{Des}(T))|} = 2^n$ which is the cardinality of $\mathrm{Mark}(T)$, so the preimage of each element must contain exactly 2^m elements. ☐

Now we are ready to prove Theorem 6.2 from the beginning of the section.

Proof of Theorem 6.2. Let S be a semi-standard k-ribbon tableaux of shape λ. We have already defined the standardization of S (Definition 5.7). Let us denote by $St'(S)$ the marked standardization of S, which is simply formed by starting with $St(S)$ and then marking all boxes that were marked in S. We will show that there is a bijection $\phi_{T'}$ between semi-standard k-ribbon tableaux S that standardize to T' and the compositions C refining $\mathrm{Des}(T')$, satisfying $X^{|C|} = X^{|\phi_{T'}(S)|}$. This will imply:

$$\begin{aligned}
\sum_{T' \in \mathrm{SShT'}^{(k)}(\lambda)} F_{\mathrm{Des}(T')}(X) &= \sum_{T'} \sum_{C \,\mathrm{refines}\, \mathrm{Des}(T')} X^C = \sum_{T'} \sum_{C} X^{|\phi_{T'}^{-1}(C)|} \\
&= \sum_{S \in \mathrm{SsShT}^{(k)}(\lambda)} X^{|S|} = RQ_\lambda^{(k)}(X)
\end{aligned}$$

where for a combination $C = (c_1, c_2, \ldots, c_t)$ we use X^C to denote $x_1^{c_1} x_2^{c_2} \cdots x_t^{c_t}$.

Assume S satisfies $St'(S) = T'$. We define $\phi_{T'}(S) = (i_1, i_2, \ldots)$ where i_m stands for the total number of boxes labelled m or m' on S. As S has n ribbons, $\phi_{T'}(S)$ will be a combination of n that satisfies $x^{|S|} = x^{|\phi_{T'}(S)|}$.

We claim that S refines $\mathrm{Des}(T')$. Consider the pre-image of ribbon R_i (the unique ribbon labeled i or i' on T') under St'. We will denote the label of this ribbon in S by $St'^{-1}(i)$. To prove that S refines $\mathrm{Des}(T')$, it is sufficient to show that if i is a descent of T', then $St'^{-1}(i)$ and $St'^{-1}(i+1)$ are not both elements of $\{m, m'\}$ for any m (Note that, by the standardization algorithm, we will have $St'^{-1}(i) \leq St'^{-1}(i+1)$ in any case).

Let i be a descent of T'. By Lemma 2.1, there are two possiblities:

- Case 1, $i \in \mathrm{Des}(T)$ and i is not marked in T': Then $St'^{-1}(i)$ is an unmarked number m. $St'^{-1}(i) \geq m$, so it can not be m'. Assume it is also m. Then, we have two ribbons labeled m, but $diag(R_i) > diag(R_{i+1})$ by the definition of standardization (Definition 5.7).
- Case 2, $i \notin \mathrm{Des}(T)$ and $i+1$ is marked in T': This means $St'^{-1}(i+1)$ is a marked number m', and $St'^{-1}(i) \geq m'$ can not be m. It can not be m' either, because as in the first case, we get two ribbons labeled m' but $diag(R_i) < diag(R_{i+1})$, which as in Case 1, contradicts the definition of standardization.

Next, we will show that for any combination C refining $\mathrm{Des}(T)$ there is a unique S that standardizes to T' with $\phi_{T'}(S) = C$. Let $C = (c_1, c_2, \ldots, c_T)$ be a combination of n that refines $\mathrm{Des}(T')$. We will define S by labeling the ribbons R_1 to R_{c_1} with 1, ribbons R_{c_1+1} to $R_{c_1+c_2}$ with 2, $R_{c_1+c_2+1}$ to $R_{c_1+c_2+c_3}$ with 3 and so on, and then marking the image of R_i iff i is marked in T. We need to show that this S is semi-simple, and it standardizes to T'. Uniqueness then, comes from the fact that the placement of the markings are preserved.

Assume ribbons R_i and R_{i+1} have the same unmarked label m. Then, $i \notin \mathrm{Des}(T')$ and i and $i+1$ are both not labeled in T', so we must have $i \notin \mathrm{Des}(T)$ by Lemma 2.1. That means, $diag(R_i) < diag(R_{i+1})$, so unmarked numbers are ordered so that their diagonals will increase in T'. Similarly, if R_i and R_{i+1} both have the same marked label m', then $i \in \mathrm{Des}(T')$ and $diag(R_i) > diag(R_{i+1})$. These mean that we have $St'(S) = T$.

Additionally, we can remove ribbon R_{i+1} before R_i. This implies that if both ribbons are labeled m, $H(R_{i+1})$ is going to be strictly to the right of $H(R_i)$ as $diag(R_i) < diag(R_{i+1})$. If they are both labeled m', $H(R_{i+1})$ is going to be strictly above $H(R_i)$ as $diag(R_i) > diag(R_{i+1})$.

This proves the expansion of the k-ribbon function in terms of descent functions. The peak function expansion follows by Proposition 6.7:

$$\begin{aligned} RQ_{\lambda}^{(k)}(X) &= \sum_{T' \in \mathrm{SShT}'^{(k)}(\lambda)} F_{\mathrm{Des}(T')} = \sum_{T} \sum_{T' \in \mathrm{Mark}(T)} F_{\mathrm{Des}(T')} \\ &= \sum_{T} 2^{|\mathrm{Peak}(\mathrm{Des}(T))|+1} G_{\mathrm{Peak}(\mathrm{Des}(T))}. \end{aligned}$$ □

7. Shifted LLT polynomials

In [5], Lascoux, Leclerc and Thibon give a q-analogue for the k-ribbon that is Schur positive functions for the unshifted case. For this, they use the spin statistic on ribbon tableaux which depends on the total height of its ribbons. In this section, we show that there is no direct way of extending the concept of height to double ribbons that will give positive structure coefficients in the shifted case. Nevertheless, we are able to give a non-trivial q-analogue for the shifted ribbon functions and prove its Schur Q-positivity.

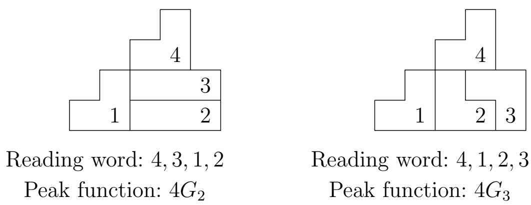
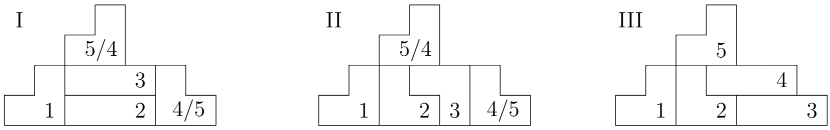


Figure 23: The two 3-ribbon tableaux of shape $\{5, 4, 2, 1\}$ and their corresponding peak functions.

Theorem 7.1. *There is no "intrinsic" definition for the height of a double ribbon, which, along with the usual definition of heights for the single ribbon, gives a Schur Q-positive or even a Schur positive function. Here by intrinsic, we mean there is no definition that only comes from the shape of the double ribbon, and is independent of its placement or the other ribbons in the shape.*

Proof. If we consider the example in Figure 23, we can see that the only difference between the two fillings is the placement of ribbons 2 and 3. For any intrinsic definition, the heights of the double ribbons 4 and 1 would match in the two fillings. The total height of 2 and 3 being higher on the shape on the right, we get a function $4q^cG_2 + 4q^dG_3$ with $c \neq d$ which is not Schur P-positive. In fact, G_2 and G_3 by themselves are not even Schur positive or symmetric functions. □

Another example where all the tableaux need to have the same cospin to obtain Schur Q-positivity is given in Figure 24. A common point of these



Reading words: 5, 3, 1, 2, 4
4, 3, 1, 2, 5
Peak function: $8G_{\{2,4\}} + 4G_{\{2\}}$

Reading words: 5, 1, 2, 3, 4
4, 1, 2, 3, 5
Peak function: $4G_{\{4\}} + 4G_{\{3\}}$

Reading word: 5, 4, 1, 2, 3
Peak function: $4G_{\{3\}}$

Figure 24: Standard 3-ribbon tableaux of shape $(7, 5, 2, 1)$ and their corresponding peak functions.

two examples with trivial cospin is that both have only one piece in their 3-quotient, which motivates a slightly technical q-analogue defined through the quotient.

Definition 7.2. *For a shifted shape λ with k-abacus representation*

$$\lambda = [\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}],$$

we define the q-analogue of the shifted k-ribbon Q-function as follows:

$$RQ_\lambda^{(k)}(X; q) = Q_{\alpha^{(k)}}(X) \sum_{T \in SRT_{\lfloor k/2 \rfloor}(\mu)} q^{cospin(T)} 2^{|\mathrm{Peak}(T)|+1} F_{\mathrm{Peak}(T)}(X) \tag{8}$$

where μ is the unshifted partition corresponding to the $\lfloor k/2 \rfloor$-quotient

$$(\mu^1, \mu^2, \ldots, \mu^{\lfloor k/2 \rfloor}),$$

with $\mu^i = a^{(i)} \diamond a^{(k-i)}$ if $i < k/2$ and $\mu_{k/2} = a^{(k/2)} \diamond \varnothing$ when k is even.

Note that when $q = 1$, we get the formulation of $RQ_\lambda^{(k)}(X)$ given in Equation 7, so $RQ_\lambda^{(k)}(X; 1) = RQ_\lambda^{(k)}(X)$ as desired.

Theorem 7.3. *The function $RQ_\lambda^{(k)}(X; q)$ expands into Schur's Q-functions with coefficients from $\mathbb{Z}^+[q]$.*

Proof. Let $[\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}]$ be the k-abacus representation for λ, and μ is the be the unshifted partition corresponding to the $\lfloor k/2 \rfloor$-quotient $(\mu^1, \mu^2, \ldots, \mu^{\lfloor k/2 \rfloor})$ as defined in Theorem 7.3. The LLT polynomial for μ satisfies

$$GF_{\mu/\mu_0}^{(\lfloor k/2 \rfloor)}(X; q) = \sum_{T \in SRT_{\lfloor k/2 \rfloor}(\mu)} q^{cospin(T)} F_{\mathrm{Des}(T)}(X) = \sum f_\gamma^\mu(q) s_\gamma(X),$$

where γ are unshifted shapes and $f_\gamma^\mu(q)$ have positive integer coefficients. As any unshifted γ can be seen as a skew-shifted shape $\gamma^+/\delta_{\ell(\gamma)}$. Multiplying by $2^{|\mathrm{Peak}(T)|+1} F_{\mathrm{Peak}(T)}(X)$ instead $F_{\mathrm{Des}(T)}(X)$ for each γ tableau T corresponds to calculating $Q_{\gamma^+/\delta_{\ell(\gamma)}}(X)$ instead of $s_\gamma(X)$. So we have:

$$
\begin{aligned}
RQ_\lambda^{(k)}(X;q) &:= Q_{\alpha^{(k)}}(X)\left(\sum_{T\in SRT_k(\mu)} q^{cospin(T)} 2^{|\mathrm{Peak}(T)|+1} F_{\mathrm{Peak}(T)}(X)\right) \\
&= Q_{\alpha^{(k)}}(X)\left(\sum f_\gamma^\mu(q) Q_{\gamma+/\delta_{\ell(\gamma)}}(X)\right).
\end{aligned}
$$

As multiplication and skewing operations on Schur's Q-functions give positive expansions into Schur's Q-functions the result follows. $\square$

Let us finish with calculating an example.

Consider the shape $(9,8,6,2)$ with the 5-ribbon quotient $\{(2,1),(2),\varnothing\}$ with standard fillings given in Figure 20. It has the Q 5-ribbon function:

$RQ^{(5)}_{(9,8,6,2)}(X) = (Q_{(3,1)/(1)} \cdot Q_{(3)/(1)})(X) = 2Q_{(5)}(X) + 4Q_{(4,1)}(X) + 3Q_{(3,2)}(X)$.

Viewed as a 2-quotient, $\{(2,1),(2)\}$ corresponds to unshifted shape $(4,4,1,1)$ with the LLT polynomial:

$$
\begin{aligned}
GF^{(2)}_{(4,4,1,1)}(X;q) &= \sum_{T\in SRT_2(4,4,1,1)} q^{cospin(T)} F_{\mathrm{Des}(T)}(X) \\
&= q^2 s_{(2,2,1)}(X) + q s_{(3,1,1)}(X) + q s_{(3,2)}(X) + s_{(4,1)}(X).
\end{aligned}
$$

Viewing $(2,2,1)$, $(3,1,1)$, $(3,2)$ and $(4,1)$ as skew shifted shapes, we get the following:

$$
\begin{aligned}
RQ^{(5)}_{(9,8,6,2)}(X;q) &= P_\varnothing(X)\big(q^2 Q_{(5,3,2)/(2,1)}(X) + q Q_{(5,4,1)/(2,1)}(X) \\
&\quad + q Q_{(4,2)/(1)}(X) + Q_{(5,1)/(1)}(X)\big) \\
&= (q+1)Q_{(5)}(X) + (q^2+2q+1)Q_{(4,1)}(X) \\
&\quad + (q^2+2q)Q_{(3,2)}(X).
\end{aligned}
$$

The Schur Q-positivity of $RQ_\lambda^{(k)}(X;q)$ implies a possible connection to representation theory. It might interesting to explore that connection in future work, as well as find a way to calculate the powers of q directly from the ribbons.

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Ezgi Kantarci Oğuz
Department of Mathematics
University of Southern California
Los Angeles, CA
USA
E-mail address: kantarci@usc.edu

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