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Kostant's weight multiplicity formula and the Fibonacci and Lucas numbers

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Consider the weight λ that is the sum of all simple roots of a simple Lie algebra $\mathfrak{g}$. Using Kostant's weight multiplicity formula we describe and enumerate the contributing terms to the multiplicity of an integral weight μ in the representation of $\mathfrak{g}$ with highest weight λ, which we denote by $L(\lambda)$. We prove that in Lie algebras of type A and B, the number of terms contributing a nonzero value in the multiplicity of the zero-weight in $L(\lambda)$ is given by a Fibonacci number, and that in the Lie algebras of type C and D, the analogous result is given by a multiple of a Lucas number. When μ is a nonzero integral weight we show that in Lie types A and B there is only one term contributing a nonzero value to the multiplicity of μ in $L(\lambda)$, and that in the Lie algebras of type C and D, all terms contribute a value of zero. We conclude by using these results to compute the q-multiplicity of an integral weight μ in the representation $L(\lambda)$ in all classical Lie algebras.

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1. Introduction

Let G be a simple linear algebraic group over $\mathbb{C}$, T a maximal algebraic torus in G of dimension r, and B, $T \subseteq B \subseteq G$, a choice of Borel subgroup. Then let $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{b}$ denote the Lie algebras of G, T, and B respectively. We let Φ denote the set of roots corresponding to $(\mathfrak{g}, \mathfrak{h})$, and $\Phi^+ \subseteq \Phi$ is the set of positive roots with respect to $\mathfrak{b}$. Let $\Delta \subseteq \Phi^+$ be the set of simple roots. The denote the set of integral and dominant integral weights by $P(\mathfrak{g})$ and $P_+(\mathfrak{g})$

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respectively. Let $W = Norm_G(T)/T$ denote the Weyl group corresponding to G and T, and for any $w \in W$, we let $\ell(w)$ denote the length of w.

We recall that with a choice of a Cartan subalgebra it is well known that the finite-dimensional irreducible representations of a Lie algebra $\mathfrak{g}$ on the vector space V can be studied by decomposing

$$V = \oplus V_\alpha \tag{1}$$

where the direct sum is indexed by a finite set of weights. Given a weight α, the corresponding subspace V_α is called a weight space and the dimension of V_α is called the multiplicity of α. Thus to study representations of $\mathfrak{g}$ it suffices to determine the multiplicity of the weights appearing in (1). For a more detailed account of this theory we refer the reader to [2].

In this work we consider the weight λ which is the sum of all simple roots of $\mathfrak{g}$. We formally use Kostant's weight multiplicity formula to compute the multiplicity of μ an integral weight in the representation with highest weight λ, which we denote by $m(\lambda, \mu)$. When μ is the zero weight, this representation is the adjoint representation in the Lie algebra of type A and the defining representation in type B; these cases were considered by Harris in [3] and [5], respectively. In the remaining Lie types it is a virtual representation: a representation arising from a nondominant integral highest weight.

One way to compute the multiplicity of a weight μ is via Kostant's weight multiplicity formula [9]:

$$m(\lambda, \mu) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \wp(\sigma(\lambda + \rho) - (\mu + \rho)), \tag{2}$$

where W denotes the Weyl group of $\mathfrak{g}$, $\wp$ denotes Kostant's partition function, and $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$, with Φ^+ denoting the set of positive roots of $\mathfrak{g}$. We recall that the Weyl group is generated by reflections about hyperplanes lying perpendicular to the simple roots of the Lie algebra $\mathfrak{g}$, and for each $\sigma \in W$, the length $\ell(\sigma)$ represents the minimum number k such that σ is a product of k reflections. Kostant's partition function $\wp : \mathfrak{h}^* \to \mathbb{Z}$ is the nonnegative integer-valued function such that for each $\xi \in \mathfrak{h}^*$, $\wp(\xi)$ counts the number of ways ξ may be written as a nonnegative $\mathbb{Z}$-linear combination of positive roots.

A challenge in using Equation (2) for weight multiplicity computations is the fact that the order of the Weyl group, indexing the sum, increases factorially as the rank of the Lie algebra considered increases. Additionally, many

Weyl group elements contribute trivially to the alternating sum, thereby yielding another source of great inefficiency. In light of this, our work focuses on describing the elements of the Weyl group that contribute a nonzero term to the multiplicity formula, which leads to the following definition.

Definition 1. For λ, μ integral weights of $\mathfrak{g}$, we define the *Weyl alternation set* by

$$\mathcal{A}(\lambda,\mu) = \{\sigma \in W : \wp(\sigma(\lambda+\rho) - (\mu+\rho)) > 0\}. \tag{3}$$

The above definition implies that $\sigma \in W$ satisfies $\sigma \in \mathcal{A}(\lambda,\mu)$ if and only if $\sigma(\lambda+\rho)-(\mu+\rho)$ can be written as a nonnegative $\mathbb{Z}$-linear combination of positive roots.

Harris, Insko, and Williams described and enumerated the Weyl alternation sets for the zero weight in the adjoint representation of the classical Lie algebras and showed that the cardinality of these sets is given by linear recurrences with constant coefficients [4, 8]. In addition, Harris, Lescinsky, and Mabie have provided visualizations for the Weyl alternation sets for different pairs of integral weights λ and μ in the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ [4, 6].

Our research continues this work by describing and enumerating the elements of the Weyl alternation sets $\mathcal{A}(\lambda,\mu)$, where λ is the sum of all the simple roots of a simple Lie algebra and μ is an integral weight. We find that when μ is the zero weight the cardinality of these Weyl alternation sets in the Lie algebras of type A and B, are given by a Fibonacci number [5] and in the Lie algebras of type C and D, the analogous result is given by a multiple of a Lucas number. Our main results are summarized in Table 1, where F_r and L_r denote the r^{th} Fibonacci and Lucas numbers, respectively. We remark that the results of Table 1 for the exceptional Lie algebras is a finite computation that was verified using the computer implementation presented in [7]. In this work, we also consider the cases where μ is a nonzero integral weight and show that in Lie types A and B the sets $\mathcal{A}(\lambda,\mu)$ consist of only the identity element of the Weyl group, while in Lie types C and D the sets $\mathcal{A}(\lambda,\mu)$ are empty.

We note that these results give a glimpse into the complicated nature of weight multiplicity computations. Although our results establish that the number of terms contributing nontrivially to $m(\lambda, 0)$ is given by either a Fibonacci or a multiple of a Lucas number, thereby reducing the computation from a factorial number of terms, to a number that grows exponentially, and one cannot reduce the computation any further. However, this reduction is enough to allow the development of new formulas for the partition function involved in the weight multiplicity formula. We present such results in

Table 1: Summary of main results

Classical Lie Algebras	$\lvert\mathcal{A}(\lambda,0)\rvert$
A_r $(r \geq 1)$	F_r
B_r $(r \geq 2)$	F_{r+1}
C_r $(r \geq 5)$	$2L_{r-2}$
D_r $(r \geq 7)$	$2L_{r-3}$

Exceptional Lie Algebras	$\lvert\mathcal{A}(\lambda,0)\rvert$
G_2	2
F_4	4
E_6	12
E_7	18
E_8	30

Section 6 by working more generally with the q-analog of Kostant's weight multiplicity formula, a polynomial valued function that when evaluated at $q = 1$ recovers (2). Our results in this section establish the following.

Theorem 1.1. Let $\lambda = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ and μ be an integral weight of a simple Lie algebra $\mathfrak{g}$. If

- $\mathfrak{g} = \mathfrak{sl}_{r+1}(\mathbb{C})$, then $m(\lambda, \mu) = \begin{cases} r & \text{if } \mu = 0 \\ 1 & \text{if } \mu \in \Phi \\ 0 & \text{otherwise} \end{cases}$
- $\mathfrak{g} = \mathfrak{so}_{2r+1}(\mathbb{C})$, then $m(\lambda, \mu) = \begin{cases} 1 & \text{if } \mu = 0 \text{ or } \mu \in W \cdot \lambda \\ 0 & \text{otherwise} \end{cases}$
  where $W \cdot \lambda$ denotes the orbit of λ under the action of the Weyl group
- $\mathfrak{g} = \mathfrak{sp}_r(\mathbb{C})$ or $\mathfrak{so}_{2r}(\mathbb{C})$, then $m(\lambda, \mu) = 0$.

This work is organized as follows: In Sections 2-5 we consider a specific Lie algebra (in alphabetical order), provide needed background and present the results regarding the Weyl alternation sets $\mathcal{A}(\lambda, \mu)$ when μ is an integral weight and $\lambda = \alpha_1 + \alpha_2 + \cdots + \alpha_r$. The results in each of these sections is separated by whether μ is the zero weight, or a nonzero integral weight. Section 6 uses the results in the previous sections to compute the q-multiplicity of μ an integral weight in the representation with highest weight λ. Thereby establishing Theorem 1.1.

2. Lie algebra of type A

In this section, we consider the Lie algebra $\mathfrak{sl}_{r+1}(\mathbb{C})$ for $r \geq 2$. In this case, the set of simple roots is given by $\Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$, and the set of positive roots is given by $\Phi^+ = \Delta \cup \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j : 1 \leq i < j \leq r\}$. The weight ρ is defined as the half sum of the positive roots, $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$, which is equivalent to $\rho = \varpi_1 + \varpi_2 + \cdots + \varpi_r$, where $\lambda, \varpi_2, \ldots, \varpi_r$ are the fundamental weights of $\mathfrak{sl}_{r+1}(\mathbb{C})$. The Weyl group elements are generated by reflections about the hyperplanes that lie perpendicular to the simple roots α_i. We denote these simple reflections by s_i, where $1 \le i \le r$, whose action on the simple roots is defined by $s_i(\alpha_j) = \alpha_j$ if $|i-j| > 1$, $s_i(\alpha_j) = -\alpha_j$ if $i = j$, and $s_i(\alpha_j) = \alpha_i + \alpha_j$ if $|i-j| = 1$. The Weyl group elements act on the fundamental weights by $s_i(\varpi_j) = \varpi_j - \delta_{i,j}\alpha_i$, where $\delta_{i,j} = 1$ when $i = j$ and 0 otherwise. We separate the results of this section into the cases where μ is the zero weight and when it is a nonzero weight.

2.1. Zero weight space

We now state the main result of this section.

Theorem 2.1. Let $\mathfrak{g} = \mathfrak{sl}_{r+1}(\mathbb{C})$ with $r \ge 2$. Then $\sigma \in \mathcal{A}(\lambda, 0)$ if and only if $\sigma = 1$ or $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$ for some collection of nonconsecutive integers $2 \le i_1, i_2, \ldots, i_k \le r-1$.

Theorem 2.1 first appeared in [3, Proposition 2.1] and its proof used the fact that the Weyl group of $\mathfrak{sl}_{r+1}$ is isomorphic to the symmetric group $\mathfrak{S}_{r+1}$. Below we present a new proof using the fact that the Weyl group is generated by the root reflections $s_1, s_2, \ldots, s_r$. In particular, this proof technique illustrates the use of the root reflection action on $\lambda + \rho$,which provides us with a more direct style of proof..

Proof of Theorem 2.1. ($\Rightarrow$) We prove this by establishing the contrapositive. Suppose that σ is neither the identity nor $s_{i_1}s_{i_2}\cdots s_{i_k}$ for some nonconsecutive integers $2 \le i_1, i_2, \ldots, i_k \le r-1$. Then σ must contain s_1, or s_r, or s_is_j for consecutive integers i and j. If $\sigma = s_1$, then we have that $s_1(\lambda+\rho) - \rho = s_1(\lambda) + s_1(\rho) - \rho = \lambda - \alpha_1 + \rho - \alpha_1 - \rho = \lambda - 2\alpha_1$, which cannot be written as a sum of positive roots given the negative coefficient of α_1. Hence, $s_1 \notin \mathcal{A}(\lambda, 0)$. Now [8, Proposition 3.4] shows that if $\sigma \notin \mathcal{A}(\lambda, 0)$, then neither is any σ' containing σ in its reduced word expression. Thus any $\sigma \in W$ containing s_1 in its reduced word expression cannot be in $\mathcal{A}(\lambda, 0)$. Similarly, if $\sigma = s_r$, then we have that $s_r(\lambda+\rho) - \rho = s_r(\lambda) + s_r(\rho) - \rho = \lambda - \alpha_r + \rho - \alpha_r - \rho = \lambda - 2\alpha_r$, which cannot be written as a sum of positive roots because of the negative coefficient of α_r. This implies that $s_r \notin \mathcal{A}(\lambda, 0)$, and so any σ containing s_r in its reduced word expression is not in $\mathcal{A}(\lambda, 0)$.

Now suppose we have an arbitrary pair of consecutive integers i, $i+1$ such that $2 \le i < r-1$. Using the property that the action of Weyl group elements on weights behave linearly, we have that $s_is_{i+1}(\lambda+\rho) - \rho = s_i(\lambda + \rho - \alpha_{i+1}) - \rho = (\lambda + \rho - \alpha_i - s_i(\alpha_{i+1})) - \rho = \lambda - 2\alpha_i - \alpha_{i+1}$, which cannot be written as a nonnegative $\mathbb{Z}$-linear combination of the positive roots and, thus, $s_is_{i+1} \notin \mathcal{A}(\lambda, 0)$. Therefore any σ containing s_is_{i+1} as a subword in its reduced word expression cannot be in $\mathcal{A}(\lambda, 0)$. A similar argument shows that $s_{i+1}s_i \notin \mathcal{A}(\lambda, 0)$. Thus, if $\sigma \in \mathcal{A}(\lambda, 0)$, then $\sigma = 1$ or $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$ for some nonconsecutive integers $2 \leq i_1, i_2, \ldots, i_k \leq r-1$, as claimed.

($\Leftarrow$) If $\sigma = 1$, then $1(\lambda+\rho)-\rho = \lambda$. Hence $1 \in \mathcal{A}(\lambda, 0)$. If $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$ for some nonconsecutive integers $2 \leq i_1, i_2, \ldots, i_k \leq r-1$ we observe that

$$s_{i_1}s_{i_2}\cdots s_{i_k}(\lambda+\rho)-\rho = \lambda - \sum_{j=1}^{k}\alpha_{i_j} \tag{4}$$

which can be written as a sum of positive roots. Thus $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$ is in $\mathcal{A}(\lambda, 0)$. $\square$

Before stating our next result, we recall that the Fibonacci numbers follow the recurrence $F_r = F_{r-1} + F_{r-2}$ with $F_1 = F_2 = 1$.

Corollary 2.1. If $r \geq 2$ and λ is the highest root of $\mathfrak{sl}_{r+1}$, then $|\mathcal{A}(\lambda, 0)| = F_r$, where F_r denotes the r^{th} Fibonacci number.

The above result first appeared in [3, Theorem 2.1], but for sake of completeness we present a proof below, which uses the description of the elements of the Weyl group as products of root reflections.

Proof of Corollary 2.1. We proceed by induction. If $r = 2$, then by Theorem 2.1 we know $\mathcal{A}_2(\lambda, 0) = \{1\}$, which shows that $|\mathcal{A}_2(\lambda, 0)| = 1 = F_2$. If $r = 3$, then $\mathcal{A}_3(\lambda, 0) = \{1, s_2\}$, which shows that $|\mathcal{A}_3(\lambda, 0)| = 2 = F_3$. Assume that for all r, with $3 \leq r \leq k$, $|\mathcal{A}_r(\lambda, 0)| = F_r$. We consider the case when $r = k+1$. Notice that all of the elements $\sigma \in W$ consisting of nonconsecutive products of the generators $s_2, s_3, \ldots, s_k$ will either contain s_k or not. If they do not contain s_k, then by our induction hypothesis, the number of Weyl group elements consisting of nonconsecutive products of the generators $s_2, s_3, \ldots, s_{k-1}$ is given by F_k. If the Weyl group element contains s_k, then we must count the number of nonconsecutive products of the reflections $s_2, s_3, \ldots, s_{k-2}$, which by our induction hypothesis is given by F_{k-1}. Therefore $|\mathcal{A}_{k+1}(\lambda, 0)| = F_{k-1} + F_k = F_{k+1}$. $\square$

2.2. Nonzero weight spaces

For the Lie algebra of type A_r we consider the case where μ is a positive root of the Lie algebra was considered in the work of Harris [3], where the following result was established.

Theorem 2.2 (Theorem 4.1 [3])**.** If $\mu \neq 0$ is a dominant integral weight of $\mathfrak{sl}_{r+1}(\mathbb{C})$ and λ is the highest root, then $\mathcal{A}(\lambda,\mu) = \begin{cases} \{1\} & \text{if } \mu = \lambda \\ \emptyset & \text{otherwise.} \end{cases}$

3. Lie algebra of type *B*

In this section, we consider the Lie algebra $\mathfrak{so}_{2r+1}(\mathbb{C})$ for $r \geq 2$. Whenever $1 \leq i \leq r$, let ε_i denote the i^{th} standard basis vector in $\mathbb{R}^r$. If $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq r-1$ and $\alpha_r = \varepsilon_r$, then the set of simple roots of $\mathfrak{so}_{2r+1}(\mathbb{C})$ is given by $\Delta = \{\alpha_1, \dots, \alpha_r\}$ and the set of positive roots is given by

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j : \ 1 \leq i < j \leq n\} \cup \{\varepsilon_i : 1 \leq i \leq r\}.$$

The fundamental weights of $\mathfrak{so}_{2r+1}(\mathbb{C})$ are defined by $\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i$ for $1 \leq i \leq r-1$, $\varpi_r = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r)$, and $\rho = \varpi_1 + \cdots + \varpi_r$. Note that $\lambda = \alpha_1 + \alpha_2 + \cdots + \alpha_r = \varepsilon_1 = \varpi_1$.

The simple root reflections act on the simple roots and fundamental weights as follows. If $1 \leq i \leq r-1$, then $s_i(\alpha_i) = -\alpha_i$, $s_i(\alpha_{i-1}) = \alpha_{i-1} + \alpha_i$, $s_i(\alpha_{i+1}) = \alpha_i + \alpha_{i+1}$, and $s_r(\alpha_r) = -\alpha_r$, $s_r(\alpha_{r-1}) = \alpha_{r-1} + 2\alpha_r$. For any $1 \leq i, j \leq r$, $s_i(\varpi_j) = \varpi_j - \delta_{i,j}\alpha_i$.

We separate the results of this section into the cases where μ is the zero weight and when it is a nonzero weight.

3.1. Zero weight space

We begin with the following technical result for $\mathfrak{so}_{2r+1}(\mathbb{C})$.

Proposition 3.1. Let $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$ where the indices of the simple reflections form a collection of nonconsecutive integers $2 \leq i_1, \dots, i_k \leq r$. Then $\sigma(\lambda + \rho) - \rho = \lambda - \sum_{j=1}^k \alpha_{i_j}$ is a nonnegative $\mathbb{Z}$-linear combination of positive roots.

Proof. Let $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$ for some collection of nonconsecutive integers $2 \leq i_1, \dots, i_k \leq r$. Note that $\sigma(\lambda) = \lambda$, and $\sigma(\rho) = \rho - \sum_{j=1}^k \alpha_{i_j}$. Thus $\sigma(\lambda + \rho) - \rho = \lambda - \sum_{j=1}^k \alpha_{i_j}$ which is a nonnegative $\mathbb{Z}$-linear combination of positive roots. $\square$

Theorem 3.1. Let $\mathfrak{g} = \mathfrak{so}_{2r+1}(\mathbb{C})$ with $r \geq 2$. Then $\sigma \in \mathcal{A}(\lambda, 0)$ if and only if $\sigma = 1$ or $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$ for some nonconsecutive integers $2 \leq i_1, \dots, i_k \leq r$.

Proof. ($\Leftarrow$) Let $\sigma = 1$, then $1(\lambda + \rho) - \rho = \lambda$ is a nonnegative $\mathbb{Z}$-linear combination of positive roots, thus $1 \in \mathcal{A}(\lambda, 0)$, and Proposition 3.1 implies that if $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$ for some nonconsecutive integers $i_1, \dots, i_k$ between and including 2 and r, then $\sigma \in \mathcal{A}(\lambda, 0)$.

($\Rightarrow$) Suppose $\sigma \in \mathcal{A}(\lambda, 0)$. We proceed by induction on $\ell(\sigma)$. If $\ell(\sigma) = 0$, then $\sigma = 1$, which satisfies the needed condition. If $\ell(\sigma) = 1$, then $\sigma = s_i$ for some $1 \leq i \leq r$. If $i = 1$, then $s_1(\lambda + \rho) - \rho = \lambda - 2\alpha_1$, which implies $s_1 \notin \mathcal{A}(\lambda, 0)$, a contradiction. Thus, $\sigma \in \mathcal{A}(\lambda, 0)$ cannot contain s_1 in its reduced word expression. If $1 < i \leq r$, then $s_i(\lambda + \rho) - \rho = \lambda - \alpha_i$, and $s_i \in \mathcal{A}(\lambda, 0)$ and s_i is of the required form.

If $\ell(\sigma) = 2$, then $\sigma = s_i s_j$ for distinct integers i, j satisfying $1 < i, j \leq r$. Without loss of generality, assume $i < j$. If i, j are consecutive integers, then $i = j - 1$, with $1 < i, j < r$ or $i = r - 1$ and $j = r$. In either case we note $s_{j-1}s_j(\lambda+\rho)-\rho = \lambda-\alpha_i-2\alpha_j$ and $s_{r-1}s_r(\lambda+\rho)-\rho = \lambda-\alpha_{r-1}-3\alpha_r$ neither of which can be written as a nonnegative $\mathbb{Z}$-linear combination of positive root. Thus, $s_{r-1}s_r$, $s_r s_{r-1}$, $s_{j-1}s_j$, $s_j s_{j-1} \notin \mathcal{A}(\lambda, 0)$, a contradiction. Moreover, any $\sigma \in W$ containing $s_j s_{j-1}$ or $s_{j-1}s_j$ in its reduced word expression cannot be in $\mathcal{A}(\lambda, 0)$ for all $2 < j \leq r$. The case were i, j are consecutive was already considered in Proposition 5.1.

Suppose that for all $\sigma \in \mathcal{A}(\lambda, 0)$ with $1 < \ell(\sigma) \leq k$, there exists some nonconsecutive integers $2 \leq i_1, \ldots, i_{\ell(\sigma)} \leq r$ such that $\sigma = s_{i_1}s_{i_2}\cdots s_{i_{\ell(\sigma)}}$. Now consider $\tau \in \mathcal{A}(\lambda, 0)$ with $\ell(\tau) = k+1$. Then $\tau = s_l\sigma$ for some $2 \leq l \leq r$ and for some $\sigma \in W$ with $\ell(\sigma) = k$. Note that in fact $\sigma \in \mathcal{A}(\lambda, 0)$, as otherwise τ would not be in $\mathcal{A}(\lambda, 0)$, giving a contradiction. Hence, by our induction hypothesis there exist nonconsecutive integers $2 \leq i_1, i_2, \cdots, i_k \leq r$ such that $\sigma = s_{i_1}\cdots s_{i_k}$. By Proposition 3.1, $\sigma(\lambda+\rho) = \lambda + \rho - \sum_{j=1}^{k}\alpha_{i_j}$. Hence $\tau(\lambda + \rho) - \rho = s_l\sigma(\lambda+\rho) - \rho = \lambda - \alpha_l - \sum_{j=1}^{k} s_l(\alpha_{i_j}) = \lambda - \alpha_l - \sum_{j=1}^{k}(\alpha_{i_j} + c_{l,i_j}\alpha_l)$ where $c_{l,i_j} = 2$ if $i_l = r$ and $i_j = r-1$, $c_{l,i_j} = 0$ if $|l - i_j| > 1$ and $c_{l,i_j} = 1$ otherwise. Observe that whenever $c_{l,j_1} = 1$ or 2, the expression $\tau(\lambda+\rho) - \rho$ contains a negative coefficient on a simple root, and thus $\tau \notin \mathcal{A}(\lambda, 0)$, a contradiction. Therefore, $l, i_1, \cdots, i_k$ must be nonconsecutive integers between and including 2 and r. $\square$

Corollary 3.1. If $r \geq 2$ and $\lambda = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ is a fundamental weight of $\mathfrak{so}_{2r+1}(\mathbb{C})$, then $|\mathcal{A}(\lambda, 0)| = F_{r+1}$, where F_{r+1} denotes the $(r+1)^{th}$ Fibonacci number.

The proof of Corollary 3.1 is analogous to that of Corollary 2.1, hence we omit it.

We remark that the results in this section first appeared in an unpublished preprint of the second author as [5, Proposition 2.1, Theorem 2.1, and Theorem 1.1], respectively. However, the proofs presented in this current manuscript are new and, as in the previous section, they use the action of root reflections on $\lambda+\rho$ without using the definition of the root reflections involving the symmetric bilinear form on $\mathfrak{h}^*$ corresponding to the trace form as in [2].

3.2. Nonzero weight spaces

Throughout this section $r \geq 2$ and as before $\lambda = \alpha_1 + \alpha_2 + \cdots + \alpha_r$.

Theorem 3.2. If $\mu \neq 0$ is a dominant integral weight of $\mathfrak{so}_{2r+1}(\mathbb{C})$, then

$$\mathcal{A}(\lambda, \mu) = \begin{cases} \{1\} & \text{if } \mu = \lambda \\ \emptyset & \text{otherwise.} \end{cases}$$

We begin by proving the following technical results from which Theorem 3.2 follows.

Proposition 3.2. If $\lambda = \sum_{\alpha \in \Delta} \alpha$ is a fundamental weight of $\mathfrak{so}_{2r+1}(\mathbb{C})$, then $\mathcal{A}(\lambda, \lambda) = \{1\}$.

Proof. Since $\lambda = \alpha_1 + \cdots + \alpha_r$, notice $\sigma(\lambda+\rho)-\rho-\lambda$ is a nonnegative $\mathbb{Z}$-linear combination of positive roots only if $\sigma(\lambda+\rho)-\rho$ is. By Theorem 3.1 we know $\sigma(\lambda+\rho)-\rho$ is a nonnegative $\mathbb{Z}$-linear combination of positive roots if and only if $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$, for some nonconsecutive integers $i_1, \ldots, i_k$ between 2 and r. Hence $\mathcal{A}(\lambda, \lambda) \subset \mathcal{A}(\lambda, 0)$. Suppose that $\sigma \in \mathcal{A}(\lambda, \lambda)$ with $\ell(\sigma) = k \geq 1$, then there exist nonconsecutive integers $i_1, \ldots, i_k$ between 2 and r such that $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$. By Proposition 3.1 we have that $\sigma(\lambda+\rho)-\rho = \lambda - \sum_{j=1}^{k} \alpha_{i_j}$. Then notice $\sigma(\lambda+\rho)-\rho-\lambda$ will not be a nonnegative $\mathbb{Z}$-linear combination of positive roots, reaching a contradiction. Thus $\ell(\sigma) = 0$ and $\sigma = 1$. $\square$

Proposition 3.3. Let $\mu \neq 0$ be a dominant integral weight of $\mathfrak{so}_{2r+1}(\mathbb{C})$. Then there exists $\sigma \in W$ such that $\wp(\sigma(\lambda+\rho)-\rho-\mu) > 0$ if and only if $\mu = \lambda$.

Proof. $(\Rightarrow)$ Let $\mu \in P_+(\mathfrak{so}_{2r+1}(\mathbb{C}))$ with $\mu \neq 0$, and assume $\sigma \in W$ such that $\wp(\sigma(\lambda+\rho)-\rho-\mu) > 0$. By [2, Proposition 3.1.19], we know that $P_+(\mathfrak{so}_{2r+1}(\mathbb{C}))$ consists of all weights $\mu = k_1\varepsilon_1 + k_2\varepsilon_2 + \cdots + k_r\varepsilon_r$, with $k_1 \geq k_2 \geq \cdots \geq k_r \geq 0$ and satisfy that $2k_i$ and $k_i - k_j$ are integers for all i, j. Now observe that

$$\begin{aligned}\sigma(\lambda+\rho)-\rho-\mu = \sigma&\left(\left(r+\frac{1}{2}\right)\varepsilon_1 + \left(r-\frac{3}{2}\right)\varepsilon_2 + \cdots + \frac{3}{2}\varepsilon_{r-1} + \frac{1}{2}\varepsilon_r\right) \\ &- \left(\left(r-\frac{1}{2}\right)\varepsilon_1 + \left(r-\frac{3}{2}\right)\varepsilon_2 + \cdots + \frac{1}{2}\varepsilon_r\right) \\ &- (k_1\varepsilon_1 + \cdots + k_r\varepsilon_r).\end{aligned}$$

Let a_i denote the coefficient of α_i in $\sigma(\lambda+\rho)-\rho-\mu$. Then

$$a_1 = \begin{cases} -i+1-k_1 & \text{if } \sigma(\varepsilon_i)=\varepsilon_1 \text{ for some } 2\le i\le r \\ -2r+i-k_1 & \text{if } \sigma(\varepsilon_i)=-\varepsilon_1 \text{ for some } 2\le i\le r \\ 1-k_1 & \text{if } \sigma(\varepsilon_1)=\varepsilon_1 \\ -2r-k_1 & \text{if } \sigma(\varepsilon_1)=-\varepsilon_1. \end{cases}$$

Since $r\ge 2$ and $a_1\in\mathbb{N}$, it must be that $\sigma(\varepsilon_1)=\varepsilon_1$ and $a_1=1-k_1$. If $k_1=0$, then $k_i=0$ for all $1\le i\le r$, and so $\mu=0$, a contradiction. Hence $k_1=1$. Since $k_i-k_j\in\mathbb{Z}$ for all i and j, and since $1=k_1\ge k_2\ge k_3\ge\cdots\ge k_r\ge 0$, we have that $k_i=0$ or 1, for all $2\le i\le r$. We want to show that $k_i=0$ for all $2\le i\le r$. It suffices to show $k_2=0$. A simple computation shows that

$$a_2 = \begin{cases} -i+2-k_2 & \text{if } \sigma(\varepsilon_i)=\varepsilon_2 \text{ for some } 3\le i\le r \\ -2r+i+1-k_2 & \text{if } \sigma(\varepsilon_i)=-\varepsilon_2 \text{ for some } 3\le i\le r \\ -k_2 & \text{if } \sigma(\varepsilon_2)=\varepsilon_2 \\ -2r+3-k_2 & \text{if } \sigma(\varepsilon_2)=-\varepsilon_2. \end{cases}$$

Since $r\ge 2$ and $a_2\in\mathbb{N}$, it must be that $\sigma(\varepsilon_2)=\varepsilon_2$, and hence $k_2=0$. Thus $\mu=\varepsilon_1=\lambda$.

($\Leftarrow$) By Proposition 3.2, we know if $\mu=\lambda$, then $\wp(\sigma(\lambda+\rho)-\rho-\lambda)>0$ when $\sigma=1$. $\square$

4. Lie algebra of type C

In this section, we consider the Lie algebra $\mathfrak{sp}_{2r}(\mathbb{C})$ for $r\ge 3$. Whenever $1\le i\le r$ let ε_i denote the i^{th} standard basis vector in $\mathbb{R}^r$. If $\alpha_i=\varepsilon_i-\varepsilon_{i+1}$ for $1\le i\le r-1$ and $\alpha_r=2\varepsilon_r$, then the set of simple roots of $\mathfrak{sp}_{2r}(\mathbb{C})$ is given by $\Delta=\{\alpha_1,\ldots,\alpha_r\}$ and the set of positive roots is given by

$$\Phi^+=\{\varepsilon_i-\varepsilon_j,\varepsilon_i+\varepsilon_j:\ 1\le i<j\le r\}\cup\{2\varepsilon_i:1\le i\le r\}.$$

The fundamental weights of $\mathfrak{sp}_{2r}(\mathbb{C})$ are $\varpi_i=\varepsilon_1+\cdots+\varepsilon_i$ for $1\le i\le r$, and $\rho=\varpi_1+\cdots+\varpi_r$. The simple root reflections act on the simple roots and fundamental weights as follows. If $1\le i\le r$, then $s_i(\alpha_j)=\alpha_j$ if $|i-j|>1$, $s_i(\alpha_j)=-\alpha_j$ if $i=j$, $s_i(\alpha_j)=\alpha_i+\alpha_j$ if $|i-j|=1$ and $i\ne r-1$, $j\ne r$, and $s_{r-1}(\alpha_r)=2\alpha_{r-1}+\alpha_r$. As before $s_i(\varpi_j)=\varpi_j-\delta_{i,j}\alpha_i$ for all $1\le i,j\le r$. Throughout this section, we let $\lambda=\alpha_1+\alpha_2+\cdots+\alpha_r$.

We separate the results of this section into the cases where μ is the zero weight and when it is a nonzero weight.

4.1. Zero weight space

We begin with the following technical result for $\mathfrak{sp}_{2r}(\mathbb{C})$.

Proposition 4.1. Let $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$ for some nonconsecutive integers $2 \le i_1, \dots, i_k \le r-1$. If σ contains s_{r-1} in its reduced word expression, then $\sigma(\lambda+\rho)-\rho = \lambda + \alpha_{r-1} - \sum_{j=1}^{k}\alpha_{i_j}$, otherwise $\sigma(\lambda+\rho)-\rho = \lambda - \sum_{j=1}^{k}\alpha_{i_j}$, both of which are nonnegative $\mathbb{Z}$-linear combinations of positive roots.

Proof. Let $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$ for some nonconsecutive integers satisfying $2 \le i_1, \dots, i_k \le r-1$. Note that

$$\begin{aligned} s_{r-1}(\lambda+\rho) &= \alpha_1 + \alpha_2 + \cdots + \alpha_{r-3} + (\alpha_{r-2}+\alpha_{r-1}) - \alpha_{r-1} \\ &\qquad + (2\alpha_{r-1}+\alpha_r) + \rho - \alpha_{r-1} \\ &= \lambda + \rho. \end{aligned}$$

Hence, if σ contains s_{r-1}, without loss of generality, let $i_k = r-1$, and observe that $\sigma(\lambda+\rho)-\rho = s_{i_1}s_{i_2}\cdots s_{i_{k-1}}s_{r-1}(\lambda+\rho)-\rho = s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\lambda+\rho)-\rho = \lambda+\alpha_{r-1}-\sum_{j=1}^{k-1}\alpha_{i_j} = \lambda+\alpha_{r-1}-\sum_{j=1}^{k}\alpha_{i_j}$. If σ does not contain s_{r-1}, then $\sigma(\lambda+\rho)-\rho = s_{i_1}s_{i_2}\cdots s_{i_k}(\lambda+\rho)-\rho = \lambda+\rho-\rho-\sum_{j=1}^{k}\alpha_{i_j} = \lambda-\sum_{j=1}^{k}\alpha_{i_j}$. Lastly, note that both expressions can be written as nonnegative integrals sum of positive roots. □

Proposition 4.2. If $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$ for some nonconsecutive integers satisfying $2 \le i_1, \dots, i_k \le r-4$, then

- $\sigma s_{r-2}s_{r-1}(\lambda+\rho)-\rho = \lambda - \left(\sum_{j=1}^{k}\alpha_{i_j}\right) - \alpha_{r-2}$
- $\sigma s_{r-1}s_{r-2}(\lambda+\rho)-\rho = \sigma s_{r-2}s_{r-1}s_{r-2}(\lambda+\rho)-\rho = \lambda - \left(\sum_{j=1}^{k}\alpha_{i_j}\right) - \alpha_{r-2} - \alpha_{r-1}$

all of which can be represented as nonnegative $\mathbb{Z}$-linear combinations of positive roots.

Proof. The result follows from Proposition 4.1 and by computing the action of the simple roots s_{r-2} and s_{r-1} on $\lambda+\rho$. □

The following result describes all of the elements of $\mathcal{A}(\lambda, 0)$ for the Lie algebra of type C.

Theorem 4.1. Let $\mathfrak{g} = \mathfrak{sp}_{2r}(\mathbb{C})$ with $r \ge 3$. Then $\sigma \in \mathcal{A}(\lambda, 0)$ if and only if

1. $\sigma = 1$ or

2. $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$ for some nonconsecutive integers $2 \leq i_1, i_2, \ldots, i_k \leq r-1$ or
3. $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k} \pi$ for some nonconsecutive integers $2 \leq i_1, i_2, \ldots, i_k \leq r-4$ and $\pi \in \{s_{r-2}s_{r-1}, s_{r-1}s_{r-2}, s_{r-2}s_{r-1}s_{r-2}\}$.

Proof. ($\Leftarrow$) Let $\sigma = 1$, then $1(\lambda+\rho)-\rho = \lambda$ is a nonnegative $\mathbb{Z}$-linear combination of positive roots, thus $1 \in \mathcal{A}(\lambda, 0)$. Propositions 4.1 and 4.2 show that if σ is of the form listed in (2) or (3) above, then then $\sigma \in \mathcal{A}(\lambda, 0)$.

($\Rightarrow$) Suppose $\sigma \in W$ is not of the three forms listed above. Then σ contains s_1 or s_r, or $s_i s_j$ where i, j are consecutive integers, but not of the forms $s_{r-2}s_{r-1}$ or $s_{r-1}s_{r-2}$. We observe that $s_1(\lambda+\rho)-\rho = (\lambda-\alpha_1+\rho-\alpha_1)-\rho = \lambda-2\alpha_1$ and $s_r(\lambda+\rho)-\rho = (\lambda-\alpha_r+\rho-\alpha_r)-\rho = \lambda-2\alpha_r$, which cannot be written as sums of positive roots because of the negative coefficient of α_1 and of α_r, respectively. This implies that $s_1, s_r \notin \mathcal{A}(\lambda, 0)$, and hence if σ contains s_1 or s_r in its reduced word expression, then $\sigma \notin \mathcal{A}(\lambda, 0)$.

For consecutive integers $1 < j-1, j < r-1$ we have $s_{j-1}s_j(\lambda+\rho)-\rho = \lambda - 2\alpha_{j-1} - \alpha_j$ and $s_j s_{j-1}(\lambda+\rho)-\rho = \lambda - \alpha_{j-1} - 2\alpha_j$, which implies that $s_{j-1}s_j, s_j s_{j-1} \notin \mathcal{A}(\lambda, 0)$. Hence if σ contains $s_i s_j$ for some consecutive integers $2 \leq i, j \leq r-2$ then $\sigma \notin \mathcal{A}(\lambda, 0)$. Thus σ must be of one of the three forms listed in the theorem in order for $\sigma \in \mathcal{A}(\lambda, 0)$. $\square$

Recall that the Lucas numbers follow the recurrence $L_r = L_{r-1} + L_{r-2}$, with $L_1 = 1$ and $L_2 = 3$. We can now connect our work with this famous sequence of integers.

Corollary 4.1. If $r \geq 3$ and $\lambda = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ is a weight of $\mathfrak{sp}_{2r}(\mathbb{C})$, then $|\mathcal{A}(\lambda, 0)| = 2L_{r-2}$, where L_k denotes the k^{th} Lucas number.

Proof. As in Corollary 2.1, we know that there are F_r Weyl group elements in $\mathcal{A}(\lambda, 0)$ arising from parts 1 and 2 of Theorem 4.1. By the same reasoning, there are F_{r-3} elements $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k} \pi$ for some nonconsecutive integers $2 \leq i_1, i_2, \ldots, i_k \leq r-4$, for each π as specified in part 3 of Theorem 4.1. This yields an additional $3F_{r-3}$ elements in $\mathcal{A}(\lambda, 0)$. Thus $|\mathcal{A}(\lambda, 0)| = F_r + 3F_{r-3}$, where F_k denotes the k^{th} Fibonacci number. The result follows from the fact that $F_r + 3F_{r-3} = 2L_{r-2}$. $\square$

4.2. Nonzero weight spaces

Throughout this section $r \geq 3$ and as before $\lambda = \alpha_1 + \alpha_2 + \cdots + \alpha_r$.

Theorem 4.2. If $\mu \neq 0$ is a dominant integral weight of $\mathfrak{sp}_{2r}(\mathbb{C})$, then $\mathcal{A}(\lambda, \mu) = \emptyset$

We begin by proving the following technical results from which Theorem 4.2 follows.

Proposition 4.3. Let $\mu \neq 0$ be a dominant integral weight of $\mathfrak{sp}_{2r}(\mathbb{C})$. If there exists $\sigma \in W$ such that $\wp(\sigma(\lambda+\rho)-\rho-\mu) > 0$, then $\mu = \varpi_1$.

Proof. ($\Rightarrow$) Let $\mu \in P_+(\mathfrak{sp}_{2r}(\mathbb{C}))$ with $\mu \neq 0$, and assume $\sigma \in W$ such that $\wp(\sigma(\lambda+\rho)-\rho-\mu) > 0$. By [2, Proposition 3.1.19], we know that $P_+(\mathfrak{sp}_{2r}(\mathbb{C}))$ consists of all weights $\mu = k_1\varepsilon_1 + k_2\varepsilon_2 + \cdots + k_r\varepsilon_r$, satisfying $k_1 \geq k_2 \geq \cdots \geq k_r \geq 0$ with k_i an integer for all i.

Now observe that

$$\begin{aligned}\sigma(\lambda+\rho)-\rho-\mu = \sigma(&(r+1)\varepsilon_1 + (r-1)\varepsilon_2 + \cdots + 3\varepsilon_{r-2} + 2\varepsilon_{r-1} + 2\varepsilon_r) \\ &- (r\varepsilon_1 + (r-1)\varepsilon_2 + \cdots + 2\varepsilon_{r-1} + \varepsilon_r) \\ &- (k_1\varepsilon_1 + k_2\varepsilon_2 + \cdots + k_r\varepsilon_r).\end{aligned}$$

Let a_i denote the coefficient of α_i in $\sigma(\lambda+\rho)-\rho-\mu$. Then

$$a_1 = \begin{cases} 1-k_1 & \text{if } \sigma(\varepsilon_1) = \varepsilon_1 \\ -2r-1-k_1 & \text{if } \sigma(\varepsilon_1) = -\varepsilon_1 \\ -i+1-k_1 & \text{if } \sigma(\varepsilon_i) = \varepsilon_1 \text{ for some } 2 \leq i \leq r-1 \\ -2r-1+i-k_1 & \text{if } \sigma(\varepsilon_i) = -\varepsilon_1 \text{ for some } 2 \leq i \leq r-1 \\ 2-r-k_1 & \text{if } \sigma(\varepsilon_r) = \varepsilon_1 \\ -2-r-k_1 & \text{if } \sigma(\varepsilon_r) = -\varepsilon_1. \end{cases} \tag{5}$$

Since $r \geq 3$ and $a_1 \in \mathbb{N}$, it must be that $\sigma(\varepsilon_1) = \varepsilon_1$ and $a_1 = 1-k_1$. Hence, $k_1 = 0$ or $k_1 = 1$. If $k_1 = 0$, then $k_i = 0$ for all $1 \leq i \leq r$, and so $\mu = 0$, a contradiction. Thus $k_1 = 1$. Since $1 = k_1 \geq k_2 \geq k_3 \geq \cdots \geq k_r \geq 0$, we have that $k_i = 0$ or 1, for all $2 \leq i \leq r$. We want to show that $k_i = 0$ for all $2 \leq i \leq r$. It suffices to show $k_2 = 0$. A simple computation shows that

$$a_2 = \begin{cases} -k_2 & \text{if } \sigma(\varepsilon_2) = \varepsilon_2 \\ -2(r-1)-k_2 & \text{if } \sigma(\varepsilon_2) = -\varepsilon_2 \\ -i+2-k_2 & \text{if } \sigma(\varepsilon_i) = \varepsilon_2 \text{ for some } 3 \leq i \leq r-1 \\ -2r+i-k_2 & \text{if } \sigma(\varepsilon_i) = -\varepsilon_2 \text{ for some } 3 \leq i \leq r-1 \\ 3-r-k_2 & \text{if } \sigma(\varepsilon_r) = \varepsilon_2 \\ -r-1-k_2 & \text{if } \sigma(\varepsilon_r) = -\varepsilon_2. \end{cases} \tag{6}$$

Since $r \geq 3$ and $a_2 \in \mathbb{N}$, it must be that either

1. $\sigma(\varepsilon_2) = \varepsilon_2$ and $a_2 = -k_2$ or
2. $r = 3$, $\sigma(\varepsilon_r) = \varepsilon_2$, and $a_2 = 3 - r - k_2 = -k_2$.

However, in either case it must be that $k_2 = 0$ in order for $a_2 \in \mathbb{N}$. This implies that $k_i = 0$ for all $2 \leq i \leq r$. Thus $\mu = \varepsilon_1 = \varpi_1$. $\square$

Proposition 4.4. If ϖ_1 is a fundamental weight of the Lie algebra $\mathfrak{sp}_{2r}(\mathbb{C})$, then $\mathcal{A}(\lambda, \varpi_1) = \emptyset$.

Proof. We begin by noting that $\varpi_1 = \varepsilon_1 = \alpha_1 + \cdots + \alpha_{r-1} + \frac{1}{2}\alpha_r$. Now we compute

$$1(\lambda + \rho) - \rho - \varpi_1 = \frac{1}{2}\alpha_r \tag{7}$$

$$s_1(\lambda + \rho) - \rho - \varpi_1 = -2\alpha_1 + \frac{1}{2}\alpha_r \tag{8}$$

$$s_i(\lambda + \rho) - \rho - \varpi_1 = -\alpha_i + \frac{1}{2}\alpha_r \quad \text{for } 2 \leq i \leq r-2 \tag{9}$$

$$s_{r-1}(\lambda + \rho) - \rho - \varpi_1 = \frac{1}{2}\alpha_r \tag{10}$$

$$s_r(\lambda + \rho) - \rho - \varpi_1 = -\frac{3}{2}\alpha_r \tag{11}$$

none of which are nonnegative $\mathbb{Z}$-linear combinations of positive roots. Hence $1 \notin \mathcal{A}(\lambda, \varpi_1)$ and $s_i \notin \mathcal{A}(\lambda, \varpi_1)$ for all $1 \leq i \leq r$. Then by [8, Proposition 3.4] it follows that since $s_i \notin \mathcal{A}(\lambda, \varpi_1)$ for any $1 \leq i \leq r$, then neither is any $\sigma \in W$ containing any s_i in its reduced word expression. This establishes that $\mathcal{A}(\lambda, \varpi_1) = \emptyset$. $\square$

5. Lie algebra of type *D*

In this section, we consider the Lie algebra $\mathfrak{g} = \mathfrak{so}_{2r}(\mathbb{C})$ for $r \geq 4$. For $1 \leq i \leq r$ let ε_i denote the i^{th} standard basis vector in $\mathbb{R}^r$. If $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq r-1$ and $\alpha_r = \varepsilon_{r-1} + \varepsilon_r$, then the set of simple roots is given by $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ and the set of positive roots is given by $\Phi^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j : 1 \leq i < j \leq r\}$. The fundamental weights of $\mathfrak{so}_{2r}(\mathbb{C})$ are $\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i$ for $1 \leq i \leq r-2$, $\varpi_{r-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r)$, $\varpi_r = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} + \varepsilon_r)$, and $\rho = \varpi_1 + \cdots + \varpi_r$. The simple root reflections act on the simple roots and fundamental weights as follows. If $1 \leq i \leq r$, then $s_i(\alpha_i) = -\alpha_i$. If $1 \leq i < j \leq r-1$ with $|i-j| = 1$ or if $i = r-2$ and $j = r$, then $s_i(\alpha_j) = s_j(\alpha_i) = \alpha_i + \alpha_j$. Lastly, $s_{r-1}(\alpha_r) = \alpha_r$, $s_r(\alpha_{r-1}) = \alpha_{r-1}$, and in all other cases $s_i(\alpha_j) = \alpha_j$. As before $s_i(\varpi_j) = \varpi_j - \delta_{i,j}\alpha_i$ for all $1 \leq i, j \leq r$. Throughout this section, we let $\lambda = \alpha_1 + \alpha_2 + \cdots + \alpha_r$.

We separate the results of this section into the cases where μ is the zero weight and when it is a nonzero weight.

5.1. Zero weight space

We begin with the following technical result for $\mathfrak{so}_{2r}(\mathbb{C})$.

Proposition 5.1. Let $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$ for some nonconsecutive integers satisfying $2 \le i_1,\dots,i_k \le r-2$. If σ contains s_{r-2}, then $\sigma(\lambda+\rho)-\rho = \lambda+\alpha_{r-2}-\sum_{j=1}^{k}\alpha_{i_j}$, otherwise $\sigma(\lambda+\rho)-\rho = \lambda-\sum_{j=1}^{k}\alpha_{i_j}$, both of which are nonnegative $\mathbb{Z}$-linear combinations of positive roots.

Proof. Let $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$ for some nonconsecutive integers satisfying $2 \le i_1,\dots,i_k \le r-2$. If σ contains s_{r-2}, then without loss of generality assume $i_k = r-2$ and note $\sigma(\lambda+\rho)-\rho = s_{i_1}s_{i_2}\cdots s_{i_{k-1}}s_{r-2}(\lambda+\rho)-\rho = s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\lambda+\rho)-\rho = \lambda-\sum_{j=1}^{k-1}\alpha_{i_j} = \lambda+\alpha_{r-2}-\sum_{j=1}^{k}\alpha_{i_j}$. However, if σ does not contain s_{r-2}, then $\sigma(\lambda+\rho)-\rho = s_{i_1}s_{i_2}\cdots s_{i_k}(\lambda+\rho)-\rho = \lambda-\sum_{j=1}^{k}\alpha_{i_j}$. Lastly, note that both expressions can be written as a nonnegative $\mathbb{Z}$-linear combination of positive roots. $\square$

Proposition 5.2. If $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$ for some nonconsecutive integers satisfying $2 \le i_1,\dots,i_k \le r-5$, then

- $\sigma s_{r-3}s_{r-2}(\lambda+\rho)-\rho = \lambda-\left(\sum_{j=1}^{k}\alpha_{i_j}\right)-\alpha_{r-3}$,
- $\sigma s_{r-2}s_{r-3}(\lambda+\rho)-\rho = \sigma s_{r-3}s_{r-2}s_{r-3}(\lambda+\rho)-\rho = \lambda-\left(\sum_{j=1}^{k}\alpha_{i_j}\right)-\alpha_{r-3}-\alpha_{r-2}$,

both of which can be written as nonnegative $\mathbb{Z}$-linear combinations of positive roots.

Proof. The result follows from Proposition 5.1 and by computing the action of the simple roots s_{r-3} and s_{r-2} on $\lambda+\rho$. $\square$

Theorem 5.1. Let $\mathfrak{g} = \mathfrak{so}_{2r}(\mathbb{C})$ with $r \ge 4$. Then $\sigma \in \mathcal{A}(\lambda,0)$ if and only if

1. $\sigma = 1$ or
2. $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$ for some nonconsecutive integers $2 \le i_1,\dots,i_k \le r-2$ or
3. $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}\pi$ for some nonconsecutive integers $2 \le i_1,\dots,i_k \le r-5$ and $\pi \in \{s_{r-3}s_{r-2},\ s_{r-2}s_{r-3},\ s_{r-3}s_{r-2}s_{r-3}\}$.

Proof. ($\Leftarrow$) Let $\sigma = 1$, then $1(\lambda+\rho)-\rho = \lambda$ is a nonnegative $\mathbb{Z}$-linear combination of positive roots. Hence $1 \in \mathcal{A}(\lambda,0)$. If $\sigma \in W$ has one of the forms listed in (2) or (3), then Propositions 5.1 and 5.2 show that $\sigma \in \mathcal{A}(\lambda,0)$.

($\Rightarrow$) Suppose $\sigma \in W$ is not of the three forms listed above. Then σ contains s_1, s_{r-1}, s_r, or consecutive reflections s_i and s_j, where $\{i,j\} \neq \{r-3, r-2\}$. Note that $s_1(\lambda+\rho)-\rho = \lambda-2\alpha_1$, $s_{r-1}(\lambda+\rho)-\rho = \lambda-2\alpha_{r-1}$, and $s_r(\lambda+\rho)-\rho = \lambda-2\alpha_r$, none of which can be written as sums of positive roots because of the negative coefficients of α_1, α_{r-1} and α_r, respectively. This implies that $s_1, s_{r-1}, s_r \notin \mathcal{A}(\lambda, 0)$, and hence if σ contains s_1, s_{r-1}, or s_r in its reduced word expression, then $\sigma \notin \mathcal{A}(\lambda, 0)$.

For consecutive integers $2 \leq j-1, j \leq r-3$ we have $s_{j-1}s_j(\lambda+\rho)-\rho = \lambda - 2\alpha_{j-1} - \alpha_j$ and $s_j s_{j-1}(\lambda+\rho)-\rho = \lambda - \alpha_{j-1} - 2\alpha_j$, which implies that $s_{j-1}s_j,\ s_j s_{j-1} \notin \mathcal{A}(\lambda, 0)$. Hence if σ contains s_i, s_j for some consecutive integers $2 \leq i, j \leq r-3$ then $\sigma \notin \mathcal{A}(\lambda, 0)$. Thus σ must be of one of the three forms listed in the theorem in order for $\sigma \in \mathcal{A}(\lambda, 0)$. $\square$

Corollary 5.1. If $r \geq 4$ and $\lambda = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ is a weight of $\mathfrak{so}_{2r}(\mathbb{C})$, then $|\mathcal{A}(\lambda, 0)| = 2L_{r-3}$, where L_k denotes the k^{th} Lucas number.

The proof of Corollary 5.1 is analogous to that of Corollary 4.1, hence we omit it.

5.2. Nonzero weight spaces

Throughout this section $r \geq 4$ and as before $\lambda = \alpha_1 + \alpha_2 + \cdots + \alpha_r$.

Theorem 5.2. If $\mu \neq 0$ is a dominant integral weight of $\mathfrak{so}_{2r}(\mathbb{C})$, then $\mathcal{A}(\lambda, \mu) = \emptyset$.

We begin by proving the following technical results from which Theorem 5.2 follows.

Proposition 5.3. Let $\mu \neq 0$ be a dominant integral weight of $\mathfrak{so}_{2r}(\mathbb{C})$. If there exists $\sigma \in W$ such that $\wp(\sigma(\lambda+\rho)-\rho-\mu) > 0$, then $\mu = \varpi_1$.

Proof. ($\Rightarrow$) Let $\mu \in P_+(\mathfrak{so}_{2r}(\mathbb{C}))$ with $\mu \neq 0$, and assume $\sigma \in W$ such that $\wp(\sigma(\lambda+\rho)-\rho-\mu) > 0$. By [2, Proposition 3.1.19], we know that $P_+(\mathfrak{so}_{2r}(\mathbb{C}))$ consists of all weights $\mu = k_1\varepsilon_1 + k_2\varepsilon_2 + \cdots + k_r\varepsilon_r$, satisfying $k_1 \geq k_2 \geq \cdots \geq |k_r|$ where $2k_i$ and $k_i - k_j$ are integers for all i. Now observe that

$$\begin{aligned}\sigma(\lambda+\rho)-\rho-\mu = \sigma(&r\varepsilon_1 + (r-2)\varepsilon_2 + (r-3)\varepsilon_3 + \cdots + 2\varepsilon_{r-2} + 2\varepsilon_{r-1}) \\ &- ((r-1)\varepsilon_1 + (r-2)\varepsilon_2 + \cdots + 2\varepsilon_{r-2} + \varepsilon_{r-1}) \\ &- (k_1\varepsilon_1 + k_2\varepsilon_2 + \cdots + k_r\varepsilon_r).\end{aligned}$$

Let a_i denote the coefficient of α_i in $\sigma(\lambda+\rho)-\rho-\mu$. Then

$$a_1 = \begin{cases} 1-k_1 & \text{if } \sigma(\varepsilon_1)=\varepsilon_1 \\ -2r+1-k_1 & \text{if } \sigma(\varepsilon_1)=-\varepsilon_1 \\ -i+1-k_1 & \text{if } \sigma(\varepsilon_i)=\varepsilon_1 \text{ for some } 2\le i\le r-1 \\ -2r+1+i-k_1 & \text{if } \sigma(\varepsilon_i)=-\varepsilon_1 \text{ for some } 2\le i\le r-1 \\ 3-r-k_1 & \text{if } \sigma(\varepsilon_r)=\varepsilon_1 \\ -1-r-k_1 & \text{if } \sigma(\varepsilon_r)=-\varepsilon_1. \end{cases} \tag{12}$$

Since $r\ge 4$ and $a_1\in\mathbb{N}$, it must be that $\sigma(\varepsilon_1)=\varepsilon_1$ and $a_1=1-k_1$. Hence, $k_1=0$ or $k_1=1$. If $k_1=0$, then $k_i=0$ for all $1\le i\le r$, so $\mu=0$, a contradiction. Thus $k_1=1$. Since $k_i-k_j\in\mathbb{Z}$ for all i and j, and since $1=k_1\ge k_2\ge k_3\ge\cdots\ge|k_r|$, we have that $k_i=0$ or 1, for all $2\le i\le r$. We want to show that $k_i=0$ for all $2\le i\le r$. It suffices to show $k_2=0$. A simple computation shows

$$a_2 = \begin{cases} -k_2 & \text{if } \sigma(\varepsilon_2)=\varepsilon_2 \\ -2(r-2)-k_2 & \text{if } \sigma(\varepsilon_2)=-\varepsilon_2 \\ -i+2-k_2 & \text{if } \sigma(\varepsilon_i)=\varepsilon_2 \text{ for some } 3\le i\le r-1 \\ -2r+i+2-k_2 & \text{if } \sigma(\varepsilon_i)=-\varepsilon_2 \text{ for some } 3\le i\le r-1 \\ 4-r-k_2 & \text{if } \sigma(\varepsilon_r)=\varepsilon_2 \\ -r-k_2 & \text{if } \sigma(\varepsilon_r)=-\varepsilon_2. \end{cases} \tag{13}$$

Since $r\ge 4$ and $a_2\in\mathbb{N}$, it must be that either

1. $\sigma(\varepsilon_2)=\varepsilon_2$ and $a_2=-k_2$ or
2. $r=4$, $\sigma(\varepsilon_r)=\varepsilon_2$, and $a_2=4-r-k_2=-k_2$.

However, in either case it must be that $k_2=0$ in order for $a_2\in\mathbb{N}$. This implies that $k_i=0$ for all $2\le i\le r$. Thus $\mu=\varepsilon_1=\varpi_1$. □

Proposition 5.4. If ϖ_1 is a fundamental weight of the Lie algebra $\mathfrak{so}_{2r}(\mathbb{C})$, then $\mathcal{A}(\lambda,\varpi_1)=\emptyset$.

Proof. We begin by noting that $\varpi_1=\varepsilon_1=\alpha_1+\cdots+\alpha_{r-2}+\frac{1}{2}\alpha_{r-1}+\frac{1}{2}\alpha_r$. Now we compute

$$1(\lambda+\rho)-\rho-\varpi_1=\frac{1}{2}(\alpha_{r-1}+\alpha_r) \tag{14}$$

$$s_1(\lambda+\rho)-\rho-\varpi_1=-2\alpha_1+\frac{1}{2}(\alpha_{r-1}+\alpha_r) \tag{15}$$

$$s_i(\lambda+\rho)-\rho-\varpi_1=-\alpha_i+\frac{1}{2}(\alpha_{r-1}+\alpha_r)\alpha_r \quad \text{for } 2\le i\le r-3 \tag{16}$$

$$s_{r-2}(\lambda+\rho)-\rho-\varpi_1=\frac{1}{2}(\alpha_{r-1}+\alpha_r) \tag{17}$$

$$s_{r-1}(\lambda+\rho)-\rho-\varpi_1=-\frac{3}{2}\alpha_{r-1}+\frac{1}{2}\alpha_r \tag{18}$$

$$s_r(\lambda+\rho)-\rho-\varpi_1=\frac{1}{2}\alpha_{r-1}-\frac{3}{2}\alpha_r \tag{19}$$

none of which are nonnegative $\mathbb{Z}$-linear combinations of positive roots. Hence $1\notin\mathcal{A}(\lambda,\varpi_1)$ and $s_i\notin\mathcal{A}(\lambda,\varpi_1)$ for all $1\le i\le r$. Then by [8, Proposition 3.4] it follows that since $s_i\notin\mathcal{A}(\lambda,\varpi_1)$ for any $1\le i\le r$, then neither is any $\sigma\in W$ containing any s_i in its reduced word expression. This establishes that $\mathcal{A}(\lambda,\varpi_1)=\emptyset$. □

6. A q-analog

The q-analog of Kostant's partition function is the polynomial valued function, $\wp_q$, defined on $\mathfrak{h}^*$ by $\wp_q(\xi)=c_0+c_1q+\cdots+c_kq^k$, where c_j equals the number of ways to write ξ as a nonnegative $\mathbb{Z}$-linear combination of exactly j positive roots, for $\xi\in\mathfrak{h}^*$. The q-analog of Kostant's weight multiplicity formula is defined, in [10], as:

$$m_q(\lambda,\mu)=\sum_{\sigma\in W}(-1)^{\ell(\sigma)}\wp_q(\sigma(\lambda+\rho)-(\mu+\rho)).$$

In the sections that follow we consider the classical Lie algebras and provide formulas for the value of $m_q(\lambda,\mu)$ when $\lambda=\alpha_1+\alpha_2+\cdots+\alpha_r$ and μ is a dominant integral weight.

6.1. Lie algebra of type A

Note $\lambda=\alpha_1+\alpha_2+\cdots+\alpha_r$ is the highest root of $\mathfrak{sl}_{r+1}(\mathbb{C})$. In this case it is known that $m_q(\lambda,0)=\sum_{i=1}^r q^i$, where $1,2,\ldots,r$ are the exponents of $\mathfrak{sl}_{r+1}(\mathbb{C})$ [10]. A combinatorial proof of this result was presented in [3]; however, we provide the results and their proofs here for sake of completeness.

Theorem 6.1. If $\lambda=\alpha_1+\alpha_2+\cdots+\alpha_r$ is the highest root of $\mathfrak{sl}_{r+1}(\mathbb{C})$, then $m_q(\lambda,0)=q+q^2+\cdots+q^r$.

In order to establish Theorem 6.1, we will make use of the following technical result.

Lemma 6.1. The cardinality of the set $\{\sigma\in\mathcal{A}(\lambda,0)\mid\ell(\sigma)=k\}$ is $\binom{r-1-k}{k}$ and $\max\{\,\ell(\sigma)\mid\sigma\in\mathcal{A}(\lambda,0)\}=\lfloor\frac{r-1}{2}\rfloor$.

Proof. This result follows from Theorem 2.1 and the fact that for any $n, k \in \mathbb{N}$ satisfying $k \leq n$

1. the number of ways to select k nonconsecutive numbers from the set $\{1, 2, \ldots, n\}$ is given by $\binom{n-k+1}{k}$ and
2. the maximum number of nonconsecutive numbers that can selected from the set $\{1, 2, \ldots, n\}$ is $\lfloor \frac{n+1}{2} \rfloor$. $\square$

We now prove the following combinatorial identity.

Proposition 6.1. If $\lambda = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ is the highest root of $\mathfrak{sl}_{r+1}(\mathbb{C})$ and $\sigma \in \mathcal{A}(\lambda, 0)$, then $\wp_q(\sigma(\lambda + \rho) - \rho) = q^{1+\ell(\sigma)}(1+q)^{r-1-2\ell(\sigma)}$.

Proof. If $\sigma \in \mathcal{A}(\lambda, 0)$ with $\ell(\sigma) = 0$, then $\sigma = 1$ and $\sigma(\lambda + \rho) - \rho = \lambda = \alpha_1 + \cdots + \alpha_r$. Since $\Phi^+ = \{\alpha_i : 1 \leq i \leq r\} \cup \{\alpha_i + \cdots + \alpha_j : 1 \leq i < j \leq r\}$, for any $i \geq 0$, we can think of c_{i+1}, the coefficient of q^{i+1} in $\wp_q(\alpha_1 + \cdots + \alpha_r)$, as the number of ways to place i lines in $r-1$ slots. Hence $c_{i+1} = \binom{r-1}{i}$ and $\wp_q(\lambda) = \sum_{i=0}^{r-1} \binom{r-1}{i} q^{i+1} = q(1+q)^{r-1}$.

If $\sigma \in \mathcal{A}(\lambda, 0)$ with $\ell(\sigma) = k \neq 0$, then Theorem 2.1 implies that $\sigma = s_1 s_2 \cdots s_k$, for some nonconsecutive integers $2 \leq i_1, i_2, \ldots, i_k \leq r-1$. Then by (4), $\sigma(\lambda + \rho) - \rho = \lambda - \sum_{j=1}^{k} \alpha_{i_j}$. Let c_j denote the coefficient of q^j in $\wp_q(\sigma(\lambda+\rho) - \rho)$. Since σ subtracts k many nonconsecutive simple roots from λ, we will at a minimum need $k+1$ positive roots to write $\lambda - \sum_{j=1}^{k} \alpha_{i_j}$. So $c_j = 0$, whenever $j < k+1$. Also observe that $\lambda - \sum_{j=1}^{k} \alpha_{i_j}$ can be written with at most $r-k$ positive roots. Hence $c_j = 0$, whenever $j > n-k$.

For $i \geq 0$, we can think of c_{k+1+i} as the number of ways to place i lines in $r-1-2k$ slots. This is because for each simple root that σ removes from λ, we lose 2 slots in which to place a line, one before and one after. So $c_{k+1+i} = \binom{r-1-2k}{i}$, whenever $0 \leq i \leq r-1-2k$. Therefore $\wp_q(\sigma(\lambda+\rho)-\rho) = \sum\limits_{i=0}^{r-1-2k} \binom{r-1-2k}{i} q^{k+1+i} = q^{1+k}(1+q)^{r-1-2k}$. $\square$

The following proposition will be used in the proof of Theorem 6.1.

Proposition 6.2. For $r \geq 1$, $\displaystyle\sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^k \binom{r-1-k}{k} q^{1+k}(1+q)^{r-1-2k} = \sum_{i=1}^{r} q^i.$

Proof. Equation (4.3.7) in [11] shows that for integers k and $n \geq 0$

$$\sum_{k \leq \frac{n}{2}} (-1)^k \binom{n-k}{k} q^k (1+q)^{n-2k} = \frac{1-q^{n+1}}{1-q}. \tag{20}$$

Suppose $r \geq 1$, and let $n = r - 1 \geq 0$. Then by (20) we have that

$$\sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^k \binom{r-1-k}{k} q^{1+k}(1+q)^{r-1-2k} = q\left(\frac{1-q^{n+1}}{1-q}\right).$$

Now observe that

$$\sum_{i=1}^{r} q^i = \sum_{i=1}^{n+1} q^i = q\sum_{i=0}^{n} q^i = q\left(\frac{1-q^{n+1}}{1-q}\right).$$

Therefore

$$\sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^k \binom{r-1-k}{k} q^{1+k}(1+q)^{r-1-2k} = \sum_{i=1}^{r} q^i. \qquad \square$$

Proof of Theorem 6.1. By Lemma 6.1 and Propositions 6.1 and 6.2, if $k = \ell(\sigma)$, then

$$\begin{aligned} m_q(\lambda, 0) &= \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \wp_q(\sigma(\lambda+\rho)-\rho) \\ &= \sum_{\sigma \in \mathcal{A}(\lambda,0)} (-1)^{\ell(\sigma)} \wp_q(\sigma(\lambda+\rho)-\rho) \\ &= \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^k \binom{r-1-k}{k} q^{1+k}(1+q)^{r-1-2k} \\ &= q + q^2 + q^3 + \cdots + q^r. \end{aligned} \qquad \square$$

We now present the multiplicity result when μ is an integral weight of $\mathfrak{sl}_{r+1}(\mathbb{C})$.

Corollary 6.1. If $\mu \in P(\mathfrak{sl}_{r+1}(\mathbb{C}))$, then $m(\lambda, \mu) = \begin{cases} r & \text{if } \mu = 0 \\ 1 & \text{if } \mu \in \Phi \\ 0 & \text{otherwise.} \end{cases}$

Proof. The fact that $m(\lambda, 0) = r$ follows from Theorem 6.1 and the fact that $m_q(\lambda, 0)|_{q=1} = m(\lambda, 0)$. To see that $m(\lambda, \mu) = 1$ when μ is a positive root, recall that if $\mu \in P(\mathfrak{sl}_{r+1}(\mathbb{C}))$, then there exists $w \in W$ and $\xi \in P_+(\mathfrak{sl}_{r+1}(\mathbb{C}))$ such that $w(\xi) = \mu$ [2, Proposition 3.1.20]. Also by [2, Proposition 3.2.27] we know that weight multiplicities are invariant under W. Thus it suffices to compute $m(\lambda,\mu)$ for $\mu \in P_+(\mathfrak{sl}_{r+1}(\mathbb{C}))$. By Theorem 2.2 we know $\mathcal{A}(\lambda,\lambda) = \{1\}$, and $\mathcal{A}(\lambda,\mu) = \emptyset$ whenever $\mu \in P_+(\mathfrak{sl}_{r+1}(\mathbb{C})) - \{0,\lambda\}$. This implies that $m(\lambda,\lambda) = \wp(1(\lambda+\rho) - \rho - \lambda) = \wp(0) = 1$, and that $m(\lambda,\mu) = 0$ whenever $\mu \in P_+(\mathfrak{sl}_{r+1}(\mathbb{C})) - \{0,\lambda\}$. $\square$

6.2. Lie algebra of type B

It is known that the multiplicity of the zero weight in the representation with highest weight $\lambda = \alpha_1 + \alpha_2 + \cdots + \alpha_r = \varpi_1$ is equal to 1, see [1]. In this section, we give a combinatorial proof of this result by proving the following.

Theorem 6.2. Let $r \geq 2$ and let $\lambda = \alpha_1 + \alpha_2 + \cdots + \alpha_r = \varpi_1$ be a fundamental weight of $\mathfrak{so}_{2r+1}(\mathbb{C})$. Then $m_q(\lambda,0) = q^r$.

Observe that the subset of positive roots of $\mathfrak{so}_{2r+1}(\mathbb{C})$ used to write $\sigma(\lambda+\rho) - \rho$, for any $\sigma \in \mathcal{A}(\lambda,0)$, is equal to the set of positive roots of $\mathfrak{sl}_{r+1}(\mathbb{C})$. Thus we state the following technical results.

Lemma 6.2. The cardinality of the sets

$$\{\sigma \in \mathcal{A}(\lambda,0) : \ell(\sigma) = k \text{ and } \sigma \text{ does not contain } s_r\ \}$$

and

$$\{\sigma \in \mathcal{A}(\lambda,0) : \ell(\sigma) = k+1 \text{ and } \sigma \text{ contains } s_r\}$$

are $\binom{r-1-k}{k}$ and $\binom{r-2-k}{k}$, respectively. Also

$$\max\{\ell(\sigma) : \sigma \in \mathcal{A}(\lambda,0) \text{ and } \sigma \text{ does not contain } s_r\} = \left\lfloor \frac{r-1}{2} \right\rfloor \quad \text{and}$$
$$\max\{\ell(\sigma) : \sigma \in \mathcal{A}(\lambda,0) \text{ and } \sigma \text{ contain } s_r\} = \left\lfloor \frac{r-2}{2} \right\rfloor.$$

Lemma 6.2 is analogous to Lemma 6.1 and hence we omit the proof.

Proposition 6.3. Let $r \geq 2$ and let $\lambda = \alpha_1 + \alpha_2 + \cdots + \alpha_r = \varpi_1$ be a fundamental weight of $\mathfrak{so}_{2r+1}(\mathbb{C})$. If $\sigma \in \mathcal{A}(\lambda,0)$, then

$$\wp_q(\sigma(\lambda+\rho) - \rho) = \begin{cases} q^{1+\ell(\sigma)}(1+q)^{r-1-2\ell(\sigma)} & \text{if } \sigma \text{ does not contain } s_r \\ q^{\ell(\sigma)}(1+q)^{r-2\ell(\sigma)} & \text{if } \sigma \text{ contains } s_r. \end{cases}$$

To see that the statement of Proposition 6.3 holds, it suffices to note that whenever σ does not contain s_r, then the count is precisely that of Proposition 6.3, while if σ contains s_r, then it reduces to Proposition 6.3 with r replaced by $r-1$ and $\ell(\sigma)$ replaced by $\ell(\sigma)-1$. With the results at hand we can now prove Theorem 6.2.

Proof of Theorem 6.2. Observe that

$$m_q(\lambda,0) = \sum_{\substack{\sigma\in\mathcal{A}(\lambda,0)\\ \text{no } s_r \text{ in } \sigma}} (-1)^{\ell(\sigma)}\wp_q(\sigma(\lambda+\rho)-\rho) + \sum_{\substack{\sigma\in\mathcal{A}(\lambda,0)\\ s_r \text{ in } \sigma}} (-1)^{\ell(\sigma)}\wp_q(\sigma(\lambda+\rho)-\rho).$$

By Lemma 6.2, Proposition 6.3 and Proposition 6.2 it follows that

$$\begin{aligned}
&\sum_{\substack{\sigma\in\mathcal{A}(\lambda,0)\\ \text{no } s_r \text{ in } \sigma}} (-1)^{\ell(\sigma)}\wp_q(\sigma(\lambda+\rho)-\rho)\\
&\quad= \sum_{k=0}^{\lfloor\frac{r-1}{2}\rfloor} (-1)^k \binom{r-1-k}{k} q^{1+k}(1+q)^{r-1-2k}\\
&\quad= \sum_{i=1}^{r} q^i, \text{ and}\\
&\sum_{\substack{\sigma\in\mathcal{A}(\lambda,0)\\ s_r \text{ in } \sigma}} (-1)^{\ell(\sigma)}\wp_q(\sigma(\lambda+\rho)-\rho)\\
&\quad= \sum_{k=0}^{\lfloor\frac{r-2}{2}\rfloor} (-1)^{1+k} \binom{r-2-k}{k} q^{1+k}(1+q)^{r-2-2k}\\
&\quad= -\sum_{i=1}^{r-1} q^i.
\end{aligned}$$

Therefore, $m_q(\lambda,0) = (q+q^2+\cdots+q^{r-1}+q^r)-(q+q^2+\cdots+q^{r-1}) = q^r$. $\square$

We can now present the following multiplicity result regarding the nonzero weight spaces.

Corollary 6.2. If $\mu \in P(\mathfrak{so}_{2r+1}(\mathbb{C}))$, then

$$m(\lambda,\mu) = \begin{cases} 1 & \text{if } \mu = 0 \text{ or } \mu \in W\cdot\lambda \\ 0 & \text{otherwise} \end{cases}$$

where $W\cdot\lambda$ denotes the orbit of λ under the action of the Weyl group.

Proof. The fact that $m(\lambda, 0) = 1$ follows directly from Theorem 6.2, since $m(\lambda, 0) = m_q(\lambda, 0)|_{q=1} = 1$. Recall that given $\mu \in P(\mathfrak{so}_{2r+1}(\mathbb{C}))$, there exists $w \in W$ and $\xi \in P_+(\mathfrak{so}_{2r+1}(\mathbb{C}))$ such that $w(\xi) = \mu$ and also recall that weight multiplicities are invariant under W [2, Propositions 3.1.20, 3.2.27]. Thus it suffices to consider $\mu \in P_+(\mathfrak{so}_{2r+1}(\mathbb{C}))$. Theorem 3.2 implies $m(\lambda, \lambda) = 1$ and hence $m(\lambda, \mu) = 1$ whenever $\mu \in W \cdot \lambda$. Moreover, Theorem 3.2 also shows $m(\lambda, \mu) = 0$, whenever $\mu \in P_+(\mathfrak{so}_{2r+1}(\mathbb{C})) \setminus \{0, \lambda\}$. ☐

6.3. Lie algebra of type *C*

In this section, we give a result regarding the multiplicity of an integral weight μ in a highest weight representation of $\mathfrak{sp}_{2r}(\mathbb{C})$ with highest weight $\lambda = \alpha_1 + \alpha_2 + \cdots + \alpha_r$.

Theorem 6.3. If $r \geq 3$ and $\lambda = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ is a fundamental weight of $\mathfrak{sp}_{2r}(\mathbb{C})$, then $m_q(\lambda, 0) = 0$.

Proof. By Corollary 4.1 we know that the number of elements in $\mathcal{A}(\lambda, 0)$ is even. Hence, we will establish this result by showing that we can pair up elements $\sigma, \tau \in \mathcal{A}(\lambda, 0)$ such that $\ell(\tau) = \ell(\sigma) \pm 1$ and $\sigma(\lambda + \rho) - \rho = \tau(\lambda + \rho) - \rho$. This implies that the value $\wp_q(\sigma(\lambda + \rho) - \rho)$ appears in $m_q(\lambda, 0)$ with opposite signs. Thus the contributions of these terms cancel in $m_q(\lambda, 0)$. By pairing all of the elements in $\mathcal{A}(\lambda, 0)$ in this way, we establish $m_q(\lambda, 0) = 0$.

To prove the claim we recall that by Theorem 4.1 the elements of $\mathcal{A}(\lambda, 0)$ consist of

1. $\sigma = 1$ or
2. $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$ for some nonconsecutive integers $2 \leq i_1, i_2, \ldots, i_k \leq r - 1$ or
3. $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k} \pi$ for some nonconsecutive integers $2 \leq i_1, i_2, \ldots, i_k \leq r - 4$ and $\pi \in \{s_{r-2}s_{r-1}, s_{r-1}s_{r-2}, s_{r-2}s_{r-1}s_{r-2}\}$.

By Propositions 4.1 and 4.2 note that if $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$ for some nonconsecutive integers $2 \leq i_1, i_2, \ldots, i_k \leq r - 3$, then

$$\sigma(\lambda + \rho) - \rho = \sigma s_{r-1}(\lambda + \rho) - \rho.$$

Thus, we pair σ with $\tau = \sigma s_{r-1}$, which satisfies $\ell(\tau) = \ell(\sigma) + 1$, and the contribution of these terms cancel each other out.

The only remaining elements in $\mathcal{A}(\lambda, 0)$ are of the form $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k} \pi$ for some nonconsecutive integers $2 \leq i_1, i_2, \ldots, i_k \leq r - 4$, where $\pi \in$

$\{s_{r-2}, s_{r-2}s_{r-1}, s_{r-1}s_{r-2}, s_{r-2}s_{r-1}s_{r-2}\}$. Then by Propositions 4.1 and 4.2 note that

$$\sigma s_{r-2}(\lambda+\rho)-\rho = \sigma s_{r-2}s_{r-1}(\lambda+\rho)-\rho$$
$$\sigma s_{r-1}s_{r-2}(\lambda+\rho)-\rho = \sigma s_{r-2}s_{r-1}s_{r-2}(\lambda+\rho)-\rho.$$

Thus, we pair σs_{r-2} with $\tau = \sigma s_{r-2}s_{r-1}$ and we pair $\sigma s_{r-1}s_{r-2}$ with $\tau = \sigma s_{r-2}s_{r-1}s_{r-2}$ which satisfy $\ell(\tau) = \ell(\sigma)+1$, and the contributions of these terms cancel each other out. This completes the proof. $\square$

Corollary 6.3. If $\mu \in P(\mathfrak{sp}_{2r}(\mathbb{C}))$, then $m(\lambda, \mu) = 0$.

The above result follows directly from Theorems 4.2 and 6.3.

6.4. Lie algebra of type *D*

In this section, we give a result regarding the multiplicity of an integral weight μ in a highest weight representation of $\mathfrak{so}_{2r}$ with highest weight $\lambda = \alpha_1+\alpha_2+\cdots+\alpha_r$.

Theorem 6.4. If $r \geq 4$ and $\lambda = \alpha_1+\alpha_2+\cdots+\alpha_r$ is a fundamental weight of $\mathfrak{so}_{2r}(\mathbb{C})$, then $m_q(\lambda, 0) = 0$.

Proof. By Corollary 5.1 we know that the number of elements in $\mathcal{A}(\lambda, 0)$ is even. Hence, we will establish this result by showing that we can pair up elements $\sigma, \tau \in \mathcal{A}(\lambda, 0)$ such that $\ell(\tau) = \ell(\sigma) \pm 1$ and $\sigma(\lambda+\rho)-\rho = \tau(\lambda+\rho)-\rho$. This implies that the value $\wp_q(\sigma(\lambda+\rho)-\rho)$ appears in $m_q(\lambda, 0)$ with opposite signs. Thus the contributions of these terms cancel in $m_q(\lambda, 0)$. By pairing all of the elements in $\mathcal{A}(\lambda, 0)$ in this way, we establish $m_q(\lambda, 0) = 0$.

To prove the claim we recall that by Theorem 5.1 the elements of $\mathcal{A}(\lambda, 0)$ consist of

1. $\sigma = 1$ or
2. $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$ for some nonconsecutive integers $2 \leq i_1, i_2, \ldots, i_k \leq r-2$ or
3. $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}\pi$ for some nonconsecutive integers $2 \leq i_1, i_2, \ldots, i_k \leq r-5$ and $\pi \in \{s_{r-3}s_{r-2},\ s_{r-2}s_{r-3},\ s_{r-3}s_{r-2}s_{r-3}\}$.

By Propositions 5.1 and 5.2 note that if $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$ for some nonconsecutive integers $2 \leq i_1, i_2, \ldots, i_k \leq r-4$, then

$$\sigma(\lambda+\rho)-\rho = \sigma s_{r-2}(\lambda+\rho)-\rho$$
$$\sigma s_{r-2}s_{r-3}(\lambda+\rho)-\rho = \sigma s_{r-3}\sigma s_{r-2}s_{r-3}(\lambda+\rho)-\rho.$$

Thus, we pair σ with $\tau = \sigma s_{r-2}$, and $\sigma s_{r-2} s_{r-3}$ with $\tau = \sigma s_{r-3} \sigma s_{r-2} s_{r-3}$. In either case $\ell(\tau) = \ell(\sigma) + 1$, and the contributions of these terms cancel each other out.

The only remaining elements in $\mathcal{A}(\lambda, 0)$ are of the form $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k} \pi$ for some nonconsecutive integers $2 \le i_1, i_2, \ldots, i_k \le r-5$, where $\pi = s_{r-3}$ or $\pi = s_{r-3} s_{r-2}$. Then by Propositions 5.1 and 5.2 note that

$$\sigma s_{r-3}(\lambda + \rho) - \rho = \sigma s_{r-3} s_{r-2}(\lambda + \rho) - \rho.$$

Thus, we pair σs_{r-3} with $\tau = \sigma s_{r-3} s_{r-2}$, which satisfies $\ell(\tau) = \ell(\sigma) + 1$, and the contributions of these terms cancel each other out. This completes the proof. □

Corollary 6.4. If $\mu \in P(\mathfrak{so}_{2r}(\mathbb{C}))$, then $m(\lambda, \mu) = 0$.

The above result follows directly from Theorems 5.2 and 6.4.

7. Future work

Determining Weyl alternation sets is a new way to describe the complexity of computing weight multiplicities, having only been defined in 2011 by the second author. The only cases where a concrete description of the elements of the Weyl alternation sets exists is in the adjoint representation of the classical Lie algebras (i.e. the representation whose highest weight is the highest root) [5, 8] and in the present work, where we considered the weight λ as the sum of the simple roots and μ and integral weight of the classical Lie algebras. Extending these techniques to other representations can be rather difficult as the Weyl group action on the highest weight of the representation is not as straight forward to describe as in these cases. However, it would be of interest to provide a classification of highest weights where the techniques presented in this manuscript can be extended to describe the elements of other Weyl alternation sets.

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