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Fixed-point-free involutions and Schur P-positivity

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The orbits of the symplectic group acting on the type A flag variety are indexed by the fixed-point-free involutions in a finite symmetric group. The cohomology classes of the closures of these orbits have polynomial representatives $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z$ akin to Schubert polynomials. We show that the *fixed-point-free involution Stanley symmetric functions* $\hat{F}^{\mathtt{FPF}}_z$, which are stable limits of the polynomials $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z$, are Schur P-positive. To do so, we construct an analogue of the Lascoux-Schützenberger tree, an algebraic recurrence that computes Schubert polynomials. As a byproduct of our proof, we obtain a Pfaffian formula of geometric interest for $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z$ when z is a fixed-point-free version of a Grassmannian permutation. We also classify the fixed-point-free involution Stanley symmetric functions that are single Schur P-functions, and show that the decomposition of $\hat{F}^{\mathtt{FPF}}_z$ into Schur P-functions is unitriangular with respect to dominance order on strict partitions. These results and proofs mirror previous work by the authors related to the orthogonal group action on the type A flag variety.

1. Introduction

Fix a positive integer n and let $B \subset \mathrm{GL}_n(\mathbb{C})$ be the Borel subgroup of lower triangular matrices in the general linear group. The orbits Ω_w of the opposite Borel subgroup of upper triangular matrices acting on the *flag variety* $\mathrm{Fl}(n) = \mathrm{GL}_n(\mathbb{C})/B$ are indexed by permutations $w \in S_n$ and their closures X_w give $\mathrm{Fl}(n)$ a CW-complex structure. The cohomology ring of $\mathrm{Fl}(n)$ has a presentation in terms of the *Schubert polynomials* $\mathfrak{S}_w$ introduced by Lascoux and Schützenberger [15]. For the precise definition of $\mathfrak{S}_w$, see Section 2.2.

Schubert polynomials are of continued interest to both algebraic geometers and combinatorialists. Computing the positive structure coefficients c^w_{uv} in the expansion $\mathfrak{S}_u\mathfrak{S}_v = \sum c^w_{uv}\mathfrak{S}_w$ remains a prominent open problem in algebraic combinatorics. Among other interesting formulas, there is a generating function-type description of $\mathfrak{S}_w$ in terms of the reduced words for w

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[3], and a determinantal formula for $\mathfrak{S}_w$ when w is *vexillary* (2143-avoiding) or *fully commutative* (321-avoiding). When w is *dominant* (132-avoiding), $\mathfrak{S}_w$ is a monomial.

Assume n is even and consider the symplectic group $\mathrm{Sp}_n(\mathbb{C})$ acting on $\mathrm{Fl}(n)$. There are again finitely many orbits, now indexed by the fixed-point-free involutions in S_n [22]. For a fixed-point-free involution $z \in S_n$, the cohomology class of the corresponding orbit closure Y_z is represented by the *fixed-point-free involution Schubert polynomial* $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z$ introduced in [30] and described precisely by Definition 2.4. In [8], we gave a generating function-type description of $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z$ in terms of reduced words and derived a simple product formula for $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z$ when z is a dominant fixed-point-free involution. In this paper, we continue to study $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z$ and related combinatorics. Some of this combinatorics also appears in representation theory when studying the quasi-parabolic Iwahori-Hecke algebra modules defined by Rains and Vazirani [21].

The groups $\mathrm{O}_n(\mathbb{C})$ and $\mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_q(\mathbb{C})$ (with $p + q = n$) also act on $\mathrm{Fl}(n)$ with finitely many orbits. This paper is a continuation of the authors' previous work on the $\mathrm{O}_n(\mathbb{C})$ case [11]. The $\mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_q(\mathbb{C})$ case has not yet been as thoroughly investigated, though there has been some recent progress in [4]; see also [5, 31].

The symmetric group S_n of permutations of $[n] = \{1, 2, \ldots, n\}$ is a Coxeter group generated by the simple transpositions $s_i = (i, i + 1)$ for $1 \leq i \leq n - 1$. For $u \in S_m$ and $v \in S_n$, we write $u \times v$ for the permutation in S_{m+n} that maps $i \mapsto u(i)$ for $i \in [m]$ and $m + i \mapsto m + v(i)$ for $i \in [n]$. The *Stanley symmetric function* of $w \in S_n$ is then the stable limit

$$F_w \overset{\mathrm{def}}{=} \lim_{m \to \infty} \mathfrak{S}_{1_m \times w}$$

where 1_m denotes the identity element of S_m. This is a well-defined homogeneous symmetric function; see Section 2.2. These functions were introduced by Stanley to enumerate reduced words [26]. Edelman and Greene showed bijectively that Stanley symmetric functions are Schur positive using an insertion algorithm [7].

A permutation is *Grassmannian* if it has exactly one descent. If $w \in S_n$ is Grassmannian then $\mathfrak{S}_w$ is a Schur polynomial and F_w is a Schur function [19, Proposition 2.6.8]. One can show algebraically that F_w is Schur positive by using the *Lascoux-Schützenberger tree* [15], an iterated recurrence for Schubert polynomials based on certain specializations of Monk's rule. The Lascoux-Schützenberger tree decomposes $\mathfrak{S}_w$ into a sum of Schubert polynomials indexed by Grassmannian permutations and other terms whose stable limits vanish.

Let FPF_n be the set of fixed-point-free involutions in S_{2n}. Define $\Theta_n = (1,2)(3,4)\dots(2n-1,2n) \in \mathsf{FPF}_n$. The *fixed-point-free involution Stanley symmetric function* of $z \in \mathsf{FPF}_n$ is the limit

$$\hat{F}^{\mathtt{FPF}}_z \overset{\text{def}}{=} \lim_{n\to\infty} \hat{\mathfrak{S}}^{\mathtt{FPF}}_{\Theta_n \times z}$$

which is a well-defined homogeneous symmetric function; see Section 2.3. We introduced these functions in [8] to study the enumeration of certain analogues of reduced words.

The odd power-sum functions $p_1, p_3, p_5, \dots$ generate a subalgebra Γ of the usual algebra of symmetric functions Λ. This subalgebra has a distinguished basis $\{P_\lambda\}$ indexed by strict integer partitions, whose elements P_λ are the so-called *Schur P-functions*. See Section 2.4 for the precise definition. In [8] we conjectured the following statement, which is proved at the end of Section 5:

Theorem 1.1. Each $\hat{F}^{\mathtt{FPF}}_z$ is *Schur P-positive*, i.e., $\hat{F}^{\mathtt{FPF}}_z \in \mathbb{N}\text{-span}\{P_\lambda : \lambda \text{ is a strict partition}\}$.

The first step in our proof of this result to identify the "fixed-point-free" analogue of a Grassmannian permutation and then prove that $\hat{F}^{\mathtt{FPF}}_z$ is a Schur P-function when z is an involution of this type. The precise definition of an *FPF-Grassmannian* involution is sightly unintuitive; for the details, see Definition 4.14. We can easily describe which Schur P-function corresponds to an FPF-Grassmannian involution, however.

The *(FPF-involution) code* of $z \in \mathsf{FPF}_n$ is the sequence $\hat{c}_{\mathtt{FPF}}(z) = (c_1, c_2, \dots, c_{2n})$ in which c_i is the number of positive integers j with $j < i < z(j)$ and $j < z(i)$. Define the *shape* of $z \in \mathsf{FPF}_n$ to be the partition $\nu(z)$ given by the transpose of the partition that sorts $\hat{c}_{\mathtt{FPF}}(z)$. For example, if $z = 2n \cdots 321 = (1,2n)(2,2n-1)\cdots(n,n+1) \in \mathsf{FPF}_n$, then $\hat{c}_{\mathtt{FPF}}(z) = (0,1,2,\dots,n-1,n-1,\dots,2,1,0)$ and $\nu(z) = (2n-2, 2n-4, \dots, 2)$. The following is proved as Theorem 4.19.

Theorem 1.2. If $z \in \mathsf{FPF}_n$ is FPF-Grassmannian, then $\nu(z)$ is strict and $\hat{F}^{\mathtt{FPF}}_z = P_{\nu(z)}$.

The second step in our proof of Theorem 1.1 is to define an analogue of the Lascoux-Schützenberger tree for fixed-point-free involutions. We do this using the transition equations that we introduced in [10]. We show that repeated applications of these transition equations always result in a sum of $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z$'s where z is FPF-Grassmannian, along with other terms whose stable limits vanish. The desired Schur P-positivity property follows from Theorem 1.2 on taking limits.

This proof can be recast as an algorithm to explicitly compute any $\hat{F}^{\mathtt{FPF}}_z$. By choosing an appropriate involution, one can use this algorithm to expand any product $P_\lambda P_\mu$ as a positive linear combination of Schur P-functions. In this way, we obtain a new Littlewood-Richardson rule for Schur P-functions from our results (see Corollary 5.24).

It remains an open problem to find a bijective proof of Theorem 1.2. Since the FPF-transition equations have a bijective interpretation [10], a bijective proof of Theorem 1.2 would, in principle, lead to a bijective proof of Theorem 1.1. A more direct way of proving Theorem 1.1 bijectively would be to find an insertion algorithm for *fixed-point-free involution words* (see Section 2.3).

A permutation $w \in S_n$ is *vexillary* if F_w is a single Schur function. Analogously, we say that $z \in \mathsf{FPF}_n$ is *FPF-vexillary* if $\hat{F}^{\mathtt{FPF}}_z$ is a single Schur P-function. FPF-Grassmannian involutions are FPF-vexillary by Theorem 1.2. Stanley showed that $w \in S_n$ is vexillary if and only if w avoids the pattern 2143. A similar result holds for involutions; see Theorem 7.8 for the full statement.

Theorem 1.3. There is a pattern avoidance condition characterizing FPF-vexillary involutions.

The *dominance order* on partitions is the partial order $\leq$ with $\lambda \leq \mu$ if $\sum_{i=1}^m \lambda_i \leq \sum_{i=1}^m \mu_i$ for all $m \in \mathbb{N}$. In Section 6, we show that the Schur P-expansion of $\hat{F}^{\mathtt{FPF}}_z$ is unitriangular with respect to dominance order, in the following sense:

Theorem 1.4. If $z \in \mathsf{FPF}_n$ then $\nu(z)$ is strict and $\hat{F}^{\mathtt{FPF}}_z \in P_\nu + \mathbb{N}\text{-span}\{P_\lambda : \lambda < \nu(z)\}$.

We mention a quick application of these results. The explicit version of Theorem 1.3 implies that the reverse permutation $2n \cdots 321 \in \mathsf{FPF}_n$ is FPF-vexillary. By Theorem 1.4, we therefore have $\hat{F}^{\mathtt{FPF}}_{2n\cdots 321} = P_{\nu(2n\cdots 321)} = P_{(2n-2,2n-4,\ldots,2)}$. In prior work, we proved that $\hat{F}^{\mathtt{FPF}}_{2n\cdots 321} = (s_{\delta_n})^2$ where s_λ is the Schur function of a partition λ and $\delta_n = (n-1,\ldots,3,2,1)$ [8, Theorem 1.4]. Combining these formulas shows that $P_{(2n-2,2n-4,\ldots,2)} = (s_{\delta_n})^2$, which is a special case of [6, Theorem V.3].

Assume $z \in \mathsf{FPF}_n$ is FPF-Grassmannian. The symmetric function $\hat{F}^{\mathtt{FPF}}_z = P_{\nu(z)}$ can then be expressed as the Pfaffian of a matrix whose entries are

Schur P-functions indexed by partitions with at most two parts. This formula is essentially Schur's original definition of P_λ in [24]. In general, the polynomial $\hat{\mathfrak{S}}^{\mathsf{FPF}}_z$ is not equal to $P_{\nu(z)}$ specialized to finitely many variables. However, $\hat{\mathfrak{S}}^{\mathsf{FPF}}_z$ has a similar Pfaffian formula which we sketch as follows.

There is an FPF-Grassmannian involution z of shape $(n-\phi_1, n-\phi_2, \dots, n-\phi_r)$ associated to each sequence of integers $1 \leq \phi_1 < \phi_2 < \cdots < \phi_r \leq n$, and we define $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_1, \phi_2, \dots, \phi_r; n] = \hat{\mathfrak{S}}^{\mathsf{FPF}}_z$ to be the FPF-involution Schubert polynomial of this element. For the precise definition, see (30). The following is restated as Theorem 8.8 and illustrated in a concrete case by Example 8.9.

Theorem 1.5. Suppose $1 \leq \phi_1 < \phi_2 < \cdots < \phi_r \leq n$ are integers. Let m be whichever of r or $r+1$ is even. Define $\mathfrak{M}$ to be the $m \times m$ skew-symmetric matrix with $\mathfrak{M}_{ij} = -\mathfrak{M}_{ji} = \hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_i, \phi_j; n]$ whenever $i < j$, where $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_i, \phi_{r+1}; n] \overset{\text{def}}{=} \hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_i; n]$. Then $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_1, \phi_2, \dots, \phi_r; n] = \operatorname{pf} \mathfrak{M}$.

Combining this identity with our Lascoux-Schützenberger tree for fixed-point-free involutions gives an algorithm for expanding any $\hat{\mathfrak{S}}^{\mathsf{FPF}}_z$ as a sum of Pfaffians. One piece is missing to make this algorithm effective as a means of computing $\hat{\mathfrak{S}}^{\mathsf{FPF}}_z$: it remains an open problem to find a simple formula for the terms $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_i, \phi_j; n]$ appearing in the matrix $\mathfrak{M}$ in Theorem 1.5. This is unexpectedly nontrivial.

There is a determinantal formula for $\mathfrak{S}_w$ which holds when $w \in S_n$ is a vexillary permutation. Analogously, there should exist a Pfaffian formula for $\hat{\mathfrak{S}}^{\mathsf{FPF}}_z$ applicable when z is any FPF-vexillary involution. Such a formula would generalize Theorem 1.5 since FPF-Grassmannian involutions are FPF-vexillary. There is also a determinantal formula for $\mathfrak{S}_w$ when w is fully commutative. This formula should have an analogue for the polynomials $\hat{\mathfrak{S}}^{\mathsf{FPF}}_z$; however, we do not yet know what the appropriate "fixed-point-free" analogue of a fully commutative permutation should be.

Knutson, Lam, and Speyer have given a geometric interpretation of the Stanley symmetric function F_w as the representative for the class of a *graph Schubert variety* in the Grassmannian $\mathrm{Gr}(n, 2n)$ [14]. It would be interesting to find a geometric interpretation of Theorem 1.1 in this vein. Schur P-functions are cohomology representatives for Schubert varieties in the orthogonal Grassmannian. We believe there is a way to adapt the construction of Knutson, Lam, and Speyer to give a subvariety of the orthogonal Grassmannian whose class is represented by $\hat{F}^{\mathsf{FPF}}_z$, resulting in a geometric proof of Theorem 1.1. A similar approach should also relate $\mathrm{O}_n(\mathbb{C})$-orbit closures to the geometry of the Lagrangian Grassmannian.

2. Preliminaries

Let $\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z}$ denote the respective sets of positive, nonnegative, and all integers. For $n \in \mathbb{P}$, let $[n] \overset{\text{def}}{=} \{1, 2, \ldots, n\}$. The *support* of a map $w : X \to X$ is the set $\operatorname{supp}(w) \overset{\text{def}}{=} \{i \in X : w(i) \neq i\}$. Define $S_\mathbb{Z}$ as the group of permutations of $\mathbb{Z}$ with finite support, and let $S_\infty \subset S_\mathbb{Z}$ be the subgroup of permutations with support contained in $\mathbb{P}$. We view S_n as the subgroup of permutations in S_∞ fixing all integers outside $[n]$.

Throughout, we let $s_i \overset{\text{def}}{=} (i, i+1) \in S_\mathbb{Z}$ for $i \in \mathbb{Z}$. Let $\mathcal{R}(w)$ be the set of *reduced words* for $w \in S_\mathbb{Z}$, i.e., the sequences $(s_{i_1}, s_{i_2}, \ldots, s_{i_p})$ of simple transpositions of shortest possible length such that $w = s_{i_1} s_{i_2} \ldots s_{i_p}$. Write $\ell(w)$ for the common length of each word in $\mathcal{R}(w)$. When $w : \mathbb{Z} \to \mathbb{Z}$ is any bijection, we let $\operatorname{Des}_R(w)$ (respectively, $\operatorname{Des}_L(w)$) denote the set of simple transpositions s_i for $i \in \mathbb{Z}$ with $w(i) > w(i+1)$ (respectively $w^{-1}(i) > w^{-1}(i+1)$). If $w \in S_\mathbb{Z}$ then $\operatorname{Des}_L(w)$ and $\operatorname{Des}_R(w)$ are the usual *right* and *left descent sets* of w, consisting of the simple transpositions s such that $\ell(sw) < \ell(w)$ and $\ell(ws) < \ell(w)$, respectively.

2.1. Divided difference operators

We recall a few properties of *divided difference operators*. Our main references are [13, 19]. Let $\mathcal{L} \overset{\text{def}}{=} \mathbb{Z}\left[x_1, x_2, \ldots, x_1^{-1}, x_2^{-1}, \ldots\right]$ be the ring of Laurent polynomials over $\mathbb{Z}$ in a countable set of commuting indeterminates, and let $\mathcal{P} \overset{\text{def}}{=} \mathbb{Z}[x_1, x_2, \ldots]$ be the subring of polynomials in $\mathcal{L}$. The group S_∞ acts on $\mathcal{L}$ by permuting variables, and one defines

$$(1) \qquad \partial_i f \overset{\text{def}}{=} (f - s_i f)/(x_i - x_{i+1}) \qquad \text{for } i \in \mathbb{P} \text{ and } f \in \mathcal{L}.$$

The *divided difference operator* ∂_i defines a map $\mathcal{L} \to \mathcal{L}$ that restricts to a map $\mathcal{P} \to \mathcal{P}$. It is clear by definition that $\partial_i f = 0$ if and only if $s_i f = f$. If $f \in \mathcal{L}$ is homogeneous and $\partial_i f \neq 0$ then $\partial_i f$ is homogeneous of degree $\deg(f) - 1$. If $f, g \in \mathcal{L}$ then $\partial_i(fg) = (\partial_i f)g + (s_i f)\partial_i g$, and if $\partial_i f = 0$, then $\partial_i(fg) = f\partial_i g$.

For $i \in \mathbb{P}$ the *isobaric divided difference operator* $\pi_i : \mathcal{L} \to \mathcal{L}$ is defined by

$$(2) \qquad \pi_i(f) \overset{\text{def}}{=} \partial_i(x_i f) = f + x_{i+1}\partial_i f \qquad \text{for } f \in \mathcal{L}.$$

Observe that $\pi_i f = f$ if and only if $s_i f = f$, in which case $\pi_i(fg) = f\pi_i(g)$ for $g \in \mathcal{L}$. If $f \in \mathcal{L}$ is homogeneous with $\pi_i f \neq 0$, then $\pi_i f$ is homogeneous of the same degree. The operators ∂_i and π_i both satisfy the braid relations for S_∞, so we may define $\partial_w = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}$ and $\pi_w = \pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ for any $(s_{i_1}, s_{i_2}, \dots, s_{i_k}) \in \mathcal{R}(w)$. Moreover, one has $\partial_i^2 = 0$ and $\pi_i^2 = \pi_i$ for all $i \in \mathbb{P}$.

2.2. Schubert polynomials and Stanley symmetric functions

Fix $n \in \mathbb{P}$ and let $w_n \overset{\text{def}}{=} n\cdots 321 \in S_n$ and $x^{\delta_n} \overset{\text{def}}{=} x_1^{n-1}x_2^{n-2}\cdots x_{n-1}^1$. The *Schubert polynomial* (see [13, 19]) of $w \in S_n$ is the polynomial

$$\mathfrak{S}_w \overset{\text{def}}{=} \partial_{w^{-1}w_n} x^{\delta_n} \in \mathcal{P}.$$

This formula for $\mathfrak{S}_w$ is independent of the choice of n such that $w \in S_n$, and we consider the Schubert polynomials to be a family indexed by S_∞. Since $\partial_i^2 = 0$, it follows that

$$\text{(3)} \qquad \mathfrak{S}_1 = 1 \quad \text{and} \quad \partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_i} & \text{if } s_i \in \mathrm{Des}_R(w) \\ 0 & \text{if } s_i \notin \mathrm{Des}_R(w) \end{cases} \quad \text{for each } i \in \mathbb{P}.$$

Conversely, one can show that $\{\mathfrak{S}_w\}_{w\in S_\infty}$ is the unique family of homogeneous polynomials indexed by S_∞ satisfying (3); see [13, Theorem 2.3] or the introduction of [2]. Each $\mathfrak{S}_w$ has degree $\ell(w)$, and the polynomials $\mathfrak{S}_w$ for $w \in S_\infty$ form a $\mathbb{Z}$-basis for $\mathcal{P}$ [19, Proposition 2.5.4].

There is a useful formula for $\mathfrak{S}_w$ as a sort of generating function over reduced words due to Billey, Jockusch, and Stanley [3]. Fix $w \in S_n$, and for each $a = (s_{a_1}, s_{a_2}, \dots, s_{a_k}) \in \mathcal{R}(w)$, let $C(a)$ be the set of sequences of positive integers $I = (i_1, i_2 \dots, i_k)$ satisfying

$$\text{(4)} \qquad i_1 \le i_2 \le \cdots \le i_k \qquad \text{and} \qquad i_j < i_{j+1} \text{ whenever } a_j < a_{j+1}.$$

We write $I \le a$ to indicate that $i_j \le a_j$ for all j and define $x_I = x_{i_1}x_{i_2}\cdots x_{i_k}$. The Schubert polynomial corresponding to $w \in S_n$ is then [3, Theorem 1.1]

$$\text{(5)} \qquad \mathfrak{S}_w = \sum_{a\in\mathcal{R}(w)} \sum_{\substack{I\in C(a) \\ I\le a}} x_I.$$

For example, since $\mathcal{R}(312) = \{(s_2, s_1)\}$ and $\mathcal{R}(1342) = \{(s_2, s_3)\}$, it holds that

$$\mathfrak{S}_{312} = x_1^2 \qquad \text{and} \qquad \mathfrak{S}_{1342} = x_1x_2 + x_1x_3 + x_2x_3.$$

As expected, one has $\partial_1\mathfrak{S}_{312} = \partial_3\mathfrak{S}_{1342} = \mathfrak{S}_{132} = x_1 + x_2$.

Write Λ for the usual subring of bounded degree *symmetric functions* in the ring of formal power series $\mathbb{Z}[[x_1, x_2, \dots]]$. A sequence of power series

$f_1, f_2, \ldots$ has a limit $\lim_{n\to\infty} f_n \in \mathbb{Z}[[x_1, x_2, \ldots]]$ if the coefficient sequence of each fixed monomial is eventually constant. For any map $w : \mathbb{Z} \to \mathbb{Z}$ and $N \in \mathbb{Z}$, let $w \gg N : \mathbb{Z} \to \mathbb{Z}$ be the map $i \mapsto w(i - N) + N$.

Definition 2.1. If $w \in S_\mathbb{Z}$ then the limit

$$F_w \overset{\text{def}}{=} \lim_{N\to\infty} \mathfrak{S}_{w\gg N} = \sum_{a\in\mathcal{R}(w)} \sum_{I\in C(a)} x_I \in \mathbb{Z}[[x_1, x_2, \ldots]]$$

is the *Stanley symmetric function* of w.

The second equality in this definition follows from (5). Stanley introduced these power series and proved that they are symmetric in [26]. (The indexing conventions of [26] differ from ours by the transformation of indices $w \mapsto w^{-1}$.) The symmetric function F_w is homogeneous of degree $\ell(w)$, and the coefficient of any square-free monomial in F_w is $|\mathcal{R}(w)|$. For example,

$$F_{321} = \sum_{i<j<k} 2x_i x_j x_k + \sum_{i<j} (x_i^2 x_j + x_i x_j^2)$$

and $|\mathcal{R}(321)| = |\{(s_1, s_2, s_1), (s_2, s_1, s_2)\}| = [x_1x_2x_3]F_{321} = 2$.

Definition 2.1 makes it clear that $F_w = F_{w\gg N}$ for any $N \in \mathbb{Z}$, but does not tell us how to efficiently compute these symmetric functions. It is well-known result of Edelman and Greene [7] that each F_w is Schur positive; for a brief account of one way to compute the corresponding Schur expansion, see [11, §4.2]. We require one other definition of F_w.

Lemma 2.2 (Macdonald [17])**.** If $w \in S_\infty$ then $F_w = \lim_{n\to\infty} \pi_{w_n} \mathfrak{S}_w$.

Proof. This is reproved in [8, §3]: the claim follows from [8, Proposition 3.37 and Theorem 3.39]. □

2.3. FPF-involution Schubert polynomials

For $n \in \mathbb{P}$, let FPF_n be the set of permutations $z \in S_n$ with $z = z^{-1}$ and $z(i) \neq i$ for all $i \in [n]$. Let FPF_∞ and $\mathsf{FPF}_\mathbb{Z}$ be the S_∞- and $S_\mathbb{Z}$-conjugacy classes of the permutation $\Theta : \mathbb{Z} \to \mathbb{Z}$ given by

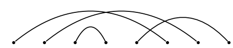
$$\Theta : i \mapsto i - (-1)^i. \tag{6}$$

We refer to elements of FPF_n, FPF_∞, and $\mathsf{FPF}_\mathbb{Z}$ as *fixed-point-free (FPF) involutions*. Note that FPF_n is empty if n is odd. For $z \in \mathsf{FPF}_\mathbb{Z}$ and $N \in \mathbb{Z}$, we see $z \gg N \in \mathsf{FPF}_\mathbb{Z}$ if and only if N is even. While technically $\mathsf{FPF}_n \not\subset \mathsf{FPF}_\infty$, there is a natural inclusion

$$\iota : \mathsf{FPF}_n \hookrightarrow \mathsf{FPF}_\infty \tag{7}$$

given by the map that sends $z \in \mathsf{FPF}_n$ to the permutation of $\mathbb{Z}$ whose restrictions to $[n]$ and to $\mathbb{Z} \setminus [n]$ coincide respectively with those of z and Θ. In symbols, we have $\iota(z) = z \cdot \Theta \cdot s_1 \cdot s_3 \cdot s_5 \cdots s_{n-1}$. We obtain $\Theta_n = (1,2)(3,4)\dots(2n{-}1,2n)$ by restricting Θ to $[2n]$.

We identify elements of FPF_n, FPF_∞, or $\mathsf{FPF}_\mathbb{Z}$ with the complete matchings on $[n]$, $\mathbb{P}$, or $\mathbb{Z}$ with distinct vertices connected by an edge whenever they form a nontrivial cycle. We depict such matchings with the vertices on a horizontal axis, ordered from left to right, and edges shown as convex curves in the upper half plane. For example,

$(1,6)(2,7)(3,4)(5,8) \in \mathsf{FPF}_8$ is represented as 

We will omit the numbers labeling the vertices in these matchings if they remain clear from context.

For each $z \in \mathsf{FPF}_\mathbb{Z}$, define

$$\begin{aligned} \mathrm{Inv}(z) &= \{(i,j) \in \mathbb{Z} \times \mathbb{Z} : i < j, z(i) > z(j)\}, \\ \mathrm{Cyc}_\mathbb{Z}(z) &= \{(i,j) \in \mathbb{Z} \times \mathbb{Z} : i < j = z(i)\}, \end{aligned} \tag{8}$$

so that $\mathrm{Des}_R(z) = \{s_i : (i,i+1) \in \mathrm{Inv}(z)\}$. In turn let

$$\mathrm{Cyc}_\mathbb{P}(z) = \mathrm{Cyc}_\mathbb{Z}(z) \cap (\mathbb{P} \times \mathbb{P}).$$

The set

$$\mathrm{Inv}_\mathsf{FPF}(z) \overset{\text{def}}{=} \mathrm{Inv}(z) - \mathrm{Cyc}_\mathbb{Z}(z) \tag{9}$$

is finite with an even number of elements, and is empty if and only if $z = \Theta$. We let $\hat\ell_\mathsf{FPF}(z) = \frac{1}{2}|\mathrm{Inv}_\mathsf{FPF}(z)|$ and

$$\mathrm{Des}_R^\mathsf{FPF}(z) = \{s_i \in \mathrm{Des}_R(z) : (i,i+1) \notin \mathrm{Cyc}_\mathbb{Z}(z)\}. \tag{10}$$

These definitions are related by the following proposition.

Proposition 2.3. If $z \in \mathsf{FPF}_\mathbb{Z}$ then

$$\hat\ell_\mathsf{FPF}(szs) = \begin{cases} \hat\ell_\mathsf{FPF}(z) - 1 & \text{if } s \in \mathrm{Des}_R^\mathsf{FPF}(z) \\ \hat\ell_\mathsf{FPF}(z) & \text{if } s \in \mathrm{Des}_R(z) - \mathrm{Des}_R^\mathsf{FPF}(z) \\ \hat\ell_\mathsf{FPF}(z) + 1 & \text{if } s \in \{s_i : i \in \mathbb{Z}\} - \mathrm{Des}_R(z). \end{cases}$$

Proof. If $s \in \mathrm{Des}_R(z) - \mathrm{Des}_R^{\mathtt{FPF}}(z)$, we have $szs = z$. When $s_i \in \mathrm{Des}_R^{\mathtt{FPF}}(z)$, we see $z(i) > z(i+1) \neq i$ so $\mathrm{Inv}_{\mathtt{FPF}}(z) = \mathrm{Inv}_{\mathtt{FPF}}(szs) \cup \{(i, i+1), (z(i+1), z(i))\}$. Then $\hat\ell_{\mathtt{FPF}}(z) = \hat\ell_{\mathtt{FPF}}(szs) + 1$. Finally, if $s \notin \mathrm{Des}_R(z)$, we see szs satisfies the previous case so $\hat\ell_{\mathtt{FPF}}(z) = \hat\ell_{\mathtt{FPF}}(szs) - 1$. □

Define $\mathcal{A}_{\mathtt{FPF}}(z)$ for $z \in \mathsf{FPF}_{\mathbb{Z}}$ as the set of permutations $w \in S_{\mathbb{Z}}$ of minimal length with $z = w^{-1}\Theta w$. This set is nonempty and finite, and its elements all have length $\hat\ell_{\mathtt{FPF}}(z)$. We define

$$\hat{\mathcal{R}}_{\mathtt{FPF}}(z) = \bigsqcup_{w \in \mathcal{A}_{\mathtt{FPF}}(z)} \mathcal{R}(w) \tag{11}$$

to be the set of *(reduced) fixed-point-free involution words* for z.

Definition 2.4. The *FPF-involution Schubert polynomial* of $z \in \mathsf{FPF}_\infty$ is

$$\hat{\mathfrak{S}}_z^{\mathtt{FPF}} \overset{\text{def}}{=} \sum_{w \in \mathcal{A}_{\mathtt{FPF}}(z)} \mathfrak{S}_w.$$

For $z \in \mathsf{FPF}_n$, we set $\mathcal{A}_{\mathtt{FPF}}(z) = \mathcal{A}_{\mathtt{FPF}}(\iota(z))$ and $\hat{\mathfrak{S}}_z^{\mathtt{FPF}} = \hat{\mathfrak{S}}_{\iota(z)}^{\mathtt{FPF}}$.

Example 2.5. We have $\iota(4321) = s_1 s_2 \Theta s_2 s_1 = s_3 s_2 \Theta s_2 s_3$ and $\mathcal{A}_{\mathtt{FPF}}(4321) = \{312, 1342\}$, so $\hat{\mathfrak{S}}_{4321}^{\mathtt{FPF}} = \mathfrak{S}_{312} + \mathfrak{S}_{1342} = x_1^2 + x_1x_2 + x_1x_3 + x_2x_3$.

The polynomials $\hat{\mathfrak{S}}_z^{\mathtt{FPF}}$ have the following characterization via divided differences.

Theorem 2.6 ([8, Corollary 3.13])**.** The FPF-involution Schubert polynomials $\{\hat{\mathfrak{S}}_z^{\mathtt{FPF}}\}_{z \in \mathsf{FPF}_\infty}$ are the unique family of homogeneous polynomials indexed by FPF_∞ such that $\hat{\mathfrak{S}}_\Theta^{\mathtt{FPF}} = 1$ and such that if $i \in \mathbb{P}$ and $s = s_i$ then

$$\partial_i \hat{\mathfrak{S}}_z^{\mathtt{FPF}} = \begin{cases} \hat{\mathfrak{S}}_{szs}^{\mathtt{FPF}} & \text{if } s \in \mathrm{Des}_R(z) \text{ and } (i, i+1) \notin \mathrm{Cyc}_{\mathbb{Z}}(z) \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Wyser and Yong first considered these polynomials in [30], where they were denoted $\Upsilon_{z;(\mathrm{GL}_n, \mathrm{Sp}_n)}$. They showed, when n is even, that the FPF-involution Schubert polynomials indexed by FPF_n are cohomology representatives for the $\mathrm{Sp}_n(\mathbb{C})$-orbit closures in the flag variety $\mathrm{Fl}(n) = \mathrm{GL}_n(\mathbb{C})/B$, with $B \subset \mathrm{GL}_n(\mathbb{C})$ denoting the Borel subgroup of lower triangular matrices. The symmetric functions $\hat F_z^{\mathtt{FPF}}$ are related to the polynomials $\hat{\mathfrak{S}}_z^{\mathtt{FPF}}$ by the following identity.

Definition 2.7. The *FPF-involution Stanley symmetric function* of $z \in \mathsf{FPF}_{\mathbb{Z}}$ is the power series

$$\hat{F}^{\mathtt{FPF}}_z \overset{\text{def}}{=} \sum_{w \in \mathcal{A}_{\text{FPF}}(z)} F_w = \lim_{N\to\infty} \hat{\mathfrak{S}}^{\mathtt{FPF}}_{z \gg 2N} \in \Lambda.$$

Lemma 2.8. If $z \in \mathsf{FPF}_\infty$ then $\hat{F}^{\mathtt{FPF}}_z = \lim_{n\to\infty} \pi_{w_n} \hat{\mathfrak{S}}^{\mathtt{FPF}}_z$.

Proof. This is immediate from Lemma 2.2. □

2.4. Schur P-functions

Our main results will relate $\hat{F}^{\mathtt{FPF}}_z$ to the *Schur P-functions* in Λ, which were introduced in work of Schur [24] and have since arisen in a variety of other contexts (see, e.g., [2, 12, 20]). Good references for these symmetric functions include [28, §6] and [18, §III.8]. For integers $0 \le m \le n$, let

$$G_{m,n} \overset{\text{def}}{=} \prod_{i\in[m]} \prod_{j\in[n-i]} \left(1 + x_i^{-1} x_{i+j}\right) \in \mathcal{L}. \tag{13}$$

For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, let $\ell(\lambda)$ denote the largest index $i \in \mathbb{P}$ with $\lambda_i \neq 0$. The partition λ is *strict* if $\lambda_i \neq \lambda_{i+1}$ for all $i < \ell(\lambda)$. Define $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_\ell^{\lambda_\ell}$ where $\ell = \ell(\lambda)$.

Definition 2.9. Let λ be a strict partition with $\ell = \ell(\lambda)$ parts. The power series

$$P_\lambda \overset{\text{def}}{=} \lim_{n\to\infty} \pi_{w_n} \left(x^\lambda G_{\ell,n}\right) \in \Lambda$$

is then a well-defined, homogeneous symmetric function of degree $\sum_i \lambda_i$, which one calls the *Schur P-function* of λ.

We present this slightly unusual definition of P_λ for its compatibility with Definition 2.1. The symmetric functions P_λ may be described more concretely as generating functions for certain shifted tableaux [18, Ex. (8.16′), §III.8]. The equivalence of the two definitions is explained in [18, Example 1, §III.8].

Whereas the Schur functions form a $\mathbb{Z}$-basis for Λ, the Schur P-functions form a $\mathbb{Z}$-basis for the subring $\Gamma = \mathbb{Q}[p_1, p_3, p_5, \dots] \cap \Lambda$ generated by the odd-indexed power sum symmetric functions [28, Corollary 6.2(b)]. Sagan [23] and Worley [29] showed independently that each Schur P-function P_λ is itself Schur positive. For more information about the positivity properties of the symmetric functions, see the discussion of [18, Eq. (8.17), §III.8] in Macdonald's book.

3. Transition formulas

The *Bruhat order* $<$ on $S_{\mathbb{Z}}$ is the weakest partial order with $w < wt$ when $w \in S_{\mathbb{Z}}$ and $t \in S_{\mathbb{Z}}$ is a transposition such that $\ell(w) < \ell(wt)$. We define the *Bruhat order* $<$ on $\mathsf{FPF}_{\mathbb{Z}}$ as the weakest partial order with $z < tzt$ when $z \in \mathsf{FPF}_{\mathbb{Z}}$ and $t \in S_{\mathbb{Z}}$ is a transposition such that $\hat{\ell}_{\mathsf{FPF}}(z) < \hat{\ell}_{\mathsf{FPF}}(tzt)$. Rains and Vazirani's results in [21] imply the following theorem from [10].

Theorem 3.1 ([10, Theorem 4.6])**.** Let $n \in 2\mathbb{P}$. The following properties hold:

(a) $(\mathsf{FPF}_{\mathbb{Z}}, <)$ is a graded poset with rank function $\hat{\ell}_{\mathsf{FPF}}$.
(b) If $y, z \in \mathsf{FPF}_n$ then $y \leq z$ holds in $(S_{\mathbb{Z}}, <)$ if and only if $\iota(y) \leq \iota(z)$ holds in $(\mathsf{FPF}_{\mathbb{Z}}, <)$.
(c) Fix $y, z \in \mathsf{FPF}_{\mathbb{Z}}$ and $w \in \mathcal{A}_{\mathsf{FPF}}(z)$. Then $y \leq z$ if and only if some $v \in \mathcal{A}_{\mathsf{FPF}}(y)$ has $v \leq w$.

Both $\iota(\mathsf{FPF}_n)$ and FPF_∞ are lower ideals in $(\mathsf{FPF}_{\mathbb{Z}}, <)$. We write $y \lessdot_{\mathsf{FPF}} z$ for $y, z \in \mathsf{FPF}_{\mathbb{Z}}$ if $\{w \in \mathsf{FPF}_{\mathbb{Z}} : y \leq w < z\} = \{y\}$. If $y, z \in \mathsf{FPF}_n$ for some $n \in 2\mathbb{P}$ and $\iota(y) \lessdot_{\mathsf{FPF}} \iota(z)$, then we write $y \lessdot_{\mathsf{FPF}} z$. For example, the set FPF_4 is totally ordered by $<$ and we have

$$\mathsf{FPF}_4 = \{(1,2)(3,4) \lessdot_{\mathsf{FPF}} (1,3)(2,4) \lessdot_{\mathsf{FPF}} (1,4)(2,3)\}.$$

Let $z \in \mathsf{FPF}_{\mathbb{Z}}$. Cycles $(a, b), (i, j) \in \mathrm{Cyc}_{\mathbb{Z}}(z)$ with $a < i$ are *crossing* if $a < i < b < j$ and *nesting* if $a < i < j < b$. One can check that $\hat{\ell}_{\mathsf{FPF}}(z) = 2n + c$ where n and c are the respective numbers of unordered pairs of nesting and crossing cycles of z. If $E \subset \mathbb{Z}$ has size $n \in \mathbb{P}$ then we write ϕ_E and ψ_E for the unique order-preserving bijections $[n] \to E$ and $E \to [n]$, and define

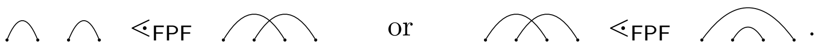
$$[z]_E \stackrel{\text{def}}{=} \psi_{z(E)} \circ z \circ \phi_E \in S_n. \tag{14}$$

The operation $z \mapsto [z]_E$ is usually called *standardization* or *flattening*.

Proposition 3.2 ([1, Corollary 2.3])**.** Let $y \in \mathsf{FPF}_{\mathbb{Z}}$. Fix integers $i < j$ and let $A = \{i, j, y(i), y(j)\}$ and $z = (i, j)y(i, j)$. Then $\hat{\ell}_{\mathsf{FPF}}(z) = \hat{\ell}_{\mathsf{FPF}}(y) + 1$ if and only if the following conditions hold:

(a) One has $y(i) < y(j)$ but no $e \in \mathbb{Z}$ exists with $i < e < j$ and $y(i) < y(e) < y(j)$.
(b) Either $[y]_A = (1,2)(3,4) \lessdot_{\mathsf{FPF}} [z]_A = (1,3)(2,4)$ or $[y]_A = (1,3)(2,4) \lessdot_{\mathsf{FPF}} [z]_A = (1,4)(2,3)$.

Remark 3.3. If condition (a) holds then $(i,j) \notin \mathrm{Cyc}_{\mathbb{Z}}(y)$ so necessarily $|A| = 4$. Condition (b) asserts that $[y]_A \lessdot_{\mathsf{FPF}} [z]_A$, which occurs if and only if $[y]_A$ and $[z]_A$ coincide with



In the first case $[(i,j)]_A \in \{(1,4),(2,3)\}$, and in the second $[(i,j)]_A \in \{(1,2),(3,4)\}$.

Define $\hat\ell_{\mathsf{FPF}}(y,z) = \hat\ell_{\mathsf{FPF}}(z) - \hat\ell_{\mathsf{FPF}}(y)$. Given $y \in \mathsf{FPF}_{\mathbb{Z}}$ and $r \in \mathbb{Z}$, let

$$\begin{aligned} \hat\Psi^+(y,r) &\overset{\text{def}}{=} \left\{ z \in \mathsf{FPF}_{\mathbb{Z}} : \hat\ell_{\mathsf{FPF}}(y,z) = 1,\ z = (r,j)y(r,j) \text{ for } j > r \right\}, \\ \hat\Psi^-(y,r) &\overset{\text{def}}{=} \left\{ z \in \mathsf{FPF}_{\mathbb{Z}} : \hat\ell_{\mathsf{FPF}}(y,z) = 1,\ z = (i,r)y(i,r) \text{ for } i < r \right\}. \end{aligned} \tag{15}$$

These sets are both nonempty, and if z belongs to either of them then $y \lessdot_{\mathsf{FPF}} z$. We can now state the transition formula for FPF-involution Schubert polynomials.

Theorem 3.4 ([10, Theorem 4.17])**.** If $y \in \mathsf{FPF}_\infty$ and $(p,q) \in \mathrm{Cyc}_{\mathbb{P}}(y)$ then

$$(x_p + x_q)\hat{\mathfrak{S}}^{\mathtt{FPF}}_y = \sum_{z \in \hat\Psi^+(y,q)} \hat{\mathfrak{S}}^{\mathtt{FPF}}_z - \sum_{z \in \hat\Psi^-(y,p)} \hat{\mathfrak{S}}^{\mathtt{FPF}}_z$$

where we set $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z = 0$ for all $z \in \mathsf{FPF}_{\mathbb{Z}} - \mathsf{FPF}_\infty$.

Example 3.5. Set $\hat\Psi^\pm(y,r) = \hat\Psi^\pm(\iota(y),r)$ if $y \in \mathsf{FPF}_n$. For

$$y = (1,2)(3,7)(4,5)(6,8) \in \mathsf{FPF}_8$$

we have

$$\begin{aligned} \hat\Psi^+(y,7) &= \{(7,8)y(7,8)\} = \{(1,2)(3,8)(4,5)(6,7)\} \\ \hat\Psi^-(y,3) &= \{(2,3)y(2,3)\} = \{(1,3)(2,7)(4,5)(6,8)\} \end{aligned}$$

so $(x_3 + x_7)\hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,2)(3,7)(4,5)(6,8)} = \hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,2)(3,8)(4,5)(6,7)} - \hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,3)(2,7)(4,5)(6,8)}$.

Taking limits and invoking Definition 2.7 gives the following identity.

Theorem 3.6. If $y \in \mathsf{FPF}_{\mathbb{Z}}$ and $(p,q) \in \mathrm{Cyc}_{\mathbb{Z}}(y)$ then

$$\sum_{z \in \hat\Psi^-(y,p)} \hat F^{\mathtt{FPF}}_z = \sum_{z \in \hat\Psi^+(y,q)} \hat F^{\mathtt{FPF}}_z.$$

Proof. We have $\hat{\Psi}^{\pm}(y \gg 2N, r+2N) = \{w \gg 2N : w \in \hat{\Psi}^{\pm}(y,r)\}$ for $y \in \mathsf{FPF}_{\mathbb{Z}}$ and $r, N \in \mathbb{Z}$, so it follows that $\sum_{z\in\hat{\Psi}^+(y,q)} \hat{F}^{\mathsf{FPF}}_z - \sum_{z\in\hat{\Psi}^-(y,p)} \hat{F}^{\mathsf{FPF}}_z = \lim_{N\to\infty}(x_{p+2N} + x_{q+2N})\hat{\mathfrak{S}}^{\mathsf{FPF}}_{y\gg 2N} = 0$. □

4. FPF-Grassmannian involutions

In this section we identify a class of "Grassmannian" elements of $\mathsf{FPF}_{\mathbb{Z}}$ for which $\hat{F}^{\mathsf{FPF}}_z$ is a Schur P-function. The *(Rothe) diagram* of a permutation $w \in S_\infty$ is the set

$$D(w) \overset{\text{def}}{=} \{(i,j) \in \mathbb{P}\times\mathbb{P} : i < w^{-1}(j) \text{ and } j < w(i)\}. \tag{16}$$

Equivalently, $D(w) = \{(i, w(j)) : (i,j) \in \mathrm{Inv}(w)\}$ where

$$\mathrm{Inv}(w) \overset{\text{def}}{=} \{(i,j) \in \mathbb{Z}\times\mathbb{Z} : i < j \text{ and } w(i) > w(j)\}.$$

Following [8, Section 3.2], the *(FPF-involution) diagram* of $z \in \mathsf{FPF}_\infty$ is the set

$$\hat{D}_{\mathsf{FPF}}(z) \overset{\text{def}}{=} \{(i,j) \in \mathbb{P}\times\mathbb{P} : j < i < z(j) \text{ and } j < z(i)\}. \tag{17}$$

One can check that $\hat{D}_{\mathsf{FPF}}(z) = \{(i, z(j)) : (i,j) \in \mathrm{Inv}_{\mathsf{FPF}}(z),\ z(j) < i\}$.

The *code* of $w \in S_\infty$ is the sequence $c(w) = (c_1, c_2, c_3, \dots)$ where c_i is the number of integers $j > i$ with $w(i) > w(j)$. The ith term of $c(w)$ is the number of positions in the ith row of $D(w)$. As in the introduction, the *(FPF-involution) code* of $z \in \mathsf{FPF}_\infty$ is the sequence $\hat{c}_{\mathsf{FPF}}(z) = (c_1, c_2, \dots)$ in which c_i is the number of positions in the ith row of $\hat{D}_{\mathsf{FPF}}(z)$, and the *shape* of z is the partition $\nu(z)$ whose transpose is the partition that sorts $\hat{c}_{\mathsf{FPF}}(z)$. For $z \in \mathsf{FPF}_n$ when $n \in 2\mathbb{P}$, we define

$$\hat{D}_{\mathsf{FPF}}(z) \overset{\text{def}}{=} \hat{D}_{\mathsf{FPF}}(\iota(z)) \qquad \text{and} \qquad \hat{c}_{\mathsf{FPF}}(z) \overset{\text{def}}{=} \hat{c}_{\mathsf{FPF}}(\iota(z)).$$

Then $\hat{D}_{\mathsf{FPF}}(z)$ is the subset of positions in $D(z)$ strictly below the diagonal.

The shifted shape of a strict partition μ is the set $\{(i, i+j-1) \in \mathbb{P}\times\mathbb{P} : 1 \le j \le \mu_i\}$. An involution z in FPF_n or FPF_∞ is *FPF-dominant* if $\{(i-1, j) : (i,j) \in \hat{D}_{\mathsf{FPF}}(z)\}$ is the transpose of the shifted shape of a strict partition (which is necessarily $\nu(z)$). (We shift up since $\hat{D}_{\mathsf{FPF}}(z)$ has no positions in row $i = 1$.) By contrast, a permutation is *dominant* if it is merely 132-avoiding.

Example 4.1. While $y = (1,8)(2,4)(3,5)(6,7)$ is FPF-dominant, $z = (1,3)(2,7)(4,8)(5,6)$ is not. The corresponding diagrams are

$$\hat{D}_{\mathtt{FPF}}(y) = \begin{array}{cccccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \times \\ \circ & \cdot & \cdot & \times & \cdot & \cdot & \cdot & \cdot \\ \circ & \circ & \cdot & \cdot & \times & \cdot & \cdot & \cdot \\ \circ & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \circ & \cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \circ & \cdot & \cdot & \cdot & \cdot & \cdot & \times & \cdot \\ \circ & \cdot & \cdot & \cdot & \cdot & \times & \cdot & \cdot \\ \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

and

$$\hat{D}_{\mathtt{FPF}}(z) = \begin{array}{cccccccc} \cdot & \cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \circ & \cdot & \cdot & \cdot & \cdot & \cdot & \times & \cdot \\ \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \circ & \cdot & \cdot & \cdot & \cdot & \cdot & \times \\ \cdot & \circ & \cdot & \circ & \cdot & \times & \cdot & \cdot \\ \cdot & \circ & \cdot & \circ & \times & \cdot & \cdot & \cdot \\ \cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \times & \cdot & \cdot & \cdot & \cdot \end{array}$$

where cells with $\circ$ are in $\hat{D}_{\mathtt{FPF}}$, $\times$ indicates a non-zero entry in the permutation matrix and $\cdot$ indicates a cell not in the diagram. Observe that $\hat{D}_{\mathtt{FPF}}$ consists of the positions below the diagonal that are not weakly below any $\times$ and not weakly right of any $\times$. The relevant codes are

$$\hat{c}_{\mathtt{FPF}}(y) = (0,1,2,1,1,1,1,0) \qquad \text{and} \qquad \hat{c}_{\mathtt{FPF}}(z) = (0,1,0,1,2,2,0,0),$$

and $\nu(y) = (6,1)$ is the transpose of $(2,1,1,1,1,1)$. The involution y is not dominant (i.e. 132-avoiding) since in one-line notation $y = 84523761$. One can show that the only elements of FPF_n for $n \in \mathbb{P}$ that are dominant in the classical sense are those of the form $(1,n+1)(2,n+2)\cdots(n,2n)$. These involutions are all FPF-dominant.

The following generalizes [8, Theorem 1.3], which applies only when $z \in \mathsf{FPF}_n$ is dominant.

Theorem 4.2. If $z \in \mathsf{FPF}_\infty$ is FPF-dominant then $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z = \prod_{(i,j)\in\hat{D}_{\mathtt{FPF}}(z)}(x_i + x_j)$.

Proof. For $z' \in \mathsf{FPF}_n$ we defined $\hat{\mathfrak{S}}^{\mathtt{FPF}}_{z'} = \hat{\mathfrak{S}}^{\mathtt{FPF}}_{\iota(z')}$, so we may as well assume $z \in \mathsf{FPF}_n$ for some n. Since $z = w_n$ is dominant, by [8, Theorem 1.3] we have

$$\hat{\mathfrak{S}}^{\mathtt{FPF}}_{w_n} = \prod_{\substack{1\le i<j\le n \\ i+j\le n}} (x_i + x_j).$$

Now assume $z \neq w_n$, and induct downward on $\hat{\ell}_{\mathtt{FPF}}(z)$. Let $j \in [n]$ be minimal such that $z(j) < n - j + 1$. The choice of j implies $z(j) + 1 \notin \{z(1), z(2), \ldots, z(j)\}$, so $z(z(j) + 1) \notin [j]$. Setting $s = s_{z(j)}$, this shows $s \notin \mathrm{Des}_R(z)$ and hence $\hat{\ell}_{\mathtt{FPF}}(szs) = \hat{\ell}_{\mathtt{FPF}}(z) + 1$ by Proposition 2.3. Given that $z < zs < szs$, it is not hard to check that

$$D(szs) = D(z) \sqcup \{(z(j), j), (j, z(j))\}. \tag{18}$$

If $z(j) < j$, then the minimality of j implies $j = z(z(j)) = n - z(j) + 1$, a contradiction; hence $z(j) > j$, so (18) implies $\hat{D}_{\mathtt{FPF}}(szs) = \hat{D}_{\mathtt{FPF}}(z) \sqcup \{(z(j), j)\}$. For example, if our involution is $z = (1,8)(2,7)(3,5)(4,6)$, then $j = 3$ and the diagrams of z and szs are

$$\hat{D}_{\mathtt{FPF}}(z) = \begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \times \\
\circ & \cdot & \cdot & \cdot & \cdot & \cdot & \times & \cdot \\
\circ & \circ & \cdot & \cdot & \times & \cdot & \cdot & \cdot \\
\circ & \circ & \circ & \cdot & \cdot & \times & \cdot & \cdot \\
\circ & \circ & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\
\circ & \circ & \cdot & \times & \cdot & \cdot & \cdot & \cdot \\
\circ & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

and

$$\hat{D}_{\mathtt{FPF}}(szs) = \begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \times \\
\circ & \cdot & \cdot & \cdot & \cdot & \cdot & \times & \cdot \\
\circ & \circ & \cdot & \cdot & \cdot & \times & \cdot & \cdot \\
\circ & \circ & \circ & \cdot & \times & \cdot & \cdot & \cdot \\
\circ & \circ & \circ & \times & \cdot & \cdot & \cdot & \cdot \\
\circ & \circ & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\
\circ & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

On the left, $\times$ is a point of the form $(i, z(i))$ and $\circ$ indicates an element of $\hat{D}_{\mathtt{FPF}}(z)$, i.e., a point above and left of a $\times$ and below the main diagonal.

The picture on the right follows the same conventions with z replaced by szs.

Let $\lambda = \nu(z)$ be the shape of z. Since $z(j) > j$ and $z(i) = n - i + 1$ for $i < j$, drawing a picture makes clear that $\lambda_j = z(j) - j - 1$ and $\lambda_i = n - 2i$ for $i < j$. The previous paragraph therefore shows that szs is FPF-dominant with shape $\nu(szs) = (\lambda_1, \cdots, \lambda_{j-1}, \lambda_j + 1, \lambda_{j+1}, \ldots)$. By induction,

$$\hat{\mathfrak{S}}^{\mathtt{FPF}}_{szs} = \prod_{(a,b)\in\hat{D}_{\mathtt{FPF}}(szs)} (x_a + x_b) = (x_{z(j)} + x_j) \prod_{(a,b)\in\hat{D}_{\mathtt{FPF}}(z)} (x_a + x_b).$$

We claim that $\prod_{(a,b)\in\hat{D}_{\mathtt{FPF}}(z)}(x_a + x_b)$ is symmetric in the variables $x_{z(j)}$ and $x_{z(j)+1}$. First, $z(j) > j$ forces column $z(j)$ of $\hat{D}_{\mathtt{FPF}}(z)$ to be empty, so any variable $x_{z(j)}$ or $x_{z(j)+1}$ in the product comes from a factor $x_a + x_b$ with $(a, b) = (z(j), b) \in \hat{D}_{\mathtt{FPF}}(z)$. The inner corners of λ (the cells rightmost in their row and bottommost in their column) appear in columns $n-1, n-2, \ldots, n-j+1, z(j)-1, \ldots$ from right to left. Thus, since $z(j)-1 < z(j) < z(j)+1 \leq n-j+1$, columns $z(j)$ and $z(j)+1$ of λ have the same length—in the figure above, these two columns appear (transposed) as rows 5 and 6 of $\hat{D}_{\mathtt{FPF}}(z)$. This implies that $(z(j), b) \in \hat{D}_{\mathtt{FPF}}(z)$ if and only if $(z(j)+1, b) \in \hat{D}_{\mathtt{FPF}}(z)$, which proves the claim. Now

$$\begin{aligned}\hat{\mathfrak{S}}^{\mathtt{FPF}}_{z} = \partial_{z(j)}\hat{\mathfrak{S}}^{\mathtt{FPF}}_{szs} &= \partial_{z(j)}\left[(x_{z(j)} + x_j) \prod_{(a,b)\in\hat{D}_{\mathtt{FPF}}(z)} (x_a + x_b)\right] \\ &= \partial_{z(j)}(x_{z(j)} + x_j) \prod_{(a,b)\in\hat{D}_{\mathtt{FPF}}(z)} (x_a + x_b) \\ &= \prod_{(a,b)\in\hat{D}_{\mathtt{FPF}}(z)} (x_a + x_b). \qquad \square\end{aligned}$$

The *lexicographic order* on S_∞ is the total order induced by identifying $w \in S_\infty$ with its one-line representation $w(1)w(2)w(3)\cdots$. For z in FPF_n or FPF_∞, we let $\beta_{\min}(z)$ denote the lexicographically minimal element of $\mathcal{A}_{\mathtt{FPF}}(z)$. The next lemma follows from [9, Theorem 6.22].

Lemma 4.3. Suppose $z \in \mathsf{FPF}_\infty$ and $\mathrm{Cyc}_{\mathbb{P}}(z) = \{(a_i, b_i) : i \in \mathbb{P}\}$ where $a_1 < a_2 < \cdots$. The lexicographically minimal element $\beta_{\min}(z) \in \mathcal{A}_{\mathtt{FPF}}(z)$ is the inverse of the permutation whose one-line representation is $a_1b_1a_2b_2a_3b_3\cdots$.

The same statement with "$a_1b_1a_2b_2\cdots$" replaced by "$a_1b_1a_2b_2\cdots a_nb_n$" holds if $z \in \mathsf{FPF}_{2n}$.

Example 4.4. If $z = (1,4)(2,3) \in \mathsf{FPF}_4$ then $a_1b_1a_2b_2 = 1423$ and $\beta_{\min}(z) = 1423^{-1} = 1342$.

Typically $\hat{D}_{\mathsf{FPF}}(z) \neq D(\beta_{\min}(z))$, but the analogous statement holds for codes.

Lemma 4.5 ([8, Lemma 3.8])**.** If $z \in \mathsf{FPF}_\infty$ then $\hat{c}_{\mathsf{FPF}}(z) = c(\beta_{\min}(z))$.

A pair $(i,j) \in \mathbb{Z} \times \mathbb{Z}$ is an *FPF-visible inversion* of $z \in \mathsf{FPF}_\mathbb{Z}$ if $i < j$ and $z(j) < \min\{i, z(i)\}$. These are precisely the involutions corresponding to the cells of $\hat{D}_{\mathsf{FPF}}(z)$.

Lemma 4.6. The set of FPF-visible inversions of $z \in \mathsf{FPF}_\infty$ is $\mathrm{Inv}(\beta_{\min}(z))$.

Proof. Suppose $(i,j) \in \mathbb{Z} \times \mathbb{Z}$ is an FPF-visible inversion of $z \in F_\infty$. Either $z(j) < i < z(i)$ or $z(j) < z(i) < i$, and in both cases j appears before i in the one-line representation of $\beta_{\min}(z)^{-1}$ so $(i,j) \in \mathrm{Inv}(\beta_{\min}(z))$. Since $|\mathrm{Inv}(\beta_{\min(z)})| = \hat{\ell}_{\mathsf{FPF}}(z) = |\hat{D}_{\mathsf{FPF}}(z)|$, this completes our proof. □

If $(i, i+1)$ is an FPF-visible inversion of $z \in \mathsf{FPF}_\mathbb{Z}$, then $i \in \mathbb{Z}$ is an *FPF-visible descent*. Let

$$\mathrm{Des}_V^{\mathsf{FPF}}(z) \overset{\mathrm{def}}{=} \{s_i : i \in \mathbb{Z} \text{ is an FPF-visible descent of } z\} \subset \mathrm{Des}_R^{\mathsf{FPF}}(z). \tag{19}$$

Since $s_i \in \mathrm{Des}_R(w)$ for $w \in S_\mathbb{Z}$ if and only if $(i, i+1) \in \mathrm{Inv}(w)$, the following is immediate.

Lemma 4.7. If $z \in \mathsf{FPF}_\infty$ then $\mathrm{Des}_V^{\mathsf{FPF}}(z) = \mathrm{Des}_R(\beta_{\min}(z))$.

The *essential set* of a subset $D \subset \mathbb{P} \times \mathbb{P}$ is the set $\mathrm{Ess}(D)$ of positions $(i,j) \in D$ such that $(i+1, j) \notin D$ and $(i, j+1) \notin D$. The following is similar to [11, Lemma 4.14].

Lemma 4.8. For $z \in \mathsf{FPF}_\infty$, the ith row of $\mathrm{Ess}(\hat{D}_{\mathsf{FPF}}(z))$ is nonempty if and only if $s_i \in \mathrm{Des}_V^{\mathsf{FPF}}(z)$.

Proof. If $s_i \in \mathrm{Des}_V^{\mathsf{FPF}}(z)$ then $(i, z(i+1)) \in \hat{D}_{\mathsf{FPF}}(z)$ but all positions of the form $(i+1, j) \in \hat{D}_{\mathsf{FPF}}(z)$ have $j < z(i+1)$, so the ith row of $\mathrm{Ess}(\hat{D}_{\mathsf{FPF}}(z))$ is nonempty. Conversely, if the ith row of this set is nonempty, then there is some $(i,j) \in \hat{D}_{\mathsf{FPF}}(z)$ with $(i+1, j) \notin \hat{D}_{\mathsf{FPF}}(z)$. This holds only if $j = z(k)$ for some $k > i$ with $z(i) > z(k)$ and $i > z(k) \geq z(i+1)$, in which case $s_i \in \mathrm{Des}_V^{\mathsf{FPF}}(z)$. □

A permutation $w \in S_\infty$ is *n-Grassmannian* if $\mathrm{Des}_R(w) = \{s_n\}$.

Proposition 4.9. For $z \in \mathsf{FPF}_\infty$ and $n \in \mathbb{P}$, the following are equivalent:

(a) $\mathrm{Des}_V^{\mathtt{FPF}}(z) = \{s_n\}$.
(b) $\hat{c}_{\mathtt{FPF}}(z)$ has the form $(0, c_2, \dots, c_n, 0, 0, \dots)$ where $c_2 \leq \cdots \leq c_n \neq 0$.
(c) $\mathrm{Ess}(\hat{D}_{\mathtt{FPF}}(z))$ is nonempty and contained in $\{(n, j) : j \in \mathbb{P}\}$.
(d) The lexicographically minimal atom $\beta_{\min}(z) \in \mathcal{A}_{\mathtt{FPF}}(z)$ is n-Grassmannian.

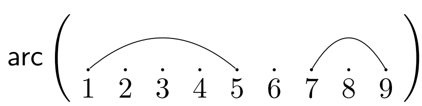
Proof. We have (a) $\Leftrightarrow$ (d) by Lemma 4.7 and (a) $\Leftrightarrow$ (c) by Lemma 4.8. Finally, Lemma 4.5 implies that (b) $\Leftrightarrow$ (d) since $w \in S_\infty$ is n-Grassmannian if and only if the first n terms of $c(w)$ are weakly increasing and the remaining entries are 0. □

The preceding conditions suggest a natural concept of a "Grassmannian" fixed-point-free involution, but this definition turns out to be slightly too restrictive. Define $\mathsf{Invol}_{\mathbb{Z}} \overset{\mathrm{def}}{=} \{w \in S_{\mathbb{Z}} : w = w^{-1}\}$. Consider the maps $\mathsf{arc} : \mathsf{Invol}_{\mathbb{Z}} \to \mathsf{FPF}_{\mathbb{Z}}$ and $\mathsf{dearc} : \mathsf{FPF}_{\mathbb{Z}} \to \mathsf{Invol}_{\mathbb{Z}}$ given as follows.

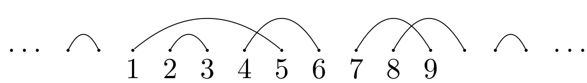
Definition 4.10. For $y \in \mathsf{Invol}_{\mathbb{Z}}$, let m be any even integer with $m < i$ for all $i \in \mathrm{supp}(y)$, write ϕ for the order-preserving bijection $\mathbb{Z} \to \mathbb{Z} \setminus \mathrm{supp}(y)$ with $\phi(0) = m$, and define $\mathsf{arc}(y)$ as the unique element of $\mathsf{FPF}_{\mathbb{Z}}$ with $\mathsf{arc}(y)(i) = y(i)$ for $i \in \mathrm{supp}(y)$ and $\mathsf{arc}(y) \circ \phi = \phi \circ \Theta$.

We use the symbol arc to denote this map since $\mathsf{arc}(y)$ is formed by "arcifying" the matching that represents y, i.e., by adding in edges to pair up all isolated vertices.

We have $\mathsf{arc}(y) = \iota(y)$ for $y \in \mathsf{FPF}_n$. The involution $\mathsf{arc}(z)$ is formed from z by turning every pair of adjacent fixed points into a cycle; there are two ways of doing this, and we choose the way that makes $(2i-1, 2i)$ into a cycle for all sufficiently large $i \in \mathbb{Z}$. For example, the value of



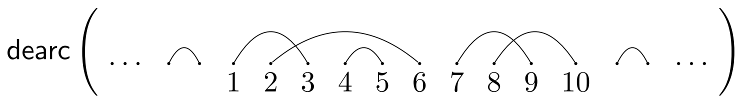
is



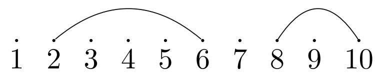
Definition 4.11. For $z \in \mathsf{FPF}_{\mathbb{Z}}$, define $\mathsf{dearc}(z) \in \mathsf{Invol}_{\mathbb{Z}}$ as the involution whose nontrivial cycles are precisely the pairs $(p, q) \in \mathrm{Cyc}_{\mathbb{Z}}(z)$ for which there exists $(a, b) \in \mathrm{Cyc}_{\mathbb{Z}}(z)$ with $p < b < q$.

We use the symbol dearc to denote this map since $\mathsf{dearc}(z)$ is formed by removing all "trivial" arcs from the matching that represents z.

The permutation $\mathsf{dearc}(z)$ is the involution that restricts to the same map as z on its support, and whose fixed points are the integers $i \in \mathbb{Z}$ such that $\max\{i, z(i)\} < z(j)$ for all $j \in \mathbb{Z}$ with $\min\{i, z(i)\} < j < \max\{i, z(i)\}$. For example, the value of



is



We see in these examples that dearc and arc restrict to maps $\mathsf{FPF}_\infty \to \mathsf{Invol}_\infty$ and $\mathsf{Invol}_\infty \to \mathsf{FPF}_\infty$.

Proposition 4.12. Let $z \in \mathsf{FPF}_\mathbb{Z}$. Then $\mathsf{dearc}(z) = 1$ if and only if $z = \Theta$.

Proof. If $z \neq \Theta$ and i is the largest integer such that $i < z(i) \neq i+1$, then necessarily $z(i+1) < z(i)$, so $(i, z(i))$ is a nontrivial cycle of $\mathsf{dearc}(z)$, which is therefore not the identity. $\square$

Proposition 4.13. The composition $\mathsf{arc} \circ \mathsf{dearc}$ is the identity map $\mathsf{FPF}_\mathbb{Z} \to \mathsf{FPF}_\mathbb{Z}$.

Proof. Fix $z \in \mathsf{Invol}_\infty$. Let $\mathcal{C}$ be the set of cycles $(p, q) \in \mathrm{Cyc}_\mathbb{Z}(z)$ such that p and q are fixed points in $\mathsf{dearc}(z)$. By definition, if (p, q) and (p', q') are distinct elements of $\mathcal{C}$ then $p < q < p' < q'$ or $p' < q' < p < q$. The claim that $\mathsf{arc} \circ \mathsf{dearc}(z) = z$ is a straightforward consequence of this fact. $\square$

An involution $y \in \mathsf{Invol}_\mathbb{Z}$ is *I-Grassmannian* if $y = 1$ or $y = (\phi_1, n+1)(\phi_2, n+2)\cdots(\phi_r, n+r)$ for some integers $r \in \mathbb{P}$ and $\phi_1 < \phi_2 < \cdots < \phi_r \leq n$. See [11, Proposition-Definition 4.16] for several equivalent characterizations of such involutions.

Definition 4.14. An involution $z \in \mathsf{FPF}_\mathbb{Z}$ is *FPF-Grassmannian* if $\mathsf{dearc}(z) \in \mathsf{Invol}_\mathbb{Z}$ is I-Grassmannian.

Define an element of FPF_n to be FPF-Grassmannian if its image under $\iota : \mathsf{FPF}_n \to \mathsf{FPF}_\infty \subset \mathsf{FPF}_\mathbb{Z}$ is FPF-Grassmannian.

Remark 4.15. The sequence $(g_n^{\mathsf{FPF}})_{n\geq 1} = (1, 3, 12, 41, 124, 350, 952, 2540, \dots)$ with g_n^{FPF} the number of FPF-Grassmannian elements of $\iota(\mathsf{FPF}_n) \subset \mathsf{FPF}_\mathbb{Z}$ seems unrelated to any sequence in [25].

Suppose $z \in \mathsf{FPF}_{\mathbb{Z}} - \{\Theta\}$ is FPF-Grassmannian, so that

$$\mathsf{dearc}(z) = (\phi_1, n+1)(\phi_2, n+2)\cdots(\phi_r, n+r) \in \mathsf{Invol}_\infty$$

for integers $r \in \mathbb{P}$ and $\phi_1 < \phi_2 < \cdots < \phi_r \leq n$. Recall from the introduction that $\nu(z)$ is the transpose of the partition given by sorting $\hat{c}_{\mathtt{FPF}}(z)$.

Lemma 4.16. In the notation just given, it holds that

$$\nu(z) = (n - \phi_1, n - \phi_2, \ldots, n - \phi_r).$$

Proof. The definitions of $\hat{D}_{\mathtt{FPF}}(y)$, $\hat{c}_{\mathtt{FPF}}(y)$ and $\nu(y)$ make sense even when $y \in \mathsf{Invol}_{\mathbb{Z}}$. Let $y = \mathsf{dearc}(z)$. It is easy to check that the only nonempty columns of $\hat{D}_{\mathtt{FPF}}(y)$ are $\phi_1, \phi_2, \ldots, \phi_r$ and that the ϕ_ith column is $\{(\phi_i + 1, \phi_i), (\phi_i + 2, \phi_i), \ldots, (n, \phi_i)\}$. Therefore $\nu(y) = (n - \phi_1, n - \phi_2, \ldots, n - \phi_r)$, since sorting $\hat{c}_{\mathtt{FPF}}(y)$ gives the transpose of this partition.

Fix positive integers $i < k$ and suppose (i, k) is a cycle in z that is not a cycle in y, so that $y(i) = i$ and $y(k) = k$. Suppose $i < j < k$. From the definition of dearc, it follows that $(j, i) \in \hat{D}_{\mathtt{FPF}}(z) \setminus \hat{D}_{\mathtt{FPF}}(y)$ and $j = \phi_l$ for some $l \in [r]$. Therefore, we have $(k, j) \in \hat{D}_{\mathtt{FPF}}(y) \setminus \hat{D}_{\mathtt{FPF}}(z)$, so

$$\hat{D}_{\mathtt{FPF}}(z) \cap [i, k]^2 = \{(p, j) \in \mathbb{P} \times \mathbb{P} : i \leq j < p < k\}$$

and

$$\hat{D}_{\mathtt{FPF}}(y) \cap [i, k]^2 = \{(p, j) \in \mathbb{P} \times \mathbb{P} : i < j < p \leq k\}.$$

If p is an integer with $i \leq p \leq k$ then

$$\{q < i : (p, q) \in \hat{D}_{\mathtt{FPF}}(z)\} = \{q < i : (p, q) \in \hat{D}_{\mathtt{FPF}}(y)\} = \{l : \phi_l < i\}.$$

With $\hat{c}_{\mathtt{FPF}}(z) = (c_1(z), c_2(z), \ldots)$ and $\hat{c}_{\mathtt{FPF}}(y) = (c_1(y), c_2(y), \ldots)$, we deduce that $c_j(z) = c_{j+1}(y)$ for $i \leq j < k$ and $c_k(z) = c_i(y)$. When j is not between the endpoints of some cycle (i, k) in z but not y, we have $c_j(y) = c_j(z)$. Therefore $\hat{c}_{\mathtt{FPF}}(z)$ and $\hat{c}_{\mathtt{FPF}}(y)$ are the same multisets, so $\nu(z) = \nu(y)$. $\square$

Example 4.17. Consider $z = (1, 4)(2, 6)(3, 7)(5, 8)$ and $y = \mathsf{dearc}(z) = (2, 6)(3, 7)(5, 8)$. Then

$$\hat{D}_{\mathtt{FPF}}(z) = \begin{array}{cccccccc}
\cdot & \cdot & \cdot & \times & \cdot & \cdot & \cdot & \cdot \\
\circ & \cdot & \cdot & \cdot & \cdot & \times & \cdot & \cdot \\
\circ & \circ & \cdot & \cdot & \cdot & \cdot & \times & \cdot \\
\times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \circ & \circ & \cdot & \cdot & \cdot & \cdot & \times \\
\cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \times & \cdot & \cdot & \cdot
\end{array}$$

and

$$\hat{D}_{\texttt{FPF}}(y) = \begin{array}{cccccccc} \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \times & \cdot & \cdot \\ \cdot & \circ & \cdot & \cdot & \cdot & \cdot & \times & \cdot \\ \cdot & \circ & \circ & \times & \cdot & \cdot & \cdot & \cdot \\ \cdot & \circ & \circ & \cdot & \cdot & \cdot & \cdot & \times \\ \cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \times & \cdot & \cdot & \cdot \end{array}.$$

The positions marked $\times$ in the respective diagrams are those of the form $(i, y(i))$ or $(i, z(i))$. We have $\hat{c}_{\texttt{FPF}}(z) = (0,1,2,0,2,0,0)$ while $\hat{c}_{\texttt{FPF}}(y) = (0,0,1,2,2,0,0)$. In addition, we observe that $c_1(z) = c_2(y)$, $c_2(z) = c_3(y)$, and $c_3(z) = c_4(y)$, as predicted in the argument for Lemma 4.16.

Given integers $a, b \in \mathbb{P}$ with $a < b$, define $\partial_{b,a} = \partial_{b-1}\partial_{b-2}\cdots\partial_a$ and $\pi_{b,a} = \pi_{b-1}\pi_{b-2}\cdots\pi_a$. For $a, b \in \mathbb{P}$ with $a \geq b$, set $\partial_{b,a} = \pi_{b,a} = \mathrm{id}$.

Lemma 4.18. Maintain the preceding setup, but assume z is an FPF-Grassmannian element of $\mathsf{FPF}_\infty - \{\Theta\}$ so that $1 \leq \phi_1 < \phi_2 < \cdots < \phi_r \leq n$. Then $\hat{\mathfrak{S}}^{\texttt{FPF}}_z = \pi_{\phi_1,1}\pi_{\phi_2,2}\cdots\pi_{\phi_r,r}\left(x^{\nu(z)}G_{r,n}\right)$.

Proof. The proof depends on the following claim, which is proved as [11, Lemma 2.2]:

Claim. If $a \leq b$ and $f \in \mathcal{L}$ are such that $\partial_i f = 0$ for $a < i < b$, then $\pi_{b,a} f = \partial_{b,a}\left(x_a^{b-a} f\right)$.

If $c_1 < c_2 < \cdots < c_k$ are the fixed points in $[n]$ of $\mathsf{dearc}(z)$, then k is even and we have $(c_1, c_2), (c_3, c_4), \ldots, (c_{k-1}, c_k) \in \mathrm{Cyc}_{\mathbb{Z}}(z)$. Hence if $\phi_i = i$ for all $i \in [r]$ then z is FPF-dominant and

$$\hat{D}_{\texttt{FPF}}(z) = \{(i + j, i) : i \in [r] \text{ and } j \in [n - i]\}.$$

In this case the lemma reduces to the formula $\hat{\mathfrak{S}}^{\texttt{FPF}}_z = x_1^{n-1}x_2^{n-2}\cdots x_r^{n-r}G_{r,n}$ which follows from Theorem 4.2.

Alternatively, suppose there exists $i \in [r]$ such that $i < \phi_i$. Assume i is minimal with this property. Then $\hat{\mathfrak{S}}^{\texttt{FPF}}_z = \partial_{\phi_i,i}\hat{\mathfrak{S}}^{\texttt{FPF}}_v$ for the FPF-Grassmannian involution $v \in \mathsf{FPF}_\infty$ with $\mathsf{dearc}(v) = (1, n+1)(2, n+2)\cdots(i, n+i)(\phi_{i+1}, n+i+1)(\phi_{i+2}, n+i+2)\cdots(\phi_r, n+r)$. By induction, it holds that

$$\hat{\mathfrak{S}}^{\texttt{FPF}}_v = \pi_{\phi_{i+1},i+1}\pi_{\phi_{i+2},i+2}\cdots\pi_{\phi_r,r}\left(x^{\nu(v)}G_{r,n}\right).$$

Since $x^{\nu(v)} = x_i^{\phi_i - i} x^{\nu(z)}$ and since multiplication by x_i commutes with π_j when $i < j$, it follows from the claim that

$$\begin{aligned}\hat{\mathfrak{S}}^{\mathsf{FPF}}_z &= \partial_{\phi_i,i}\hat{\mathfrak{S}}^{\mathsf{FPF}}_v \\ &= \partial_{\phi_i,i}(x_i^{\phi_i-i}\pi_{\phi_{i+1},i+1}\pi_{\phi_{i+2},i+2}\cdots\pi_{\phi_r,r}(x^{\nu(z)}G_{r,n})) \\ &= \pi_{\phi_i,i}\pi_{\phi_{i+1},i+1}\pi_{\phi_{i+2},i+2}\cdots\pi_{\phi_r,r}(x^{\nu(z)}G_{r,n}).\end{aligned}$$

The last expression is $\pi_{\phi_1,1}\cdots\pi_{\phi_r,r}(x^{\nu(z)}G_{r,n})$ since we assume $\pi_{\phi_1,1} = \cdots = \pi_{\phi_{i-1},i-1} = \mathrm{id}$. ☐

Theorem 4.19. If $z \in \mathsf{FPF}_{\mathbb{Z}}$ is FPF-Grassmannian, then $\hat{F}^{\mathsf{FPF}}_z = P_{\nu(z)}$.

Proof. Since $\hat{F}^{\mathsf{FPF}}_z = \hat{F}^{\mathsf{FPF}}_{z\gg N}$ for $N \in 2\mathbb{Z}$, we may assume that $z \in \mathsf{FPF}_\infty$ and that $\mathsf{dearc}(z)$ is I-Grassmannian. Since $\pi_{w_n}\pi_i = \pi_{w_n}$ for $i \in [n-1]$, Lemma 4.18 implies that if $\nu(z)$ has r parts and $n \geq r$ then $\pi_{w_n}\hat{\mathfrak{S}}^{\mathsf{FPF}}_z = \pi_{w_n}\left(x^{\nu(z)}G_{r,n}\right)$. Now take the limit as $n \to \infty$ and apply Lemma 2.8. ☐

Let us clarify the difference between FPF-Grassmannian involutions and elements of $\mathsf{FPF}_{\mathbb{Z}}$ with at most one FPF-visible descent. Define $\mathsf{Invol}_\infty \overset{\text{def}}{=} S_\infty \cap \mathsf{Invol}_{\mathbb{Z}}$ and for each $y \in \mathsf{Invol}_\infty$ let

(20) $$\mathrm{Des}_V(y) \overset{\text{def}}{=} \{i \in \mathbb{Z} : z(i+1) \leq \min\{i, z(i)\}.$$

Elements of $\mathrm{Des}_V(y)$ are *visible descents* of y.

Lemma 4.20. Let $z \in \mathsf{FPF}_\infty$ and $E = \{i \in \mathbb{P} : |z(i) - i| \neq 1\}$. Suppose $y \in \mathsf{Invol}_\infty$ is the involution with $y(i) = z(i)$ if $i \in E$ and $y(i) = i$ otherwise. Then $z = \mathsf{arc}(y)$ and $\mathrm{Des}^{\mathsf{FPF}}_V(z) = \mathrm{Des}_V(y)$.

Proof. It is evident that $z = \mathsf{arc}(y)$. Suppose $s_i \in \mathrm{Des}_V(y)$. Since $y(i+1) \neq i$ for all $i \in \mathbb{P}$ by definition, we must have $y(i+1) < \min\{i, y(i)\}$, so $i+1 \in E$, and therefore either $i \in E$ or $z(i) = i-1$. It follows in either case that $z(i+1) < \min\{i, z(i)\}$ so $s_i \in \mathrm{Des}^{\mathsf{FPF}}_V(z)$. Conversely, suppose $s_i \in \mathrm{Des}^{\mathsf{FPF}}_V(z)$ so that $i+1 \in E$. If $i \in E$ then $s_i \in \mathrm{Des}_V(y)$ holds immediately, and if $i \notin E$ then $z(i+1) < z(i) = i-1$, in which case $y(i+1) = z(i+1) < i = y(i)$ so $s_i \in \mathrm{Des}_V(y)$. ☐

In our previous work, we showed that $y \in \mathsf{Invol}_{\mathbb{Z}}$ is I-Grassmannian if and only if $|\mathrm{Des}_V(y)| \leq 1$ [11, Proposition-Definition 4.16]. Using this fact, we deduce the following:

Proposition 4.21. An involution $z \in \mathsf{FPF}_{\mathbb{Z}}$ has $|\mathrm{Des}_V^{\mathtt{FPF}}(z)| \leq 1$ if and only if z is FPF-Grassmannian and $\nu(z)$ is a strict partition whose consecutive parts each differ by odd numbers.

Proof. We may assume that $z \in \mathsf{FPF}_\infty - \{\Theta\}$. If z is FPF-Grassmannian and the consecutive parts of $\nu(z)$ differ by odd numbers then one can check that $|\mathrm{Des}_V^{\mathtt{FPF}}(z)| \leq 1$. Conversely, define $y \in \mathsf{Invol}_\infty$ as in Lemma 4.20 so that $z = \mathsf{arc}(y)$. We have $\mathrm{Des}_V^{\mathtt{FPF}}(z) = \mathrm{Des}_V(y) = \{s_n\}$ if and only if $y = (\phi_1, n+1)(\phi_2, n+2)\cdots(\phi_r, n+r)$ for integers $r \in \mathbb{P}$ and $0 = \phi_0 < \phi_1 < \phi_2 < \cdots < \phi_r \leq n$. If y has this form then each $\phi_i - \phi_{i-1}$ is necessarily odd, and $\mathsf{dearc}(z) = y$ or $\mathsf{dearc}(z) = (\phi_2, n+2)(\phi_3, n+3)\cdots(\phi_r, n+r)$, so z is FPF-Grassmannian and the consecutive parts of $\nu(z)$ differ by odd numbers. □

Remark 4.22. Using the previous result, one can show that the number k_n of elements of FPF_n with at most one FPF-visible descent satisfies the recurrence $k_{2n} = 2k_{2n-2} + 2n - 3$ for $n \geq 2$. The corresponding sequence $(k_{2n})_{n\geq 1} = (1, 3, 9, 23, 53, 115, 241, 495, \dots)$ is [25, A183155].

5. Schur *P*-positivity

In this section we describe a recurrence for expanding $\hat{F}_z^{\mathtt{FPF}}$ into FPF-Grassmannian summands, and use this to deduce that each $\hat{F}_z^{\mathtt{FPF}}$ is Schur P-positive. Our strategy is similar to the one used in [11, §4.2], though with some added technical complications.

Order the set $\mathbb{Z} \times \mathbb{Z}$ lexicographically. Recall that $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ is an FPF-visible inversion of $z \in \mathsf{FPF}_{\mathbb{Z}}$ if $i < j$ and $z(j) < \min\{i, z(i)\}$, and that $i \in \mathbb{Z}$ is an FPF-visible descent of z if $(i, i+1)$ is an FPF-visible inversion. By Lemma 4.7, every $z \in \mathsf{FPF}_{\mathbb{Z}} - \{\Theta\}$ has an FPF-visible descent.

Lemma 5.1. Let $z \in \mathsf{FPF}_{\mathbb{Z}} - \{\Theta\}$ and suppose $j \in \mathbb{Z}$ is the smallest integer such that $z(j) < j - 1$. Then $j - 1$ is the minimal FPF-visible descent of z.

Proof. By hypothesis, either $z(j) < j-2 = z(j-1)$ or $z(j) < j-1 < z(j-1)$, so $j-1$ is an FPF-visible descent of z. If $k-1$ is another FPF-visible descent of z, then $z(k) < k - 1$ so $j \leq k$. □

Lemma 5.2. Suppose $z \in \mathsf{FPF}_{\mathbb{Z}} - \{\Theta\}$. Let $(q, r) \in \mathbb{Z} \times \mathbb{Z}$ be the lexicographically maximal FPF-visible inversion of z. Suppose m is the largest even integer such that $z(m) \neq m - 1$. Then:

(a) The number q is the maximal FPF-visible descent of z.
(b) The number r is the maximal integer with $z(r) < \min\{q, z(q)\}$.

(c) It holds that $z(q+1) < z(q+2) < \cdots < z(m) \leq q$.
(d) Either $z(q) < q < r \leq m$ or $q < z(q) = r+1 = m$.

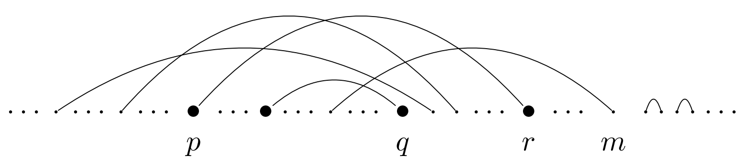
Proof. Since $(q+1, r)$ is not an FPF-visible inversion of z, we must have $\min\{q+1, z(q+1)\} \leq z(r) < \min\{q, z(q)\}$. These inequalities can only hold if $z(q+1) < q+1$, so q is an FPF-visible descent of z. Since $(i, i+1)$ is not an FPF-visible inversion of z for any $i > q$, we conclude that q is the maximal FPF-visible descent of z. This prove part (a). Parts (b) and (c) follow similarly from the assumption that (q, r) is the lexicographically maximal FPF-visible inversion.

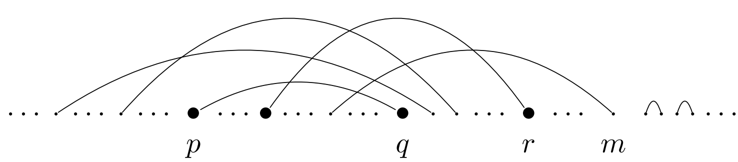
If $z(q) < q$, then $z(q) < r \leq m$ since (q, r) is an FPF-visible inversion. Assume $q < z(q)$. To prove (d), it remains to show that $z(q) = r+1 = m$. It cannot hold that $r < z(q)-1$, since then either $(q, r+1)$ or $(r+1, z(q))$ would be an FPF-visible inversion of z, contradicting the maximality of (q, r). It also cannot hold that $z(q) < r$, as then $(z(q), r)$ would be an FPF-visible inversion of z. Hence $r = z(q) - 1$. If $j > z(q)$, then since $z(i) < q$ for all $q < i < z(q)$ and since $(z(q), j)$ cannot be an FPF-visible inversion of z, we must have $z(j) > z(q)$. From this observation and the fact that z has no FPF-visible descents greater than q, we deduce that $z(j) = \Theta(j)$ for all $j > z(q)$, which implies that $z(q) = m$ as required. ☐

Definition 5.3. Let $\eta_{\mathtt{FPF}} : \mathsf{FPF}_{\mathbb{Z}} - \{\Theta\} \to \mathsf{FPF}_{\mathbb{Z}}$ be the map $\eta_{\mathtt{FPF}} : z \mapsto (q, r)z(q, r)$ where (q, r) is the maximal FPF-visible inversion of z.

Remark 5.4. Suppose $z \in \mathsf{FPF}_{\mathbb{Z}} - \{\Theta\}$ has maximal FPF-visible inversion (q, r). Let $p = z(r)$ and $y = \eta_{\mathtt{FPF}}(z) = (q, r)z(q, r)$ and write m for the largest even integer such that $z(m) \neq m-1$. The two cases of Lemma 5.2 (d) correspond to the following pictures:

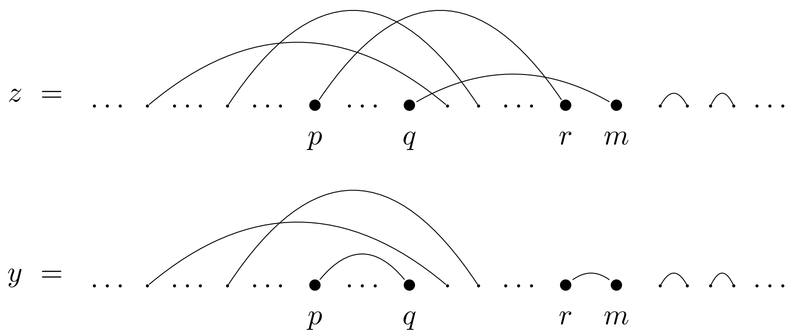
(a) If $z(q) < q < r \leq m$ then y and z may be represented as

$z =$ 

$y =$ 

We have $z(q+1) < z(q+2) < \cdots < z(r) < z(q)$, and if $r < m$ then $z(q) < z(r+1) < z(r+2) < \cdots < z(m) < q$.

(b) If $q < z(q) = r + 1 = m$ then y and z may be represented as



Here, we have $z(q+1) < z(q+2) < \cdots < z(r) = p < q$, so $z(i) < q$ whenever $p < i < q$.

Recall the definition of $\beta_{\min}(z)$ from Lemma 4.3.

Proposition 5.5. If (q, r) is the maximal FPF-visible inversion of $z \in \mathsf{FPF}_\infty - \{\Theta\}$ and $w = \beta_{\min}(z)$ is the minimal element of $\mathcal{A}_{\mathsf{FPF}}(z)$, then $w(q, r) = \beta_{\min}(\eta_{\mathsf{FPF}}(z))$ is the minimal atom of $\eta_{\mathsf{FPF}}(z)$.

Proof. Let $\mathrm{Cyc}_{\mathbb{P}}(z) = \{(a_i, b_i) : i \in \mathbb{P}\}$ and $\mathrm{Cyc}_{\mathbb{P}}(\eta_{\mathsf{FPF}}(z)) = \{(c_i, d_i) : i \in \mathbb{P}\}$ where $a_1 < a_2 < \dots$ and $c_1 < c_2 < \dots$. By Lemma 4.3, it suffices to show that interchanging q and r in the word $a_1 b_1 a_2 b_2 \cdots$ gives $c_1 d_1 c_2 d_2 \cdots$, which is straightforward from Remark 5.4. □

Recall the definition of the sets $\hat{\Psi}^+(y, r)$ and $\hat{\Psi}^-(y, r)$ from (15).

Lemma 5.6. If $z \in \mathsf{FPF}_\mathbb{Z} - \{\Theta\}$ has maximal FPF-visible inversion (q, r) then $\hat{\Psi}^+(\eta_{\mathsf{FPF}}(z), q) = \{z\}$.

Proof. This holds by Proposition 3.2, Remark 5.4, and the definitions of $\eta_{\mathsf{FPF}}(z)$ and $\hat{\Psi}^+(y, q)$. □

For $z \in \mathsf{FPF}_\mathbb{Z}$ let

$$\hat{\mathfrak{T}}_1^{\mathsf{FPF}}(z) \overset{\text{def}}{=} \begin{cases} \varnothing & \text{if } z \text{ is FPF-Grassmannian} \\ \hat{\Psi}^-(y, p) & \text{otherwise} \end{cases} \tag{21}$$

where in the second case, we define $y = \eta_{\mathsf{FPF}}(z)$ and $p = y(q)$ where q is the maximal FPF-visible descent of z.

Definition 5.7. The *FPF-involution Lascoux-Schützenberger tree* $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$ of $z \in \mathsf{FPF}_\mathbb{Z}$ is the tree with root z, in which the children of any vertex $v \in \mathsf{FPF}_\mathbb{Z}$ are the elements of $\hat{\mathfrak{T}}_1^{\mathsf{FPF}}(v)$.

Remark 5.8. As the name suggests, our definition is inspired by the classical construction of the *Lascoux-Schützenberger tree* for ordinary Stanley symmetric functions; see [15, 16] or [11, §4.2].

For $z \in \mathsf{FPF}_n$ we define $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z) = \hat{\mathfrak{T}}^{\mathsf{FPF}}(\iota(z))$. A given involution is allowed to correspond to more than one vertex in $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$. All vertices v in $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$ satisfy $\hat{\ell}_{\mathsf{FPF}}(v) = \hat{\ell}_{\mathsf{FPF}}(z)$ by construction, so if $z \neq \Theta$ then Θ is not a vertex in $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$. An example tree $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$ is shown in Figure 1.

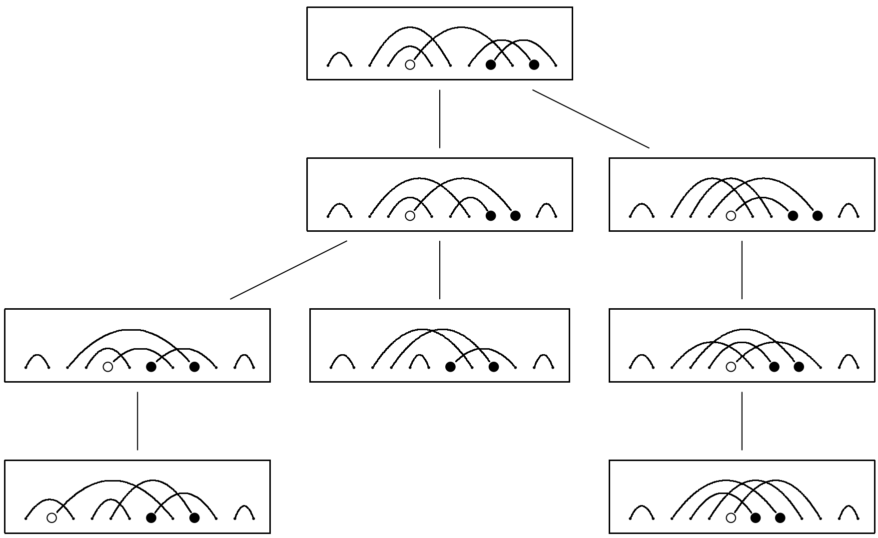


Figure 1: The tree $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$ for $z = (1,2)(3,7)(4,6)(5,10)(8,11)(9,12) \in \mathsf{FPF}_{12} \hookrightarrow \mathsf{FPF}_{\mathbb{Z}}$. We draw all vertices as elements of $\mathsf{FPF}_{12} \subset \mathsf{Invol}_{12}$ for convenience. The maximal FPF-visible inversion of each vertex is marked with $\bullet$, and the minimal FPF-visible descent is marked with $\circ$ (when this is not also maximal). By Theorem 4.19 and Corollary 5.9, we have $\hat{F}^{\mathsf{FPF}}_z = P_{(5,2)} + P_{(4,3)} + P_{(4,2,1)}$.

Corollary 5.9. Suppose $z \in \mathsf{FPF}_{\mathbb{Z}}$ is a fixed-point-free involution that is not FPF-Grassmannian, whose maximal FPF-visible descent is $q \in \mathbb{Z}$. The following identities then hold:

(a) $\hat{\mathfrak{S}}^{\mathsf{FPF}}_z = (x_p + x_q)\hat{\mathfrak{S}}^{\mathsf{FPF}}_y + \sum_{v \in \hat{\mathfrak{T}}^{\mathsf{FPF}}_1(z)} \hat{\mathfrak{S}}^{\mathsf{FPF}}_v$ where $y = \eta_{\mathsf{FPF}}(z)$ and $p = y(q)$.

(b) $\hat{F}^{\mathsf{FPF}}_z = \sum_{v \in \hat{\mathfrak{T}}^{\mathsf{FPF}}_1(z)} \hat{F}^{\mathsf{FPF}}_v$.

Proof. The result follows from Theorems 3.4 and 3.6 and Lemma 5.6. ☐

We would like to show that the intervals between the minimal and maximal FPF-visible descents of the vertices in $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$ form a descending chain as one moves down the tree. This fails, however: a child in the tree may have strictly smaller FPF-visible descents than its parent. A similar property does hold if we instead consider the visible descents of the image of $z \in \mathsf{FPF}_{\mathbb{Z}}$ under the map $\mathsf{dearc} : \mathsf{FPF}_{\mathbb{Z}} \to \mathsf{Invol}_{\mathbb{Z}}$ from Definition 4.11. Recall that a visible descent for $y \in \mathsf{Invol}_{\mathbb{Z}}$ is an integer $i \in \mathbb{Z}$ with $z(i+1) \leq \min\{i, z(i)\}$. The following is [11, Lemma 4.24].

Lemma 5.10 (See [11])**.** Let $z \in \mathsf{Invol}_{\mathbb{Z}} - \{1\}$ and suppose $j \in \mathbb{Z}$ is the smallest integer such that $z(j) < j$. Then $j-1$ is the minimal visible descent of z.

Lemma 5.11. Let $z \in \mathsf{FPF}_{\mathbb{Z}} - \{\Theta\}$ and suppose $(i,j) \in \mathrm{Cyc}_{\mathbb{Z}}(z)$ is the cycle with j minimal such that $i < b < j$ for some $(a,b) \in \mathrm{Cyc}_{\mathbb{Z}}(z)$. Then $j-1$ is the minimal visible descent of $\mathsf{dearc}(z)$.

Proof. The claim follows by the preceding lemma since j is minimal such that $\mathsf{dearc}(z)(j) < j$. □

Lemma 5.12. Let $z \in \mathsf{FPF}_{\mathbb{Z}}$. A number $i \in \mathbb{Z}$ is a visible descent of $\mathsf{dearc}(z)$ if and only if one of the following conditions holds:

(a) $z(i+1) < z(i) < i$.
(b) $z(i) < z(i+1) < i$ and $\{t \in \mathbb{Z} : z(i) < t < i\} \subset \{z(t) : i < t\}$.
(c) $z(i+1) < i < z(i)$ and $\{t \in \mathbb{Z} : z(i+1) < t < i+1\} \not\subset \{z(t) : i+1 < t\}$.

Proof. It is straightforward to check that $i \in \mathbb{Z}$ is a visible descent of $\mathsf{dearc}(z)$ if and only if either (a) $z(i+1) < z(i) < i$; (b) $z(i) < z(i+1) < i$ and i is a fixed point of $\mathsf{dearc}(z)$; or (c) $z(i+1) < i < z(i)$ and $i+1$ is not a fixed point of $\mathsf{dearc}(z)$. The given conditions are equivalent to these statements. □

Corollary 5.13. Let $y, z \in \mathsf{FPF}_{\mathbb{Z}}$ and $i, j \in \mathbb{Z}$ with $i < j$. Suppose $y(t) = z(t)$ for all integers $t > i$. Then j is a visible descent of $\mathsf{dearc}(y)$ if and only if j is a visible descent of $\mathsf{dearc}(z)$.

Proof. By Lemma 5.12, whether or not j is a visible descent of $\mathsf{dearc}(z)$ depends only on the action of z on integers greater than or equal to j. □

Corollary 5.14. Let $z \in \mathsf{FPF}_{\mathbb{Z}}$ and suppose i is a visible descent of $\mathsf{dearc}(z)$. Then either i or $i-1$ is an FPF-visible descent of z. Therefore, if j is the maximal FPF-visible descent of z, then $i \leq j+1$.

Proof. It follows from Lemma 5.12 that i is an FPF-visible descent of z unless $z(i) < z(i+1) < i$ and $\{t \in \mathbb{Z} : z(i) < t < i\} \subset \{z(t) : i < t\}$, in which case $i-1$ is an FPF-visible descent of z. □

The following statement is the first of two key technical lemmas in this section.

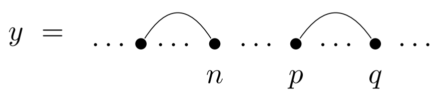
Lemma 5.15. Let $y \in \mathsf{FPF}_{\mathbb{Z}} - \{\Theta\}$ and $(p, q) \in \mathrm{Cyc}_{\mathbb{Z}}(y)$, and suppose $z = (n, p)y(n, p) \in \hat{\Psi}^{-}(y, p)$.

(a) If $i \in \mathbb{Z}\backslash\{n, y(n), p, q\}$ is such that $\mathsf{dearc}(y)(i) = i$, then $\mathsf{dearc}(z)(i) = i$.
(b) If j and k are the minimal visible descents of $\mathsf{dearc}(y)$ and $\mathsf{dearc}(z)$ and $j \leq q - 1$, then $j \leq k$.

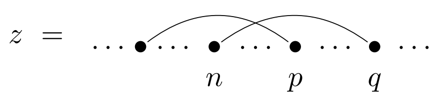
Remark 5.16. Part (b) is false if $j \geq q$: consider $y = (6, 7)\Theta(6, 7)$ and $(n, p, q) = (2, 3, 4)$. There is no analogous inequality governing the minimal FPF-visible descents of y and z.

Proof. Since $y <_{\mathsf{FPF}} z = (n, p)y(n, p) \in \hat{\Psi}^{-}(y, p)$, it follows from Proposition 3.2 that either $y(n) < n < p < q$, in which case $n < p < z(p) < q = z(n)$ and y and z correspond to the diagrams

(22)

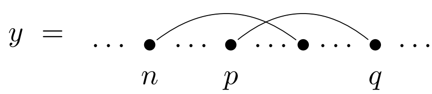


and

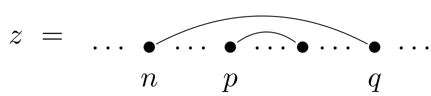


or $n < p < y(n) < q$, in which case $n < p < z(p) < q = z(n)$ and we instead have

(23)



and



Let $A = \{n, y(n), p, q\} = \{n, p, z(p), q\}$ and note that $y(i) = z(i)$ for all $i \in \mathbb{Z} \setminus A$. Suppose $(a, b) \in \mathrm{Cyc}_{\mathbb{Z}}(y)$ is such that $b \notin A$ and $b < y(i)$ for all $a < i < b$, so that a and b are both fixed points of $\mathsf{dearc}(y)$. Then (a, b) is also a cycle of z, and to prove part (a) it suffices to check that $b < z(i)$ for all $i \in A$ with $a < i < b$. This holds if $i \in \{n, y(n)\}$ since then $y(i) < z(i)$, and we cannot have $a < q < b$ since $y(q) < q$. Suppose $a < p < b$; it remains to show that $b < z(p)$. Since $b < y(i)$ for all $a < i < b$ by hypothesis, it follows that if y and z are as in (22) then $n < a < p < b < q$, and that if y are z are as in (23) then $a < p < b < y(n)$. The first of these cases cannot occur in view of Proposition 3.2(a), since $y \lessdot_{\mathsf{FPF}} z$. In the second case $y(n) = z(p)$ so $b < z(p)$ as needed.

To prove part (b), note that $\Theta \notin \{y, z\}$ so neither $\mathsf{dearc}(y)$ nor $\mathsf{dearc}(z)$ is the identity. Let j and k be the minimal visible descents of $\mathsf{dearc}(y)$ and $\mathsf{dearc}(z)$ and assume $j \leq q-1$. Write S_y for the set of integers $i \in \mathbb{Z} \setminus A$ such that $\mathsf{dearc}(y)(i) < i$, and let $T_y = S_y \setminus A$ and $U_y = S_y \cap A$. Define S_z, T_z, and U_z similarly. Lemma 5.10 implies that $j \leq k$ if and only if $\min S_y \leq \min S_z$. Since $j \leq q-1$ we have $\min S_y \leq q$. It follows from part (a) that $T_z \subset T_y$, so $\min T_y \leq \min T_z$.

There are two cases to consider. First suppose $y(n) < n < p < q$ and $z(p) < n < p < q = z(n)$. It is then evident from (22) that $\{q\} \subset U_z \subset \{p, q\}$. Since $\min S_y \leq q$ by hypothesis, to prove that $\min S_y \leq \min S_z$ it suffices to show that if $p \in U_z$ then $\min S_y < p$. Since $y \lessdot_{\mathsf{FPF}} z$, neither y nor z can have any cycles (a, b) with $y(n) < a < p$ and $n < b < p$. It follows that if $p \in U_z$ then y and z share a cycle (a, b) with either (i) $a < b$ and $y(n) < b < n$, or (ii) $a < y(n) < n < b < p$. If (i) occurs then $n \in U_y$ while if (ii) occurs then $\min T_y < p$, so $\min S_y < p$ as desired.

Suppose instead that $n < p < y(n) < q$ and $n < p < z(p) < q = z(n)$. In view of (23), we then have $\{q\} \subset U_z \subset \{y(n), q\}$. As $\min S_y \leq q$, to prove that $\min S_y \leq \min S_z$ it now suffices to show that if $y(n) \in U_z$ then $y(n) \in U_y$. This implication is clear from (23), since if $y(n) = z(p) \in U_z$ then y and z must share a cycle (a, b) with $a < b$ and $p < b < y(n)$. □

Lemma 5.17. Let $y \in \mathsf{FPF}_{\mathbb{Z}} - \{\Theta\}$ and $(p, q) \in \mathrm{Cyc}_{\mathbb{Z}}(y)$ and suppose $z = (q, r)y(q, r) \in \hat{\Psi}^+(y, q)$. The involution $\mathsf{dearc}(y)$ has a visible descent less than $q - 1$ if and only if $\mathsf{dearc}(z)$ does, and in this case the minimal visible descents of $\mathsf{dearc}(y)$ and $\mathsf{dearc}(z)$ are equal.

Proof. Let $\mathcal{C}_w$ for $w \in \mathsf{FPF}_{\mathbb{Z}}$ be the set of cycles $(a, b) \in \mathrm{Cyc}_{\mathbb{Z}}(w)$ with $b < q$. By Lemma 5.11, the set $\mathcal{C}_w$ determines whether or not $\mathsf{dearc}(w)$ has a visible descent less than $q-1$ and, when this occurs, the value of $\mathsf{dearc}(w)$'s smallest visible descent. Since $q < r$ we have $\mathcal{C}_y = \mathcal{C}_z$, so the result follows. □

Our second key technical lemma is the following.

Lemma 5.18. Suppose $z \in \mathsf{FPF}_{\mathbb{Z}}$ is not FPF-Grassmannian, so that $\eta_{\mathsf{FPF}}(z) \neq \Theta$. Let (q, r) be the maximal FPF-visible inversion of z and define $y = \eta_{\mathsf{FPF}}(z) = (q, r)z(q, r)$.

(a) The maximal visible descent of $\mathsf{dearc}(z)$ is q or $q + 1$.
(b) The maximal visible descent of $\mathsf{dearc}(y)$ is at most q.

(c) The minimal visible descent of $\mathsf{dearc}(y)$ is equal to that of $\mathsf{dearc}(z)$, and is at most $q-1$.

Proof. Adopt the notation of Remark 5.4. To prove the first two parts, let j and k be the maximal visible descents of $\mathsf{dearc}(y)$ and $\mathsf{dearc}(z)$, respectively. In case (a) of Remark 5.4, it follows by inspection that $j \leq q = k$, with equality unless $r = q+1$ and there exists at least one cycle $(a,b) \in \mathrm{Cyc}_{\mathbb{Z}}(z)$ such that $p < b < q$. In case (b) of Remark 5.4, one of the following occurs:

- If $p = q-1 = r-2$, then $j < q-1 < k = q+1$.
- If $p = q-1 < r-2$, then $j = q$ and $k \in \{q, q+1\}$.
- If $p < q-1$, then $j = k = q$.

We conclude that $j \leq q$ and $k \in \{q, q+1\}$ as required.

Let j and k now be the minimal visible descents of $\mathsf{dearc}(y)$ and $\mathsf{dearc}(z)$, respectively. Part (c) is immediate from Lemmas 5.6 and 5.17 if $j < q-1$ or $k < q-1$, so assume that j and k are both at least $q-1$. Suppose $z(q) < q < r \leq m$ so that we are in case (a) of Remark 5.4, when q is the maximal visible descent of $\mathsf{dearc}(z)$. Since z is not FPF-Grassmannian, we must have $k = q-1$, so by Lemma 5.11 there exists $(a,b) \in \mathrm{Cyc}_{\mathbb{Z}}(z)$ with $z(q) < b < q$. Since $y(q) = p < z(q)$, it follows that $j \leq q-1$; as the reverse inequality holds by hypothesis, we get $j = k = q-1$ as desired.

Suppose instead that we are in case (b) of Remark 5.4. Since $q < z(q)$, it cannot hold that $q-1$ is a visible descent of $\mathsf{dearc}(z)$, so we must have $k \geq q$. As z is not FPF-Grassmannian, it follows from part (a) that $k = q$ and that $q+1$ is the maximal visible descent of $\mathsf{dearc}(z)$. This is impossible, however, since we can only have $k = q$ if there exists $(a,b) \in \mathrm{Cyc}_{\mathbb{Z}}(z)$ with $z(q+1) < b < q+1$, while $q+1$ can only be a visible descent of $\mathsf{dearc}(z)$ if no such cycle exists. □

Lemma 5.19. Suppose $z \in \mathsf{FPF}_{\mathbb{Z}}$ is not FPF-Grassmannian and $v \in \hat{\mathfrak{T}}_1^{\mathsf{FPF}}(z)$. Let i and j be the minimal and maximal visible descents of $\mathsf{dearc}(z)$. If d is a visible descent of $\mathsf{dearc}(v)$, then $i \leq d \leq j$.

Proof. Let (q,r) be the maximal FPF-visible descent of z, set $y = (q,r)z(q,r) = \eta_{\mathsf{FPF}}(z)$ and $p = y(q) = z(r)$, and let $n < p < q$ be the unique integer such that $v = (n,p)y(n,p)$. Since $y \lessdot_{\mathsf{FPF}} v$, it must hold that $y(n) < q$, so $v(t) = y(t)$ for all $t > q$. The maximal visible descent of $\mathsf{dearc}(y)$ is at most $q \leq j$ by Lemma 5.18, so the same is true of the maximal visible descent of $\mathsf{dearc}(v)$ by Corollary 5.13. On the other hand, the minimal visible descent of $\mathsf{dearc}(y)$ is $i \leq q-1$ by Lemma 5.18, so by Lemma 5.15 the minimal visible descent of $\mathsf{dearc}(v)$ is at least i. □

For $z \in \mathsf{FPF}_{\mathbb{Z}}$, let $\hat{\mathfrak{T}}^{\mathsf{FPF}}_0(z) \overset{\text{def}}{=} \{z\}$ and $\hat{\mathfrak{T}}^{\mathsf{FPF}}_n(z) \overset{\text{def}}{=} \bigcup_{v\in\hat{\mathfrak{T}}^{\mathsf{FPF}}_{n-1}(z)} \hat{\mathfrak{T}}^{\mathsf{FPF}}_1(v)$.

Lemma 5.20. Suppose $z \in \mathsf{FPF}_{\mathbb{Z}}$ and $v \in \hat{\mathfrak{T}}^{\mathsf{FPF}}_1(z)$. Let (q, r) be the maximal FPF-visible inversion of z, and let (q_1, r_1) be any FPF-visible inversion of v. Then $q_1 < q$ or $r_1 < r$. Hence, if $n \geq r - q$ then the maximal FPF-visible descent of every element of $\hat{\mathfrak{T}}^{\mathsf{FPF}}_n(z)$ is strictly less than q.

Proof. It is considerably easier to track the FPF-visible inversions of z and v than the visible inversions of $\mathsf{dearc}(z)$ and $\mathsf{dearc}(v)$, and this result follows essentially by inspecting Remark 5.4. In more detail, let $y = \eta_{\mathsf{FPF}}(z) = (q, r)z(q, r)$ and $p = z(r) = y(q)$. Since $y \lessdot_{\mathsf{FPF}} v = (n, p)y(n, p)$ for some $n < p$, we must have $v(i) = y(i)$ for all $i > q$, and so it is apparent from Remark 5.4 that $q_1 \leq q$. If $q_1 = q$, then necessarily $v(q) < p < v(i)$ for all $i \geq r$, and it follows that $r_1 < r$. □

Theorem 5.21. The FPF-involution Lascoux-Schützenberger tree $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$ is finite for $z \in \mathsf{FPF}_{\mathbb{Z}}$, and $\hat{F}^{\mathsf{FPF}}_z = \sum_v \hat{F}^{\mathsf{FPF}}_v$ where the sum is over the finite set of leaf vertices v in $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$.

Proof. By induction, Corollary 5.14, and Lemmas 5.19 and 5.20, we deduce that for a sufficiently large n either $\hat{\mathfrak{T}}^{\mathsf{FPF}}_n(z) = \varnothing$ or all elements of $\hat{\mathfrak{T}}^{\mathsf{FPF}}_n(z)$ are FPF-Grassmannian, whence $\hat{\mathfrak{T}}^{\mathsf{FPF}}_{n+1}(z) = \varnothing$. The tree $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$ is therefore finite, so the identity $\hat{F}^{\mathsf{FPF}}_z = \sum_v \hat{F}^{\mathsf{FPF}}_v$ holds by Corollary 5.9. □

Corollary 5.22. If $z \in \mathsf{FPF}_{\mathbb{Z}}$ then

$$\hat{F}^{\mathsf{FPF}}_z \in \mathbb{N}\text{-span}\left\{\hat{F}^{\mathsf{FPF}}_y : y \in \mathsf{FPF}_{\mathbb{Z}} \text{ is FPF-Grassmannian}\right\}$$

and this symmetric function is consequently Schur P-positive.

This leads immediately to a proof of Theorem 1.1 from the introduction.

Proof of Theorem 1.1. Since $\hat{F}^{\mathsf{FPF}}_z$ is a Schur P-function if $z \in \mathsf{FPF}_{\mathbb{Z}}$ is FPF-Grassmannian by Theorem 4.19, Corollary 5.22 implies that every $\hat{F}^{\mathsf{FPF}}_z$ is Schur P-positive. □

We close this section by applying Theorem 1.1 to compute the product of two Schur P-functions. Given $u \in S_m$ and $v \in S_n$, write $u \times v \in S_{m+n}$ for the permutation mapping $i \mapsto u(i)$ for $i \in [m]$ and $m + i \mapsto m + v(i)$ for $i \in [n]$. It is well known that $F_{u\times v} = F_u F_v$; for instance, this follows by applying stabilization to [15, Proposition 1.2]. An analogous result holds for FPF-involutions.

Proposition 5.23. Let $y \in \mathtt{FPF}_m$ and $z \in \mathtt{FPF}_n$. Then $\hat{F}^{\mathsf{FPF}}_{y\times z} = \hat{F}^{\mathsf{FPF}}_y \hat{F}^{\mathsf{FPF}}_z$.

Proof. Since $\mathcal{A}_{\mathtt{FPF}}(y \times z) = \{u \times v : (u,v) \in \mathcal{A}_{\mathtt{FPF}}(y) \times \mathcal{A}_{\mathtt{FPF}}(z)\}$, this follows from Definition 2.7. □

As a corollary, we obtain a new rule for multiplying Schur-P functions.

Corollary 5.24. Suppose ρ and μ are strict partitions. Let y and z be FPF-Grassmannian involutions with $\nu(y) = \rho$ and $\nu(z) = \mu$. Then $P_\rho P_\mu = \sum_\lambda C^\lambda_{\rho\mu} P_\lambda$ where $C^\lambda_{\rho\mu}$ is the number of FPF-Grassmannian involutions with shape λ appearing as leaves in $\hat{\mathfrak{T}}^{\mathtt{FPF}}(y \times z)$.

Proof. The result follows immediately from Proposition 5.23 and Theorem 5.21. □

Remark 5.25. A similar rule can be constructed for both Schur-P and Schur-Q functions using the results in [11, §4.2].

6. Triangularity

We can show that the expansion of $\hat{F}^{\mathtt{FPF}}_z$ into Schur P-functions is unitriangular with respect to the dominance order $\leq$ on (strict) partitions. As in the introduction, define $\nu(z)$ for $z \in \mathsf{FPF}_\infty$ to be the transpose of the partition given by sorting $\hat{c}_{\mathtt{FPF}}(z)$, and let $\nu(z) = \nu(\iota(z))$ for $z \in \mathsf{FPF}_n$.

Example 6.1. Let $y = (1,8)(2,4)(3,5)(6,7)$ and $z = (1,3)(2,7)(4,8)(5,6)$ be as in as Example 4.1. Then sorting $\hat{c}_{\mathtt{FPF}}(y)$ gives $(2,1,1,1,1,1,0,0)$ so the shape of y is $\nu(y) = (6,1)$. Similarly, sorting $\hat{c}_{\mathtt{FPF}}(z)$ gives $(2,2,1,1,0,0,0,0)$ so the shape of z is $\nu(z) = (4,2)$.

This construction is consistent with our earlier definition of $\nu(z)$ when $z \in \mathsf{FPF}_\infty$ is FPF-Grassmannian. Define $<_{\mathcal{A}_{\mathtt{FPF}}}$ on S_∞ as the transitive relation generated by setting $v <_{\mathcal{A}_{\mathtt{FPF}}} w$ when the one-line representation of v^{-1} can be transformed to that of w^{-1} by replacing a consecutive subsequence starting at an odd index of the form $adbc$ with $a < b < c < d$ by $bcad$, or equivalently when it holds for an odd number $i \in \mathbb{P}$ that

$$s_i v > v > s_{i+1} v > s_{i+2} s_{i+1} v = s_i s_{i+1} w < s_{i+1} w < w < s_i w. \tag{24}$$

For example,

$$235164 = (412635)^{-1} <_{\mathcal{A}_{\mathtt{FPF}}} (413526)^{-1} = 253146,$$

but $(12534)^{-1} \not<_{\mathcal{A}_{\mathtt{FPF}}} (13425)^{-1}$. Recall the definition of $\beta_{\min}(z)$ from Lemma 4.3. In earlier work, we showed [9, Theorem 6.22] that $<_{\mathcal{A}_{\mathtt{FPF}}}$ is a partial order and that $\mathcal{A}_{\mathsf{FPF}}(z) = \{w \in S_\infty : \beta_{\min}(z) \leq_{\mathcal{A}_{\mathsf{FPF}}} w\}$ for all $z \in \mathsf{FPF}_\infty$.

Write λ^T for the transpose of a partition λ. Then $\lambda \leq \mu$ if and only if $\mu^T \leq \lambda^T$ [18, Eq. (1.11), §I.1]. The *shape* of $w \in S_\infty$ is the partition $\lambda(w)$ given by sorting $c(w)$.

Lemma 6.2. Let $z \in \mathsf{FPF}_\infty$. If $v, w \in \mathcal{A}_{\mathsf{FPF}}(z)$ and $v <_{\mathcal{A}_{\mathsf{FPF}}} w$, then $\lambda(v) < \lambda(w)$.

Proof. Suppose $v, w \in \mathcal{A}_{\mathsf{FPF}}(z)$ are such that $s_i v > v > s_{i+1} v > s_{i+2}s_{i+1}v = s_i s_{i+1} w < s_{i+1} w < w < s_i w$ for an odd number $i \in \mathbb{P}$, so that $v <_{\mathcal{A}_{\mathsf{FPF}}} w$. Define $a = w^{-1}(i+2)$, $b = w^{-1}(i)$, $c = w^{-1}(i+1)$, and $d = w^{-1}(i+3)$ so that $a < b < c < d$. The diagram $D(v^{-1})$ is then given by permuting rows i, $i+1$, $i+2$, and $i+3$ of $D(w^{-1}) \cup \{(i+3, b), (i+3, c)\} - \{(i, a), (i+1, a)\}$, and so $\lambda(v)$ is given by sorting $\lambda(w) - 2e_j + e_k + e_l$ for some indices $j < k < l$ with $\lambda(w)_j - 2 \geq \lambda(w)_k \geq \lambda(w)_l$. One checks in this case that $\lambda(v) < \lambda(w)$, as desired. □

Theorem 6.3. Let $z \in \mathsf{FPF}_\infty$ and $\nu = \nu(z)$. Then $\nu^T \leq \nu$. If $\nu^T = \nu$ then $\hat{F}^{\mathsf{FPF}}_z = s_\nu$ and otherwise $\hat{F}^{\mathsf{FPF}}_z \in s_{\nu^T} + s_\nu + \mathbb{N}\text{-span}\left\{s_\lambda : \nu^T < \lambda < \nu\right\}$.

Proof. It follows from [26, Theorem 4.1] that if $w \in S_\infty$ then $\lambda(w) \leq \lambda(w^{-1})^T$, and if equality holds then $F_w = s_{\lambda(w)}$ while otherwise $F_w \in s_{\lambda(w)} + s_{\lambda(w^{-1})^T} + \mathbb{N}\text{-span}\{s_\nu : \lambda(w) < \nu < \lambda(w^{-1})^T\}$. Lemma 4.5 implies that $\nu(z)^T = \lambda(\beta_{\min}(z))$, so by Lemma 6.2 we have $\hat{F}^{\mathsf{FPF}}_z = \sum_{w \in \mathcal{A}_{\mathsf{FPF}}(z)} F_w \in s_{\nu(z)^T} + \mathbb{N}\text{-span}\{s_\mu : \nu(z)^T < \mu\}$. The result follows since $\hat{F}^{\mathsf{FPF}}_z$ is Schur P-positive and each P_μ is fixed by the linear map $\omega : \Lambda \to \Lambda$ with $\omega(s_\mu) = s_{\mu^T}$ for partitions μ [18, Example 3(a), §III.8]. □

We may finally prove Theorem 1.4 from the introduction.

Proof of Theorem 1.4. One has $P_\lambda \in s_\lambda + \mathbb{N}\text{-span}\{s_\nu : \nu < \lambda\}$ for any strict partition λ [18, Eq. (8.17)(ii), §III.8]. Since $\hat{F}^{\mathsf{FPF}}_z$ is Schur P-positive, the result follows by Theorem 6.3. □

Strangely, we do not know of an easy way to show directly that $\nu(z)$ is a strict partition.

7. FPF-vexillary involutions

Define an element z of FPF_n or $\mathsf{FPF}_\mathbb{Z}$ to be *FPF-vexillary* if $\hat{F}^{\mathsf{FPF}}_z = P_\mu$ for a strict partition μ. In this section, we derive a pattern avoidance condition classifying such involutions.

Remark 7.1. All FPF-Grassmannian involutions, as well as all elements of FPF_n for $n \in \{2,4,6\}$, are FPF-vexillary. The sequence $(v^{\mathsf{FPF}}_{2n})_{n\geq 1} = (1,3,15,92,617,4354,\dots)$, with v^{FPF}_n counting the FPF-vexillary elements of FPF_n, again seems unrelated to any existing entry in [25].

In this section, we require the following variant of (14). For $z \in \mathsf{FPF}_{\mathbb{Z}}$, define

$$[[z]]_E \overset{\text{def}}{=} \iota([z]_E) \in \mathsf{FPF}_\infty \tag{25}$$

for each finite set $E \subset \mathbb{Z}$ with $z(E) = E$.

Lemma 7.2. If $z \in \mathsf{FPF}_{\mathbb{Z}}$ is FPF-Grassmannian and $E \subset \mathbb{Z}$ is a finite set with $z(E) = E$, then the fixed-point-free involution $[[z]]_E$ is also FPF-Grassmannian.

Proof. Suppose $z \in \mathsf{FPF}_{\mathbb{Z}}$ is FPF-Grassmannian and $E \subset \mathbb{Z}$ is finite and z-invariant. We may assume that $z \in \mathsf{FPF}_\infty$ and $E \subset \mathbb{P}$. Fix a set $F = \{1,2,\dots,2n\}$ where $n \in \mathbb{P}$ is large enough that $E \subset F$ and $[[z]]_F = z$. Note that for any z-invariant set $D \subset E$ we have $[[z]]_D = [[z']]_{D'}$ for $z' = [[z]]_E$ and $D' = \psi_E(D)$. Inductively applying this property, we see that it suffices to show that $[[z]]_E$ is FPF-Grassmannian when $E = F \setminus \{a,b\}$ with $\{a,b\} \subset F$ a nontrivial cycle of z. In this special case, it is a straightforward exercise to check that $\mathsf{dearc}([[z]]_E)$ is either $[\mathsf{dearc}(z)]_E$ or the involution formed by replacing the leftmost cycle of $[\mathsf{dearc}(z)]_E$ by two fixed points. In either case it is easy to see that $\mathsf{dearc}([[z]]_E)$ is I-Grassmannian, so $[[z]]_E$ is FPF-Grassmannian as needed. ☐

We fix the following notation in Lemmas 7.3, 7.5, and 7.6. Let $z \in \mathsf{FPF}_{\mathbb{Z}} - \{\Theta\}$ and write $(q,r) \in \mathbb{Z}\times\mathbb{Z}$ for the maximal FPF-visible inversion of z. Set $y = \eta_{\mathsf{FPF}}(z) = (q,r)z(q,r) \in \mathsf{FPF}_{\mathbb{Z}}$ and define $p = y(q) < q$ so that $\hat{\mathfrak{T}}^{\mathsf{FPF}}_1(z) = \hat{\Psi}^-(y,p)$ if z is not FPF-Grassmannian.

Lemma 7.3. Let $E \subset \mathbb{Z}$ be a finite set with $\{q,r\} \subset E$ and $z(E) = E$. Then $(\psi_E(q), \psi_E(r))$ is the maximal FPF-visible inversion of $[[z]]_E$. Moreover, it holds that $[[\eta_{\mathsf{FPF}}(z)]]_E = \eta_{\mathsf{FPF}}([[z]]_E)$.

Proof. The first assertion holds since the set of FPF-visible inversions of z contained in $E \times E$ and the set of all FPF-visible inversions of $[[z]]_E$ are in bijection via the order-preserving map $\psi_E \times \psi_E$. The second claim follows from the definition of η_{FPF} since $\{q, r, z(q), z(r)\} \subset E$. ☐

Define

$$L^{\mathsf{FPF}}(z) \overset{\text{def}}{=} \{i \in \mathbb{Z} : i < p \text{ and } (i,p)y(i,p) \in \hat{\Psi}^-(y,p)\}. \tag{26}$$

For any $E \subset \mathbb{Z}$ we define

$$\mathfrak{C}^{\mathsf{FPF}}(z,E) \overset{\text{def}}{=} \left\{(i,p)y(i,p) : i \in E \cap L^{\mathsf{FPF}}(z)\right\}. \tag{27}$$

Also let $\mathfrak{C}^{\mathsf{FPF}}(z) \overset{\text{def}}{=} \mathfrak{C}^{\mathsf{FPF}}(z,\mathbb{Z})$, so that $\mathfrak{C}^{\mathsf{FPF}}(z) = \hat{\mathfrak{T}}_1^{\mathsf{FPF}}(z)$ if z is not FPF-Grassmannian. The following shows that $\mathfrak{C}^{\mathsf{FPF}}(z)$ is always nonempty.

Lemma 7.4. If $z \in \mathsf{FPF}_{\mathbb{Z}} - \{\Theta\}$ is FPF-Grassmannian, then $|\mathfrak{C}^{\mathsf{FPF}}(z)| = 1$.

Proof. Assume $z \in \mathsf{FPF}_{\mathbb{Z}} - \{\Theta\}$ is FPF-Grassmannian. By Proposition 4.13 we have $z = \mathsf{arc}(g)$ for an I-Grassmannian involution $g \in \mathsf{Invol}_{\mathbb{Z}}$. Using this fact and the observations in Remark 5.4, one checks that $\mathfrak{C}^{\mathsf{FPF}}(z) = \{(i,p)y(i,p)\}$ where i is the greatest integer less than p such that $y(i) < q$. □

Lemma 7.5. Let $E \subset \mathbb{Z}$ be a finite set such that $\{q,r\} \subset E$ and $z(E) = E$.

(a) The restriction of $v \mapsto [[v]]_E$ is an injective map $\mathfrak{C}^{\mathsf{FPF}}(z,E) \to \mathfrak{C}^{\mathsf{FPF}}([[z]]_E)$.
(b) If E contains $L^{\mathsf{FPF}}(z)$, then the injective map in (a) is a bijection.

Proof. Part (a) is straightforward from the definition of $\mathfrak{C}^{\mathsf{FPF}}(z)$ given Lemma 7.3. We prove the contrapositive of part (b). Suppose $a < b = \psi_E(p)$ and $(a,b)[[y]]_E(a,b)$ belongs to $\mathfrak{C}^{\mathsf{FPF}}([[z]]_E)$ but is not in the image of $\mathfrak{C}^{\mathsf{FPF}}(z,E)$ under the map $v \mapsto [[v]]_E$. Suppose $a = \psi_E(i)$ for $i \in E$. Then $(a,b)[[y]]_E(a,b) = [[(i,p)y(i,p)]]_E$, and it follows from Proposition 3.2 that $[[y]]_E(a) < [[y]]_E(b)$, so we likewise have $y(i) < y(p)$. Since $(i,p)y(i,p) \notin \mathfrak{C}^{\mathsf{FPF}}(z,E)$, there must exist an integer j with $i < j < p$ and $y(i) < y(j) < y(p)$. Let j be maximal with this property and set $k = z(j)$. One can check using Proposition 3.2 that either j or k belongs to $L^{\mathsf{FPF}}(z)$ but not E, so $E \not\supset L^{\mathsf{FPF}}(z)$. □

We say that $z \in \mathsf{FPF}_{\mathbb{Z}}$ *contains a bad FPF-pattern* if there is a finite set $E \subset \mathbb{Z}$ with $z(E) = E$ and $|E| \leq 12$, such that $[[z]]_E$ is not FPF-vexillary. We refer to E as a *bad FPF-pattern* for z.

Lemma 7.6. If $z \in \mathsf{FPF}_{\mathbb{Z}}$ is such that $|\hat{\mathfrak{T}}_1^{\mathsf{FPF}}(z)| \geq 2$, then z contains a bad FPF-pattern.

Proof. If $u \neq v$ and $\{u,v\} \subset \hat{\mathfrak{T}}_1^{\mathsf{FPF}}(z)$, then u, v, and z agree outside a set $E \subset \mathbb{Z}$ of size 8 with $z(E) = E$. It follows by Lemmas 7.4 and 7.5 that E is a bad FPF-pattern for z. □

Lemma 7.7. Suppose $z \in \mathsf{FPF}_{\mathbb{Z}}$ is such that $\hat{\mathfrak{T}}_1^{\mathtt{FPF}}(z) = \{v\}$ is a singleton set. Then z contains no bad FPF-patterns if and only if v contains no bad FPF-patterns.

Proof. By definition, z and v agree outside a set $A \subset \mathbb{Z}$ of size 6 with $v(A) = z(A) = A$. If z (respectively, v) contains a bad FPF-pattern that is disjoint from A, then the other involution clearly does also. If z contains a bad FPF-pattern B that intersects A, then $E = A \cup B$ has size at most 16 since $|B| \leq 12$ and both A and B are z-invariant. In this case, $[[z]]_E$ contains a bad FPF-pattern and Lemma 7.5(b) shows that $\mathfrak{C}^{\mathtt{FPF}}([[z]]_E) = \{[[v]]_E\}$, and if $[[v]]_E$ contains a bad FPF-pattern then v does also. By similar arguments, it follows that if v contains a bad FPF-pattern B that intersects A, then $E = A \cup B$ has size at most 16, $[[v]]_E$ contains a bad FPF-pattern, $\mathfrak{C}^{\mathtt{FPF}}([[z]]_E) = \{[[v]]_E\}$, and v contains a bad FPF-pattern if $[[v]]_E$ does.

These observations show that to prove the lemma, it suffices to consider the case when z belongs to the image of $\iota : \mathsf{FPF}_{16} \hookrightarrow \mathsf{FPF}_{\mathbb{Z}}$. Using a computer, we have checked that if z is such an involution and $\mathfrak{C}^{\mathtt{FPF}}(z) = \{v\}$ is a singleton set, then z contains no bad FPF-patterns if and only if v contains no bad FPF-patterns. There are 940,482 possibilities for z, a sizeable but tractable number. $\square$

Theorem 7.8. An involution $z \in \mathsf{FPF}_{\mathbb{Z}}$ is FPF-vexillary if and only if $[[z]]_E$ is FPF-vexillary for all sets $E \subset \mathbb{Z}$ with $z(E) = E$ and $|E| = 12$.

Proof. Let $\mathcal{X} \subset \mathsf{FPF}_{\mathbb{Z}}$ be the set that contains $z \in \mathsf{FPF}_{\mathbb{Z}}$ if and only if z is FPF-Grassmannian or $\hat{\mathfrak{T}}_1^{\mathtt{FPF}}(z) = \{v\}$ and $v \in \mathcal{X}$. It follows from Corollary 5.9(b) that $\mathcal{X}$ is the set of all FPF-vexillary involutions in $\mathsf{FPF}_{\mathbb{Z}}$. On the other hand, Lemmas 7.2, 7.6, and 7.7 show that $\mathcal{X}$ is the set of involutions $z \in \mathsf{FPF}_{\mathbb{Z}}$ that contain no bad FPF-patterns. Thus $z \in \mathsf{FPF}_{\mathbb{Z}}$ is FPF-vexillary if and only if z has no bad FPF-patterns, which is equivalent to the theorem statement. $\square$

Corollary 7.9. An involution $z \in \mathsf{FPF}_{\mathbb{Z}}$ is FPF-vexillary if and only if for all finite sets $E \subset \mathbb{Z}$ with $z(E) = E$ the involution $[z]_E$ is not any of the following sixteen permutations:

$(1,3)(2,4)(5,8)(6,7)$, $(1,3)(2,5)(4,7)(6,8)$, $(1,3)(2,5)(4,8)(6,7)$, $(1,3)(2,6)(4,8)(5,7)$, $(1,4)(2,3)(5,7)(6,8)$, $(1,4)(2,3)(5,8)(6,7)$, $(1,5)(2,3)(4,7)(6,8)$, $(1,5)(2,3)(4,8)(6,7)$, $(1,5)(2,4)(3,7)(6,8)$, $(1,5)(2,4)(3,8)(6,7)$, $(1,6)(2,3)(4,8)(5,7)$, $(1,6)(2,4)(3,8)(5,7)$, $(1,6)(2,5)(3,8)(4,7)$, $(1,3)(2,4)(5,7)(6,9)(8,10)$, $(1,3)(2,5)(4,6)(7,9)(8,10)$, $(1,3)(2,4)(5,7)(6,8)(9,11)(10,12)$.

Proof. It follows by a computer calculation using the formulas in Theorems 4.19 and 5.21 that $z \in \iota(\mathsf{FPF}_{12}) \subset \mathsf{FPF}_\infty$ is not FPF-vexillary if and only if there is a z-invariant subset $E \subset \mathbb{Z}$ such that $[z]_E$ is one of the given involutions. The corollary follows from this fact by Theorem 7.8. □

8. Pfaffian formulas

The *Pfaffian* of a skew-symmetric $n \times n$ matrix A is

$$\text{pf}\, A \overset{\text{def}}{=} \sum_{z \in \mathsf{FPF}_n} (-1)^{\hat{\ell}_{\text{FPF}}(z)} \prod_{z(i)<i\in[n]} A_{z(i),i}. \tag{28}$$

It is a classical fact that $\det A = (\text{pf}\, A)^2$. Since $\det A = 0$ when A is skew-symmetric but n is odd, the definition (28) is consistent with the fact that the set FPF_n of fixed-point-free involutions in S_n is nonempty only if n is even. If $A = (a_{ij})$ is a 2×2 skew-symmetric matrix then $\text{pf}\, A = a_{12} = -a_{21}$. If $A = (a_{ij})$ is a 4×4 skew-symmetric matrix then $\text{pf}\, A = a_{21}a_{43} - a_{31}a_{42} + a_{41}a_{32}$.

Both $\hat{\mathfrak{S}}^{\text{FPF}}_z$ and $\hat{F}^{\text{FPF}}_z$ can be expressed by certain Pfaffian formulas when z is FPF-Grassmannian. We fix the following notation for the duration of this section: first, let

$$n, r \in \mathbb{P} \qquad \text{and} \qquad \phi \in \mathbb{P}^r \text{ with } 0 < \phi_1 < \phi_2 < \cdots < \phi_r < n. \tag{29}$$

Set $\phi_i = 0$ for $i > r$. Define $y = (\phi_1, n+1)(\phi_2, n+2)\cdots(\phi_r, n+r) \in \mathsf{Invol}_\infty$ and $z = \mathsf{arc}(y)$. Let

$$\hat{\mathfrak{S}}^{\text{FPF}}[\phi_1, \phi_2, \dots, \phi_r; n] \overset{\text{def}}{=} \hat{\mathfrak{S}}^{\text{FPF}}_z \quad \text{and} \quad \hat{F}^{\text{FPF}}[\phi_1, \phi_2, \dots, \phi_r; n] \overset{\text{def}}{=} \hat{F}^{\text{FPF}}_z. \tag{30}$$

In the case that r is odd, we set $\hat{\mathfrak{S}}^{\text{FPF}}[\phi_1, \phi_2, \dots, \phi_r, 0; n] \overset{\text{def}}{=} \hat{\mathfrak{S}}^{\text{FPF}}_z$ and $\hat{F}^{\text{FPF}}[\phi_1, \phi_2, \dots, \phi_r, 0; n] \overset{\text{def}}{=} \hat{F}^{\text{FPF}}_z$.

Proposition 8.1. In the notation just given, $z \in \mathsf{FPF}_\infty$ is FPF-Grassmannian with shape $\nu(z) = (n - \phi_1, n - \phi_2, \dots, n - \phi_r)$. Moreover, each FPF-Grassmannian element of $\mathsf{FPF}_\infty - \{\Theta\}$ occurs as such an involution z for a unique choice of $n, r \in \mathbb{P}$ and $\phi \in \mathbb{P}^r$ as in (29).

Proof. Let $X = [n] \setminus \{\phi_1, \phi_2, \dots, \phi_r\}$ so that $n \in X$. If $|X|$ is even then $\mathsf{dearc}(z) = y$. If $|X|$ is odd and at least 3, then $\mathsf{dearc}(z) = y \cdot (n, n+r+1)$. If $|X| = 1$, finally, then $\phi = (1, 2, \dots, n-1)$ and $\mathsf{dearc}(z) = (2, n+2)(3, n+3)\cdots(n, 2n)$. In each case, $\nu(z) = (n - \phi_1, n - \phi_2, \dots, n - \phi_r)$ as desired. The second assertion holds since an FPF-Grassmannian element of FPF_∞

is uniquely determined by its image under $\mathsf{dearc} : \mathsf{FPF}_\infty \to \mathsf{Invol}_\infty$, which must be I-Grassmannian with an even number of fixed points in $[n]$ and not equal to $(i+1, n+1)(i+2, n+2)\cdots(n, 2n-i)$ for any $i \in [n]$. □

Let $\ell^+(\phi)$ be whichever of r or $r+1$ is even, and let $[a_{ij}]_{1\le i<j\le n}$ denote the skew-symmetric matrix with a_{ij} in position (i,j) and $-a_{ij}$ in position (j,i) for $i<j$ (and zeros on the diagonal).

Corollary 8.2. In the setup of (29),

$$\hat{F}^{\mathsf{FPF}}[\phi_1, \phi_2, \ldots, \phi_r; n] = \mathrm{pf}\left[\hat{F}^{\mathsf{FPF}}[\phi_i, \phi_j; n]\right]_{1\le i<j\le \ell^+(\phi)}.$$

Proof. If λ is a strict partition then $P_\lambda = \mathrm{pf}[P_{\lambda_i\lambda_j}]_{1\le i<j\le\ell^+(\lambda)}$ by [18, Eq. (8.11), §III.8]. Given this fact and the preceding proposition, the result follows from Theorem 4.19. □

Our goal is to prove that the identity in this corollary holds with $\hat{F}^{\mathsf{FPF}}[\cdots; n]$ replaced by $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\cdots; n]$. In the following lemmas, we let

$$\mathfrak{M}^{\mathsf{FPF}}[\phi; n] = \mathfrak{M}^{\mathsf{FPF}}[\phi_1, \phi_2, \ldots, \phi_r; n] \overset{\text{def}}{=} \left[\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_i, \phi_j; n]\right]_{1\le i<j\le \ell^+(\phi)} \tag{31}$$

denote the $\ell^+(\phi)\times\ell^+(\phi)$ skew-symmetric matrix with $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_i, \phi_j; n]$ is position (i,j) for $i<j$.

Lemma 8.3. Maintain the notation of (29), and suppose $p \in [n-1]$. Then

$$\partial_p\left(\mathrm{pf}\,\mathfrak{M}^{\mathsf{FPF}}[\phi; n]\right) = \begin{cases} \mathrm{pf}\,\mathfrak{M}^{\mathsf{FPF}}[\phi + e_i; n] & \text{if } p = \phi_i \notin \{\phi_2 - 1, \ldots, \phi_r - 1\} \\ & \text{for some } i \in [r] \\ 0 & \text{otherwise} \end{cases}$$

where $e_i = (0, \ldots, 0, 1, 0, 0, \ldots)$ is the standard basis vector whose ith coordinate is 1.

Proof. Let $\mathfrak{M} = \mathfrak{M}^{\mathsf{FPF}}[\phi; n]$. If $1 \le i < j \le \ell^+(\phi)$ then (12) implies that $\partial_p\mathfrak{M}_{ij} = \partial_p\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_i, \phi_j; n]$ is $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_i+1, \phi_j]$ if $p = \phi_i \neq \phi_j - 1$, $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_i, \phi_j+1]$ if $p = \phi_j$, and 0 otherwise. Thus if $p \notin \{\phi_1, \phi_2, \ldots, \phi_r\}$ then $\partial_p(\mathrm{pf}\,\mathfrak{M}) = 0$. Suppose $p = \phi_k$. Then $\partial_p\mathfrak{M}_{ij} = 0$ unless $i = k$ or $j = k$, so $\partial_p(\mathrm{pf}\,\mathfrak{M}) = \mathrm{pf}\,\mathfrak{N}$ where $\mathfrak{N}$ is the matrix formed by applying ∂_p to the entries in the kth row and kth column of $\mathfrak{M}$. If $k < r$ and $\phi_k = \phi_{k+1} - 1$, then columns k and $k+1$ of $\mathfrak{N}$ are identical, so $\mathrm{pf}\,\mathfrak{M} = \mathrm{pf}\,\mathfrak{N} = 0$. If $k = r$ or if $k < r$ and $\phi_k \neq \phi_{k+1} - 1$, then $\mathfrak{N} = \mathfrak{M}^{\mathsf{FPF}}[\phi + e_k; n]$. □

Lemma 8.4. Let $n \geq 2$ and $D = (x_1 + x_2)(x_1 + x_3) \cdots (x_1 + x_n)$. Then $\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[1; n] = D$, and if $b \in \mathbb{P}$ is such that $1 < b < n$, then $\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[1, b; n]$ is divisible by D.

Proof. Theorem 4.2 implies that $\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[1; n] = D$ and, when $n > 2$, that $\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[1, 2; n] = (x_2+x_3) \cdots (x_2+x_n)D$. If $2 < b < n$ then $\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[1, b; n] = \partial_{b-1}(\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[1, b-1; n])$ by the previous lemma. Since D is symmetric in x_{b-1} and x_b, the desired property holds by induction. □

If $i : \mathbb{P} \to \mathbb{N}$ is a map with $i^{-1}(\mathbb{P}) \subset [n]$, then let $x^i = x_1^{i(1)} x_2^{i(2)} \cdots x_n^{i(n)}$. Given a nonzero polynomial $f = \sum_{i:\mathbb{P}\to\mathbb{N}} c_i x^i \in \mathbb{Z}[x_1, x_2, \dots]$, let $j : \mathbb{P} \to \mathbb{N}$ be the lexicographically minimal index such that $c_j \neq 0$ and define $\mathrm{lt}(f) = c_j x^j$. We refer to $\mathrm{lt}(f)$ as the *least term* of f. Set $\mathrm{lt}(0) = 0$, so that $\mathrm{lt}(fg) = \mathrm{lt}(f)\,\mathrm{lt}(g)$ for any polynomials f, g. The following is [8, Proposition 3.14].

Lemma 8.5 (See [8])**.** If $z \in \mathsf{FPF}_\infty$ then $\mathrm{lt}(\hat{\mathfrak{S}}^{\mathsf{FPF}}_z) = x^{\hat{c}_{\mathsf{FPF}}(z)} = \prod_{(i,j) \in \hat{D}_{\mathsf{FPF}}(z)} x_i$.

Let $\mathscr{M}$ denote the set of monomials $x^i = x_1^{i(1)} x_2^{i(2)} \cdots$ for maps $i : \mathbb{P} \to \mathbb{N}$ with $i^{-1}(\mathbb{P})$ finite. Define $\prec$ as the "lexicographic" order on $\mathscr{M}$, that is, the order with $x^i \prec x^j$ when there exists $n \in \mathbb{P}$ such that $i(t) = j(t)$ for $1 \leq t < n$ and $i(n) < j(n)$. Note that $\mathrm{lt}(\hat{\mathfrak{S}}^{\mathsf{FPF}}_z) \in \mathscr{M}$. Also, observe that if $a, b, c, d \in \mathscr{M}$ and $a \preceq c$ and $b \preceq d$, then $ab \preceq cd$ with equality if and only if $a = c$ and $b = d$.

Lemma 8.6. Let $i, j, n \in \mathbb{P}$. The following identities then hold:

(a) If $i < n$ then $\mathrm{lt}(\hat{\mathfrak{S}}^{\mathsf{FPF}}[i; n]) \succeq x_{i+1} x_{i+2} \cdots x_n$, with equality if and only if i is odd.

(b) If $i < j < n$ then $\mathrm{lt}(\hat{\mathfrak{S}}^{\mathsf{FPF}}[i, j; n]) \succeq (x_{i+1} x_{i+2} \cdots x_n)(x_{j+1} x_{j+2} \cdots x_n)$, with equality if and only if i is odd and j is even.

Proof. The result follows by routine calculations using Lemma 8.5. For example, suppose $i < j < n$ and let $y = (i, n+1)(j, n+2)$ and $z = \mathsf{arc}(y)$, so that $\hat{\mathfrak{S}}^{\mathsf{FPF}}[i, j; n] = \hat{\mathfrak{S}}^{\mathsf{FPF}}_z$. If i is even and $j = i + 1$, then $\hat{D}_{\mathsf{FPF}}(z) = \{(i, i-1), (i+1, i-1)\} \cup \{(i+1, i), (i+3, i), \dots, (n, i)\} \cup \{(i+3, i+1), \dots, (n, i+1)\}$ so $\mathrm{lt}(\hat{\mathfrak{S}}^{\mathsf{FPF}}[i, j; n]) = (x_i x_{i+1} x_{i+3} \cdots x_n)(x_j x_{j+2} \cdots x_n)$. The other cases follow by similar analysis. □

Lemma 8.7. If $n \in \mathbb{P}$ and $r \in [n-1]$ then

$$\hat{\mathfrak{S}}^{\mathsf{FPF}}[1, 2, \dots, r; n] = \mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[1, 2, \dots, r; n].$$

Proof. The proof is similar to that of [11, Lemma 4.77]. Let $D_i = (x_i + x_{i+1})(x_i + x_{i+2}) \cdots (x_i + x_n)$ for $i \in [n-1]$ and $\mathfrak{M} = \mathfrak{M}^{\mathsf{FPF}}[1, 2, \dots, r; n]$.

Theorem 4.2 implies that $\hat{\mathfrak{S}}^{\mathsf{FPF}}[1,2,\dots,r;n] = D_1 D_2 \cdots D_r$. Lemma 8.3 implies that $\operatorname{pf}\mathfrak{M}$ is symmetric in $x_1, x_2, \dots, x_r$. Lemma 8.4 implies that every entry in the first column of $\mathfrak{M}$, and therefore also $\operatorname{pf}\mathfrak{M}$, is divisible by D_1. Since $s_i(D_i)$ is divisible by D_{i+1}, it follows that $\operatorname{pf}\mathfrak{M}$ is divisible by $\hat{\mathfrak{S}}^{\mathsf{FPF}}[1,2,\dots,r;n]$. To prove the lemma, it suffices to show that $\operatorname{pf}\mathfrak{M}$ and $\hat{\mathfrak{S}}^{\mathsf{FPF}}[1,2,\dots,r;n]$ have the same least term.

Let $m \in \mathbb{P}$ be whichever of r or $r+1$ is even and choose $z \in \mathsf{FPF}_m$. By Lemma 8.6,

$$\operatorname{lt}\left(\prod_{z(i)<i\in[m]} \mathfrak{M}_{z(i),i}\right) \succeq (x_2\cdots x_n)(x_3\cdots x_n)\cdots(x_{r+1}\cdots x_n)$$
$$= \operatorname{lt}(\hat{\mathfrak{S}}^{\mathsf{FPF}}[1,2,\dots,r;n]),$$

with equality if and only if i is odd and j is even whenever $i < j = z(i)$. The only element $z \in \mathsf{FPF}_m$ with the latter property is the involution $z = (1,2)(3,4)\cdots(m-1,m) = \Theta_m$, so we deduce from (28) that $\operatorname{lt}(\operatorname{pf}\mathfrak{M}) = \operatorname{lt}(\hat{\mathfrak{S}}^{\mathsf{FPF}}[1,2,\dots,r;n])$ as needed. $\square$

Let $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi;n] = \hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_1,\phi_2,\dots,\phi_r;n]$. The following is the main result of this section.

Theorem 8.8. It holds that $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi;n] = \operatorname{pf}\mathfrak{M}^{\mathsf{FPF}}[\phi;n]$.

Proof. If $\phi = (1,2,\dots,r)$ then $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi;n] = \operatorname{pf}\mathfrak{M}^{\mathsf{FPF}}[\phi;n]$ by the previous lemma. Otherwise, there exists a smallest $i \in [r]$ such that $i < \phi_i$. If $p = \phi_i - 1$ then $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi;n] = \partial_p \hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi - e_i;n]$ by (12) and $\operatorname{pf}\mathfrak{M}^{\mathsf{FPF}}[\phi;n] = \partial_p(\operatorname{pf}\mathfrak{M}^{\mathsf{FPF}}[\phi - e_i;n])$ by Lemma 8.3. We may assume that $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi - e_i;n] = \operatorname{pf}\mathfrak{M}^{\mathsf{FPF}}[\phi - e_i;n]$ by induction, so the result follows. $\square$

Example 8.9. For $\phi = (1,2,3)$ and $n = 4$, the theorem implies that the polynomial $\hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,5)(2,6)(3,7)(4,8)}$ is equal to the Pfaffian

$$\operatorname{pf}\begin{pmatrix} 0 & \hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,5)(2,6)(3,4)} & \hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,5)(2,4)(3,6)} & \hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,5)(2,3)(4,6)} \\ -\hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,5)(2,6)(3,4)} & 0 & \hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,4)(2,5)(3,6)} & \hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,3)(2,5)(4,6)} \\ -\hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,5)(3,6)(2,4)} & -\hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,4)(2,5)(3,6)} & 0 & \hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,2)(3,5)(4,6)} \\ -\hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,5)(2,3)(4,6)} & -\hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,3)(2,5)(4,6)} & -\hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,2)(3,5)(4,6)} & 0 \end{pmatrix}$$

where for $z \in \mathsf{FPF}_n$ we define $\hat{\mathfrak{S}}^{\mathsf{FPF}}_z = \hat{\mathfrak{S}}^{\mathsf{FPF}}_{\iota(z)}$. By Theorem 4.2, both of these expressions evaluate to $(x_1+x_2)(x_1+x_3)(x_1+x_4)(x_2+x_3)(x_2+x_4)(x_3+x_4)$.

Appendix A. Index of symbols

The tables below list our non-standard notations, with references to definitions where relevant.

Symbol	Meaning	Reference
$\mathbb{N}$	The set of nonnegative integers	
$\mathbb{P}$	The set of positive integers	
$[n]$	The set of positive integers $\{1, 2, \dots, n\}$	
ϕ_E	The unique order-preserving bijection $[n] \to E$ for $E \subset \mathbb{Z}$	
ψ_E	The unique order-preserving bijection $E \to [n]$ for $E \subset \mathbb{Z}$	
$S_\mathbb{Z}$	The group of permutations of $\mathbb{Z}$ with finite support	
$\mathsf{Invol}_\mathbb{Z}$	The set $\{w \in S_\mathbb{Z} : w = w^{-1}\}$ of involutions in $S_\mathbb{Z}$	
S_∞	Subgroup of permutations in $S_\mathbb{Z}$ fixing all numbers outside $\mathbb{P}$	
Invol_∞	The set $\{w \in S_\infty : w = w^{-1}\}$ of involutions in S_∞	
S_n	Subgroup of permutations in S_∞ fixed all numbers outside $[n]$	
Θ	The permutation of $\mathbb{Z}$ given by $i \mapsto i - (-1)^i$	(6)
Θ_n	The permutation $(1, 2)(3, 4) \dots (2n - 1, 2n) \in S_{2n}$	
FPF_n	The set of fixed-point-free involutions in S_{2n}	
FPF_∞	The S_∞-conjugacy class of Θ	§2.3
$\mathsf{FPF}_\mathbb{Z}$	The $S_\mathbb{Z}$-conjugacy class of Θ	§2.3
ι	The natural inclusion $\mathsf{FPF}_n \hookrightarrow \mathsf{FPF}_\infty$	(7)
arc	A certain map $\mathsf{Invol}_\mathbb{Z} \to \mathsf{FPF}_\mathbb{Z}$	Def. 4.10
dearc	A certain map $\mathsf{FPF}_\mathbb{Z} \to \mathsf{Invol}_\mathbb{Z}$	Def. 4.11
η_{FPF}	A certain map $\mathsf{FPF}_\mathbb{Z} - \{\Theta\} \to \mathsf{FPF}_\mathbb{Z}$	Def. 5.3
w_n	The longest permutation $n \cdots 321 \in S_n$	
$[w]_E$	The standardization of w to the subset $E \subset \mathbb{Z}$	(14)
$[[w]]_E$	The element $\iota([w]_E) \in \mathsf{FPF}_\infty$ for $E \subset \mathbb{Z}$ with $w(E) = E$	(25)
$w \gg N$	The map $\mathbb{Z} \to \mathbb{Z}$ given by $i \mapsto w(i - N) + N$	
$\mathcal{R}(w)$	The set of reduced words for $w \in W$	§2
$\mathcal{A}_{\mathsf{FPF}}(z)$	The set of minimal length elements $w \in S_\mathbb{Z}$ with $z = w^{-1}\Theta w$	
$\hat{\mathcal{R}}_{\mathsf{FPF}}(z)$	The disjoint union $\hat{\mathcal{R}}_{\mathsf{FPF}}(z) = \bigsqcup_{w \in \mathcal{A}_{\mathsf{FPF}}(z)} \mathcal{R}(w)$	(11)
$\beta_{\min}(z)$	The minimal atom in $\mathcal{A}_{\mathsf{FPF}}(z)$ for $z \in \mathsf{FPF}_\infty$	Lem. 4.3
$\mathrm{Cyc}_\mathbb{Z}(z)$	The set $\{(i, j) \in \mathbb{Z} \times \mathbb{Z} : i < j = z(i)\}$ for $z \in \mathsf{FPF}_\mathbb{Z}$	(9)
$\mathrm{Cyc}_\mathbb{P}(z)$	The intersection $\mathrm{Cyc}_\mathbb{Z}(z) \cap (\mathbb{P} \times \mathbb{P})$	
$\mathrm{Inv}(z)$	The inversion set $\{(i, j) \in \mathbb{Z} \times \mathbb{Z} : i < j \text{ and } z(i) > z(j)\}$	
$\mathrm{Inv}_{\mathsf{FPF}}(z)$	The set $\mathrm{Inv}(z) - \mathrm{Cyc}_\mathbb{Z}(z)$ for $z \in \mathsf{FPF}_\mathbb{Z}$	(9)
$\hat{\ell}_{\mathsf{FPF}}$	The FPF-involution length function $\mathsf{FPF}_\mathbb{Z} \to \mathbb{N}$	(10)
$\mathrm{Des}_R^{\mathsf{FPF}}(z)$	A modified right descent set for $z \in \mathsf{FPF}_\mathbb{Z}$	(10)
$\mathrm{Des}_V^{\mathsf{FPF}}(z)$	The set of FPF-visible descents of $z \in \mathsf{FPF}_\mathbb{Z}$	(19)
$\mathrm{Des}_V(z)$	The set of visible descents of $z \in \mathsf{Invol}_\mathbb{Z}$	(20)
$\mathfrak{S}_w$	The Schubert polynomial of $w \in S_n$	(3)
$\hat{\mathfrak{S}}_z^{\mathsf{FPF}}$	The FPF-involution Schubert polynomial $\sum_{w \in \mathcal{A}_{\mathsf{FPF}}(z)} \mathfrak{S}_w$	Def. 2.4

Symbol	Meaning	Reference
F_w	The Stanley symmetric function of $w \in S_n$	Def. 2.1
$\hat{F}^{\mathsf{FPF}}_z$	The FPF-involution symmetric function $\sum_{w\in\mathcal{A}_{\mathsf{FPF}}(z)} F_w$	Def. 2.7
$<$	The Bruhat order on $S_{\mathbb{Z}}$ or $\mathsf{FPF}_{\mathbb{Z}}$	§3
$\lessdot_{\mathsf{FPF}}$	The covering relation for the Bruhat order on $\mathsf{FPF}_{\mathbb{Z}}$	§3
$<_{\mathcal{A}_{\mathsf{FPF}}}$	A certain partial order of $\mathcal{A}_{\mathsf{FPF}}(z)$	(24)
$D(w)$	The Rothe diagram $\{(i, w(j)) : (i,j) \in \mathrm{Inv}(w)\}$	(16)
$\hat{D}_{\mathsf{FPF}}(z)$	The involution Rothe diagram of $z \in \mathsf{FPF}_\infty$	(17)
$c(w)$	The code of $w \in S_\infty$	§4
$\hat{c}_{\mathsf{FPF}}(z)$	The involution code of $w \in \mathsf{FPF}_\infty$	§4
$\lambda(w)$	The partition given by sorting $c(w)$ for $w \in S_\infty$	§6
$\nu(z)$	The shape of $w \in \mathsf{FPF}_\infty$	§6
δ_n	The partition $(n-1, n-2, \dots, 3, 2, 1)$	
λ^T	The transpose of a partition λ	
$\mathcal{P}$	The polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$	
$\mathcal{L}$	The Laurent polynomial ring $\mathbb{Z}\left[x_1, x_2, \dots, x_1^{-1}, x_2^{-1}, \dots\right]$	
∂_i	The ith divided difference operator	(1)
π_i	The ith isobaric divided difference operator	(2)
$G_{m,n}$	A certain element of $\mathcal{L}$	(13)
Λ	The Hopf algebra of symmetric functions over $\mathbb{Z}$	[27]
s_λ	The Schur function indexed by a partition λ	[27]
P_λ	The Schur P-function indexed by a strict partition λ	Def. 2.9
$\hat{\Psi}^{\pm}(y,r)$	Index sets for sums in transition formula Theorem 3.4	(15)
$\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$	The FPF-involution Lascoux-Schützenberger tree	Def. 5.7
$L^{\mathsf{FPF}}(z)$	The set $\{i \in \mathbb{Z} : i < p \text{ and } (i,p)y(i,p) \in \hat{\Psi}^-(y,p)\}$	(26)
$\mathfrak{C}^{\mathsf{FPF}}(z,E)$	The set $\left\{(i,p)y(i,p) : i \in E \cap L^{\mathsf{FPF}}(z)\right\}$	(27)
$\mathrm{pf}\, A$	The Pfaffian of a skew-symmetric matrix A	(28)
$\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi; n]$	An instance of $\hat{\mathfrak{S}}^{\mathsf{FPF}}_z$ where z is FPF-Grassmannian	(30)
$\hat{F}^{\mathsf{FPF}}[\phi; n]$	An instance of $\hat{F}^{\mathsf{FPF}}_z$ where z is FPF-Grassmannian	(30)
$\mathfrak{M}^{\mathsf{FPF}}[\phi; n]$	A certain skew-symmetric matrix	(31)

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