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Gamma-positivity of variations of Eulerian polynomials

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An identity of Chung, Graham and Knuth involving binomial coefficients and Eulerian numbers motivates our study of a class of polynomials that we call binomial-Eulerian polynomials. These polynomials share several properties with the Eulerian polynomials. For one thing, they are h-polynomials of simplicial polytopes, which gives a geometric interpretation of the fact that they are palindromic and unimodal. A formula of Foata and Schützenberger shows that the Eulerian polynomials have a stronger property, namely γ-positivity, and a formula of Postnikov, Reiner and Williams does the same for the binomial-Eulerian polynomials. We obtain q-analogs of both the Foata-Schützenberger formula and an alternative to the Postnikov-Reiner-Williams formula, and we show that these q-analogs are specializations of analogous symmetric function identities. Algebro-geometric interpretations of these symmetric function analogs are presented.

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1. Introduction

In [10], Chung, Graham, and Knuth give several proofs of the following interesting symmetry involving Eulerian numbers $a_{n,j}$ and binomial coefficients. For nonnegative integers r, s,

$$\sum_{m=1}^{r+s} \binom{r+s}{m} a_{m,r-1} = \sum_{m=1}^{r+s} \binom{r+s}{m} a_{m,s-1}. \tag{1.1}$$

A q-analog of this identity was subsequently obtained independently by Chung and Graham [9] and Han, Lin, and Zeng [20].

Equation (1.1) is equivalent to palindromicity of the polynomial

$$\tilde{A}_n(t) = \sum_{j=0}^{n} \tilde{a}_{n,j} t^j := 1 + t \sum_{m=1}^{n} \binom{n}{m} A_m(t),$$

for all $n \geq 0$, where $A_m(t)$ is the Eulerian polynomial. We refer to $\tilde{A}_n(t)$ as a binomial-Eulerian polynomial and $\tilde{a}_{n,j}$ as a binomial-Eulerian number. It is well known and easy to prove that the Eulerian polynomials are palindromic as well. Hence it is natural to ask whether the binomial-Eulerian polynomials share any other properties with the Eulerian polynomials, such as unimodality.

A polynomial $A(t) = \sum_{j=0}^{d} a_j t^j \in \mathbb{R}[t]$ is said to be *palindromic* if $a_j = a_{d-j}$ for all $j = 0, \ldots, d$, and it is said to be *positive* and *unimodal* if for some c

$$0 \leq a_0 \leq a_1 \leq \cdots \leq a_c \geq \cdots \geq a_{d-1} \geq a_d \geq 0.$$

For example, $A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4$ is clearly palindromic, positive, and unimodal. Many important polynomials arising in algebra, combinatorics, and geometry are palindromic, positive and unimodal, see e.g., [35, 36, 6].

One can easily see that $A(t)$ is palindromic if and only if there exist $\gamma_0, \ldots, \gamma_{\lfloor \frac{d}{2} \rfloor} \in \mathbb{R}$ such that

$$A(t) = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_k t^k (1+t)^{d-2k}. \tag{1.2}$$

The palindromic polynomial $A(t)$ is said to be *γ-positive* if $\gamma_k \geq 0$ for all k. It is well known and not difficult to see that γ-positivity implies unimodality.

The Eulerian polynomials $A_n(t)$ are γ-positive as is evident from the Foata-Schützenberger formula [14, Theorem 5.6],

$$A_n(t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}\, t^k (1+t)^{n-1-2k}, \tag{1.3}$$

where $\gamma_{n,k} = |\Gamma_{n,k}|$ and $\Gamma_{n,k}$ is the set of permutations $\sigma \in \mathfrak{S}_n$ with

- no double descents[1],
- no final descent,
- $\operatorname{des}(\sigma) = k$.

For example $A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4$ is γ-positive since

$$A_5(t) = 1t^0(1+t)^4 + 22t^1(1+t)^2 + 16t^2(1+t)^0.$$

Recent interest in γ-positivity stems from Gal's strengthening [16] of the Charney-Davis conjecture [8] by asserting that the h-polynomial of every flag simplicial sphere is γ-positive[2]. Since, as is well known, the Eulerian polynomials are the h-polynomials of dual permutohedra, the Foata-Schützenberger formula confirms Gal's conjecture for dual permutohedra.

The permutohedron is an example of a chordal nestohedron. In [26, Section 11.2], Postnikov, Reiner, and Williams confirm Gal's conjecture for all dual chordal nestohedra by giving explicit combinatorial formulae for the γ-coefficients. Another example of a chordal nestohedron, discussed in [26, Section 10.4], is the stellohedron, and the h-polynomial of its dual turns out to be equal to $\tilde{A}_n(t)$. It follows that palindromicity of $\tilde{A}_n(t)$ is equivalent to the Dehn-Sommerville equations for the dual stellohedron.

The γ-positivity formula of Postnikov, Reiner, and Williams in the case of the stellohedron says that

$$\tilde{A}_n(t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \bar{\gamma}_{n,k}\, t^k (1+t)^{n-2k}, \tag{1.4}$$

where $\bar{\gamma}_{n,k}$ is the number of $\sigma \in \mathfrak{S}_{n+1}$ such that σ has no double descents, no final descent, $\sigma(1) < \sigma(2) < \cdots < \sigma(m) = n+1$, for some $m \geq 1$, and $\operatorname{des}(\sigma) = k$.

[1] The terminology used here is defined in Section 2.
[2] The terminology used here is defined in Section 5.

Here we obtain a γ-positivity formula[3] for $\tilde{A}_n(t)$ that is somewhat simpler than the Postnikov-Reiner-Williams formula and is similar to the Foata-Schutzenberger formula for $A_n(t)$. For all $n \geq 1$,

$$\tilde{A}_n(t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \tilde{\gamma}_{n,k}\, t^k (1+t)^{n-2k}, \tag{1.5}$$

where $\tilde{\gamma}_{n,k} = |\tilde{\Gamma}_{n,k}|$ and $\tilde{\Gamma}_{n,k}$ is the set of permutations $\sigma \in \mathfrak{S}_n$ with

- no double descents,
- $\mathrm{des}(\sigma) = k$.

(A nice bijection between $\tilde{\Gamma}_{n,k}$ and the set of permutations enumerated in the Postnikov-Reiner-Williams formula (1.4) was obtained by Ellzey [11].) Moreover, we present q-analogs of this γ-positivity formula (1.5) and of the Foata-Schützenberger formula (1.3), and observe that they are specializations of analogous symmetric function identities. Algebro-geometric interpretations of these symmetric function analogs are also presented, which suggest an equivariant version of the Gal phenomenon.

The q-analogues of the Eulerian numbers and Eulerian polynomials that we consider were first examined in previous work [30, 31] of the authors on the joint distribution of the excedance statistic and the major index[4]. They are used in the Chung-Graham, Han-Ling-Zeng q-analog of (1.1) mentioned above. The q-analog $a_{n,j}(q)$ of the Eulerian number $a_{n,j}$ and the q-analog $A_n(q,t)$ of the Eulerian polynomial $A_n(t)$ are polynomials in $\mathbb{Z}[q]$ and $\mathbb{Z}[q][t]$, respectively, defined by

$$A_n(q,t) = \sum_{j=0}^{n-1} a_{n,j}(q) t^j := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}, \tag{1.6}$$

for $n \geq 1$, and $A_n(q,t) := 1$, for $n = 0$. For example,

$$A_2(q,t) = 1 + t$$
$$A_3(q,t) = 1 + (2 + q + q^2)t + t^2$$
$$A_4(q,t) = 1 + (3 + 2q + 3q^2 + 2q^3 + q^4)t + (3 + 2q + 3q^2 + 2q^3 + q^4)t^2 + t^3.$$

[3] An alternative proof of (1.5) using poset topological techniques will appear in [19].

[4] The permutation statistics terminology is defined in Section 2.

Another combinatorial description of $A_n(q,t)$ is given in more recent work [32, 33] of the authors.

In [30, 31], the authors obtain a q-analog of Euler's formula for the exponential generating function of the Eulerian polynomials,

$$\sum_{n\geq 0} A_n(q,t)\frac{z^n}{[n]_q!} = \frac{\exp_q(z)(1-t)}{\exp_q(tz)-t\exp_q(z)}. \tag{1.7}$$

(As is standard, $[n]_q! := \prod_{j=1}^n [j]_q$, where $[j]_q := \sum_{i=0}^{j-1} q^i$. Also, $\exp_q(z) := \sum_{n\geq 0}\frac{z^n}{[n]_q!}$.)

The q-analog $\tilde{a}_{n,j}(q)$ of the binomial-Eulerian number $\tilde{a}_{n,j}$ and the q-analog $\tilde{A}_n(q,t)$ of the binomial-Eulerian polynomial $\tilde{A}_n(t)$ are polynomials in $\mathbb{Z}[q]$ and $\mathbb{Z}[q][t]$, respectively, defined by

$$\tilde{A}_n(q,t) = \sum_{j=0}^{n} \tilde{a}_{n,j}(q)t^j := 1 + t\sum_{m=1}^{n}\binom{n}{m}_q A_m(q,t).$$

For example,

$$\begin{aligned}\tilde{A}_2(q,t) &= 1+(2+q)t+t^2\\ \tilde{A}_3(q,t) &= 1+(3+2q+2q^2)t+(3+2q+2q^2)t^2+t^3.\end{aligned}$$

The following q-analog of (1.3) is proved in [23, Equations (1.4) and (6.1)] and also appears in Lin and Zeng [22] (with a different proof). For $n\geq 1$,

$$A_n(q,t) = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \gamma_{n,k}(q)\, t^k(1+t)^{n-1-2k}, \tag{1.8}$$

where

$$\gamma_{n,k}(q) := \sum_{\sigma\in\Gamma_{n,k}} q^{\mathrm{inv}(\sigma)}.$$

Here we give an alternative derivation[5] of (1.8) and we derive the q-analog of (1.5),

$$\tilde{A}_n(q,t) = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} \tilde{\gamma}_{n,k}(q)\, t^k(1+t)^{n-2k}, \tag{1.9}$$

[5]This approach is discussed in earlier work [31, Remark 5.5] of the authors, though the $\gamma_{n,k}(q)$ are not given.

where

$$\tilde{\gamma}_{n,k}(q) := \sum_{\sigma \in \tilde{\Gamma}_{n,k}} q^{\mathrm{inv}(\sigma)}.$$

We derive (1.8) and (1.9) by specializing analogous symmetric function identities. These identities involve the symmetric function polynomials $Q_n(\mathbf{x}, t)$ and $\tilde{Q}_n(\mathbf{x}, t)$, which specialize to $A_n(q, t)$ and $\tilde{A}_n(q, t)$, respectively, and are defined as follows. For $\mathbf{x} = (x_1, x_2, \dots)$, let

$$\sum_{n \geq 0} Q_n(\mathbf{x}, t) z^n := \frac{(1-t)H(z)}{H(tz) - tH(z)}, \tag{1.10}$$

where

$$H(z) := \sum_{n \geq 0} h_n(\mathbf{x}) z^n,$$

and $h_n(\mathbf{x})$ is the complete homogeneous symmetric function of degree n. For $n \geq 0$, let

$$\tilde{Q}_n(\mathbf{x}, t) := h_n(\mathbf{x}) + t \sum_{m=1}^{n} h_{n-m}(\mathbf{x}) Q_m(\mathbf{x}, t). \tag{1.11}$$

For all $n \geq 1$ and $k \geq 0$, let

$$\gamma_{n,k}(\mathbf{x}) := \sum_{D \in \mathcal{H}_{n,k}} s_D(\mathbf{x}),$$

where $s_D(\mathbf{x})$ is the skew Schur function of shape D and $\mathcal{H}_{n,k}$ is the set of skew hooks of size n for which k columns have size 2 and the remaining $n-2k$ columns, including the last column, have size 1. From an interpretation of $Q_n(\mathbf{x}, t)$ due to Gessel [18], we have the identity,

$$Q_n(\mathbf{x}, t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}(\mathbf{x})\, t^k (1+t)^{n-1-2k}, \tag{1.12}$$

for all $n \geq 1$. We use (1.12) to derive the identity

$$\tilde{Q}_n(\mathbf{x}, t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \tilde{\gamma}_{n,k}(\mathbf{x})\, t^k (1+t)^{n-2k}, \tag{1.13}$$

where

$$\tilde{\gamma}_{n,k}(\mathbf{x}) = \sum_{H \in \tilde{\mathcal{H}}_{n,k}} s_H(\mathbf{x})$$

and $\tilde{\mathcal{H}}_{n,k}$ is the set of skew hooks of size n for which k columns have size 2 and the remaining $n-2k$ columns have size 1.

It was shown by Danilov and Jurkiewicz (see [36, eq. (26)]) that the h-polynomial of a simplicial polytope is equal to the Poincaré polynomial of the toric variety associated with the polytope. In [36] Stanley, using a formula of Procesi [27], gives a representation theoretic interpretation of $Q_n(\mathbf{x},t)$ involving the toric variety associated with the dual permutohedron. This and an equivariant version of the hard Lefschetz theorem yield a geometric proof that $Q_n(\mathbf{x},t)$ is palindromic, Schur-positive and Schur-unimodal. Here we give an analogous interpretation for $\tilde{Q}_n(\mathbf{x},t)$ involving the dual stellohedron. This leads to the formulation of an equivariant version of the Gal phenomenon, with the symmetric group actions on the dual permutohedron and the dual stellohedron exhibiting this phenomenon.

The paper is organized as follows. In Section 2, we recall some basic facts about Eulerian polynomials, permutation statistics, q-analogs, and symmetric functions. The formulae (1.12) and (1.13) are obtained in Section 3 and direct proofs of palindromicity and Schur-unimodality of $Q_n(\mathbf{x},t)$ and $\tilde{Q}_n(\mathbf{x},t)$ are given. In Section 4, we show how these formulae specialize to (1.8) and (1.9), respectively. Algebro-geometic interpretations of the results in Section 3 are presented in Section 5. In Section 6, we discuss derangement analogs of the results of the previous sections.

2. Preliminaries

While investigating divergent series in [12], Euler showed that, for each positive integer n, there is a monic polynomial $A_n(t) \in \mathbb{Z}[t]$ of degree $n-1$ such that

$$\sum_{k\geq 0}(k+1)^n t^k = \frac{A_n(t)}{(1-t)^{n+1}}.$$

Let us write

$$A_n(t) = \sum_{j=0}^{n-1} a_{n,j} t^j.$$

The coefficients $a_{n,j}$ of the *Eulerian polynomial* $A_n(t)$ are called *Eulerian numbers*.

For a permutation $\sigma \in \mathfrak{S}_n$, the *descent set* of σ is

$$\mathrm{DES}(\sigma) := \{i \in [n-1] : \sigma(i) > \sigma(i+1)\}$$

and the *descent number* of σ is

$$\mathrm{des}(\sigma) := |\mathrm{DES}(\sigma)|.$$

The fact that

$$\sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = A_n(t) \tag{2.1}$$

for all n seems to have been observed first by Riordan in [28]. Earlier, MacMahon had shown in [24, Vol. I, p.186] that, with the *excedance number* of $\sigma \in \mathfrak{S}_n$ defined as

$$\mathrm{exc}(\sigma) := |\{i \in [n-1] : \sigma(i) > i\}|,$$

the equation

$$\sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} \tag{2.2}$$

holds for all n.

Recall that the *q-binomial coefficients* are defined by

$$\binom{n}{k}_q := \begin{cases} \frac{[n]_q!}{[k]_q![n-k]_q!} & 0 \le k \le n, \\ 0 & \text{otherwise.} \end{cases}$$

There are two additional fundamental permutation statistics, the *major index*

$$\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{DES}(\sigma)} i$$

and the *inversion number*

$$\mathrm{inv}(\sigma) := |\{(i,j) : 1 \le i < j \le n, \sigma(i) > \sigma(j)\}|.$$

MacMahon [24] introduced the major index and proved the first equality in

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} = [n]_q! = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)}$$

after the second equality had been obtained in [29] by Rodrigues.

In [30, 31], the authors define a fixed point version of the q-Eulerian polynomial, which refines the q-Eulerian polynomial given in (1.6). For $n \geq 1$, let

$$A_n(q,t,r) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)} r^{\mathrm{fix}(\sigma)},$$

where $\mathrm{fix}(\sigma)$ is the number of fixed points of σ, and let $A_0(q,t,r) := 1$. So $A_n(q,t,1) = A_n(q,t)$ for all $n \geq 0$. In [30, 31], the refinement of (1.7),

$$\sum_{n \geq 0} A_n(q,t,r) \frac{z^n}{[n]_q!} = \frac{\exp_q(rz)(1-t)}{\exp_q(tz) - t\exp_q(z)} \tag{2.3}$$

is derived.

As mentioned in the introduction, the Foata-Schutzenberger formula (1.3) establishes γ-positivity of the Eulerian polynomials and the Postnikov-Reiner-Williams formula (1.4) establishes γ-positivity of the binomial-Eulerian polynomials. We now give precise definitions of the terminology used in these formulas. We say $\sigma \in \mathfrak{S}_n$ has

- a *double descent* if there exists $i \in [n-2]$ such that $\sigma(i) > \sigma(i+1) > \sigma(i+2)$
- an *initial descent* if $\sigma(1) > \sigma(2)$
- a *final descent* if $\sigma(n-1) > \sigma(n)$.

We say that a polynomial $f(q) \in \mathbb{R}[q]$ is q-positive if its coefficients are nonnegative. Given polynomials $f(q), g(q) \in \mathbb{R}[q]$ we say that $f(q) \leq_q g(q)$ if $g(q) - f(q)$ is q-positive. More generally, let R be an algebra over $\mathbb{R}$ with basis b. An element $s \in R$ is said to be b-*positive* if the expansion of s in the basis b has nonnegative coefficients. Given $r, s \in R$, we say that $r \leq_b s$ if $s - r$ is b-positive.

The $\mathbb{R}$-algebras considered in this paper are $R = \mathbb{R}, \mathbb{R}[q]$, and the algebra Λ of symmetric functions over $\mathbb{R}$. If $R = \mathbb{R}$ and $b = \{1\}$ then b-positive is the same as positive and $<_b$ is the usual numerical $<$ relation. If $R = \mathbb{R}[q]$ and $b = \{q^i : i \in \mathbb{N}\}$ then b-positive is what we called q-positive above and $<_b$ is the same as $<_q$. For $R = \Lambda$, we consider the basis of Schur functions $\{s_\lambda(\mathbf{x}) : \lambda \in \cup_{n\geq 0} \mathrm{Par}(n)\}$ and the basis of complete homogeneous symmetric functions $\{h_\lambda(\mathbf{x}) : \lambda \in \cup_{n \geq 0} \mathrm{Par}(n)\}$, where $\mathrm{Par}(n)$ is the set of partitions of n. It is a basic fact that h-positive implies Schur-positive (see for example [38, Proposition 7.18.7]).

Definition 2.1. Let R be an $\mathbb{R}$-algebra with basis b. We say that a polynomial $A(t) := a_0 + a_1t + \cdots + a_nt^n \in R[t]$ is

- *b-positive* if each coefficient a_i is b-positive,
- *b-unimodal* if for some c,

$$a_0 \leq_b a_1 \leq_b \cdots \leq_b a_c \geq_b a_{c+1} \geq_b a_{c+2} \geq_b \cdots \geq_b a_n,$$

- *palindromic* with center of symmetry $\frac{n}{2}$ if $a_j = a_{n-j}$ for $0 \leq j \leq n$,
- *b-γ-positive* if there exist b-positive $\gamma_0, \dots, \gamma_{\lfloor \frac{d}{2} \rfloor} \in R$ such that

$$A(t) = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_k t^k (1+t)^{d-2k}.$$

The following results are well known, at least in the case that $R = \mathbb{R}$ (see [33, Appendix B]).

Proposition 2.2 (see [36, Proposition 1])**.** *Let R be an $\mathbb{R}$-algebra with basis b. Let $A(t)$ and $B(t)$ be palindromic, b-positive, b-unimodal polynomials in $R[t]$ with respective centers of symmetry c_A and c_B. Then*

1. *$A(t)B(t)$ is palindromic, b-positive, b-unimodal with center of symmetry $c_A + c_B$.*
2. *If $c_A = c_B$ then $A(t) + B(t)$ is palindromic, b-positive, b-unimodal with center of symmetry c_A.*

Corollary 2.3. *If $A(t) \in R[t]$ is b-γ-positive then $A(t)$ is palindromic, b-positive, and b-unimodal.*

3. Schur-γ-positivity

In this section we establish Schur-γ-positivity of the symmetric function analogs $Q_n(\mathbf{x}, t)$ and $\tilde{Q}_n(\mathbf{x}, t)$ given in (1.10) and (1.11), and we present combinatorial formulae for the γ-coefficients. We also present direct proofs of palindromicity, Schur-positivity, and Schur-unimodality, which don't rely on Schur-γ-positivity.

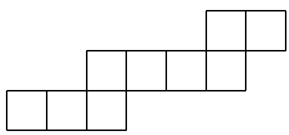
It is an easy consequence of the following result of Gessel that $Q_n(\mathbf{x}, t)$ is Schur-γ-positive. Let $\mathbb{P}_n$ be the set of words of length n over the alphabet of positive integers $\mathbb{P}$. Given a word $w \in \mathbb{P}_n$, we let w_i denote its ith letter. That is, $w = w_1w_2 \dots w_n$. Just as for permutations, let $\operatorname{des}(w)$ equal the number of $i \in [n-1]$ such that $w_i > w_{i+1}$. A word w is said to have a double descent if there exists an $i \in [n-2]$ such that $w_i > w_{i+1} > w_{i+2}$. Let NDD_n be the set of words in $\mathbb{P}_n$ with no double descents. For $w \in \mathbb{P}_n$, let $\mathbf{x}_w := x_{w_1}x_{w_2}\dots x_{w_n}$.

Theorem 3.1 (Gessel [18], see [31, Theorem 7.3])**.**

$$1+\sum_{n\geq 1} z^n \sum_{\substack{w\,\in\, NDD_n\\ w_{n-1}\,\leq\, w_n}} \mathbf{x}_w\, t^{\mathrm{des}(w)}(1+t)^{n-1-2\mathrm{des}(w)} = \frac{(1-t)H(z)}{H(zt)-tH(z)}, \tag{3.1}$$

where $w_0 = 0$.

The symmetric function polynomial $Q_n(\mathbf{x},t)$ defined in (1.10) can now be given an explicit expansion which establishes Schur-γ-positivity. The γ-coefficients are described in terms of hook shaped skew Schur functions. A *skew hook* is a connected skew diagram with no 2×2 square. Let $\mathcal{H}_{n,k}$ be the set of skew hooks of size n for which k columns have size 2 and the remaining $n-2k$ columns, including the last column, have size 1. For example,

 $\in \quad \mathcal{H}_{9,2}.$

Corollary 3.2. *Let*

$$\gamma_{n,k}(\mathbf{x}) := \sum_{D\in\mathcal{H}_{n,k}} s_D(\mathbf{x}), \tag{3.2}$$

where $s_D(\mathbf{x})$ *is the skew Schur function of shape* D*. Then*

$$Q_n(\mathbf{x},t) = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \gamma_{n,k}(\mathbf{x})\, t^k(1+t)^{n-1-2k}. \tag{3.3}$$

Consequently the polynomial $Q_n(\mathbf{x},t)$ *is Schur-*γ*-positive.*

Proof. By (3.1), for $n\geq 1$,

$$Q_n(\mathbf{x},t) = \sum_{\substack{w\,\in\, NDD_n\\ w_{n-1}\,\leq\, w_n}} \mathbf{x}_w\, t^{\mathrm{des}(w)}(1+t)^{n-1-2\mathrm{des}(w)}.$$

Note that the semistandard tableaux of hook shape in $\mathcal{H}_{n,k}$ correspond bijectively to words $w\in NDD_n$ with $w_{n-1}\leq w_n$ and with k descents. Indeed, by reading the entries of such a semistandard tableau from southwest to northeast, one gets such a word. For example, the semistandard tableau

$$\begin{array}{ccccccc} & & & & & \boxed{2} & \boxed{8} \\ & & \boxed{1} & \boxed{1} & \boxed{8} & \boxed{9} & \\ \boxed{2} & \boxed{5} & \boxed{5} & & & & \end{array}$$

corresponds to the word $255118928 \in NDD_9$, which has 2 descents. It follows that

$$\sum_{\substack{w \in NDD_n \\ w_{n-1} \leq w_n \\ \mathrm{des}(w) = k}} \mathbf{x}_w = \sum_{D \in \mathcal{H}_{n,k}} s_D(\mathbf{x}).$$

The consequence follows from the fact that skew Schur functions are Schur-positive. □

Next we derive an analogous Schur-γ-positivity result for $\tilde{Q}_n(\mathbf{x}, t)$, which was defined in (1.11). We begin with a generating function formula.

Proposition 3.3.

$$\sum_{n \geq 0} \tilde{Q}_n(\mathbf{x}, t) z^n = \frac{(1-t)H(z)H(tz)}{H(tz) - tH(z)}. \tag{3.4}$$

Equivalently, for all $n \geq 0$,

$$\tilde{Q}_n(\mathbf{x}, t) = \sum_{m=0}^{n} h_{n-m}(\mathbf{x}) Q_m(\mathbf{x}, t) t^{n-m}. \tag{3.5}$$

Proof. By the definitions (1.11) and (1.10),

$$\begin{aligned} \sum_{n \geq 0} \tilde{Q}_n(\mathbf{x}, t) z^n &= H(z)(1 + t \sum_{n \geq 1} Q_n(\mathbf{x}, t) z^n) \\ &= H(z)(1 + t(\frac{(1-t)H(z)}{H(tz) - tH(z)} - 1)) \\ &= H(z) \frac{H(tz)(1-t)}{H(tz) - tH(z)}. \end{aligned}$$

□

Let $\tilde{\mathcal{H}}_{n,k}$ be the set of skew hooks of size n for which k columns have size 2 and the remaining $n - 2k$ columns have size 1.

Theorem 3.4. *Let*

$$\tilde{\gamma}_{n,k}(\mathbf{x}) := \sum_{D \in \tilde{\mathcal{H}}_{n,k}} s_D(\mathbf{x}), \tag{3.6}$$

where $s_D(\mathbf{x})$ is the skew Schur function of shape D. Then

$$\tilde{Q}_n(\mathbf{x},t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \tilde{\gamma}_{n,k}(\mathbf{x})\, t^k (1+t)^{n-2k}. \tag{3.7}$$

Consequently the polynomial $\tilde{Q}_n(\mathbf{x},t)$ is Schur-γ-positive.

Proof. For $n \geq 1$, let

$$W_n(\mathbf{x},t) := \sum_{\substack{w \,\in\, NDD_n \\ w_{n-1} \,\leq\, w_n}} \mathbf{x}_w \, t^{\mathrm{des}(w)} (1+t)^{n-1-2\mathrm{des}(w)}$$

and let

$$\tilde{W}_n(\mathbf{x},t) := \sum_{w \in NDD_n} \mathbf{x}_w \, t^{\mathrm{des}(w)} (1+t)^{n-2\mathrm{des}(w)}.$$

By (3.1), we have

$$W_n(\mathbf{x},t) = Q_n(\mathbf{x},t). \tag{3.8}$$

We will show that

$$\tilde{W}_n(\mathbf{x},t) = h_n(\mathbf{x}) t^n + \sum_{m=1}^{n} h_{n-m}(\mathbf{x}) W_m(\mathbf{x},t) t^{n-m}. \tag{3.9}$$

It follows from this, (3.5), and (3.8) that $\tilde{W}_n(\mathbf{x},t) = \tilde{Q}_n(\mathbf{x},t)$. This is equivalent to the desired result since the semistandard tableaux of skew hook shape in $\tilde{\mathcal{H}}_{n,k}$ correspond bijectively to words in NDD_n with k descents.

Let I_n be the set $\{\alpha \in \mathbb{P}_n : \alpha_1 \leq \cdots \leq \alpha_n\}$ of weakly increasing words of length n. The right side of (3.9) equals

$$\sum_{u\in I_n} t^n \mathbf{x}_u + \sum_{m=1}^{n} \sum_{\substack{w\,\in\, NDD_m \\ w_{m-1}\,\le\, w_m}} t^{\operatorname{des}(w)}(1+t)^{m-1-2\operatorname{des}(w)}\mathbf{x}_w \sum_{u\in I_{n-m}} t^{n-m}\mathbf{x}_u$$

$$= \sum_{u\in I_n} t^n \mathbf{x}_u + \sum_{m=1}^{n} \sum_{\substack{w\,\in\, NDD_m \\ w_{m-1}\,\le\, w_m \\ u\,\in\, I_{n-m}}} t^{\operatorname{des}(w)+(n-m)}(1+t)^{m-1-2\operatorname{des}(w)}\mathbf{x}_{w\cdot u}$$

where $w \cdot u$ denotes concatenation of words w and u.

For $v \in \mathbb{P}_n$ we seek the coefficient of $\mathbf{x}_v$. Note that the coefficient is 0 if v has a double descent. For $v \in NDD_n$, let j be the smallest integer such that $v_j \le v_{j+1} \le \cdots \le v_n$. So $j-1$ is either 0 (when v is weakly increasing) or the position of the last descent. Each $m \in \{j+1,\dots,n\}$ determines a decomposition of v into $w\cdot u$, where $w \in NDD_m$, $w_{m-1} \le w_m$ and $u \in I_{n-m}$. Note that $\operatorname{des}(v) = \operatorname{des}(w)$.

The only other value of m that determines a decomposition of v into $w \cdot u$ for which $w \in NDD_m$, $w_{m-1} \le w_m$ and $u \in I_{n-m}$, is $m = j-1$. In this case, if $j-1 > 0$ we have $\operatorname{des}(v) = \operatorname{des}(w)+1$. It follows that if $j > 1$, the coefficient c_v of $\mathbf{x}_v$ is given by

$$c_v = t^{\operatorname{des}(v)+n-j}(1+t)^{j-2\operatorname{des}(v)} + \sum_{m=j+1}^{n} t^{\operatorname{des}(v)+n-m}(1+t)^{m-1-2\operatorname{des}(v)}.$$

We have

$$\begin{aligned}
&\sum_{m=j+1}^{n} t^{\operatorname{des}(v)+n-m}(1+t)^{m-1-2\operatorname{des}(v)} \\
&\qquad = t^{\operatorname{des}(v)+n-j-1}(1+t)^{j-2\operatorname{des}(v)} \sum_{k=0}^{n-j-1}\left(\frac{1+t}{t}\right)^k \\
&\qquad = t^{\operatorname{des}(v)+n-j}(1+t)^{j-2\operatorname{des}(v)} \left(\left(\frac{1+t}{t}\right)^{n-j} - 1\right) \\
&\qquad = t^{\operatorname{des}(v)}(1+t)^{n-2\operatorname{des}(v)} - t^{\operatorname{des}(v)+n-j}(1+t)^{j-2\operatorname{des}(v)}, \qquad (3.10)
\end{aligned}$$

from which we conclude that $c_v = t^{\operatorname{des}(v)}(1+t)^{n-2\operatorname{des}(v)}$.

Now if $j = 1$ then v is a weakly increasing word and the coefficient of $\mathbf{x}_v$ is given by

$$c_v = t^n + \sum_{m=1}^{n} t^{n-m}(1+t)^{m-1}.$$

A simple computation shows that the summation is equal to $(1+t)^n - t^n$. Hence $c_v = (1+t)^n = t^{\mathrm{des}(v)}(1+t)^{n-2\mathrm{des}(v)}$, as in the previous case. We have therefore shown that the right hand side of (3.9) is equal to

$$\sum_{v \in NDD_n} t^{\mathrm{des}(v)}(1+t)^{n-2\mathrm{des}(v)}\mathbf{x}_v,$$

which by definition is the left side of (3.9). $\square$

Remark 3.5. It was pointed out to us by González D'León that another identity of Gessel [17, Theorem 4.2] can be used to give an alternative proof of Theorem 3.4, or equivalently of $\tilde{Q}_n(\mathbf{x},t) = \tilde{W}_n(\mathbf{x},t)$. By inverting (3.9), one can conclude from this that $Q_n(\mathbf{x},t) = W_n(\mathbf{x},t)$, which is equivalent to Gessel's unpublished result (3.1). Hence [17, Theorem 4.2] can be used to prove (3.1). Gessel [18] has a more direct proof of (3.1) however.

The following result for $Q_n(\mathbf{x},t)$ was first obtained by Stanley [36] from the algebro-geometric interpretation of $Q_n(\mathbf{x},t)$ given in (5.1).

Corollary 3.6. *For all $n \geq 0$, the symmetric function polynomials $Q_n(\mathbf{x},t)$ and $\tilde{Q}_n(\mathbf{x},t)$ are palindromic, Schur-positive, and Schur-unimodal.*

Proof. Use Corollary 2.3. $\square$

A stronger result for $Q_n(\mathbf{x},t)$ was proved by Stembridge [39], namely h-positivity and h-unimodality of $Q_n(\mathbf{x},t)$. A simpler proof of this result given in [33, Corollary C.5] relies on the formula

$$\sum_{n\geq 0} Q_n(\mathbf{x},t)z^n = 1 + \frac{\sum_{n\geq 1}[n]_t h_n z^n}{1 - t\sum_{n\geq 2}[n-1]_t h_n z^n} \tag{3.11}$$

and Proposition 2.2. Here we give an alternative proof of Corollary 3.6 for $\tilde{Q}_n(\mathbf{x},t)$ that does not rely on Theorem 3.4.

Alternative proof of Corollary 3.6 for $\tilde{Q}_n(\mathbf{x},t)$. Let $Q_n^0(\mathbf{x},t)$ be defined by

$$\sum_{n\geq 0} Q_n^0(\mathbf{x},t)z^n = \frac{1-t}{H(tz) - tH(z)} = \frac{1}{1 - t\sum_{n\geq 2}[n-1]_t h_n z^n}.$$

It follows from Proposition 2.2 that $Q_n^0(\mathbf{x},t)$ is palindromic, h-positive and h-unimodal with center of symmetry $\frac{n}{2}$. By Proposition 3.3,

$$\tilde{Q}_n(\mathbf{x},t) = \sum_{k\geq 0}\left(\sum_{j=0}^{k} t^j h_j h_{k-j}\right) Q_{n-k}^0(\mathbf{x},t). \tag{3.12}$$

It is easy to see that $\sum_{j=0}^{k} t^j h_j h_{k-j}$ is palindromic with center of symmetry $\frac{k}{2}$. It is clearly h-positive, which implies that it is Schur-positive. We claim that it is also Schur-unimodal. If $j \leq k-j$ then by Pieri's rule $h_j h_{k-j} = \sum_{i=0}^{j} s_{k-i,i}$. From this we can see that $\sum_{j=0}^{k} t^j h_j h_{k-j}$ is Schur-unimodal. By Proposition 2.2, we have that $\left(\sum_{j=0}^{k} t^j h_j h_{k-j}\right) Q_{n-k}^0(\mathbf{x},t)$ is palindromic, Schur-positive, and Schur unimodal with center of symmetry equal to $\frac{k}{2}+\frac{n-k}{2}=\frac{n}{2}$. Again by Proposition 2.2, we can conclude from (3.12) that $\tilde{Q}_n(\mathbf{x},t)$ is palindromic, Schur-positive, and Schur-unimodal with center of symmetry $\frac{n}{2}$. $\square$

4. q-γ-positivity of the q-Eulerian and q-binomial-Eulerian polynomials

It this section we use the results of the previous section to prove that the q-Eulerian polynomials

$$A_n(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

and q-binomial-Eulerian polynomials

$$\tilde{A}_n(q,t) := 1 + t\sum_{m=1}^{n}\binom{n}{m}_q A_m(q,t)$$

are q-γ-positive.

From any symmetric function $G(x_1,x_2,\dots)$ one obtains a power series in a single variable q by the *stable principal specialization*, in which each x_i is replaced by q^{i-1}. Let

$$\mathrm{ps}_q(G) := G(1,q,q^2,\dots).$$

This definition can be extended to polynomials in $\Lambda[t]$ by defining,

$$\mathrm{ps}_q\left(\sum_{i=o}^{d} G_i(\mathbf{x})t^i\right) := \sum_{i=0}^{d} \mathrm{ps}_q(G_i(\mathbf{x}))t^i.$$

Let SYT_D denote the set of standard Young tableaux of skew shape D. For $T \in SYT_D$ (written in English notation), let $\mathrm{DES}(T)$ be the set of entries i of T for which i is in a higher row than $i+1$, and let $\mathrm{maj}(T) = \sum_{i \in \mathrm{DES}(T)} i$. It is well known (see [38, Proposition 7.19.11]) that

$$\mathrm{ps}_q(s_D) = \frac{\sum_{T \in SYT_D} q^{\mathrm{maj}(T)}}{(1-q)\ldots(1-q^n)}, \tag{4.1}$$

where n is the number of cells of D. It follows from this (and is easy to see directly) that

$$\mathrm{ps}_q(h_n) = \frac{1}{(1-q)\ldots(1-q^n)}.$$

By taking stable principal specialization of both sides of (1.10), one can see that the following result is equivalent to (1.7). In fact, in [31] this result was used to prove (1.7).

Theorem 4.1 (Shareshian and Wachs [31])**.** *For all $n \geq 0$,*

$$\mathrm{ps}_q(Q_n(\mathbf{x},t)) = \frac{A_n(q,t)}{(1-q)\ldots(1-q^n)}$$

An analogous result holds for the q-binomial-Eulerian polynomials.

Corollary 4.2. *For all $n \geq 0$,*

$$\mathrm{ps}_q(\tilde{Q}_n(\mathbf{x},t)) = \frac{\tilde{A}_n(q,t)}{(1-q)\ldots(1-q^n)}.$$

Proof. Starting with the definition of $\tilde{Q}_n(\mathbf{x},t)$ given in (1.11), we have

$$\begin{aligned}
\mathrm{ps}_q(\tilde{Q}_n(\mathbf{x},t)) &= \mathrm{ps}_q(h_n) + t\sum_{m=1}^{n} \mathrm{ps}_q(h_{n-m})\mathrm{ps}_q(Q_m(\mathbf{x},t)) \\
&= \frac{1}{\prod_{i=1}^{n}(1-q^i)} + t\sum_{m=1}^{n} \frac{A_m(q,t)}{\prod_{i=1}^{m}(1-q^i)\prod_{i=1}^{n-m}(1-q^i)}
\end{aligned}$$

$$= \frac{1 + t\sum_{m=1}^{n}\binom{n}{m}_q A_m(q,t)}{\prod_{i=1}^{n}(1-q^i)}$$
$$= \frac{\tilde{A}_n(q,t)}{\prod_{i=1}^{n}(1-q^i)},$$

with the second equality following from Theorem 4.1. □

By taking the stable principal specialization of both sides of (3.4), one gets the following result. The consequences follow from (1.7) and (2.3), respectively.

Proposition 4.3.

$$\sum_{n\geq 0}\tilde{A}_n(q,t)\frac{z^n}{[q]_n!} = \frac{(1-t)\exp_q(z)\exp_q(tz)}{\exp_q(tz) - t\exp_q(z)}.$$

Consequently

$$\tilde{A}_n(q,t) = \sum_{m=0}^{n}\binom{n}{m}_q A_m(q,t)t^{n-m}$$

and

$$\tilde{A}_n(q,t) = \sum_{m=0}^{n}\binom{n}{m}_q A_m(q,t,t).$$

In [31, Remark 5.5], the authors mention that (3.1) can be used to establish q-γ-positivity of $A_n(q,t)$. Now we carry this out by using (3.3) to obtain the γ-coefficients. The following result is proved in [23, Equations (1.4) and (6.1)] without the use of (3.1).

Theorem 4.4. *Let $\Gamma_{n,k}$ be the set of permutations $\sigma \in \mathfrak{S}_n$ with no double descents, no final descent, and with* $\mathrm{des}(\sigma) = k$*, and let*

$$\gamma_{n,k}(q) := \sum_{\sigma\in\Gamma_{n,k}} q^{\mathrm{inv}(\sigma)} \quad \Big(= \sum_{\sigma\in\Gamma_{n,k}} q^{\mathrm{maj}(\sigma^{-1})}\Big).$$

Then

$$A_n(q,t) = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\gamma_{n,k}(q)\,t^k(1+t)^{n-1-2k}. \tag{4.2}$$

Consequently the q-Eulerian polynomials $A_n(q,t)$ are q-γ-positive.

Proof. By applying stable principal specialization to both sides of (3.3) we have

$$\mathrm{ps}_q(Q_n(\mathbf{x},t)) = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \mathrm{ps}_q(\gamma_{n,k}(\mathbf{x}))\, t^k(1+t)^{n-1-2k}. \tag{4.3}$$

By (3.2) and (4.1), we have

$$\begin{aligned} \mathrm{ps}_q(\gamma_{n,k}(\mathbf{x})) &= \sum_{D\in\mathcal{H}_{n,k}} \mathrm{ps}_q(s_D(\mathbf{x})) \\ &= \sum_{D\in\mathcal{H}_{n,k}} \frac{\sum_{T\in SYT_D} q^{\mathrm{maj}(T)}}{(1-q)\dots(1-q^n)}. \end{aligned} \tag{4.4}$$

If D is a skew hook then SYT_D corresponds bijectively to the set of permutations in $\mathfrak{S}_n$ with a fixed descent set determined by D. Indeed, by reading the entries of $T \in SYT_D$ from southwest to northeast, one gets a permutation $\varphi(T) \in \mathfrak{S}_n$. Descents are encountered whenever one goes up a column. So $\mathrm{DES}(\varphi(T))$ equals the set of all $i \in [n-1]$ such that the ith cell of D (ordered from southwest to northeast) is directly below the $(i+1)$st cell of D. It follows that if $D \in \mathcal{H}_{n,k}$ and $T \in SYT_D$ then $\varphi(T) \in \Gamma_{n,k}$.

Note also that for $T \in SYT_D$, $\mathrm{DES}(T) = \mathrm{DES}(\varphi(T)^{-1})$. We can now conclude that

$$\sum_{D\in\mathcal{H}_{n,k}} \sum_{T\in SYT_D} q^{\mathrm{maj}(T)} = \sum_{\sigma\in\Gamma_{n,k}} q^{\mathrm{maj}(\sigma^{-1})}. \tag{4.5}$$

For each $J \subset [n-1]$, the *descent class* of J is the set $\{\sigma \in \mathfrak{S}_n : \mathrm{DES}(\sigma) = J\}$. Note that $\Gamma_{n,k}$ is a union of descent classes. By the Foata-Schützenberger result [15, Theorem 1] that $\mathrm{inv}(\sigma)$ and $\mathrm{maj}(\sigma^{-1})$ are equidistributed on descent classes, we have

$$\sum_{\sigma\in\Gamma_{n,k}} q^{\mathrm{maj}(\sigma^{-1})} = \sum_{\sigma\in\Gamma_{n,k}} q^{\mathrm{inv}(\sigma)}.$$

Combining this with (4.5) and substituting in (4.4) results in

$$\mathrm{ps}_q(\gamma_{n,k}(\mathbf{x})) = \frac{\sum_{\sigma\in\Gamma_{n,k}} q^{\mathrm{inv}(\sigma)}}{(1-q)\dots(1-q^n)}.$$

It follows that the right side of (4.3) equals

$$\frac{\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{\sigma \in \Gamma_{n,k}} q^{\mathrm{inv}(\sigma)}\, t^k (1+t)^{n-1-2k}}{(1-q)\dots(1-q^n)},$$

while, by Theorem 4.1, the left side equals

$$\frac{A_n(q,t)}{(1-q)\dots(1-q^n)},$$

thereby completing the proof. □

By taking the stable principal specialization of both sides of equation (3.7) and using an argument analogous to the proof of Theorem 4.4, we obtain the following result.

Theorem 4.5. *Let* $\tilde{\Gamma}_{n,k}$ *be the set of permutations* $\sigma \in \mathfrak{S}_n$ *with no double descents and with* $\mathrm{des}(\sigma) = k$*, and let*

$$\tilde{\gamma}_{n,k}(q) := \sum_{\sigma \in \tilde{\Gamma}_{n,k}} q^{\mathrm{inv}(\sigma)} \quad \Big(= \sum_{\sigma \in \tilde{\Gamma}_{n,k}} q^{\mathrm{maj}(\sigma^{-1})}\Big). \tag{4.6}$$

Then

$$\tilde{A}_n(q,t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \tilde{\gamma}_{n,k}(q)\, t^k (1+t)^{n-2k}. \tag{4.7}$$

Consequently, the q*-binomial-Eulerian polynomials* $\tilde{A}_n(q,t)$ *are* q*-*γ*-positive.*

The following result for $A_n(q,t)$ was first obtained by the authors in [31].

Corollary 4.6. *For all* $n \geq 0$*, the polynomials* $A_n(q,t)$ *and* $\tilde{A}_n(q,t)$ *are palindromic and* q*-unimodal.*

Just as for Corollary 3.6, an alternative proof of Corollary 4.6 can be given which doesn't make use of Theorems 4.4 and 4.5. For $A_n(q,t)$ a simple proof is given in Appendix C.1 of [33] by using the formula

$$1 + \sum_{n \geq 1} A_n(q,t) \frac{z^n}{[n]_q!} = 1 + \frac{\sum_{n\geq 1} [n]_t \frac{z^n}{[n]_q!}}{1 - t \sum_{n \geq 2} [n-1]_t \frac{z^n}{[n]_q!}}$$

obtained by manipulating (1.7).

Alternative proof of Corollary 4.6 for $\tilde{A}_n(q,t)$. By (2.3) and Proposition 4.3,

$$\tilde{A}_n(q,t) = \sum_{k\geq 0}\left(\sum_{j=0}^{k}\binom{k}{j}_q t^j\right)A_{n-k}(q,t,0).$$

Since

$$\sum_{n\geq 0}A_n(q,t,0)\frac{z^n}{[n]_q!} = \frac{(1-t)}{\exp_q(tz)-t\exp_q(z)} = \frac{1}{1-t\sum_{n\geq 2}[n-1]_t\frac{z^n}{[n]_q!}},$$

it follows from Proposition 2.2 that $A_n(q,t,0)$ is palindromic and q-unimodal with center of symmetry $\frac{n}{2}$. It is well known that $\sum_{j=0}^{k}\binom{k}{j}_q t^j$ is palindromic and q-unimodal with center of symmetry $\frac{k}{2}$. Note that this follows from taking the stable principal specialization of $\sum_{j=0}^{k}h_jh_{k-j}t^j$, which we observed to be Schur-unimodal in the alternative proof of Corollary 3.6. By Proposition 2.2, $\tilde{A}_n(q,t)$ is a sum of palindromic, q-positive, q-unimodal polynomials with center of symmetry $\frac{k}{2}+\frac{n-k}{2}$. It therefore follows again from Proposition 2.2 that $\tilde{A}_n(q,t)$ is palindromic and q-unimodal. □

Note that palindromicity of $\tilde{A}_n(q,t)$ is equivalent to the following q-analog of (1.1).

Corollary 4.7 (Chung-Graham [9] and Han-Lin-Zeng [20])**.** *For positive integers r,s,*

$$\sum_{m=1}^{r+s}\binom{r+s}{m}_q a_{m,r-1}(q) = \sum_{m=1}^{r+s}\binom{r+s}{m}_q a_{m,s-1}(q).$$

A symmetric function analog is given by the following result, which is equivalent to palindromicity of $\tilde{Q}_n(\mathbf{x},t)$. (A more general result appears as Theorem 2 in the preprint [21] of Z. Lin.)

Corollary 4.8. *For positive integers r,s,*

$$\sum_{m=1}^{r+s}h_{r+s-m}Q_{m,r-1} = \sum_{m=1}^{r+s}h_{r+s-m}Q_{m,s-1}.$$

5. Geometric interpretation: equivariant Gal phenomenon

In this section, we will present interpretations of results in Section 3 using geometry and representation theory. The idea behind such interpretations was, to our knowledge, first employed by Stanley, and is discussed in [36].

Herein, a *polytope* is the convex hull of a finite set of points in some $\mathbb{R}^d$. A polytope is *simplicial* if every proper face is a simplex. Let P be a d-dimensional simplicial polytope. Associated with P is the h-polynomial defined by

$$h_P(t) := \sum_{j=0}^{d} f_{j-1}(t-1)^{d-j},$$

where f_i is the number of faces of P of dimension i. It is well known that the h-polynomial of every simplicial polytope is palindromic and unimodal. Indeed, palindromicity is equivalent to the Dehn-Sommerville equations, and unimodality was proved by Stanley [34] as part of the g-Theorem of Billera, Lee and Stanley (see e.g., [35, 4]).

A simplicial complex is said to be *flag* if it is the clique complex of its 1-skeleton; that is, its faces are the cliques of its 1-skeleton. Examples of flag simplicial complexes include barycentric subdivisions of simplicial complexes, or more generally order complexes of posets. Gal formulated the following strengthening of the long standing Charney-Davis conjecture [8].

Conjecture 5.1 (Gal [16])**.** *If P is a flag simplicial polytope (or more generally a flag simplicial sphere) then $h_P(t)$ is γ-positive.*

Gal's conjecture has been proved for certain special classes and examples; see [25, Section 10.8]. One such example is the dual of the permutohedron. The permutohedron P_n is the convex hull of the set $\{(\sigma(1), \dots, \sigma(n)) : \sigma \in \mathfrak{S}_n\}$. The dual permutohedron P_n^* is combinatorially equivalent to the barycentric subdivision of the boundary of the $(n-1)$-simplex. Clearly P_n^* is a flag simplicial polytope. It is well known that

$$h_{P_n^*}(t) = A_n(t).$$

Hence by (1.3), $h_{P_n^*}(t)$ is γ-positive.

We will say that a flag simplicial polytope P *exhibits Gal's phenomenon* if $h_P(t)$ is γ-positive. So P_n^* exhibits Gal's phenomenon. The permutohedron and another polytope called the stellohedron belong to a class of polytopes called chordal nestohedra. In [26, Section 11.2] Postnikov, Reiner, and Williams show that the duals of chordal nestohedra exhibit Gal's phenomenon and they give a combinatorial formula for the γ_i.

Let Δ_n be the simplex in $\mathbb{R}^n$ with vertices $0, e_1, \dots, e_n$, where e_i is the i^{th} standard basis vector. The *stellohedron* St_n is obtained from Δ_n by truncating all faces not containing 0 in an order such that if F, G are such faces and $\dim F < \dim G$ then F is truncated before G. Stellohedra are discussed in various papers, including [26, Section 10.4] and [7].

Stellohedra are simple polytopes. Therefore, each dual polytope St_n^* is a simplicial polytope. If F is a face of a polytope P and P_F is obtained from P by truncating F, then P_F^* is obtained from P^* by stellar subdivision of the dual face F^* (see for example [13, Theorem 2.4]). Therefore, St_n^* is (combinatorially equivalent to) the polytope obtained from Δ_n through stellar subdivision of all faces not contained in the convex hull of $\{e_1, \ldots, e_n\}$ in an order such that if F, G are such faces and $\dim F < \dim G$ then F is subdivided after G.

Postnikov, Reiner, and Williams [26, Section 10.4] observe that

$$h_{St_n^*}(t) = \tilde{A}_n(t).$$

Hence γ-positivity of $\tilde{A}_n(t)$ is a consequence of their general result on chordal nestohedra, as is their formula (1.4).

Associated to any simplicial polytope P is a toric variety $X(P)$. Danilov and Jurkiewicz (see [36, eq. (26)]) showed that for any simplicial polytope P,

$$h_P(t) = \sum_{j \geq 0} \dim H^{2j}(X(P)) t^j,$$

where $H^i(X(P))$ is the degree i singular cohomology of $X(P)$ over $\mathbb{C}$. From this, one has the algebro-geometric interpretation of the Eulerian and binomial-Eulerian polynomials given by,

$$A_n(t) = \sum_{j=0}^{n-1} \dim H^{2j}(X(P_n^*)) t^j$$

and

$$\tilde{A}_n(t) = \sum_{j=0}^{n} \dim H^{2j}(X(St_n^*)) t^j.$$

The purpose of this section is to discuss equivariant versions of these interpretations.

Any simplicial action of a finite group G on P determines an action of G on $X(P)$ and thus a representation of G on each cohomology group of $X(P)$. If G is the symmetric group $\mathfrak{S}_n$, the *Frobenius characteristic*, denoted by ch herein, assigns to each representation (up to isomorphism) of G a symmetric function, as discussed in [38, Section 7.18]. The symmetric group $\mathfrak{S}_n$ acts simplicially on P_n^* and St_n^*. For $P = P_n^*$, Stanley [36], using a recurrence of Procesi [27] obtained the interpretation,

$$Q_n(\mathbf{x}, t) = \sum_{j=0}^{n-1} \mathrm{ch}(H^{2j}(X(P_n^*))t^j. \tag{5.1}$$

From this interpretation, Stanley concluded that palindromicity and Schur-unimodality of $Q_n(\mathbf{x}, t)$ are consequences of an equivariant version of the hard Lefschetz theorem. Here, using (5.1) and Procesi's technique, we obtain an analogous result for $\tilde{Q}_n(\mathbf{x}, t)$, which enables us to also interpret palindromicity and unimodality of $\tilde{Q}_n(\mathbf{x}, t)$ as a consequence of the equivariant version of the hard Lefschetz theorem.

Theorem 5.2. *For all $n \geq 1$,*

$$\tilde{Q}_n(\mathbf{x}, t) = \sum_{j=0}^{n} \mathrm{ch}(H^{2j}(X(St_n^*))t^j.$$

Proof. Let Δ_n be the n-simplex with vertex set $\{0, e_1, \dots, e_n\}$. Let $\mathcal{F}_i$ be the set of i-dimensional faces of Δ_n containing 0. Let $T_n = \Delta_n$ and, for $1 \leq i \leq n-1$, let T_i be the polytope obtained from T_{i+1} by simultaneous stellar subdivision of all faces in $\mathcal{F}_i$. Note that if $F \in \mathcal{F}_i$ then F is a indeed face of T_{i+1}. Moreover, the link L_F of F in the boundary complex of T_{i+1} has one vertex for each face of the boundary of Δ_n strictly containing F. Indeed, when applying stellar subdivision to such a face E, we remove E and add a cone over the boundary of E. Call the vertex of this cone $\phi(E)$. The vertices of L_F are all such $\phi(E)$, and a set $\{\phi(E_i)\}$ of such vertices forms a face of L_F if and only if $\{E_i\}$ is a chain in the face poset of the boundary of Δ_n. Thus L_F is isomorphic to the barycentric subdivision of the link of F in the boundary of Δ_n, which is equal to $\bar{L}_{F\setminus\{0\}}$, the barycentric subdivision of the link of $F \setminus \{0\}$ in the boundary of the $(n-1)$-simplex with vertex set $\{e_1, \dots, e_n\}$.

Note that $T_1 = St_n^*$. The action of $\mathfrak{S}_n$ on $\{e_1, \dots, e_n\}$ by permutation of indices induces a simplicial action on each T_i. Thus we can consider the representations of $\mathfrak{S}_n$ on the cohomology groups of the varieties $X(T_i)$. If $F = \{0, e_{i_1}, \dots, e_{i_k}\}$, where $1 \leq i_1 < \cdots < i_k \leq n$, then $\mathfrak{S}_{[n]\setminus\{i_1,\dots,i_k\}}$ acts simplicially on L_F and this action is equivalent to the action of $\mathfrak{S}_{[n]\setminus\{i_1,\dots,i_k\}}$ on $\bar{L}_{F\setminus\{0\}}$. By viewing L_F and $\bar{L}_{F\setminus\{0\}}$ as simplicial polytopes, we have that these actions induce isomorphic representations of $\mathfrak{S}_{[n]\setminus\{i_1,\dots,i_k\}}$ on cohomology of the corresponding varieties $X(L_F)$ and $X(\bar{L}_{F\setminus\{0\}})$.

For $1 \le i \le n$, we write X_i for $X(T_i)$. Then X_n is the projective space $\mathbb{P}^n$. As explained in [13, Section VI.7], X_i is obtained from X_{i+1} by a series of equivariant blowups. For each $i \in \{1, \dots, n\}$ and each $F \in \mathcal{F}_i$, let L_F be the link of F in the boundary complex of T_{i+1}, as above. As discussed in [27, Section 3], there is an isomorphism of graded vector spaces,

$$H^*(X_i) \cong H^*(X_{i+1}) \oplus \bigoplus_{F \in \mathcal{F}_i} H^*(X(L_F)) \otimes H^+(\mathbb{P}^i), \tag{5.2}$$

where $H^+(\mathbb{P}^k) := \oplus_{j>0} H^{2j}(\mathbb{P}^k)$.

In fact, we can extend (5.2) to an isomorphism of $\mathfrak{S}_n$-representations. Note that $\mathfrak{S}_n$ acts transitively on $\mathcal{F}_i$, with the stabilizer of the face $F_i := \mathsf{conv}\{0, e_1, \dots, e_i\}$ being the subgroup $G_i := \mathfrak{S}_{\{1,\dots,i\}} \times \mathfrak{S}_{\{i+1,\dots,n\}}$. The factor $\mathfrak{S}_{\{i+1,\dots,n\}}$ in G_i acts on $H^*(X(L_{F_i}))$ as it does on $H^*(X(\bar{L}_{F_i \setminus \{0\}}))$, as mentioned above. This is equivalent to the representation of $\mathfrak{S}_{n-i}$ on $H^*(X(P^*_{n-i}))$. The factor $\mathfrak{S}_{\{1,\dots,i\}}$ acts trivially on $H^+(\mathbb{P}^i)$, as explained in [27, Section 3].

We see now that the representation of $\mathfrak{S}_n$ on $H^*(X_i)$ is the direct sum of the representation on $H^*(X_{i+1})$ with the representation induced from that of G_i on $H^*(X(L_{F_i})) \otimes H^+(\mathbb{P}^i)$ determined by the representations of $\mathfrak{S}_{\{i+1,\dots,n\}}$ and $\mathfrak{S}_{\{1,\dots,i\}}$ on the respective tensor factors. Recalling the well known fact that $H^{2j}(\mathbb{P}^i)$ has dimension one for $1 \le j \le i$ and taking Frobenius characteristics, we obtain, for $1 \le i \le n$,

$$R_i(\mathbf{x}, t) = R_{i+1}(\mathbf{x}, t) + t[i]_t h_i(\mathbf{x}) \sum_{j=0}^{n-i-1} \mathrm{ch}(H^{2j}(X(P^*_{n-i}))t^j,$$

where

$$R_i(\mathbf{x}, t) := \sum_{j \ge 0} \mathrm{ch}(H^{2j}(X_i))t^j.$$

By (5.1) we may conclude that

$$R_i(\mathbf{x}, t) = R_{i+1}(\mathbf{x}, t) + t[i]_t h_i(\mathbf{x}) Q_{n-i}(\mathbf{x}, t). \tag{5.3}$$

By induction, we have

$$R_i(\mathbf{x}, t) = h_n(\mathbf{x})[n+1]_t + \sum_{m=i}^{n-1} t[m]_t h_m(\mathbf{x}) Q_{n-m}(\mathbf{x}, t).$$

Setting $i = 1$ yields,

$$\sum_{j=0}^{n} \mathrm{ch}(H^{2j}(X(St_n^*))t^j = h_n(\mathbf{x})[n+1]_t + \sum_{m=1}^{n-1} t[n-m]_t h_{n-m}(\mathbf{x}) Q_m(t). \tag{5.4}$$

We will manipulate the symmetric function on the right side of (5.4) to obtain the desired result. Setting $r = 1$ in [31, Corollary 4.1], we obtain

$$Q_n(\mathbf{x}, t) = h_n(\mathbf{x}) + \sum_{k=0}^{n-2} Q_k(\mathbf{x}, t) h_{n-k}(\mathbf{x}) t[n-k-1]_t. \tag{5.5}$$

Now

$$\begin{aligned}
h_n(\mathbf{x})[n+1]_t &+ \sum_{m=1}^{n-1} h_{n-m}(\mathbf{x}) Q_m(\mathbf{x}, t) t[n-m]_t \\
&= h_n(\mathbf{x})[n+1]_t + h_1(\mathbf{x}) Q_{n-1}(\mathbf{x}, t)t - h_n(\mathbf{x}) t[n]_t \\
&\quad + \sum_{m=0}^{n-2} h_{n-m}(\mathbf{x}) Q_m(\mathbf{x}, t) t[n-m]_t \\
&= h_n(\mathbf{x}) + h_1(\mathbf{x}) Q_{n-1}(\mathbf{x}, t)t + \sum_{m=0}^{n-2} h_{n-m}(\mathbf{x}) Q_m(\mathbf{x}, t) t[n-m-1]_t \\
&\quad + \sum_{m=0}^{n-2} h_{n-m}(\mathbf{x}) Q_m(\mathbf{x}, t) t^{n-m} \\
&= Q_n(\mathbf{x}, t) + h_1(\mathbf{x}) Q_{n-1}(\mathbf{x}, t)t + \sum_{m=0}^{n-2} h_{n-m}(\mathbf{x}) Q_m(\mathbf{x}, t) t^{n-m} \\
&= \sum_{m=0}^{n} h_{n-m}(\mathbf{x}) Q_m(\mathbf{x}, t) t^{n-m},
\end{aligned}$$

the third equality following from (5.5). The result now follows from (5.4) and Proposition 3.3. ☐

Corollary 5.3. *For $0 \le j \le n-1$,*

$$\mathrm{ps}_q(\mathrm{ch}(H^{2j}(X(P_n^*))) = \frac{a_{n,j}(q)}{(1-q)\dots(1-q^n)}$$

and for $0 \leq j \leq n$,

$$\mathrm{ps}_q(\mathrm{ch}(H^{2j}(X(St_n^*))) = \frac{\tilde{a}_{n,j}(q)}{(1-q)\dots(1-q^n)}.$$

Proof. The first equation is a consequence of (5.1) and Theorem 4.1, while the second equation is a consequence of Theorem 5.2 and Corollary 4.2. ☐

The next result follows from combining (5.1) with Corollary 3.2 and combining Theorem 5.2 with Theorem 3.4.

Corollary 5.4. *For* $P \in \{P_n^*, St_{n-1}^*\}$, *the polynomial* $\sum_{j=0}^{n-1} \mathrm{ch}(H^{2j}(X(P))t^j$ *is Schur-*γ*-positive.*

Corollary 5.4 suggests an equivariant version of Gal's phenomenon.

Definition 5.5. Let P be a flag simplicial d-dimensional polytope on which a finite group G acts simplicially. The action of G induces a graded representation of G on cohomology of the associated toric variety $X(P)$. We say that (P, G) exhibits the *equivariant Gal phenomenon* if there exist G-modules $\Gamma_{P,k}$ such that

$$\sum_{j=0}^{d} H^{2j}(X(P))t^j = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \Gamma_{P,k}\, t^k(1+t)^{d-2k}.$$

Corollary 5.4 says that $(P_n^*, \mathfrak{S}_n)$ and $(St_n^*, \mathfrak{S}_n)$ both exhibit the equivariant Gal phenomenon.

It is not the case that every group action on a flag simplicial polytope exhibits the equivariant Gal phenomenon. Indeed, for $i \in [n]$, let e_i be the i^{th} standard basis vector in $\mathbb{R}^n$. Consider the cross-polytope CP^n, which is the convex hull of $\{\pm e_i : i \in [n]\}$. It is straightforward to see that (the boundary of) CP^n is a flag simplicial polytope. The convex hull of some set S of vertices of CP^n is a boundary face if and only if there is no i such that S contains both e_i and $-e_i$.

Let $T \leq GL_n(\mathbb{R})$ be the group of all diagonal matrices whose nonzero entries are 1 or -1 and let $S \leq GL_n(\mathbb{R})$ be the set of all $n \times n$ permutation matrices. The semidirect product $W = S \ltimes T$ preserves CP^n. It is well known and not hard to see that the h-polynomial of CP^n is $(1+q)^n$. The action of W on $H^0(X(CP^n))$ is trivial. It follows that if $G \leq W$ and (CP^n, G) exhibits the equivariant Gal phenomenon, then G acts trivially on $H^*(X(CP^n))$.

Consider the element $c \in W$ satisfying $e_1c = e_2$, $e_2c = -e_1$ and $e_ic = -e_i$ for $3 \leq i \leq n$. Note that c and c^3 fix no boundary face of CP^n and that c^2

fixes those boundary faces not including any of $\pm e_1$, $\pm e_2$. It follows that the action of C on CP^n is proper, that is, the stabilizer in C of any face F of CP^n fixes F pointwise. This allows us to apply results of Stembridge. We observe that

$$\det(I - qc) = (1 + q^2)(1 + q)^{n-2}.$$

On the other hand, according to Theorem 1.4 and Corollary 1.6 of [40], any $w \in W$ not having 1 as an eigenvalue and acting trivially on $H^*(X(CP^n))$ satisfies

$$\det(I - qw) = (1 + q)^n.$$

(Indeed, using the notation from [40], any such w satisfies $P_{\Delta^w}(q) = 1$ and $\delta(w) = 0$.)

We see that if $G \leq W$ contains (any conjugate of) c, then (CP^n, G) does not exhibit the equivariant Gal phenomenon. It would be interesting to find classes, beyond $(P_n^*, \mathfrak{S}_n)$ and $(St_n^*, \mathfrak{S}_n)$ that exhibit the equivariant Gal phenomenon.

6. Remarks on derangement polynomials

One can modify the q-Eulerian polynomials $A_n(q,t)$ and q-Eulerian numbers $a_{n,j}(q)$ by summing over all derangements in $\mathfrak{S}_n$ instead of over all permutations in $\mathfrak{S}_n$. That is, let $\mathcal{D}_n$ be the set of derangements in $\mathfrak{S}_n$ and let

$$D_n(q,t) := \sum_{\sigma \in \mathcal{D}_n} q^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)},$$

for $n \geq 1$, and let $D_n(q,t) := 1$ for $n = 0$. Since $D_n(q,t) = A_n(q,t,0)$, it follows from (2.3) that

$$\sum_{n \geq 0} D_n(q,t) \frac{z^n}{[n]_q!} = \frac{1-t}{\exp_q(tz) - t \exp_q(z)}. \tag{6.1}$$

Recall from the alternative proof of Corollary 4.6 that $D_n(q,t)$ is palindromic and q-unimodal. (This result was first noted by the authors in [31] and the $q = 1$ case was proved earlier by Brenti [5].) There is an analogous symmetric function result conjectured by Stanley [36] and proved by Brenti [5]. The analogous symmetric function result says that the symmetric function polynomial $Q_n^0(\mathbf{x}, t)$ is palindromic, Schur-positive and Schur-unimodal, where $Q_n^0(\mathbf{x}, t)$ is defined by

$$\sum_{n\geq 0} Q_n^0(\mathbf{x},t)z^n := \frac{1-t}{H(tz)-tH(z)}. \tag{6.2}$$

An algebro-geometric interpretation of this result was given subsequently by Stanley (see [37, page 825]), who determined the representation of the symmetric group on the graded local face module associated with the barycentric subdivision of the simplex.

A formula of Gessel shows that $Q_n^0(\mathbf{x},t)$ is, in fact, Schur-γ positive (see [31, Equation (7.9)]). Let

$$\gamma_{n,k}^0(\mathbf{x}) := \sum_{D\in\mathcal{H}_{n,k}^0} s_D(\mathbf{x}),$$

where $\mathcal{H}_{n,k}^0$ is the set of skew hooks of size n for which k columns have size 2 and the remaining $n-2k$ columns, including the first and last column, have size 1. Gessel's formula is equivalent to

$$Q_n^0(\mathbf{x},t) = \sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor} \gamma_{n,k}^0(\mathbf{x})\, t^k(1+t)^{n-2-2k}, \tag{6.3}$$

for all $n\geq 1$.

It is mentioned in [31, Remark 5.5] that Gessel's formula can be used to establish q-γ-positivity of $D_n(q,t)$. However, an explicit description of the γ-coefficients is not given there. By applying stable principal specialization to (6.3), one obtains the following description of the γ-coefficients. This result is proved in [23, Equation (1.3) and Theorem 3.3] without the use of Gessel's formula. It appears also in [22].

Theorem 6.1. *For $0\leq k\leq n$, let $\Gamma_{n,k}^0$ be the set of permutations $\sigma\in\mathfrak{S}_n$ with no double descents, no intial descent, no final descent, and with* $\mathrm{des}(\sigma)=k$. *Let*

$$\gamma_{n,k}^0(q) := \sum_{\sigma\in\Gamma_{n,k}^0} q^{\mathrm{inv}(\sigma)} \quad \Big(= \sum_{\sigma\in\Gamma_{n,k}^0} q^{\mathrm{maj}(\sigma^{-1})}\Big).$$

Then

$$D_n(q,t) = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} \gamma_{n,k}^0(q)\, t^k(1+t)^{n-2k}. \tag{6.4}$$

Consequently, $D_n(q,t)$ is q-γ-positive.

As discussed in Stanley [37], the Poincaré polynomial of the graded local face module associated with a certain type of subdivision of a simplicial complex is equal to the local h-polynomial associated with the subdivision, which in the case of the barycentric subdivision of the $(n-1)$-simplex is equal to $D_n(1,t)$. In [1] Athanasiadis considers γ-positivity of local h-polynomials and formulates a generalization of Gal's conjecture for local h-polynomials, which would provide a geometric interpretation of γ-positivity of $D_n(1,t)$; see also [2, 3]. One could also consider an equivariant version of Gal's phenomenon in the local setting.

We remark that in [23] the authors and Linusson consider multiset versions of the Eulerian polynomial $A_n(t)$ and the derangement polynomial $D_n(1,t)$ and show that they are γ-positive. A generalization of (1.3) is given in [23, Equation (5.4)] and a generalization of the $q=1$ case of (6.4) is given in [23, Equation (5.3)].

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