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Signature Catalan combinatorics

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The Catalan numbers constitute one of the most important sequences in combinatorics. Catalan objects have been generalized in various directions, including the classical Fuss-Catalan objects and the rational Catalan generalization of Armstrong-Rhoades-Williams. We propose a wider generalization of these families indexed by a composition s which is motivated by the combinatorics of planar rooted trees; when $s = (2, ..., 2)$ and $s = (k + 1, ..., k + 1)$ we recover the classical Catalan and Fuss-Catalan combinatorics, respectively. Furthermore, to each pair (a, b) of relatively prime numbers, we can associate a signature that recovers the combinatorics of rational Catalan objects. We present explicit bijections between the resulting s-Catalan objects, and a fundamental recurrence that generalizes the fundamental recurrence of the classical Catalan numbers. Our framework allows us to define signature generalizations of parking functions which coincide with the generalized parking functions studied by Pitman-Stanley and Yan, as well as generalizations of permutations which coincide with the notion of Stirling multipermutations introduced by Gessel-Stanley. Some of our constructions differ from the ones of Armstrong-Rhoades-Williams, however as a byproduct of our extension, we obtain the additional notions of rational permutations and rational trees.

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1. Introduction

A *permutation* σ of the set $[n] := \{1, 2, \ldots, n\}$ is a bijection $\sigma : [n] \to [n]$. A permutation can be represented in the one-line notation $\sigma_1\sigma_2 \cdots \sigma_n$, where $\sigma_i := \sigma(i)$. We denote the set of permutations of $[n]$ by $\mathfrak{S}_n$. This set has the structure of a group under composition of permutations and it is commonly known as the *symmetric group*. It is an introductory exercise in enumerative combinatorics to show that the cardinality of $\mathfrak{S}_n$ is given by the factorial numbers $n! := 1 \cdot 2 \cdots (n-1) \cdot n$. There are other families of objects that are in bijection with permutations. For example, there is a bijection between $\mathfrak{S}_n$ and the set $\mathcal{IT}_n$ of binary trees drawn in the plane with a distinguished vertex or *root* and with labeled internal nodes that increase whenever we walk away from the root. The trees in $\mathcal{IT}_n$ are also known as *increasing planar rooted binary trees* (see [45, Chapter 1] for the bijection and Figure 1 for an example of the bijection when $n = 3$). We will be denoting by $\mathcal{S}_n$ a generic family of objects in bijection with $\mathfrak{S}_n$, that is, $|\mathcal{S}_n| = n!$.

We say that a permutation σ is 312-*avoiding* if there are no indices $i < j < k$ such that $\sigma_j < \sigma_k < \sigma_i$. We denote by $\mathfrak{S}_n(312)$ the set of 312-avoiding permutations in $\mathfrak{S}_n$. It is known that the set $\mathfrak{S}_n(312)$ is in bijection with the set $\mathcal{DP}_n$ of lattice paths in $\mathbb{N} \times \mathbb{N}$ from $(0, 0)$ to (n, n) taking only north steps $(0, 1)$ and east steps $(1, 0)$, and such that at any given point the number of north steps taken is greater than or equal to the number of east steps taken (see Figure 1 for an example of the bijection when $n = 3$). These lattice paths are also known as *Dyck paths*. Any family of objects in bijection with $\mathcal{DP}_n$ is known as a *Catalan family*, named after the Belgian mathematician Eugène Charles Catalan who studied them.

Catalan objects are among the most intriguing objects in combinatorics since they appear in connection with numerous fields of mathematics in many different forms. The Catalan objects are counted by the *Catalan numbers*, which constitute one of the most important sequences in combinatorics:

$$\begin{array}{cccccccccccc} C_0, & C_1, & C_2, & C_3, & C_4, & C_5, & C_6, & C_7, & C_8, & C_9, & C_{10}, & C_{11}, & \ldots \\ 1, & 1, & 2, & 5, & 14, & 42, & 132, & 429, & 1430, & 4862, & 16796, & 58786, & \ldots \end{array}$$

The n-th term of this sequence is commonly denoted by C_n. It is given by the simple formula $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{(2n)!}{(n+1)!n!}$, and is completely

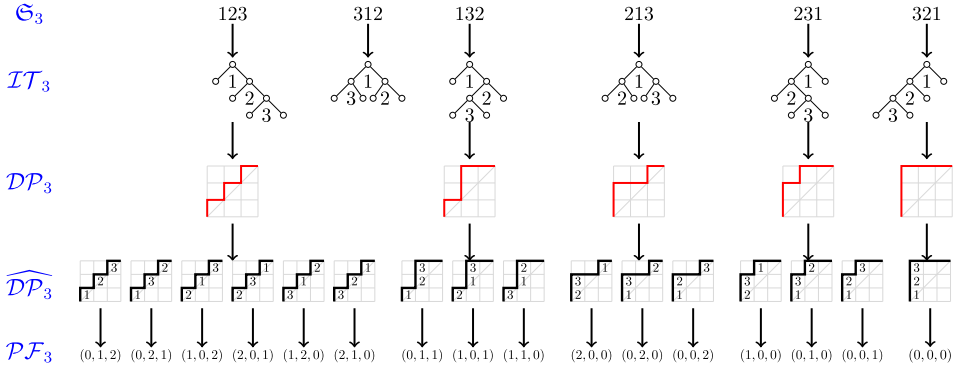


Figure 1: Maps between classical combinatorial objects in type A.

determined by the fundamental recurrence

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad C_0 = 1.$$

The Catalan numbers are known to count a wide variety of objects [46]. Among the most remarkable ones are:

1. planar rooted binary trees with n internal nodes;
2. Dyck paths in an $n \times n$ grid;
3. 312-avoiding permutations of $[n]$;
4. noncrossing partitions of $[n]$;
5. noncrossing matchings of $[2n]$;
6. triangulations of an $(n+2)$-gon;
7. parenthesizations of $n+1$ consecutive letters.

Starting from objects in $\mathcal{DP}_n$ we can create a different family $\widehat{\mathcal{DP}}_n$. These objects are obtained after decorating the north steps of a Dyck path with a permutation of $[n]$ in such a way that the consecutive sequences of north steps (also known as *runs*) have increasing values from bottom to top (see Figure 1). The objects obtained in this way are known as *decorated Dyck paths* and happen to be in bijection with another famous family of combinatorial objects known as *parking functions*. The set $\mathcal{PF}_n$ of parking functions of n consists of sequences of n nonnegative integers $(p_1, p_2, \ldots, p_n)$ with the property that after being rearranged in weakly increasing order $p_{j_1} \leq p_{j_2} \leq \cdots \leq p_{j_n}$ they satisfy $p_{j_i} < i$ for all $i \in [n]$. Figure 1 shows an

example of the bijection between decorated Dyck paths and parking functions for $n = 3$. Parking functions were studied by Konheim and Weiss [25] using a different equivalent description. There are other popular families of objects in bijection with $\mathcal{PF}_n$. For example, the set of (nonplanar nonrooted) trees on the vertex set $\{0\} \cup [n]$ is in bijection with $\mathcal{PF}_n$ (see [23, 26, 37]). Let us denote any generic family of objects in bijection with $\mathcal{PF}_n$ by $\mathcal{P}_n$. It is also known that $|\mathcal{P}_n| = (n+1)^{n-1}$, see for example [44].

The families counted by the sequences

$$n!, \quad \frac{1}{n+1}\binom{2n}{n}, \quad (n+1)^{n-1}$$

for $n \geq 0$ have puzzled mathematicians for centuries and we could describe their relation using the following idea.

Idea 1. *A family $\mathcal{C}_n$ is obtained as a sub-family of $\mathcal{S}_n$ satisfying a suitable restriction. A family $\mathcal{P}_n$ can be obtained by extending a family $\mathcal{C}_n$ after decorating its objects with the elements of $[n]$, also subject to a suitable restriction on the decoration.*

Idea 1 is the underlying idea in the story of the classical combinatorial objects mentioned above. As mentioned in [3], there are two general directions in which we can generalize this story. The first considers that all the families of objects discussed, $\mathcal{S}_n$, $\mathcal{C}_n$ and $\mathcal{P}_n$, are related directly to the combinatorics of the symmetric group that happens to be the Weyl group of Coxeter type A. Hence, we can wonder if there are corresponding objects for other Coxeter types. Work in this direction has been done by some authors like Reiner [41], Athanasiadis [4], Fomin and Reading [18], Armstrong [1], Williams [48] and others, and it is currently an active area of research.

The second direction to generalize these families of objects is to consider Catalan objects as a phenomenon that depends on two relatively prime numbers a, b such that a family $\mathcal{C}_{a,b}$ parametrized by these numbers has cardinality $\frac{1}{a+b}\binom{a+b}{b}$, also known as the *rational Catalan number*. The non-relatively prime case leads to the study of rectangular Catalan combinatorics, which is motivated by related work on rectangular diagonal harmonics and Macdonald polynomials [5, 6, 8, 9]. When $a = n$ and $b = n+1$ we recover the classical Catalan numbers, and when $a = n$ and $b = kn+1$ we obtain another classical generalization known as the *Fuss-Catalan numbers*. Even though ingredients of the rational Catalan story have appeared previously in the literature (see for example [11]), Armstrong, Rhodes and Williams started a more systematic study of the rational Catalan objects $\mathcal{C}_{a,b}$ in [3].

In this generalization of the classical story there are rational versions $\mathcal{P}_{a,b}$ of the parking objects in $\mathcal{P}_n$ (see for example [2]), but it is not clear what is the generalization for the permutation objects in $\mathcal{S}_n$ to a rational version $\mathcal{S}_{a,b}$.

1.1. s-Catalan objects

The main purpose of this paper is to start a systematic study of a wider generalization of the Catalan objects indexed by a composition of the form $s = (s(1), s(2), \ldots, s(a))$, which we call a *signature*. The members of these generalized families will be referred to as *s-Catalan objects*. When $s = (2, \ldots, 2)$ and $s = (k+1, \ldots, k+1)$ we recover the classical Catalan and Fuss-Catalan objects respectively. The rational (and rectangular) Catalan objects are obtained by the signature whose i-th entry is the number of boxes in the i-th row of an $b \times a$ grid that are crossed by the main diagonal of the grid.

The central idea for our generalization of Catalan objects relies on the combinatorics of planar rooted trees. For a given signature s we associate a family $\mathcal{T}_s$ of planar rooted trees that we call *s-trees*. Based on their structural properties, one can obtain generalizations of other classical Catalan objects such as Dyck paths, 312-avoiding permutations, noncrossing partitions, complete noncrossing matchings, triangulations of a polygon, and parenthesizations. The resulting s-Catalan objects can be described very naturally in terms of any composition s, which encodes certain combinatorial information. For instance, planar rooted binary trees can be generalized to arbitrary planar rooted trees, where the signature encodes the number of children of each internal node of the tree when reading in a certain order (preorder). Another example is that of Dyck paths, which can be generalized to lattice paths that lie weakly above a certain ribbon shape determined by s. The precise definitions and motivations for the generalized s-Catalan families will be presented in the coming sections. Some examples of s-Catalan objects are illustrated in Figure 3.

It is worth mentioning that some s-Catalan objects have been explicitly studied in the literature before. For instance, Préville-Ratelle and Viennot [40] introduced a poset structure, the ν-Tamari lattice, on the set of lattice paths lying weakly above a given path ν. In that context, s-Dyck paths are the elements of the ν-Tamari lattice for $\nu = NE^{s(1)-1} \ldots NE^{s(a)-1}$. The s-Dyck paths are also the vertices of ν-associahedra [14], and play an important role in certain Hopf algebras and some applications to multivariate diagonal harmonics [10].

As we shall see in Sections 2 and 3, the s-Catalan families have the same intrinsic combinatorial structure. They are all counted by the same determinantal formula (Section 4), and are determined by the same fundamental

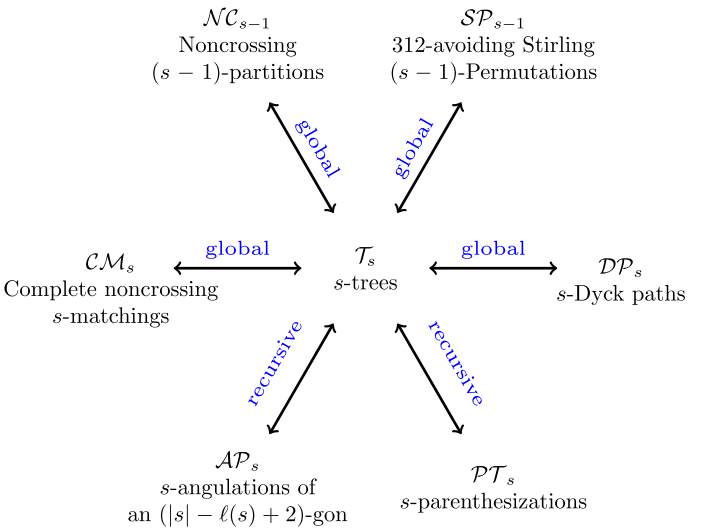


Figure 2: Bijections between s-Catalan objects.

recurrence of s-Catalan structures:

$$C_s = \sum_{(s)} C_{s_1} C_{s_2} \cdots C_{s_{s(1)}},$$

where the sum is over all sequences $(s_1, s_2, \ldots, s_{s(1)})$ of compositions such that $s = (s(1)) \oplus s_1 \oplus s_2 \oplus \cdots \oplus s_{s(1)}$ and where $C_\emptyset = 1$ (see Section 3). The s-Catalan numbers also give a refined enumeration of Catalan objects indexing with compositions (such as block sizes of a noncrossing partition). Enumeration by type (or integer partitions) has already been studied by Kreweras [26] and others (see for example [43]).

One of our main results is to show that the generalized s-Catalan families are bijectively equivalent (see Figure 2).

Theorem 1.1. *The following familes are in bijective correspondence:*

1. *s-trees;*
2. *s-Dyck paths;*
3. *312-avoiding Stirling $(s-\mathbf{1})$-permutations;*
4. *noncrossing $(s-\mathbf{1})$-partitions;*
5. *complete noncrossing s-matchings;*
6. *s-angulations of an $(|s|-\ell(s)+2)$-gon;*
7. *s-parenthesizations;*

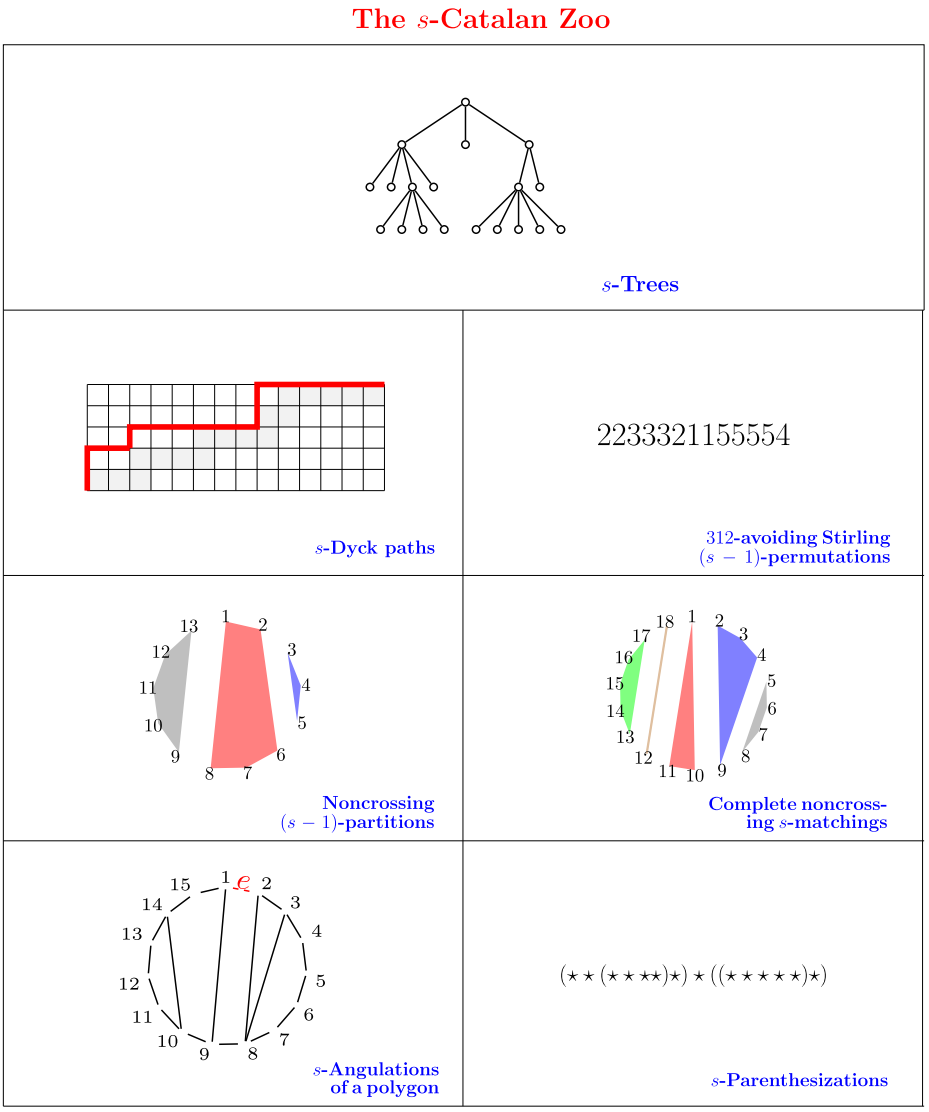


Figure 3: Examples of s-Catalan objects for $s = (3, 4, 4, 2, 5)$.

In items (3) and (4) it is assumed that $s(i) \geq 2$ for all i.

The bijections between s-trees (1) and items (2)-(5) are presented in Section 2. The idea behind these bijections is always essentially the same: we assign some labels to the tree and then read them in preorder. Figure 4 presents a quick descriptive illustration of all the bijections. We include

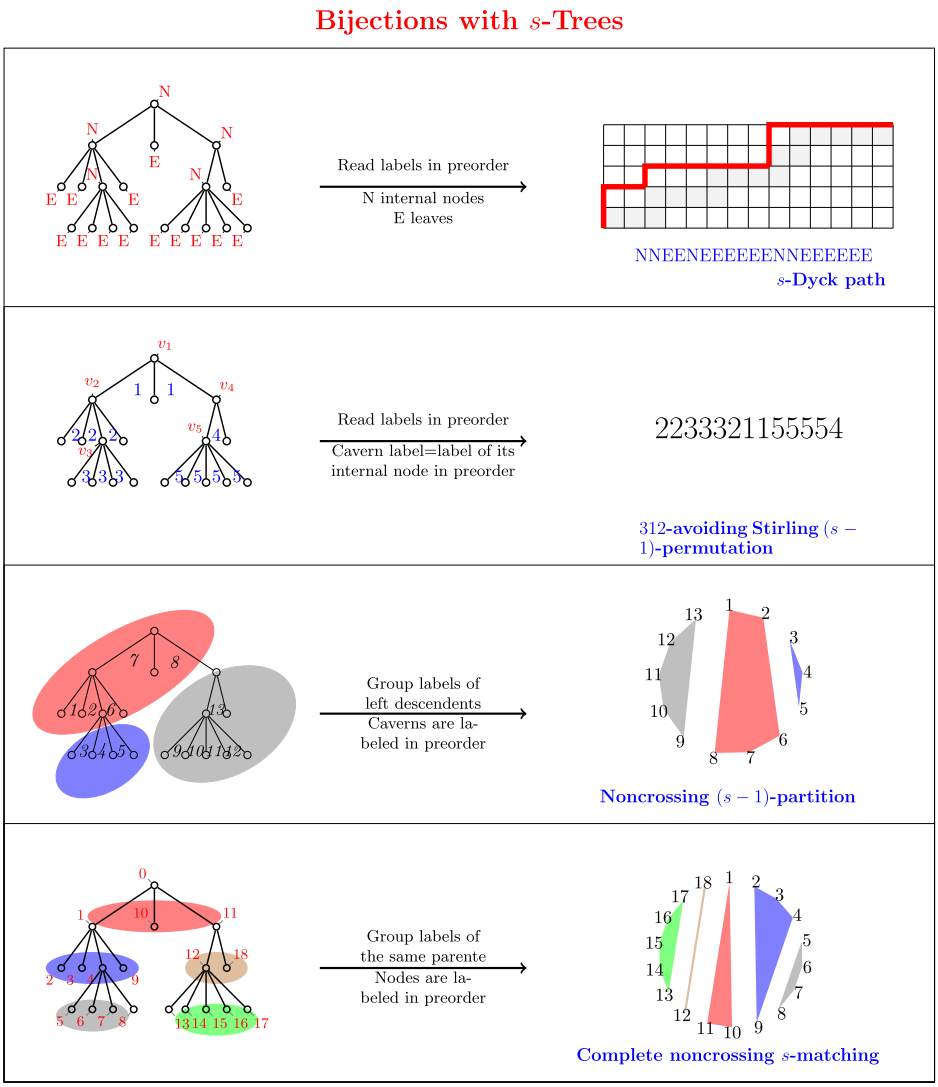


Figure 4: Examples of the bijections with s-trees.

detailed proofs explaining facts about these bijections for the sake of completeness, but some readers may like to skip some of these proofs and let themselves be convinced from the descriptions given in Figure 4. The bijections between s-trees (1) and items (6)-(7) are explained using the fundamental recurrence in Section 3. Bijections with s-Dyck paths are explained in the Appendix A. See Figure 25 for a descriptive illustration.

Our bijections generalize classical bijections between planar rooted binary trees and other Catalan objects, by naturally extending them in the context of planar rooted trees.[1] There are also many known bijections between planar rooted trees (also counted by the Catalan numbers) and other Catalan objects. Our bijections are not the restriction of those to the set of trees of a given signature in general. The only case where this holds is for our bijection between s-trees and complete noncrossing s-matchings, which coincides with a restriction of Dershowitz-Zaks's bijection between planar rooted trees and noncrossing partitions in [16].

The enumerative properties of s-Catalan objects, the s-Catalan and s-Narayana numbers are discussed in Section 4. We also propose generalizations of permutations and parking functions objects in this general setup in Section 5. The generalized permutations are encoded by the family of increasingly labeled planar rooted trees. These labeled trees are in bijection with a generalization of the Stirling permutations studied by Gessel and Stanley back in the seventies [21] (see also [36, 35, 34, 22, 24, 28, 33, 17, 42]). The generalized parking functions are obtained as appropriate decorations of s-Catalan objects as it is done in the classical case of rational parking functions [2]. These also coincide with the generalized parking functions originally studied by Pitman and Stanley [47] and Yan [50, 49]. Finally, we present the relation to the rational constructions of Armstrong, Rhoades and Williams in Section 6.

2. The s-Catalan zoo and bijections

Before introducing some s-Catalan families, let us recall some basic preliminaries about compositions and weak compositions. We denote by $\mathbb{P}$ the set of positive integers and by $\mathbb{N}$ the set of nonnegative integers. A *weak composition* is a finite sequence $\mu = (\mu(1), \mu(2), \ldots, \mu(\ell))$ of numbers $\mu(i) \in \mathbb{N}$. For a weak composition μ we define its *sum* $|\mu| := \sum_i \mu(i)$ and its *length* $\ell(\mu) := \ell$. For example, for $\mu = (2, 0, 3, 4, 0, 1)$ we have that $\ell(\mu) = 6$ and $|\mu| = 10$. If $|\mu| = n$ for some $n \in \mathbb{N}$, we say that μ is a *weak composition of* n. We denote by $\mathcal{WC}omp$ the set of weak compositions and $\mathcal{WC}omp_n$ the set of weak compositions of n. A *composition* is a weak composition μ such that $\mu(i) \neq 0$ for all $i \in [\ell]$, in other words, a composition is a finite sequence of entries in $\mathbb{P}$. We denote by $\mathcal{C}omp$ the set of compositions and $\mathcal{C}omp_n$ the set

[1] We are not aware of a bijection in the literature between planar rooted trees and noncrossing partitions in the form presented here, but our description coincides with the composition of two classical bijections from planar rooted binary trees to Dyck paths to noncrossing partitions.

of compositions of n. An *(integer) partition* λ of n (denoted $\lambda \vdash n$) is a composition of n whose entries are nonincreasing, i.e., $\lambda = (\lambda(1) \geq \lambda(2) \geq \cdots)$. We denote by $\mathcal{P}ar$ the set of partitions and $\mathcal{P}ar_n$ the set of partitions of n.

We can define several partial orders on the set of (weak) compositions. We introduce the two orderings that we will be using in this article. For $\mu, \nu \in \mathcal{WC}omp$ we say that μ is a *refinement* of ν if ν can be obtained from μ by adding adjacent parts. For example, $(3, 2, 1, 1, 5, 2, 2)$ is a refinement of $(6, 1, 7, 2)$ since $(6, 1, 7, 2) = (3 + 2 + 1, 1, 5 + 2, 2)$. Refinement defines a partial order in $\mathcal{WC}omp$ (also in $\mathcal{C}omp$) and we say that $\mu \leq \nu$ if μ is a refinement of ν.

For $\mu, \nu \in \mathcal{WC}omp$ we say that $\mu \leq_{\mathrm{dom}} \nu$ in *dominance order* if $\mu(1) + \mu(2) + \cdots + \mu(i) \leq \nu(1) + \nu(2) + \cdots + \nu(i)$ for all i. For example, $(1, 1, 4, 2) \leq_{\mathrm{dom}} (1, 2, 3, 3)$ since $1 \leq 1$, $1+1 \leq 1+2$, $1+1+4 \leq 1+2+3$ and $1+1+4+2 \leq 1+2+3+3$. For $\mu, \nu \in \mathcal{WC}omp_n$ for some n, such that $\mu \leq_{\mathrm{dom}} \nu$, the *dominance difference* $\nu \setminus_{\mathrm{dom}} \mu$ of ν and μ is the weak composition of length $\ell(\nu) - 1$ defined $(\nu \setminus_{\mathrm{dom}} \mu)(i) := (\nu(1) + \cdots + \nu(i)) - (\mu(1) + \cdots + \mu(i))$ for $i = 1, \ldots, \ell(\nu) - 1$.

We also consider a natural operation in $\mathcal{WC}omp$. For $\mu, \nu \in \mathcal{WC}omp$ we let $\mu \oplus \nu$ be the weak composition formed by the *concatenation* of μ and ν. For example, $(3, 2, 1) \oplus (1, 5, 2, 2) = (3, 2, 1, 1, 5, 2, 2)$. Sometimes we will use the notation μ^{+i} or μ^{-i} to denote the weak composition obtained from μ by adding or substracting i to the last part of μ respectively.

2.1. Planar rooted trees

A tree is a graph that has no loops or cycles. We say that a tree is rooted if one of its nodes is specially marked and called the *root*. For two nodes x and y on a rooted tree T, x is said to be the *parent* of y (and y the *child* of x) if x is the node that follows y in the unique path from y to the root. We call y a *descendant* of x if x belongs to the unique path from y to the root. A node is called a *leaf* if it has no children, otherwise is said to be *internal*. The *degree* $\deg(x)$ of a node x in a rooted tree T is defined to be the cardinality of the set of children of x. A rooted tree T is said to be planar if for every internal node x of T the set of children of x is totally ordered. For $n \geq 1$ we denote by $\mathcal{T}_n$ the set of all planar rooted trees with n internal nodes and by $\mathcal{T}$ the set of all planar rooted trees. There is a unique tree $\bullet$ that has only one node that is at the same time its leaf and its root, we call this tree the *identity tree*.

For trees $T_1, T_2, \ldots, T_k \in \mathcal{T}$ with roots $r_1, r_2, \ldots, r_k$ respectively, we denote by $[T_1, T_2, \ldots, T_k]$ the tree in $\mathcal{T}$ constructed by adding a new root r

and attaching the trees $T_1, T_2, \ldots, T_k$ to r in a way that the ith children of r is r_i for $i \in [k]$ (see Figure 5). We are going to fix an order in which to read the nodes of a tree $T = [T_1, T_2, ..., T_k] \in \mathcal{T}$. Let r be the root of $\mathcal{T}$, we say that T is read or traversed in *preorder* if we visit first the root r and then we visit in preorder each of the trees T_i in a way that T_i is traversed before T_j if $i < j$ for $i, j \in [k]$.

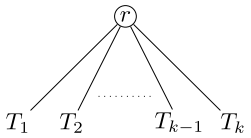


Figure 5: Example of $[T_1, T_2, \ldots, T_k]$.

Let $v_1, v_2, \ldots, v_a$ be the internal nodes of T listed in preorder, we define the *signature* of T to be the composition

$$\text{signat}(T) := (\deg(v_1), \deg(v_2), \ldots, \deg(v_a)).$$

The signature of the identity tree is defined as the empty composition, i.e., $\text{signat}(\bullet) := \emptyset$.

Definition 2.1 (s-trees)**.** For any $s \in \mathcal{C}omp$, an s-tree is a planar rooted tree with signature s. We denote by

$$\mathcal{T}_s = \{T \in \mathcal{T} \mid \text{signat}(T) = s\}$$

the set of *s-trees*.

Figure 3 illustrates an s-tree T with signature $s = (3, 4, 4, 2, 5)$.

2.2. Dyck paths

We consider *lattice paths* in $\mathbb{N} \times \mathbb{N}$ starting at $(0, 0)$ and taking only *east steps* E (in the direction $(1, 0)$) and *north steps* N (in the direction $(0, 1)$). For positive relatively prime integers a and b, a *rational Dyck path* or an *(a, b)-Dyck path* is a lattice path with endpoint (b, a) wich stays above the diagonal $y = \frac{a}{b}x$. See Figure 6 for an example of a rational Dyck path with $(a, b) = (5, 8)$.

Armstrong, Rhoades and Williams in [3] started the systematic combinatorial study of rational Catalan combinatorics. However, paths in a rectangular grid (non-relatively prime case) have appeared in the literature

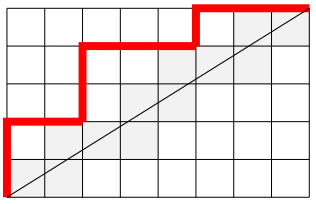


Figure 6: Example of a rational Dyck path with $(a, b) = (5, 8)$.

before [11], and leads to the study of the more general rectangular Catalan combinatorics in connection with rectangular diagonal harmonics [5, 6, 8, 9]. The family of rational Dyck paths contains as special cases the classical *Dyck paths* that occur for $n \geq 1$ when $a = n$ and $b = n + 1$ and more generally the *Fuss-Catalan* generalization of Dyck paths that occur when $a = n$ and $b = kn + 1$ for a fixed $k \in \mathbb{P}$.

Note that in the $b \times a$ grid the diagonal $y = \frac{a}{b}x$ crosses certains cells. Those cells define a ribbon shape that cannot be crossed by any rational Dyck path in order to stay above the diagonal (see the gray shape in Figure 6). Let $s \in \mathcal{C}omp$ be defined such that $s(i)$ is the number of cells in row i crossed by the diagonal for each $i \in [a]$, we call s the *degree sequence* associated to the pair (a, b). For any rational pair (a, b) with residue $r = b - \lfloor \frac{b}{a} \rfloor a$ it is not difficult to check that the degree sequence is given by

$$s(i) = \left\lceil \frac{b}{a} \right\rceil + \chi(0 < ir \pmod{\mathrm{a}} < r), \tag{2.1}$$

where $\lfloor x \rfloor$ and $\lceil x \rceil$ are the floor and ceiling functions respectively, and χ is the function

$$\chi(S) = \begin{cases} 1: & S \text{ is true} \\ 0: & \text{otherwise.} \end{cases}$$

For example, for $(5, 8)$ the degree sequence is $(2, 3, 2, 3, 2)$ since $\lceil \frac{8}{5} \rceil = 2$, $r = 8 - \lfloor \frac{8}{5} \rfloor 5 = 3$ and $3(1, 2, 3, 4, 5) = (3, 1, 4, 2, 0) \pmod 5$ (see Figure 6).

Every lattice path can be encoded with a string of the letters N and E representing the north steps and east steps respectively, where the reading order of the path starts at the origin $(0, 0)$. For example, the path D of Figure 6 can be represented as $D = NNEENNEEENEEE$. If the lattice path is a rational Dyck path then it must have exactly a north steps and it must start with a north step and finish with an east step, so it is of the form $D = NE^{i_1}NE^{i_2} \cdots NE^{i_a}E$ where E^i indicates a string of i occurrences of the letter E and then it can be uniquely associated to the weak

composition $\mu(D) := (i_1, i_2, \ldots, i_a)$. So (a, b)-Dyck paths are in bijection to weak compositions of $b-1$ of length a. For the example in Figure 6 we have that $\mu(D) = (0, 2, 0, 3, 2)$ (note that the last E is not counted). The *identity path* $D^{\emptyset} = E$ is the lattice path with exactly one east step and without any north steps. By convention, this is considered to be the unique $(0, 1)$-Dyck path. We also have that $\mu(D^{\emptyset}) = \emptyset$ (the empty weak composition). It is not hard to see that in dominance order there is a maximal (a, b)-Dyck path $\mathcal{D}$ that is defined by the boundary of the forbidden ribbon shape. This path has $\mu(\mathcal{D}) = s - \mathbf{1} := (s(1)-1, s(2)-1, \ldots, s(a)-1)$. An equivalent definition of a rational Dyck path is a path D such that $\mu(D) \leq_{dom} \mu(\mathcal{D}) = s - \mathbf{1}$.

Note that the information recorded by the rational pair (a, b) can also be completely recovered from the degree sequence s of the associated rational ribbon shape. Indeed, $a = \ell(s)$ and $b = |s| - \ell(s) + 1$. Hence we can use the set of rational degree sequences to index the rational families.

Definition 2.2 (s-Dyck paths)**.** For any signature $s \in \mathcal{C}omp$, we define an *s-Dyck path* D to be a lattice path starting at $(0, 0)$ and ending at $(|s| - \ell(s) + 1, \ell(s))$ such that

$$\mu(D) \leq_{dom} s - \mathbf{1}.$$

We denote by $\mathcal{DP}_s$ the set of s-Dyck paths, and refer to an s-Dyck path as a Dyck path when s is clear from the context.

An example of a $(3, 4, 4, 2, 5)$-Dyck path D with $\mu(D) = (0, 2, 6, 0, 5)$ is illustrated in Figure 3. We will say that a signature s is *rational* when s is obtained as in Equation 2.1 from a pair (a, b) of relatively prime numbers.

We define the *area vector* of an s-Dyck path D to be the weak composition $\mathrm{Area}(D) := (0) \oplus (s - \mathbf{1}) \setminus_{\mathrm{dom}} \mu(D)$. Its entries are the number of boxes, counted by rows, between D and the ribbon shape determined by s. The *area* of D is $\mathrm{area}(D) := |\mathrm{Area}(D)|$. In the example of Figure 3, $\mathrm{Area}(D) = (0, 2, 3, 0, 1)$ and $\mathrm{area}(D) = 6$.

2.2.1. Bijection between s-trees and s-Dyck paths Let T be an s-tree. Label the internal nodes of T with N's and the leaves with E's. The lattice path $\xi(T)$ associated to T is obtained by reading the sequence of N's and E's on the nodes of T in preorder. Note that the function ξ maps the identity tree $\bullet$ to the identity path E. An example of this bijection is illustrated in Figure 7.

We define the *area labeling* of a tree as the labeling of the nodes such that the root is labeled 0 and the ith child of an internal node x, from right to left, is labeled with the number obtained after adding $i - 1$ to the label of x. This is illustrated for our example in Figure 8. It turns out that the area vector of $D = \xi(T)$ can be read directly from T.

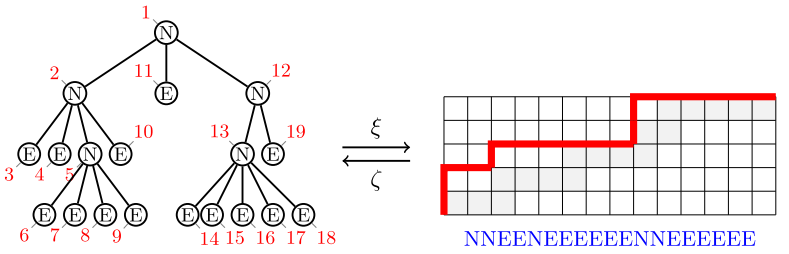


Figure 7: Example of the bijection between s-trees and s-Dyck paths.

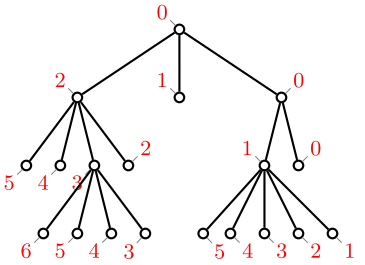


Figure 8: The area labeling of a tree T. The area vector of $D = \xi(T)$ is the sequence of labels of the internal nodes listed preorder.

Lemma 2.3. *The entries of the area vector of the Dyck path $D = \xi(T)$ are the labels in the area labeling of T of the internal nodes listed in preorder.*

Proof. The root of the tree corresponds to the first north step of the path, which contributes a zero entry in the area vector. Let x and y be two internal nodes of T such that y is the ith child from right to left of x. Let $a(x)$ and $a(y)$ be the entries of the area vector of D of the corresponding north steps in D. We need to show that $a(y) = a(x)+i-1$. Let $u_1, \ldots, u_j$ be the internal nodes strictly between x and y in T in preorder, and denote $u_0 = x$. Then,

$$a(y) = a(x) + \sum_{k=0}^{j} (\deg(u_k) - 1) - E(x, y),$$

where the sum of the degrees minus one is the increase of area determined by the ribbon shape and $E(x, y)$ denotes the number of east steps (or leaves) between x and y. Since $E(x, y) = \sum_{k=0}^{j} \deg(u_k) - j - i$, then $a(y) = a(x) - (j + 1) + j + i = a(x) + i - 1$ as desired. □

Lemma 2.4. *The path $\xi(T)$ is an s-Dyck path.*

Proof. By definition, all the entries of the area labeling of the tree T are nonnegative. Lemma 2.3 then implies that the area vector of $\xi(T)$ has nonnegative entries and therefore that it is an s-Dyck path. □

We will see below that the map ξ is actually a bijection between s-trees and s-Dyck paths. The inverse of this map is described below. Let D be an s-Dyck path and $\mu = \mu(D)$. Define the tree $\zeta(D)$ associated to D recursively as follows.

1. First, start with a root $r = v_1$ with $s(1)$ children.
2. Attach an internal node v_2 with $s(2)$ children at the $(\mu(1)+1)$th leaf, ordered from left to right.
3. In general, at the ith step of the process we have a tree τ_i with internal nodes $v_1, \ldots, v_i$ in preorder. The $(i+1)$th tree τ_{i+1} is obtained from τ_i by replacing the $(\mu(i)+1)$th leaf that appears after $v_i \in \tau_i$ in preorder by an internal node v_{i+1} with $s(i+1)$ children.
4. The process finishes after attaching the a internal nodes $v_1, \ldots, v_a$, where $a = \ell(s)$.

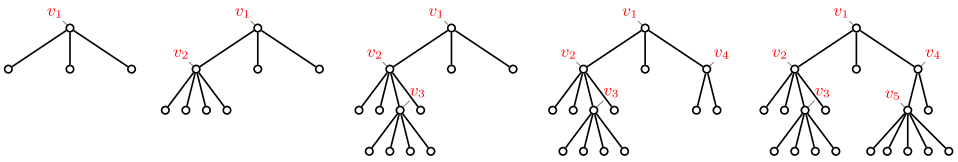


Figure 9: The recursive construction of the tree $\zeta(D)$ for the Dyck path with signature $s(D) = (3, 4, 4, 2, 5)$ and composition $\mu(D) = (0, 2, 6, 0, 5)$.

Lemma 2.5. *Let D be an s-Dyck path. The map ζ is well-defined and $\zeta(D)$ is an s-tree.*

Proof. In order to show that the map ζ is well-defined, we need to show that for $i < a$ there are at least $\mu(i)+1$ leaves in the tree τ_i that appear after the internal node $v_i \in \tau_i$ in preorder. This guarantees that a new node v_{i+1} can be added at the right place and that the tree τ_{i+1} in the recursive definition can be constructed. For $i = 1$ this is clear since v_1 has $s(1) \geq \mu(1) + 1$ children. In general, since D is a path that lies above the ribbon shape defined by s, for $i < a$ we have

$$\mu(1) + \cdots + \mu(i) \leq (s(1) - 1) + \cdots + (s(i) - 1). \tag{2.2}$$

The total number of leaves in τ_i is equal to $\sum_{k=1}^{i}(s(k)-1)+1$, from which $\sum_{k=1}^{i-1}\mu(k)$ are before v_i in preorder. Combining these two equations with the inequality (2.2) we obtain that the number of leaves in τ_i after v_i in preorder is

$$\sum_{k=1}^{i}(s(k)-1)+1-\sum_{k=1}^{i-1}\mu(k)\geq\mu(i)+1.$$

This shows that the map is well-defined. The signature of the tree $\zeta(D)$ is by construction equal to $s=(s(1),\ldots,s(a))$. □

Theorem 2.6. *The map $\xi:\mathcal{T}_s\to\mathcal{DP}_s$ is a bijection between s-trees and s-Dyck paths. The inverse of ξ is ζ.*

Proof. We will show that $D=\xi(T)$ if and only if $T=\zeta(D)$. This implies that both maps are bijections and that they are inverses to each other.

Let $D=\xi(T)$ and $\mu=\mu(D)$. Denote by $v_1,\ldots,v_a$ the internal nodes of T in preorder. By the definition of ξ, there are exactly $\mu(i)$ leaves between v_i and v_{i+1} traveling T in preorder. This condition completely characterizes T and is satisfied by $\zeta(D)$. Therefore, $T=\zeta(D)$.

On the other hand, if $T=\zeta(D)$ then T is the unique tree with internal nodes $v_1,\ldots,v_a$ in preorder such that there are exactly $\mu(i)$ leaves between v_i and v_{i+1}, for $i<a$. Therefore, $D=\xi(T)$. □

2.3. 312-avoiding Stirling permutations

Let $s=(s(1),s(2),\ldots,s(a))$ be a composition. An *s-permutation* is any multiset permutation (*multipermutation*) of the multiset $\{1^{s(1)},2^{s(2)},\ldots,a^{s(a)}\}$, where there are $s(i)$ copies of each i. Let us denote the set of s-permutations by $\mathfrak{S}_s$. We then have that

$$|\mathfrak{S}_s|=\binom{|s|}{s}:=\frac{|s|!}{s(1)!s(2)!\cdots s(a)!}.$$

For multipermutations τ and σ, we say that σ contains the pattern τ if there is a subword of σ where the elements have the same relative order as the elements in τ. For example, the multipermutation $\sigma=1132235544$ contains the pattern $\tau=212$ because the elements in the subword 323 of σ are in the same relative order than the elements in τ. A multipermutation can contain many patterns and multiple or no occurrences of a particular pattern. For example, $\sigma=1132235544$ contains no occurrence of the pattern 321. We say that σ *avoids* τ or is *τ-avoiding* if it does not contain any occurrence of the pattern τ.

A particular set of examples of pattern avoiding multipermutations come from the 212-avoiding s-permutations introduced by Gessel and Stanley in [21] when they studied a generalization of Euler's formula for Eulerian polynomials. They considered the descent enumerator of 212-avoiding multipermutations of $\{1,1,2,2,\cdots,n,n\}$, i.e., multipermutations in which all the numbers between the two occurrences of any fixed number m are larger than m. To this family belongs, for example, the permutation 12234431 but not the permutation 11322344 since 2 is less than 3 and 2 is between the two occurrences of 3. They called these multipermutations *Stirling Permutations* and they have been studied by many other authors, see for example [28, 24, 36, 35, 34, 22, 17, 42].

The set of s-permutations have also already appeared in the work on Hopf algebras of Novelli and Thibon [33]. They study generalizations of the Malvenuto-Reutenauer Hopf algebra of permutations [32] to multipermutations of $\{1^m, 2^m, \ldots, n^m\}$, and of the Loday-Ronco Hopf algebra of planar rooted binary trees [31] to $(m+1)$-ary trees. The authors introduce a notion of metasylvester congruence on permutations that allows them to obtain Hopf algebras based on decreasing trees. The element representatives of the congruence classes are 121-avoiding s-permutations for $s = (m, m, \ldots, m)$, see [33, Proposition 3.10]. These 121-avoiding s-permutations and their associated lattices, called metasylvester lattices, are studied by Pons in [39]. The metasylvester lattices are related to the combinatorics of the m-Tamari lattices introduced by Bergeron [7].

Definition 2.7 (312-avoiding Stirling s-permutations)**.** A *Stirling s- permutation* is a multipermutation of $\{1^{s(1)}, 2^{s(2)}, \ldots, a^{s(a)}\}$ that avoids the pattern 212. A 312-avoiding Stirling s-permutation is a multipermutation of the same multiset that avoids both patterns 212 and 312. We denote by $\mathcal{SP}_s$ the set of Stirling s-permutations, and by $\mathcal{SP}_s(312)$ the set of 312-avoiding Stirling s-permutations.

For instance, 1223321 and 2332211 are 312-avoiding Stirling $(2,3,2)$-permutations, while 3312221 is a Stirling $(2,3,2)$-permutation that is not 312-avoiding. See Figure 10 for the example of $\mathcal{SP}_{(2,3,2)}$ and $\mathcal{SP}_{(2,3,2)}(312)$.

Remark 2.8. Replacing the numbers from 1 to a for the numbers from a to 1 respectively and reversing the permutation transforms bijectively $(212, 312)$-avoiding s-permutations into $(121, 231)$-avoiding $\overleftarrow{s}$-permutations, where the composition $\overleftarrow{s} = (s(a), \ldots, s(1))$ is obtained by reading s in reverse order. This family of $(121, 231)$-avoiding $\overleftarrow{s}$-permutations is a generalization of 231-avoiding permutations, which is also an s-Catalan family.

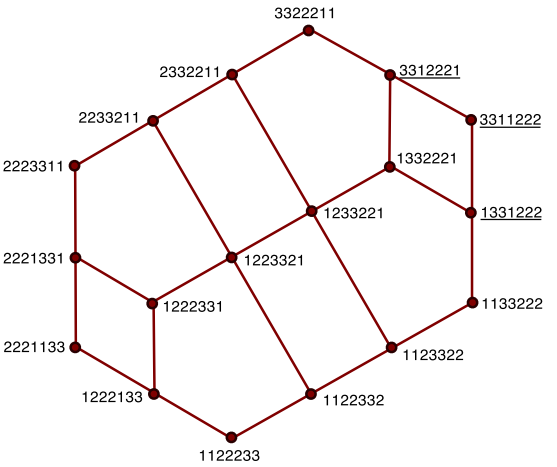


Figure 10: All Stirling s-permutations for $s = (2, 3, 2)$ (drawn as the elements of a lattice similarly as in [39]). The 312-avoiding are the non-underlined ones. There are 15 of them, which are in bijection with the 15 $(3, 4, 3)$-trees.

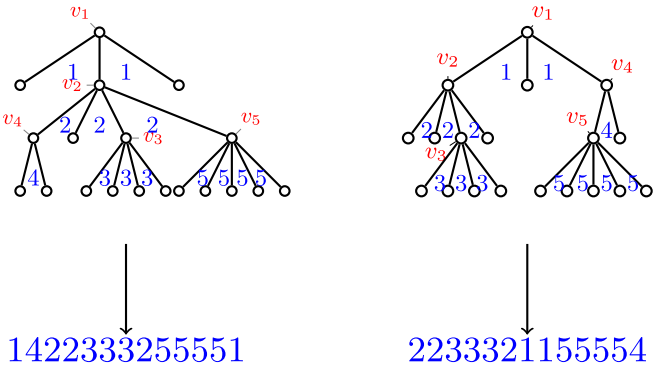


Figure 11: Example of the map $\Sigma : \mathcal{IT}_s \to \mathcal{SP}_{s-\mathbf{1}}$.

2.3.1. Bijection between s-increasing trees and Stirling $(s-1)$-permutations An *increasing tree* is a planar rooted tree whose internal nodes have been decorated with the elements of $[a]$ (where a is the number of internal nodes) such that any path of internal nodes away from the root has strictly increasing labels. An s-increasing tree is an increasing tree such that the internal node with label i has exactly $s(i)$ children. We denote by $\mathcal{IT}_s$ the set of s-increasing trees. Figure 11 shows two examples of $(3, 4, 4, 2, 5)$-increasing trees.

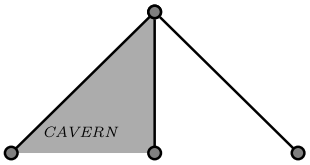


Figure 12: A cavern.

Gessel provided a simple bijection between the sets $\mathcal{IT}_s$ and $\mathcal{SP}_{s-\mathbf{1}}$ that we describe below (Gessel's result is unpublished, but can be found for example in [24]). Before explaining this bijection, we need to introduce the notion of a cavern in a planar rooted tree.

Let $T \in \mathcal{T}$ and x an internal node of T. A *cavern* of T at x is defined to be a pair of consecutive children of x. We really want to think of a cavern as the "space" between any two consecutive children of x (see Figure 12). If the ordered set of children of x is $\{c_1, c_2, \ldots, c_k\}$ then the set of caverns determined by x is

$$\text{caverns}(x) = \{\{c_1, c_2\}, \{c_2, c_3\}, \ldots, \{c_{k-1}, c_k\}\}.$$

We also define $\text{caverns}(T) := \cup_{i=1}^{a} \text{caverns}(v_a)$. Any internal node v of $\deg(v) = k$ defines exactly $k-1$ caverns and hence there are in total $|\,\text{caverns}(T)| = \sum_{i=1}^{a} \deg(v_i) - a$ caverns in a tree $T \in \mathcal{T}$ with a internal nodes.

The set of caverns has a very natural ordering. We can associate each cavern $C = \{c_i, c_{i+1}\}$ of x to the largest children c_{i+1} of x in C. In that way we can think of the caverns as the set of nonminimal children of internal nodes in T (being c_1 the minimal child of x). The order in the set caverns(T) is the one induced by preorder in the set of nonminimal children in T. (see Figure 15). We refer to this order as the *preorder order of the caverns* of T.

For $T \in \mathcal{IT}_s$ we label all the caverns with the label of its corresponding internal node. Reading the caverns in preorder gives rise to an $(s-\mathbf{1})$-permutation $\Sigma(T)$. Since T is an increasing tree, the caverns with label i are read before or after all the caverns with label j for $i < j$. Hence, the resulting $(s-\mathbf{1})$-permutation is 212-avoiding, so a Stirling permutation.

The process described above maps an increasing tree $T \in \mathcal{IT}_s$ to a Stirling $(s-\mathbf{1})$-permutation $\Sigma(T)$. In fact, every Stirling $(s-\mathbf{1})$-permutation can be attained uniquely by exactly one such tree. We construct the inverse map as follows: let σ be a Stirling $(s-\mathbf{1})$-permutation. Put a root labeled 1 with $s(1)$ children labelled with the $s(1)$ permutations that are separated by

the ones in σ (some of which might be empty). Repeat the process with each of the non-empty permutations appearing in the process with the smallest value in the permutation instead of the one. The resulting tree $\Lambda(\sigma)$ at the end of the process is the desired increasing tree, see Figure 13 for an example.

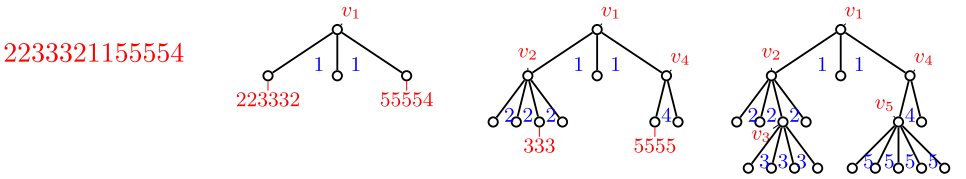


Figure 13: Example of the map $\Lambda : \mathcal{SP}_{s-\mathbf{1}} \to \mathcal{IT}_s$.

Proposition 2.9 (Gessel c.f. [24]). *The map $\Sigma : \mathcal{IT}_s \to \mathcal{SP}_{s-\mathbf{1}}$ is a bijection between increasing trees and Stirling $(s-\mathbf{1})$-permutations. The inverse of Σ is Λ.*

2.3.2. Bijection between s-trees and 312-avoiding Stirling $(s-1)$-permutations Note that if we decorate the internal nodes of an s-tree T with the values of $[a]$ in preorder we obtain an increasing tree $\tilde{T} \in \mathcal{IT}_s$, hence there is an injection $\mathcal{T}_s \hookrightarrow \mathcal{IT}_s$. It turns out that the image of the composition $\mathcal{T}_s \hookrightarrow \mathcal{IT}_s \to \mathcal{SP}_{s-\mathbf{1}}$ is precisely the set $\mathcal{SP}_{s-\mathbf{1}}(312)$. We denote the map determined by this composition by $\tilde{\Sigma}$. We also denote by $\tilde{\Lambda}(T)$ the s-tree obtained by removing the labels from the increasing tree $\Lambda(T)$. The Stirling permutation 2233321155554 that appear in Figure 13 is 312-avoiding, giving an example of the discussed bijection.

Theorem 2.10. *The map $\tilde{\Sigma} : \mathcal{T}_s \to \mathcal{SP}_{s-\mathbf{1}}(312)$ is a bijection between s-trees and 312-avoiding Stirling $(s-\mathbf{1})$-permutations. The inverse of $\tilde{\Sigma}$ is $\tilde{\Lambda}$.*

Proof. It remains to show:

1. For any s-tree T, its image $\tilde{\Sigma}(T) \in \mathcal{SP}_{s-\mathbf{1}}(312)$.
2. For any $\sigma \in \mathcal{SP}_{s-\mathbf{1}}(312)$, the tree $\tilde{\Lambda}(\sigma)$ is an s-tree.

Proof of (1). Label the internal nodes of an s-tree T in preorder, and the caverns with the label of its corresponding internal node. By Proposition 2.9 we know that reading the caverns in preorder gives rise to a Stirling $(s-\mathbf{1})$-permutation $\Sigma(T)$. Moreover, we will see next that it is also 312-avoiding. We proceed by contradiction; assume the permutation has a subsequence $k \dots i \dots j$ with $i < j < k$. The internal node k (labeled in preorder) necessarily has to be a descendant of the node i that is not coming from its

rightmost child. Similarly, the node k is a descendant of the node j that is not coming from its rightmost child. Drawing the unique path from node k to the root visits both nodes i and j. Since $i < j$, then j has to be a descendant of i. From these observations, it is straightforward to see from the tree that no cavern i appears in preorder between two caverns k and j. Thus the pattern 312 is impossible.

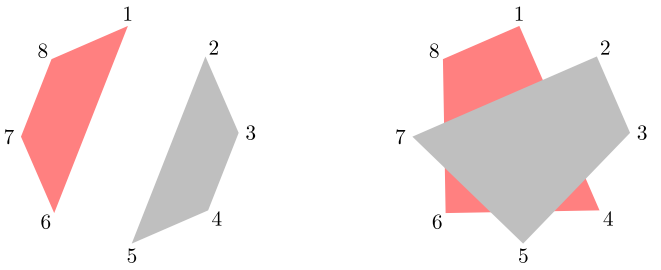
Proof of (2). Let $\sigma \in \mathcal{SP}_{s-\mathbf{1}}(312)$. By Proposition 2.9, we know that the tree $\Lambda(\sigma)$ is increasing. We need to show that the labeling of its internal nodes is given by preorder. We proceed by contradiction; assume that the labeling in $\Lambda(\sigma)$ violates preorder. Consider the minimal pair v_1, v_2 in lexicographic preorder such that $v_1 < v_2$ in preorder but the label of v_1 is larger than the label of v_2. Let v_0 be the smallest common ancestor of v_1 and v_2. Since the tree is increasing, v_2 can not be a descendant of v_1. Therefore $v_0 \neq v_1, v_2$, furthermore it has smaller label than v_1 and v_2 by lexicographic minimality of the pair v_1, v_2. If r_0, r_1, r_2 are the labels of v_0, v_1, v_2 respectively, then the permutation σ corresponding to the tree contains a subsequence r_1, r_0, r_2 which satisfies $r_0 < r_2 < r_1$, and so it is not 312-avoiding. □

2.4. Noncrossing partitions

A *partition* $\pi = \{\pi_1, \pi_2, ..., \pi_k\}$ of $[n]$ is a collection of disjoint subsets $\pi_i \subseteq [n]$ such that $[n] = \cup_i \pi_i$. We call each π_i a *block* of π and denote by $|\pi|$ the number of blocks in π. We are also defining a standard order in which to read the blocks of a partition π. We read the parts in increasing order of its minimal elements, that is, $\min(\pi_i) < \min(\pi_j)$ if and only if $i < j$. We call this order the *minimal order* of the blocks of π.

We say that a partition π of $[n]$ is *noncrossing* if there are no $x < y < z < w \in [n]$ such that $x, z \in \pi_i$ and $y, w \in \pi_j$ with $i \neq j$. See Figure 14 for an example of noncrossing and crossing partitions of 8; note that in Figure 14 (B) the subsets $\{3, 5\}$ and $\{4, 6\}$ belong to different blocks. We denote by $\mathcal{NC}_n$ the set of noncrossing partitions of $[n]$. In addition, let π_i and π_j be two blocks of π, we say that $\pi(j)$ is *nested* inside π_i if there is a bipartition $\pi_i = B_1 \cup B_2$ with both B_1 and B_2 nonempty such that $x \leq y \leq z$ for all $x \in B_1$, $y \in \pi_j$ and $z \in B_2$.

Definition 2.11 (Noncrossing s-partitions)**.** Let $\pi = \{\pi_1, \pi_2, ..., \pi_k\}$ be a partition of $[n]$ where the blocks are ordered according its minimal order. We can associate to π the composition $\mu(\pi) := (|\pi_1|, |\pi_2|, ..., |\pi_k|)$. For $s \in \mathcal{C}omp$, we say that π is an *s-partition* if $s \leq \mu(\pi)$, that is, if s is a refinement of $\mu(\pi)$. We denote the set of noncrossing s-partitions by $\mathcal{NC}_s$.



(a) Noncrossing (b) Crossing

Figure 14: Example of noncrossing and crossing partitions of 8.

For example, $\mathcal{NC}_{(1^n)}$ is the set of noncrossing partitions and $\mathcal{NC}_{(k^n)}$ is the set of *k-divisible* noncrossing partitions of $[nk]$, that is, the set of noncrossing partitions of $[nk]$ whose blocksizes are all divisible by k. The noncrossing partition in Figure 3 is a $(2,3,3,1,4)$-partition, $(1,1,2,1,3,5)$-partition, (1^{13})-partition and so on; but, for example, is not a $(3,3,3,4)$-partition since $(5,3,5)$ is not refined by $(3,3,3,4)$.

2.4.1. Bijection between s-trees and noncrossing $(s-1)$-partitions Let $s \in \mathcal{C}omp$ be such that $s_i \geq 2$ for all i and recall that $s - \mathbf{1} := (s(1) - 1, s(2) - 1, \ldots, s(\ell(s)) - 1)$. We also define $\emptyset - \mathbf{1} := \emptyset$. We will define a bijection $\phi : \mathcal{T}_s \to \mathcal{NC}_{s-\mathbf{1}}$ by labeling the caverns of the tree in preorder and grouping the labels according to a certain rule.

Recall that given two nodes x and y of T, y is called a *descendant* of x if x belongs to the unique path from y to the root. A node y is called a *left descendant* of x if y is the minimal child of x or the minimal child of a left descendant of x. We denote by $\mathrm{left}(x)$ the set of left descendant internal nodes of T union with $\{x\}$.

Let $C_1, C_2, \ldots, C_{|s-\mathbf{1}|}$ be the preorder of the caverns of $T \in \mathcal{T}_s$. We define the function ϕ as follows: Let $v_1, \ldots, v_a$ be the internal nodes of T in preorder, and let $v_{i_1}, \ldots, v_{i_k}$ be the internal nodes that are not a minimal child of another internal node. In particular, $v_{i_1} = v_1$ is the root of the tree. Denote by

$$\pi_j := \bigcup_{v \in \mathrm{left}(v_{i_j})} \mathrm{caverns}(v),$$

where we are identifying a cavern C_i with its index i. The partition associated to T is defined

$$\phi(T) := \{\pi_1, \ldots, \pi_k\}.$$

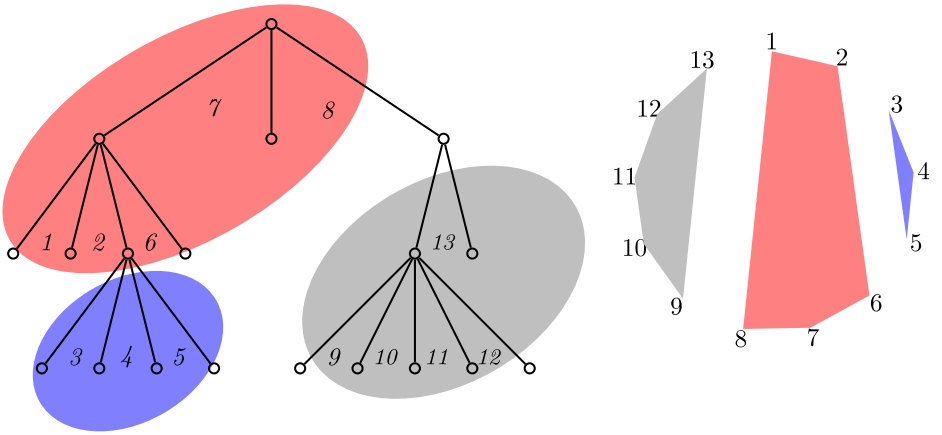


Figure 15: Example of the bijection between s-trees and noncrossing $(s-\mathbf{1})$-partitions. The caverns of the tree are ordered in preorder.

An example of this map is illustrated in Figure 15.

For simplicity, we denote by $\{B_1, \ldots, B_k\}$ the partition of $[a]$ determined by the indices $i_1, \ldots, i_k$ in the following way:

$$B_j = \begin{cases} [i_j, i_{j+1}) & \text{for} \quad 1 \leq j < a \\ [i_k, a] & \text{for} \quad j = k. \end{cases} \tag{2.3}$$

Lemma 2.12. *For $1 \leq j \leq k$, the internal nodes of* $\text{left}(v_{i_j})$ *are the nodes* $\{v_\ell\}_{\ell \in B_j}$.

Proof. Let $1 \leq j < k$. The set $\text{left}(v_{i_j})$ is obtained from v_{i_j} by consecutively adding in preorder left minimal children whenever possible. The process stops when following preorder we find the first internal node that is not a minimal child. Since this internal node is $v_{i_{j+1}}$ then the internal nodes of $\text{left}(v_{i_j})$ are exactly the nodes $\{v_\ell\}_{\ell \in [i_j, i_{j+1})}$. If $j = k$, the process does not stop until covering all remaining internal nodes of T, and so, the internal nodes of $\text{left}(v_{i_k})$ are the nodes $\{v_\ell\}_{\ell \in [i_k, a]}$. □

Lemma 2.13. *The partition $\phi(T)$ is a noncrossing $(s-\mathbf{1})$-partition.*

Proof. To see that $\phi(T)$ is noncrossing consider $1 \leq r < s \leq k$. Since v_{i_r} comes before v_{i_s} in preorder we have two possibilities, either v_{i_s} is a descendant of v_{i_r} or they are unrelated. And note that in these two cases preorder implies that π_s is either nested in π_r or that all the caverns in π_r come in preorder before the ones in π_s.

To see that $\phi(T)$ is an $(s-\mathbf{1})$-partition note that an internal node v_ℓ of T has exactly $s(\ell)-1$ caverns. Lemma 2.12 then implies that $|\pi_j| = \sum_{\ell\in B_j}(s(\ell)-1)$ for $1 \le j \le k$. Since the sets B_j form a partition of $[a]$ with increasing adjacent consecutive parts, then $s-\mathbf{1}$ is a refinement of $\mu(\pi)$. □

Given a noncrossing $(s-\mathbf{1})$-partition $\pi = \{\pi_1 < \pi_2 < \cdots < \pi_k\}$ we will construct a tree $T = \psi(\pi)$ such that $\phi(T) = \pi$. Since π is an $(s-\mathbf{1})$-partition we have that $(|\pi_1|, |\pi_2|, \ldots |\pi_k|)$ is refined by $(s-\mathbf{1})$ meaning that s can be written as $s = \tilde{s}_1 \oplus \tilde{s}_2 \oplus \cdots \tilde{s}_k$ with $|\pi_j| = |\tilde{s}_j - \mathbf{1}|$.

We say that a tree is a *left descendant tree* if every internal node is either the root or the minimal (leftmost) child of another internal node. For $1 \le j \le k$, let τ_j be the (unique) left descendant tree with signature $\tilde{s}_j$, and label the caverns of τ_j in preorder with the elements of π_j (note this is possible since the number of caverns of τ_j is exactly equal to $|\pi_j|$).

Lemma 2.14. *Let $\{\tau_1, \ldots, \tau_k\}$ be a set of left descendant trees such that the caverns of tree τ_j have been labeled in preorder with the set π_j where $\pi = \{\pi_1 < \pi_2 < \cdots < \pi_k\}$ is a noncrossing $(s-\mathbf{1})$-partition of $[n]$. Then there is a unique way to glue together the trees τ_j such that they form a planar rooted tree T satisfying:*

1. *No pair of trees τ_i and τ_j together form a larger left descendant subtree in T.*
2. *The caverns of T are labeled in preorder.*
3. $\mathrm{signat}(T) = s$.

Proof. By condition (1), we cannot glue two left descendant trees, say τ_i and τ_j with $i < j$, by attaching the root of τ_i to the left-most leaf of τ_j since they will form a larger left descendant tree. Now condition (2) and the fact of the partition π is noncrossing implies that for $i < j$ either τ_j is nested in τ_i or all labels of τ_j are larger than all the labels of τ_i. So given tree $T_1 = \tau_1$ (whose $\min \pi_1 = 1$) there is a unique way of gluing τ_2 to obtain T_2, its root needs to be glued in preorder to the leaf of τ_1 immediately after the cavern labeled $\min \pi_2 - 1$ (if $\min \pi_2 - 1$ were not a cavern of τ_1 then π would not be a partition of $[n]$) we repeat this process inductively by gluing τ_j to T_{j-1} to obtain T_j, its root glued in preorder to the leaf of T_{j-1} immediately after the cavern labeled $\min \pi_j - 1$. This construction preserves the preorder labeling and always glues τ_j to a leaf of T_{j-1} that is not the leftmost leaf of another tree τ_i. Now note that $\mathrm{signat}(\tau_1) = \tilde{s}_1$ and since $T_1 = \tau_1$ is a left descendant tree then when attaching τ_2 to form T_2 all the internal nodes of T_1 come in preorder before the internal nodes of τ_2 hence $\mathrm{signat}(T_2) = \tilde{s}_1 \oplus \tilde{s}_2$. In general when attaching τ_j to T_{j-1} to obtain T_j, all the internal nodes of τ_j come

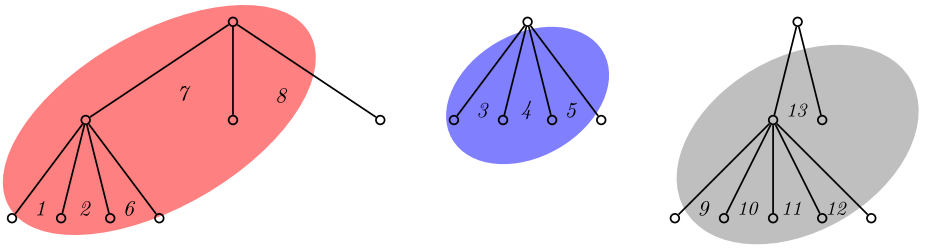


Figure 16: The recursive construction of the tree $T = \psi(\pi)$ for the noncrossing $(s - \mathbf{1})$- partition $\pi = \{\{1, 2, 6, 7, 8\}, \{3, 4, 5\}, \{9, 10, 11, 12, 13\}\}$ with $s = (3, 4, 4, 2, 5)$.

last in preorder and hence $\text{signat}(T_j) = \text{signat}(T_{j-1}) \oplus \tilde{s}_j$, this inductively implies condition (3). □

Given the conditions of Lemma 2.14 where the labels of the caverns in tree τ_j are the elements of π_j, we define $T := \psi(\pi)$ where T is the unique tree that the lemma predicts.

Theorem 2.15. *Let $s \in \mathcal{C}omp$ be such that $s_i \geq 2$ for all i. The map $\phi : \mathcal{T}_s \to \mathcal{NC}_{s-\mathbf{1}}$ is a bijection between s-trees and noncrossing $(s - \mathbf{1})$-partitions. The inverse of ϕ is ψ.*

Proof. If $\pi = \phi(T)$ then by the definition of ϕ every block $\pi_j \in \pi$ gets its labels from the caverns of a maximal left descendant subtree τ_j in T. If τ_j has $\text{signat}(\tau_j) = \tilde{s}_j$ then $s = \tilde{s}_1 \oplus \tilde{s}_2 \oplus \cdots \tilde{s}_k$ since all of these left descendant subtrees are consecutive in preorder by Lemma 2.12. But with these conditions together $\tilde{s}_j$ and π_j recover τ_j uniquely and Lemma 2.14 says that there is a unique way to glue the τ_j together so $\psi(\pi) = T$. Similarly, if $T = \psi(\pi)$ then by construction a maximal left descendant subtree of T have its caverns labeled by a block of π and by Lemma 2.14 the labels of the caverns of T are in preorder. Since each block of $\phi(T)$ is determined by the set of labels in the caverns of a maximal left descendant subtree when the caverns of T have been labeled in preorder then $\phi(T) = \pi$. □

2.5. Noncrossing matchings

A *complete matching* in the *complete graph* K_{2n} (the graph (V, E) with vertex set $V = [2n]$ and with edge set $E = \{S \subseteq [2n] \mid |S| = 2\}$) is a partition of the set $[2n]$ in blocks of cardinality 2. We say that the matching is *noncrossing* if this partition is *noncrossing*. It is known that the set of complete noncrossing matchings in K_{2n} is a Catalan family (see for example

[46]). For any $n \geq 1$ the *complete hypergraph* $\mathcal{K}_n$ on n vertices is the pair (V, E) where $V = [n]$ is the set of *vertices* and $E = \{S \subseteq [2n] \mid S \neq \emptyset\}$ the set of *edges* (Note that in this definition we consider vertices as edges of cardinality one).

Definition 2.16 (Complete noncrossing s-matchings)**.** For $s \in \mathcal{C}omp$, a *complete noncrossing s-matching* in $\mathcal{K}_{|s|}$ is a noncrossing partition M of the set $[|s|]$ such that M satisfies $|M_i| = s(i)$ when we order the blocks of $M = \{M_1 < M_2 < \cdots < M_a\}$ in the minimal order (such that $\min M_i < \min M_j$ whenever $i < j$). We denote by $\mathcal{CM}_s$ the set of complete noncrossing s-matchings.

Figure 3 illustrates an example of a complete noncrossing $(3, 4, 4, 2, 5)$-matching.

Remark 2.17. Note that for $s \in \mathcal{C}omp$ both, $(s - \mathbf{1})$-partitions and complete s-matchings are partitions of different sets ($[|s| - \ell(s) + 1]$ and $[|s|]$ respectively). We use the name of complete s-matchings to denote the later type of partitions because they provide with a suitable generalization of the concept of complete matching in K_{2n}. Also note that a given partition can be a ν-paritition for different values of ν however a complete matching M in $\mathcal{K}_{|s|}$ can only be a complete s-matching for the particular signature $s = (|M_1|, |M_2|, \ldots, |M_a|)$.

2.5.1. Bijection between s-trees and noncrossing s-matchings For $T \in \mathcal{T}_s$ let $v_0, v_1, v_2, \ldots, v_{|s|}$ be the listing of the nodes of T in preorder (where v_0 is the root of T) and let $v_{i_1}, v_{i_2}, \ldots, v_{i_a}$ be the sublist of internal nodes in preorder. For each internal node x let $\mathrm{child}(x)$ be the set of children of x. The matching $\varphi(T) := \{M_1, M_2, \cdots, M_a\}$ is defined as the partition with blocks $M_k = \{j \mid v_j \in \mathrm{child}(v_{i_k})\}$. See Figure 17 for an example of the map φ.

Lemma 2.18. *Let v_{i_k} be the k-th internal node of T in preorder then*

$$\min M_k = i_k + 1.$$

Consequently, $M_1, M_2, \ldots, M_a$ is the minimal order listing of $\varphi(T)$.

Proof. An internal node and its leftmost child are consecutive in preorder, then $\min M(k) = \min\{j \mid v_j \in \mathrm{child}(v_{i_k})\} = i_k + 1$. □

Lemma 2.19. *The partition $\varphi(T)$ is a complete noncrossing s-matching.*

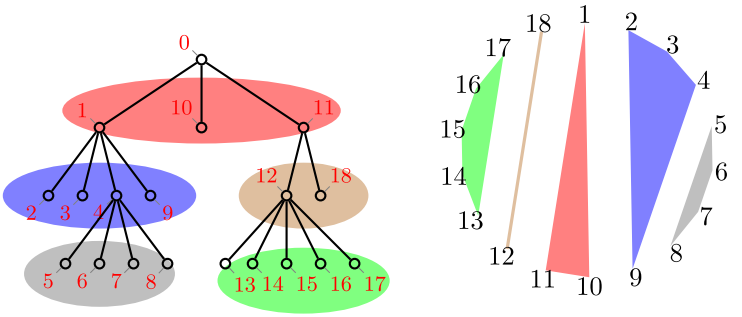


Figure 17: Example of the bijection between s-trees and noncrossing s-matchings. The nodes of the tree are ordered in preorder.

Proof. $\varphi(T)$ **is noncrossing:** Two different blocks of $\varphi(T)$ come from two different internal nodes v_{i_k} and v_{i_l}, say with $v_{i_k} < v_{i_l}$ in preorder. If v_{i_k} and v_{i_l} are unrelated all the nodes in child(v_{i_k}) occur before the nodes in child(v_{i_l}) in preorder. If v_{i_k} is an ancestor of v_{i_l} then all the nodes in child(v_{i_l}) are also descendant of exactly one child of v_{i_k}, hence they occur in between two preorder consecutive elements of child(v_{i_k}) and M_l is nested in M_k.

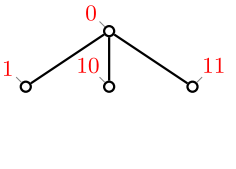
$\varphi(T)$ **is a complete s-matching:** By Lemma 2.18 $M_1, M_2 \ldots, M_a$ is the minimal order listing of $\varphi(T)$ and $|M_k| = \deg(v_{i_k}) = s(k)$ then

$$(|M_1|, |M_2| \ldots, |M_a|) = (s(1), \ldots, s(a)) = s.$$

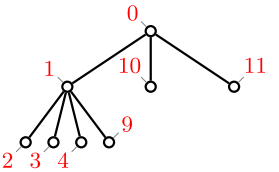
□

The inverse map $\gamma : \mathcal{CM}_s \to \mathcal{T}_s$ is defined as follows. Consider a complete noncrossing s-matching $M = \{M_1, M_2, \ldots, M_a\} \in \mathcal{CM}_s$ whose blocks are ordered according to the minimal order and recall that by definition $|M_i| = s(i)$. Define the tree $\gamma(M)$ as follows (see Figure 18 for an example):

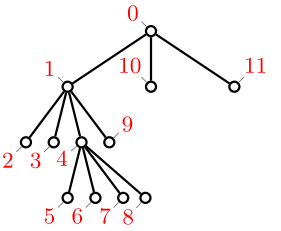
1. First, start with the root $r = v_0$ with $|M_1|$ children with increasing left-to-right labels $\{v_j \mid j \in M_1\}$ and call the resulting tree τ_1.
2. Attach an internal node to τ_1 in $v_{\min M_2 - 1}$ with $s(2)$ children with increasing left-to-right labels $\{v_j \mid j \in M_2\}$ and call the resulting tree τ_2.
3. In general, at the jth step of the process we have a tree τ_{j-1} with $j-1$ internal nodes $v_{i_1}, \ldots, v_{i_j}$ ordered in preorder. The jth tree τ_j is obtained from τ_{j-1} by attaching an internal node in $v_{\min M_j - 1}$ with $s(j)$ children with increasing left-to-right labels $\{v_j \mid j \in M_j\}$.
4. The process finishes after attaching the a internal nodes $v_{i_1}, \ldots, v_{i_a}$.



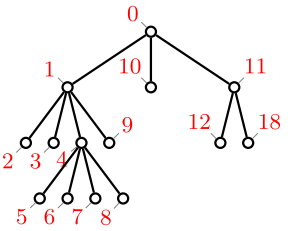
$M = \{\{1, 10, 11\}\}$



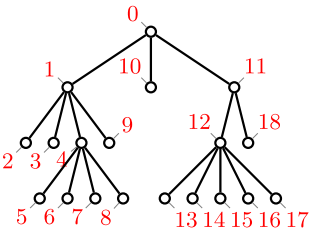
$M = \{\{1, 10, 11\}, \{2, 3, 4, 9\}\}$



$M = \{\{1, 10, 11\}, \{2, 3, 4, 9\}, \{5, 6, 7, 8\}\}$



$M = \{\{1, 10, 11\}, \{2, 3, 4, 9\}, \{5, 6, 7, 8\}, \{12, 18\}\}$



$M = \{\{1, 10, 11\}, \{2, 3, 4, 9\}, \{5, 6, 7, 8\}, \{12, 18\}, \{13, 14, 15, 16, 17\}\}$

Figure 18: The recursive construction of the tree $\gamma(M)$ for the complete noncrossing $(3, 4, 4, 2, 5)$-matching $M = \{\{1, 10, 11\}, \{2, 3, 4, 9\}, \{5, 6, 7, 8\}, \{12, 18\}, \{13, 14, 15, 16, 17\}\}$.

We will use the following simple lemma.

Lemma 2.20. *If $\{M_1, M_2, \dots M_k\}$ is a noncrossing partition whose blocks are ordered according to the minimal order, then M_k and $\bigcup_{i=1}^{k-1} M_i$ are also noncrossing.*

Lemma 2.21. *$\gamma(M)$ is an s-tree with all its nodes labeled in preorder.*

Proof. To see that the map is well-defined note that in the step j the node $v_{\min M_j - 1}$ has already been attached to the tree τ_{j-1} otherwise there is a block M_k with $\min M_j - 1 \in M_k$ and $k > j$ implying $\min M_k < \min M_i$ contradicting the minimal ordering of M.

In the step j we are attaching leaves with the labels in M_j. Since M is noncrossing by Lemma 2.20 M_j and $\bigcup_{i=1}^{j-1} M_i$ are noncrossing implying that the labels of M_j are either nested or are all greater than the labels in $\bigcup_{i=1}^{j-1} M_i$. This implies that when we attach the vertices $\{v_i \mid i \in M_j\}$ as children of $v_{\min M_j - 1}$ in τ_j, the labels of the nodes in τ_j are still consistent with preorder. □

Theorem 2.22. *The map $\varphi : \mathcal{T}_s \to \mathcal{CM}_s$ is a bijection between s-trees and complete noncrossing s-matchings. The inverse of φ is γ.*

Proof. After labeling the nodes of T in preorder. Lemma 2.18 and the constructive definition of γ imply that the labels of the internal nodes in T and $\gamma(\varphi(T))$ are the same. Also, the constructive definition of both φ and γ imply that for each internal node v_{i_k} the labeled set of children in T and $\gamma(\varphi(T))$ are equal. These two conditions determine T completely, therefore $T = \gamma(\varphi(T))$.

Similarly, in the construction of the tree $\gamma(M)$ the labels of a block M_k are the children of an internal node. Since every block of $\varphi(T)$ is the set of labels of the children of an internal node then we have that $\varphi(\gamma(M)) = M$. □

3. The fundamental recurrence

Consider a signature $s = (s(1), s(2), \ldots, s(k))$. Trees have an inherent recursive nature, in the case of s-trees the recursion is very natural: Any s-tree T can be constructed as $T = [T_1, T_2, ..., T_{s(1)}]$ for $s(1)$ suitable trees $T_1, T_2, \ldots, T_{s(1)}$ satisfying

$$\text{signat}(T) = (s(1)) \oplus \text{signat}(T_1) \oplus \cdots \oplus \text{signat}(T_{s(1)}).$$

From this we can obtain a recursion for the number C_s of s-Catalan structures:

$$C_s = \sum_{(s)} C_{s_1} C_{s_2} \cdots C_{s_{s(1)}}, \tag{3.1}$$

where the sum is over all sequences $[s_1, s_2, \ldots, s_{s(1)}]$ of compositions such that $s = (s(1)) \oplus s_1 \oplus s_2 \oplus \cdots \oplus s_{s(1)}$ and where $C_\emptyset = 1$.

Recursion (3.1) is at the heart of s-Catalan combinatorics and it is a generalization of the classical recursion for the number C_n of Catalan objects. By letting $C_n := C_{(2^n)}$ and $C_0 = 1$ in (3.1) we obtain the classical recursion:

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}. \tag{3.2}$$

3.1. Catalan decompositions

For a family $\mathcal{A}_s$ of s-Catalan objects recursion (3.1) can be realized by obtaining a rule of decomposition of an s-object $A = [A_1, A_2, \ldots, A_{s(1)}]$

into $s(1)$ objects $A_1 \in \mathcal{A}_{s_1}$, $A \in \mathcal{A}_{s_2}, \ldots, A_{s(1)} \in \mathcal{A}_{s_{s(1)}}$ such that $s = (s(1)) \oplus s_1 \oplus \cdots \oplus s_{s(1)}$. We call such a rule a *Catalan decomposition*.

Whenever we have a pair of families of objects $\mathcal{A}_n$ and $\mathcal{B}_n$ with $\mathcal{A}_\emptyset = \{A_\emptyset\}$ and $\mathcal{B}_\emptyset = \{B_\emptyset\}$ both with known Catalan decomposition rules. Then a family of bijections $\xi_s : \mathcal{A}_s \to \mathcal{B}_s$ are easily defined recursively by:

(ξ1) $\xi_\emptyset(A_\emptyset) = B_\emptyset$.
(ξ2) Otherwise, if $A = [A_1, A_2, \cdots, A_k]$ with $A_i \in \mathcal{A}_{s_i}$ then

$$\xi_s(A) := [\xi_{s_1}(A_1), \xi_{s_2}(A_2), \ldots, \xi_{s_k}(A_k)].$$

Such a family of bijections sometimes can be described in a nicer and direct way without using recursion. We illustrate in Section 3.3 and Section 3.4 the use of the recursion (3.1) with the examples of angulations of a convex polygon and parenthesizations of a word.

On the other hand, if there is a family of bijections $\xi_s : \mathcal{A}_s \to \mathcal{B}_s$ where the Catalan decomposition rule for the objects in $\mathcal{A}_s$ is known, then we might be able to transport this rule to the objects in $\mathcal{B}_s$ using the maps ξ_s. This idea can be used to describe Catalan decompositions for other families of s-Catalan objects, like s-Dyck paths, 312-avoiding Stirling $(s - \mathbf{1})$-permutations, noncrossing s-partitions, and complete noncrossing s-matchings.

3.2. Catalan decomposition for Dyck paths

The Catalan decomposition of an s-Dyck path D is relatively simple. We remove the first north step, which corresponds to the root of the corresponding tree and we label the lattice points of the resulting path $\bar{D}$ (except the last one) by their horizontal distance to the Ribbon shape determined by s. That is, a lattice point p is labeled by the maximal number of east steps that can be added to p without crossing the Ribbon shape.[2] For the example in Figure 19 we get the sequence

$$\mathbf{2}, 5, 4, 3, 2, 5, 4, 3, 2, \mathbf{1}, \mathbf{0}, 1, 5, 4, 3, 2, 1, 0.$$

We cut the path $\bar{D}$ at the first appearances of $s(1) - 1, s(1) - 2, \ldots, 1, 0$ (marked in red in the sequence and highlighted in Figure 19). The resulting sequence $(D_1, D_2, \cdots, D_{s(1)})$ of Dyck paths (some of them possibly equal to

[2]We are adding an extra "imaginary" box on top of the top-right box of the Ribbon shape to make this definition work for points on top of the grid.

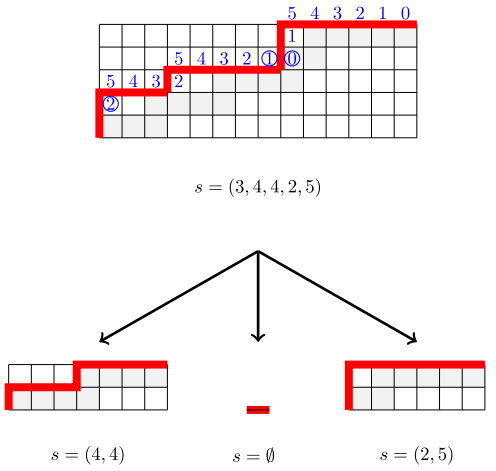


Figure 19: Example of the Catalan decomposition of an s-Dyck path.

the identity Dyck path E) form the Catalan decomposition of D, where the signature of D_i is defined as the sub-sequence of s determined by the rows of the north steps in D_i. Their composition is defined as

$$[D_1, D_2, \ldots, D_k] := ND_1D_2 \cdots D_k.$$

The validity of this decomposition follows from the correspondence between s-trees and s-Dyck paths, and the area labeling in Section 2.2.1.

3.3. Angulations of a polygon

For $n \in \mathbb{N}$ we denote by $P(n+2)$ a convex polygon with $n+2$ vertices. A *diagonal* of $P(n+2)$ is a straight line joining any two nonadjacent vertices of the polygon. An *angulation* of $P(n+2)$ is a partition of the interior of $P(n+2)$ into smaller convex polygons using a set of noncrossing diagonals. See Figure 3 for an example of an angulation of $P(15)$ into 5 parts.

Label all the vertices of $P(n+2)$ clockwise let e be the edge between the vertices 1 and 2 and let A be an angulation of $P(n+2)$. It is enough to remove e to reveal the Catalan decomposition of A into angulations of smaller polygons. An edge is considered as the polygon $P(2)$ with a unique triangulation. Then the set $\mathcal{AP}$ of angulations of polygons has the Catalan recursive structure. It is then a simple task to define the signature of an angulation A. Indeed, if removing e gives angulations $A_1, A_2, \ldots, A_k$ traveling

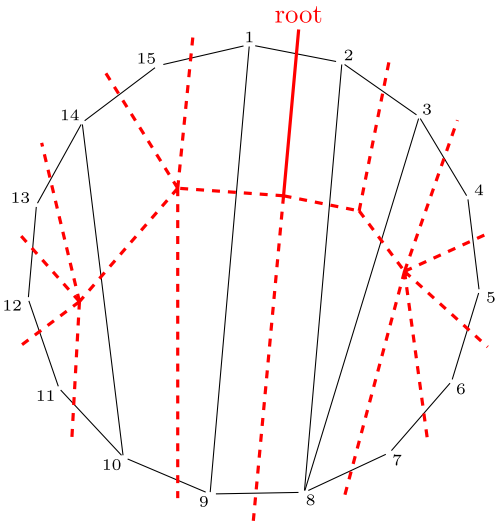


Figure 20: Example of the bijection between s-trees and s-angulations.

counter-clockwise then

$$\text{signat}(A) := (k) \oplus \text{signat}(A_1) \oplus \text{signat}(A_2) \oplus \cdots \oplus \text{signat}(A_k),$$

where the signature of a single edge is signat(▬) $= \emptyset$. We denote by $\mathcal{AP}_s$ the set of s-angulations of $P(|s| - \ell(s) + 2)$, that is the set of angulations A with signat$(A) = s$.

Using this recursive construction we can easily find a bijection between the s-trees and s-angulations of $P(|s|-\ell(s)+2)$ using the strategy described above. We leave the proof to the interested reader, but illustrate in Figure 20 an example of this bijection when $s = (3, 4, 4, 2, 5)$ and the tree T is the one in our example of Figure 3.

3.4. Parenthesizations

Starting with a word w with b letters consider the word w' of length $2a + b$ obtained by inserting a left parenthesis and a right parenthesis. For example, let $a = 3$ and $b = 5$, so for the word $w = *****$ we can insert left and right parenthesis in any way, say $w'_1 =) * ((*) * (**)$ or $w'_2 = (*((**)*)*)$. A *(proper) parenthesization* of a word w with b letters is the insertion of the same number a of left and right parenthesis with the conditions that for any $i \in [2a + b]$ the prefix $w(1)w(2)\cdots w(i)$ contains at least as many

left parenthesis as right parenthesis and that there cannot be an i such that $w(i)w(i+1) = ()$.

The word w'_1 above is not a good parenthesization since the prefix of length 1 already has more left parenthesis than right parenthesis. However, the reader can easily check that the word w'_2 is an example of a good parenthesization. Another example of a good parenthesization is the word

$$w = (**(****)*)*((*****)*). \tag{3.3}$$

A *segment* u of a word $w = w(1)w(2)\cdots w(k)$ is a subword of w of the form $u = w(i+1)w(i+2)\cdots w(i+\ell)$, i.e., all the letters of u are adjacent in w. A *block* in a properly parenthesized word w is a properly parenthesized segment u that is inclusion maximal with respect to the property that any proper prefix in u of the form $u(1)u(2)\cdots u(r)$ with $r < \ell$ contains strictly more left parenthesis than right parenthesis. For the example (3.3) there are three blocks, namely $(**(****)*)$, $*$ and $((*****)*)$.

It is easy to see out of the definition above that every properly parenthesized word factors into blocks in the form $w = B_1B_2\cdots B_k$ (see example (3.3)) and that there is a unique parenthesization of the word $*$ with $b = 1$ and $a = 0$. So parenthesizations satisfy the recursion (3.1) and are then Catalan objects. Define $\mathrm{signat}(*) = \emptyset$ and for the parenthesized word $w = B_1B_2\cdots B_k$ define its *signature* $\mathrm{signat}(w)$ recursively as

$$\mathrm{signat}(w) = (k) \oplus \mathrm{signat}(B_1) \oplus \cdots \oplus \mathrm{signat}(B_k).$$

For w of example (3.3) we have that $\mathrm{signat}(w) = (3,4,4,2,5)$. For $s \in \mathcal{C}omp$, an *s-parenthesization* is a properly parenthesized word w such that $\mathrm{signat}(w) = s$. We denote $\mathcal{PT}_s$ the set of s-Parenthesizations of a word of length $|s| - \ell(s) + 1$.

As in the case of s-angulations, s-trees can be recursively mapped bijectively to s-parenthesizations using their recursive constructions. Again, we leave the proof to the interested reader and illustrate the bijection in Figure 21 when $s = (3,4,4,2,5)$ and the tree T is the one in our running example of Figure 3. Note from the figure that at the end of the process we always omit the outermost parenthesis since it is redundant.

4. Enumeration

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell)$ be a partition. We say that a partition $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_\ell)$ *fits* λ if $\lambda'_i \leq \lambda_i$ for all i.

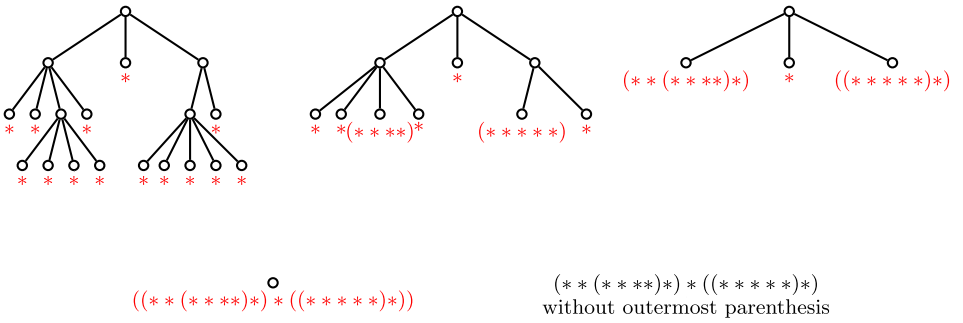


Figure 21: Example of the bijection between s-trees and s-parenthesizations.

Theorem 4.1 (Kreweras [27])**.** *The number $P(\lambda)$ of partitions fitting a partition λ is counted by the determinant formula*

$$P(\lambda) = \det\left(\begin{matrix}\lambda_j + 1 \\ j - i + 1\end{matrix}\right).$$

In particular for staircase partitions, the numbers $P(\lambda)$ recover the classical Catalan numbers.

4.1. s-Catalan numbers

Note that every composition $s = (s_1, \ldots, s_a)$ defines a partition $\lambda^s = (\lambda^s_1 \geq \cdots \geq \lambda^s_{a-1})$ such that $\lambda^s_j = \sum_{i=1}^{a-j}(s_i - 1)$, that is basically the shape of the upper shape that is left when in a $(|s| - 1) \times a$ grid we remove the central ribbon determined by s. In the example of Figure 3 we have that $\lambda^{(3,4,4,2,5)} = (9, 8, 5, 2)$. It is not hard to see that in the same manner as the ribbon shape determined by s describes a partition λ^s, every s-Dyck path describes a partition $\lambda' \leq \lambda^s$. Theorem 4.1 gives then a determinantal formula for the number of s-Dyck paths.

Corollary 4.2 (Kreweras)**.** *The s-Catalan number is counted by the determinant formula*

$$|\mathcal{C}_s| = \det\left(\begin{matrix}\sum_{i=1}^{a-j}(s_i - 1) + 1 \\ j - i + 1\end{matrix}\right).$$

4.2. s-Narayana numbers

The Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ can be refined by the Narayana numbers $N(n, k)$. These numbers are indexed by an extra parameter $1 \leq k \leq n$, can

be explicitly computed by the simple formula

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1},$$

and satisfy

$$C_n = N(n,1) + N(n,2) + \cdots + N(n,n).$$

The Narayana numbers count Catalan objects that satisfy certain property which depends on the family that they belong to. For instance, $N(n,k)$ is the number of (see [38, Chapter 2] for further information):

1. planar rooted binary trees with n internal nodes and k most left leaves;
2. Dyck paths in an $n \times n$ grid with k peaks;
3. 312-avoiding permutations of $[n]$ with $k-1$ ascents;
4. noncrossing partitions of $[n]$ with k blocks;
5. noncrossing matchings of $[2n]$ containing k matched pairs of the form $(i, i+1)$.

All these combinatorial interpretations can be naturally extended to the generalized s-Catalan families. The *s-Narayana number* is defined as the number of elements in the following families. We remark that they correspond to each other under the bijections presented in Section 2 and Appendix A.

1. s-trees with k most left leaves;
2. s-Dyck paths with k peaks;
3. 312-avoiding Stirling $(s-\mathbf{1})$-permutations with $k-1$ ascents;
4. noncrossing $(s-\mathbf{1})$-partitions with k blocks;
5. Complete noncrossing s-matchings with k parts satisfying $\min M_i + 1 \in M_i$.

As far as we know there are no simple formulas to count these combinatorial objects in this general setting. However, refinements of the s-Narayana numbers have already been studied in the literature, for instance in the case of Fuss-Narayana numbers and rational Narayana numbers [3, 29, 30].

5. Signature generalizations of permutations and parking functions

5.1. s-generalized permutations

As mentioned in the introduction, permutations of $[n]$ are in bijection with increasingly labeled planar rooted binary trees with n internal nodes. A natural generalization of permutations in terms of signatures is the collection

of permutations corresponding to $(s + \mathbf{1})$-increasing trees. As explained in Section 2.3.1, such permutations are in correspondence with Stirling s-permutations, which were already considered and studied by Gessel and Stanley [21]. As described above, these are multipermutations of the multiset $\{1^{s(1)}, 2^{s(s(2))}, \ldots, a^{s(a)}\}$ that avoid the pattern 212; see [28, 24, 36, 35, 34, 22, 17, 42] for further studies on such permutations.

The cardinality of the set $\mathcal{SP}_s$ of Stirling s-permutations generalizes the factorial numbers. To be more precise, from a Stirling permutation in $\mathcal{SP}_{(s(1),s(2),\ldots,s(k-1))}$ we can obtain a permutation in $\mathcal{SP}_{s\oplus(s(k))}$ by inserting the $s(k)$ consecutive occurrence of the label $kk\cdots k$ in any of the $s(1)+s(2)+\cdots+s(k-1)+1$ possible positions. Inductively and starting in $\mathcal{SP}_{(s(1))} = \{11\cdots1\}$ this implies that

$$|\mathcal{SP}_s| = 1\cdot(s(1)+1)\cdot(s(1)+s(2)+1)\cdot\cdots\cdot(s(1)+s(2)+\cdots+s(a-1)+1) =: |s|!^s,$$

and we read $|s|!^s$ as $|s|$ *s-factorial*. Note that if $s = (1^n)$ then $|s|!^s = n!$ and if $s = (2^n)$ then $|s|!^s = 1\cdot3\cdots(2n-1) = (2n-1)!!$, the factorial and double-factorial numbers.

In forthcoming work [15], the first author and Pons study a generalization of the weak order on permutations for the set of Stirling s-permutations. They call this generalized order the *s-weak order* and show that it has the structure of a lattice. They also investigate geometric realizations of the s-weak order in terms of polytopal subdivisions of the classical permutahedron.

5.2. *s*-generalized parking functions

A *parking function* of n is a sequence of nonnegative integers $(p_1, p_2, \ldots, p_n)$ with the property that after being rearranged in weakly increasing order $p_{j_1} \leq p_{j_2} \leq \cdots \leq p_{j_n}$ they satisfy $p_{j_i} < i$ for all $i \in [n]$. We denote by $\mathcal{PF}_n$ the set of parking functions of n. Parking functions were studied by Konheim and Weiss [25] were they used the following equivalent description to define them:

Let $C_1, \ldots, C_n$ be a collection of cars to be parked in order, car C_i before car C_{i+1}, in a one-way street in a consecutive sequence of parking spots marked $P_0, \ldots, P_{n-1}$. The driver of car C_i has a preferred spot p_i where he would like to park. In step i the driver of car C_i drives up to the spot p_i, parking there if the spot is available, or otherwise continuing until the next available spot. If the spot p_i and all the following parking spots are occupied then C_i is unable to park. A *parking function* is a sequence $(p_1, p_2, \ldots, p_n)$ of parking preferences where all the cars are able to park.

It is not difficult to verify that the two definitions of parking function stated above are equivalent. There is another way to construct parking func-

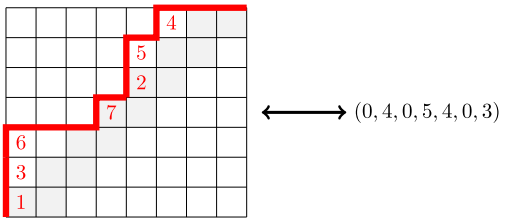


Figure 22: Example of a decorated Dyck path and its corresponding parking function.

tions. We can obtain parking functions by decorating Dyck paths with the elements of $[n]$.

A *decorated Dyck path* is a Dyck path in which the north steps have been decorated from left to right with a permutation of $[n]$ in such a way that consecutive north steps have increasing labels (see Figure 22). Let $\widehat{\mathcal{DP}}_n$ denote the set of decorated Dyck paths of $[n]$. In Figure 1 the reader can see the example of $\widehat{\mathcal{DP}}_3$.

It turns out that the sets $\mathcal{PF}_n$ and $\widehat{\mathcal{DP}}_n$ are in bijection and that decorated Dyck paths are a convenient way to encode parking functions. The bijection we will describe is common in the literature, see for example [19]. The function $P : \widehat{\mathcal{DP}}_n \to \mathcal{PF}_n$ that provides a parking function $P(D)$ starting with a decorated Dyck path D is defined as follows: the value of p_i is the number of east steps that are in D to the left of the north step decorated with the letter i. The function P is bijective and its inverse is also easy to describe. In the example of Figure 22 we have, for example, that $p_1 = 0$ because there are no east steps before the north step labeled 1 and $p_4 = 5$ since there are 5 east step before the north step labeled 4.

Perhaps the easiest way to generalize the concept of parking function is using the concept of a decorated Dyck path. For a composition s let $\widehat{\mathcal{DP}}_s$ be the set of s-Dyck paths whose north steps have been decorated with a permutation of $[a]$ in such a way that consecutive north steps have increasing labels. The discussion above illustrates that the set $\widehat{\mathcal{DP}}_s$ is also in bijection with a more general set of parking functions. For a weak composition μ a *μ-parking function* is a sequence of nonnegative integers $(p_1, p_2, \cdots, p_a)$ such that after being rearranged in weakly increasing order $p_{j_1} \leq p_{j_2} \leq \cdots \leq p_{j_a}$ they satisfy $p_{j_1} = 0$ and

$$p_{j_i} \leq \sum_{i=1}^{i-1} \mu(i) \text{ for all } i > 1.$$

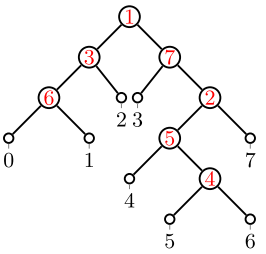


Figure 23: Example of the decorated s-tree corresponding to the parking function of Figure 22.

We denote by $\mathcal{PF}_\mu$ the set of μ-parking functions. By the same reasoning as above we have that there is a bijection $\widehat{\mathcal{DP}}_s \to \mathcal{PF}_{s-\mathbf{1}}$. μ-parking functions have already appear in the literature with the name of generalized parking functions, vector parking functions or μ-parking functions. They were originally studied by Pitman and Stanley [47] and Yan [50, 49] in a slightly more general form. This more general form can be obtained within this framework if we allow 0 values in our signatures and what we call a μ-parking function is what appears as a $(1)\oplus\mu$-vector parking function in the literature. The case $s-\mathbf{1}=(\mathbf{1^n})$ recovers the classical concept of parking functions counted by $(n+1)^{n-1}$ and in general if s is a rational signature for (a,b) relatively prime then Armstrong, Loehr and Warrington [2] showed that $|\mathcal{PF}_{a,b}|=b^{a-1}$, this includes the case $a=n$ and $b=kn+1$ or $s-\mathbf{1}=(\mathbf{k^n})$ of Fuss-Catalan parking functions. The rational signatures are the cases where a nice product formula is known and to find formulas for general signatures seems to be a difficult task. In [20] Gaydarov and Hopkins generalize the work in [37] to provide a bijection between vector parking functions and a set of trees that they call rooted plane trees.

We finish this section with the remark that even though in the literature it is most common to represent parking functions as decorated Dyck paths, we argue that is natural and probably more convenient from the recursive point of view to use the perspective of s-trees to represent a parking function. A *decorated s-tree* is an s-tree such that all the internal nodes are decorated with distinct elements of $[a]$ in such a way that an internal node that is the left-most child of its parent has a larger label. Let $\widehat{\mathcal{T}}_s$ be the set of decorated s-trees, then is clear from Theorem 2.6 that we have a bijection $\widehat{\mathcal{T}}_s \cong \widehat{\mathcal{DP}}_s \cong \mathcal{PF}_{s-\mathbf{1}}$. In Figure 23 we illustrate an example of a decorated s-tree corresponding to the (1^7)-parking function $(0,4,0,5,4,0,3)$. Note that the associated parking function can be obtained

by labeling the leaves left-to-right from 0 to $|s - \mathbf{1}| + 1$, then p_i is equal to the label of the leaf that is the leftmost descendant of the internal node decorated i.

6. Relation to the rational constructions of Armstrong, Rhoades and Williams

The constructions in this article generalize the rational Dyck paths of Armstrong, Rhodes and Williams [3] to more general signatures s. A natural question that arises is whether our constructions for noncrossing s-partitions, complete noncrossing s-matchings and s-angulations of a $(|s| - \ell(s) + 2)$-gon specialize to their rational constructions for the same type of objects in [3] when s is a rational signature. The answer to this question happens to be on the negative side. Even though in some examples the two constructions coincide they are in general different. In this section, we discuss the difference between the constructions and, in particular, we illustrate the situation in the case of rational noncrossing partitions.

Let (a, b) be a pair of relatively prime positive numbers and s the corresponding rational signature defined as in Equation 2.1. In [3], the authors define a family of inhomogeneous (a, b)-noncrossing partitions (that we will denote here by $\widetilde{\mathcal{NC}}_{(a,b)}$), a family of homogeneous (a, b)-noncrossing partitions (denoted here by $\widetilde{\mathcal{CM}}_{(a,b)}$) and a family of rational dissections of a polygon (denoted here by $\widetilde{\mathcal{AP}}_{(a,b)}$). The constructions of $\widetilde{\mathcal{NC}}_{(a,b)}$, $\widetilde{\mathcal{CM}}_{(a,b)}$ and $\widetilde{\mathcal{AP}}_{(a,b)}$ rely on a "laser" construction. A *laser* through the point (x_0, y_0) is the half-ray in the grid $b \times a$ with starting point (x_0, y_0) and slope $\frac{a}{b}$, i.e., a half-ray given by the equation $y = y_0 + \frac{a}{b}(x - x_0)$ with $x \geq x_0$. See the Figure 24 for examples of lasers that go through the points $(0, 0)$, $(2, 2)$, $(6, 3)$ and $(10, 4)$ with slope $\frac{5}{13}$.

We now describe the construction of $\widetilde{\mathcal{NC}}_{(a,b)}$. For a given (a, b)-Dyck path D we label the right ends of the east steps of D from left to right with the labels $1, 2, \ldots, (b - 1)$. For every consecutive sequence of north steps in D fire a laser through the lower point where the sequence starts. The lasers give a topological decomposition of the reqion between D and the diagonal $y = \frac{a}{b}x$ and then we can define a partition $\pi(D)$ by letting i and j to be in the same block if they belong to the same connected component. In [3, Proposition 6.1] it is shown that $\pi(D)$ is a noncrossing partition of $[b-1]$ and that π is an injective map, which allow them to define rational noncrossing partitions. In Figure 24 we illustrate an example of this construction starting with the $(5, 13)$-Dyck path (or $(3, 4, 3, 4, 3)$-Dyck path in our language)

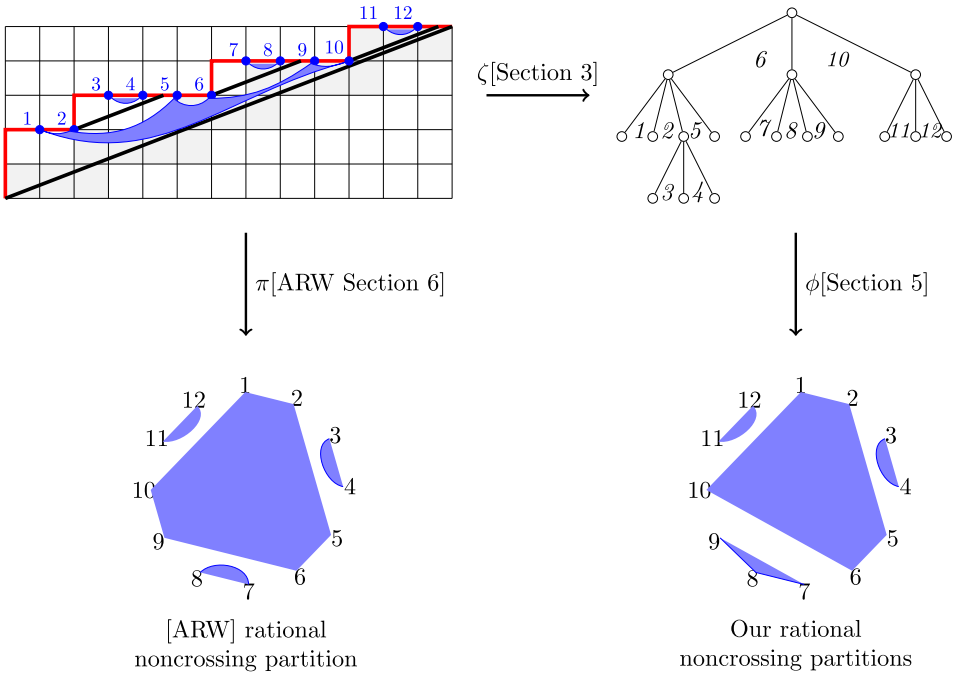


Figure 24: Difference between $\widetilde{\mathcal{NC}}_{(a,b)}$ and $\mathcal{NC}_s$ for a rational s.

$D = N^2E^2NE^4NE^4NE^3$. The noncrossing partition obtained from the construction in [3] is $\pi(D) = \{\{1,2,5,6,9,10\},\{3,4\},\{7,8\},\{11,12\}\}$ while the noncrossing partition obtained from D using our constructions, by first identifying the $(3,4,3,4,3)$-tree $\zeta(D)$ and then applying ϕ, is the noncrossing $(3,4,3,4,3)$-partition $\phi(\zeta(D)) = \{\{1,2,5,6,10\},\{3,4\},\{7,8,9\},\{11,12\}\}$. This shows that the two constructions arrive to different objects. Note that not only the noncrossing partitions associated to D are different but also the sets $\widetilde{\mathcal{NC}}_{(5,13)}$ and $\mathcal{NC}_{(2,3,2,3,2)}$ of rational noncrossing partitions are different. The partition $\pi(D) = \{\{1,2,5,6,9,20\},\{3,4\},\{7,8\},\{11,12\}\}$ has parts of sizes $\mu(\pi(D)) = (6,2,2,2)$, so in particular, it is not refined by $s - \mathbf{1} = (2,3,2,3,2)$ and hence $\pi(D) \notin \mathcal{NC}_{(2,3,2,3,2)}$. A similar situation occurs with the families $\widetilde{\mathcal{CM}}_{(a,b)}$ and $\widetilde{\mathcal{AP}}_{(a,b)}$ since they are also defined by the laser construction and our constructions have definitions that reflect the underlying tree structure of the objects instead.

In [3] Armstrong, Rhodes and Williams pose as an open problem to give characterizations of the families $\widetilde{\mathcal{NC}}_{(a,b)}$, $\widetilde{\mathcal{CM}}_{(a,b)}$ and $\widetilde{\mathcal{AP}}_{(a,b)}$ that are somewhat more natural and do not rely on a laser construction (One such

characterization of $\widetilde{\mathcal{NC}}_{(a,b)}$ was given in [13]). One advantage of our constructions is precisely that they are independent of the laser construction and that they happen to reflect the natural tree structure of s-Catalan families. One advantage in the laser constructions, however, is that they happen to have certain rotational symmetry. Indeed, it was proven in [3, Proposition 5.2] that $\widetilde{\mathcal{CM}}_{(a,b)}$ is closed under rotation, and the same property was proven for $\widetilde{\mathcal{NC}}_{(a,b)}$ in [13] for pairs $a < b$ and in [12] for all coprime pairs (a, b). It is not difficult to check that the families $\mathcal{CM}_s$ and $\mathcal{NC}_{s-\mathbf{1}}$ are in general not closed under rotation unless $s = (k^a)$ for some value of k, since the rotation operation also have the effect of permuting the entries of the signature s. Hence, if we want to have rotational symmetry, we need to go to a larger family of objects that includes all the possible permutations of the signature. Finally, a clear advantage of our constructions is that we produce, as a byproduct, a family of (a, b)-trees that is central in the new rational picture and that it was not defined before in the literature. We claim that this central object could play an important role when studying families of rational Catalan objects for its inherent recursive nature. For example, the (a, b)-trees also allow us to obtain the notion of Stirling (a, b)-permutations that were not considered before inside this picture.

Appendix A. Bijections with s-Dyck paths

In this appendix, we describe further bijections between s-Catalan objects and s-Dyck paths. As in the case of s-trees, these bijections turn out to be elegant and simple (see Figure 25). They are obtained by assigning some labels to an s-Dyck path and then reading or grouping the labels according to a certain rule. The bijections presented here are in accordance with those presented in Section 2. More precisely, if D is the s-Dyck path corresponding to an s-tree T (according to Section 2.2.1), then an s-Catalan object associated to D in this appendix is the same s-Catalan object associated to T according to the corresponding bijection in Section 2. All the details about these bijections are simple and left to be filled by the interested reader.

A.1. 312-Avoiding Stirling $(s - 1)$-permutations

Let σ be an $(s - \mathbf{1})$-permutation. We represent the permutation in a $(|s| - \ell(s) + 1) \times \ell(s)$ rectangle by placing a dot in the square at position (i, j) if $\sigma(i) = j$. Note that all the columns of the rectangle except the last one are occupied. The top part of Figure 25 illustrates an example for

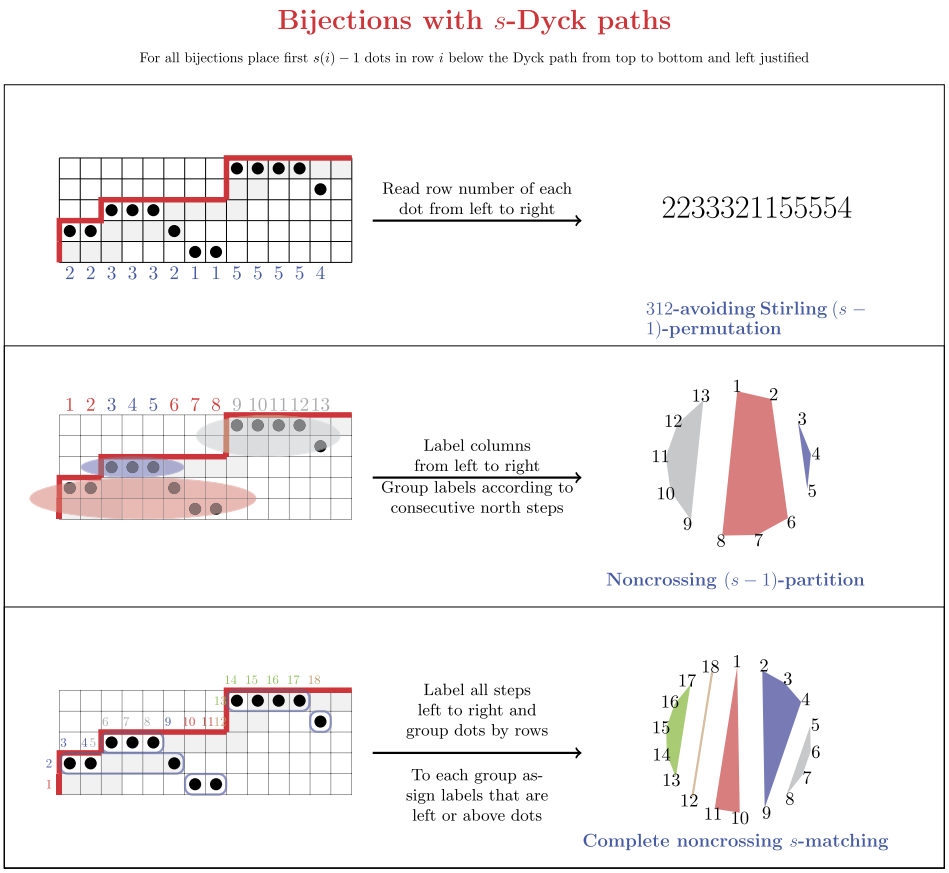


Figure 25: Examples of the bijections with s-Dyck paths.

the $(2,3,3,1,4)$-permutation $\sigma = 2233321155554$. Given such representation of σ, we shade all the boxes in the rectangle that are weakly below and weakly to the right of some dotted box. It is not hard to see that the shaded boxes bound an s-Dyck path $D(\sigma)$. Note that many permutations may give rise the same Dyck path. However, if we restrict to 312-avoiding Stirling $(s-\mathbf{1})$-permutations there is exactly one permutation for each Dyck path. This permutation is constructed as follows: place $s(j)-1$ dots, from left to right, in the highest unoccupied row j that is below the path D, such that no previously occupied columns are taken. The permutation $\sigma(D)$ is defined as the permutation with the resulting rectangular representation.

A.2. Noncrossing $(s-1)$-partitions

Given an s-Dyck path D we place $s(j)-1$ dots in row j for each j as above, and number the columns of the grid (except the last one) from 1 to $|s|-\ell(s)$. We make a horizontal strip to the right of each maximal consecutive chain of north steps in D. The columns of the dots in each strip form the blocks of the noncrossing $(s-\mathbf{1})$-partition $\pi(D)$ associated to D. The middle part of Figure 25 illustrates an example. Note that the number of blocks of the partition is equal, by definition, to the number of peaks of the Dyck path. The inverse map can be easily described as follows: let $\pi = \{\pi_1, \ldots, \pi_k\}$ be a noncrossing $(s-\mathbf{1})$-partition whose blocks are ordered according to the order of their minimal elements. Partition the sequence $s-\mathbf{1}$ in k consecutive blocks such that the sum of the values in block i is equal to the size $|\pi_i|$. In the middle of Figure 25, $s-\mathbf{1} = (\mathbf{2},\mathbf{3},\mathbf{3},\mathbf{1},\mathbf{4})$ and

$$\pi = \{\{1,2,6,7,8\},\{3,4,5\},\{9,10,11,12,13\}\}.$$

The sizes of the blocks of π are $(5,3,5)$, and so we partition $s-\mathbf{1}$ as $|2,3|3|1,4|$. For $1 \le i \le k$, define $x_i = \min \pi_i - 1$ and y_i equal to the position of the last number in block i of the partitioned $s-\mathbf{1}$. In our running example,

$$(x_1,x_2,x_3) = (1-1, 3-1, 9-1) = (0,2,8),$$
$$(y_1,y_2,y_3) = (2,3,5).$$

The s-Dyck path corresponding to π is the path with peaks at coordinates (x_i, y_i) for $i \in [k]$.

A.3. Complete noncrossing s-matchings

Given an s-Dyck path D, we place $s(j)-1$ dots in row j for each j as above, and number the steps of D (except the last one) from 1 to $|s|$. For each $1 \le i \le \ell(s)$, define M_i to be the set containing the label of the north step of D in row i, together with the labels of the east steps in the columns of the dots in that row. The complete noncrossing s-matching associated to D is defined by $M(D) := \{M_1, \ldots, M_{\ell(s)}\}$. The bottom part of Figure 25 illustrates an example. The inverse is determined by the positions of the north steps in the path, which are the minimal elements of the blocks in M.

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