Journal of Combinatorics
Volume 10, Number 4, 675–709, 2019

A family of symmetric functions associated with Stirling permutations

Rafael S. González D'León*

We present exponential generating function analogues to two classical identities involving the ordinary generating function of the complete homogeneous symmetric functions. After a suitable specialization the new identities reduce to identities involving the first and second order Eulerian polynomials. The study of these identities led us to consider a family of symmetric functions associated with a class of permutations introduced by Gessel and Stanley, known in the literature as Stirling permutations. In particular, we define certain type statistics on Stirling permutations that refine the statistics of descents, ascents and plateaux and we show that their refined versions are equidistributed, generalizing a result of Bóna. The definition of this family of symmetric functions extends to the generality of r-Stirling permutations. We discuss some occurrences of these symmetric functions in the cases of $r = 1$ and $r = 2$.

Keywords and phrases: Stirling permutations, symmetric functions, Lagrange inversion, permutation statistics.

1. Preliminaries and notation

We denote $\mathbb{N}$ the set of nonnegative integers, $\mathbb{P}$ the set of positive integers and $\mathbb{Q}$ the set of rational numbers. A *weak composition* is an infinite sequence $\mu = (\mu(1), \mu(2), \dots)$ of numbers $\mu(i) \in \mathbb{N}$ such that its *sum* $|\mu| := \sum_i \mu(i)$ is finite. If $|\mu| = n$ for some $n \in \mathbb{N}$, we say that μ is a *weak composition of* n. We denote wcomp the set of weak compositions and wcomp_n the set of weak compositions of n. An *(integer) partition* λ of n (denoted $\lambda \vdash n$) is a weak composition of n whose entries are nonincreasing, i.e., $\lambda = (\lambda(1) \geq \lambda(2) \geq \cdots)$. If $\nu \in$ wcomp is obtained by permuting the entries of another $\mu \in$ wcomp we say that ν is a *rearrangement* of μ. We denote

arXiv: 1506.01628

*Supported by NSF Grant DMS 1202755 and by Colciencias (Departamento Administrativo de Ciencia, Tecnología e Innovación de Colombia).

the set of rearrangements of μ by wcomp_μ. Let $\mathbf{x} = x_1, x_2, \dots$ be an infinite set of variables and throughout this document we denote $x^\mu := \prod_i x_i^{\mu_i}$ for $\mu \in \mathrm{wcomp}$. We denote $\Lambda = \Lambda_{\mathbb{Q}}$ the ring of symmetric functions in $\mathbf{x}$ with rational coefficients, that is, the ring of power series on $\mathbf{x}$ of bounded degree that are invariant under permutation of the variables. Let $h_\lambda(x)$ and $e_\lambda(x)$ respectively denote the complete homogeneous symmetric function and the elementary symmetric function indexed by a partition λ. It is known that for $n \geq 0$, the sets $\{h_\lambda \mid \lambda \vdash n\}$ and $\{e_\lambda \mid \lambda \vdash n\}$ are bases for the n-th homogeneous graded component of $\Lambda_{\mathbb{Q}}$. For information not presented here regarding symmetric functions the reader could go to [21] and [28, Chapter 7].

For a sequence $(a_0, a_1, \dots)$ of elements in a ring R (containing $\mathbb{Q}$) the *ordinary generating function* (or *o.g.f.*) of (a_n) is the formal power series $\sum_{n\geq 0} a_n y^n \in R[[y]]$ and the *exponential generating function* (or *e.g.f.*) of (a_n) is the formal power series $\sum_{n\geq 0} a_n \frac{y^n}{n!} \in R[[y]]$ (cf. [29]). In all the following f^{-1} denotes the multiplicative inverse and $f^{\langle -1\rangle}$ denotes the compositional inverse of $f \in R[[y]]$ whenever any of these inverses exist.

2. Introduction

We consider the ring $\Lambda[[y]]$ of power series in the variable y with coefficients in Λ. The following two identities are classical results in the study of symmetric functions.

Proposition 2.1 (cf. [21] Equation (2.6))**.** *We have*

$$\left(\sum_{n\geq 0}(-1)^n h_n(\mathbf{x}) y^n\right)^{-1} = \sum_{n\geq 0} e_n(\mathbf{x}) y^n.$$

Proposition 2.2 (cf. [27])**.** *We have*

$$\left(\sum_{n\geq 1}(-1)^{n-1} h_{n-1}(\mathbf{x}) y^n\right)^{\langle -1\rangle} = \sum_{n\geq 1} \omega\, \mathrm{PF}_{n-1}(\mathbf{x}) y^n,$$

where ω is the involution in Λ defined by $\omega(h_n(\mathbf{x})) = e_n(\mathbf{x})$ and $\mathrm{PF}_{n-1}(\mathbf{x})$ is Garsia-Haiman's parking function symmetric function, see [17] and [8].

It is known that $\omega\,\mathrm{PF}_{n-1}(\mathbf{x})$ has positive coefficients when expressed in the elementary basis (see [17] and [27]), a property known as *e-positivity*.

Proposition 2.3 (cf. [27]). *For $n \geq 0$,*

$$\omega\,\mathrm{PF}_n(\mathbf{x}) = \sum_{\pi \in \mathsf{NC}_n} e_{\lambda(\pi)}(\mathbf{x}),$$

where NC_n is the set of noncrossing partitions of $[n]$ and $\lambda(\pi)$ is the integer partition of n whose parts are the sizes of the blocks of the set partition π.

A common feature of Propositions 2.1 and 2.2 is the e-positivity of the coefficients of the power series in the right-hand side.

In this work we find exponential generating function analogues to Propositions 2.1 and 2.2. We prove the following theorems.

Theorem 2.4. *We have*

$$\left(\sum_{n\geq 0}(-1)^n h_n(\mathbf{x})\frac{y^n}{n!}\right)^{-1} = \sum_{n\geq 0}\sum_{\sigma\in\mathfrak{S}_n} e_{\lambda(\sigma)}(\mathbf{x})\frac{y^n}{n!}, \tag{1}$$

where $\mathfrak{S}_n$ is the set of permutations of $[n] =: \{1,2,\ldots,n\}$ and $\lambda(\sigma)$ is the consecutive ascending type of $\sigma \in \mathfrak{S}_n$ (defined in Section 4.1).

Theorem 2.5. *We have*

$$\left(\sum_{n\geq 1}(-1)^{n-1} h_{n-1}(\mathbf{x})\frac{y^n}{n!}\right)^{\langle -1\rangle} = \sum_{n\geq 1}\sum_{\theta\in\mathcal{Q}_{n-1}} e_{\lambda(\theta)}(\mathbf{x})\frac{y^n}{n!}, \tag{2}$$

where $\mathcal{Q}_n$ is the set of Stirling (multi)permutations of the multiset $\{1,1,2,2,\ldots,n,n\}$ (defined by Gessel and Stanley in [10]) and where $\lambda(\theta)$ is one of the types λ^{AA}, λ^{DA}, λ^{TN} and λ^{IN} (defined in Section 4.1) for $\theta \in \mathcal{Q}_n$.

Theorem 2.5 was first derived by the author in [12] using poset topology techniques applied to a poset of partitions weighted by weak compositions. The coefficient of $\frac{y^n}{n!}$ in the power series of the right-hand side of equation (2) is the generating function for the dimensions of the reduced (co)homology of the maximal intervals of the poset of weighted partitions and also for the dimensions of the multilinear components of the free Lie algebra with multiple compatible brackets on n generators. Here we take a different approach that does not involve poset theoretic techniques. We provide a different combinatorial proof of Theorem 2.5 using an interpretation by B. Drake of the compositional inverse of an exponential generating function in [5].

Drake's technique is a combinatorial interpretation of the compositional inverse of a power series that can be used under certain conditions, namely when the power series is a generating function for a family of trees constructed out of basic building blocks and involves only quadratic instructions (two blocks at a time) on how to compose building blocks. There is a great deal of literature related to the inversion of power series with respect to the operation of composition, also known as Lagrange inversion. In particular, Joyal's theory of combinatorial species provides a combinatorial framework for Lagrange inversion with minimal assumed conditions. We refer the reader to the book of Bergeron, Labelle and Leroux [1] for an account on the subject. There exist also various generalizations of Lagrange inversion, see for example the q-Lagrange inversions studied by Garsia [7], Gessel [9] and Garsia and Haiman [8]. The advantage of Drake's approach is that whenever the generating function satisfies certain conditions the interpretation for the compositional inverse becomes simple, uses the same type of combinatorial objects as the ones that are being counted and does not involve any cancellations due to signs. We remark here that, because of the independence of the symmetric functions $h_n(\mathbf{x})$ in the ring Λ, equation (2) can be seen as a Lagrange inversion formula for exponential power series. This work however is not about inversion of power series but instead is about investigating a family of symmetric functions that naturally appear when studying these inversion formulas.

In order to apply Drake's theorem we study a subset of the set of planar leaf-labeled binary trees that we call *normalized*. The normalization condition means that in any subtree the smallest label is in the leftmost leaf. This is equivalent to considering non-planar leaf-labeled binary trees (or binary phylogenetic trees) and the normalization condition is just a particular choice on how to draw these trees in the plane. Using Drake's technique we prove a version of Theorem 2.5 in terms of normalized trees (instead of Stirling permutations) and use this result and a bijection used in [12] between normalized trees and Stirling permutations (a bijection that appeared first in [4]) to derive Theorem 2.5.

We then generalize the symmetric functions that appear as coefficients of the power series of the right-hand side of equation (2) to the generality of the family $\mathcal{Q}_n(r)$ of r-Stirling permutations, where Stirling permutations correspond to the case $r = 2$ and the classical permutations in the symmetric group to the case $r = 1$. We consider the family of symmetric functions

$$\mathrm{SP}_n^{(r)}(\mathbf{x}) = \sum_{\theta \in \mathcal{Q}_n(r)} e_{\lambda(\theta)}(\mathbf{x}), \tag{3}$$

where $\lambda(\theta)$ is any of various types of θ (defined in Section 4.2).

It turns out that the case $r = 1$ is the family of symmetric functions that appear in the right hand side of equation (1). In order to prove Theorem 2.4, this time we use an interpretation of the multiplicative inverse of an exponential generating function that can be derived from a more general result discovered by Fröberg [6], Carlitz-Scoville-Vaughan [3] and Gessel [11]. As in the case of Theorem 2.5 the author provides in [13] a second proof of Theorem 2.4 using poset topology techniques over a poset of subsets weighted by weak compositions. Some of the context of that proof is discussed in Section 7.2.

We note that after the simple specialization $e_i \mapsto t$ we have that the function $\mathrm{SP}_n^{(r)}(\mathbf{x})$ reduces to $A_n^{(r)}(t)$, the r-th order Eulerian polynomial, that is the descent generating polynomial of the family of r-Stirling permutations (defined later). In the case $r = 1$, $\mathrm{SP}_n^{(1)}(\mathbf{x})$ specializes to the classical Eulerian polynomial $A_n(t) := A_n^{(1)}(t)$, that is the descent generating polynomial of $\mathfrak{S}_n$, and equation (1) specializes to the following classical result.

Theorem 2.6 (Riordan [26])**.** *We have*

$$\frac{1-t}{1-te^{(1-t)y}} = \sum_{n\geq 0} A_n(t)\frac{y^n}{n!}.$$

In the case $r = 2$, we obtain the following analogous result.

Theorem 2.7. *We have*

$$\left(\frac{(1-t)y + (1-e^{y(1-t)})t}{(1-t)^2}\right)^{\langle -1\rangle} = \sum_{n\geq 1} A_{n-1}^{(2)}(t)\frac{y^n}{n!}.$$

The paper is organized as follows: in Section 3 we discuss Drake's interpretation of compositional inverses of exponential generating functions and use this interpretation to give a version of Theorem 2.5 in terms of the family of normalized labeled binary trees. In Section 4 we use a bijection between normalized labeled binary trees and Stirling permutations to prove Theorem 2.5. We then consider the natural generalization (3) of the symmetric functions that appear as coefficients in the right-hand side of equation (2) and show in Section 5 that in the base case they are precisely the family of symmetric functions that appear in the right-hand side of equation (1). We discuss an interpretation of multiplicative inverses of exponential generating functions and use it to prove Theorem 2.4. In Section 6 we show that under a simple specialization Theorems 2.4 and 2.5 reduce to expressions involving first and second order Eulerian polynomials. Finally, in Section 7 we briefly present other contexts where the symmetric functions $\mathrm{SP}_n^{(1)}(\mathbf{x})$ and $\mathrm{SP}_n^{(2)}(\mathbf{x})$

make an appearance. In particular, these symmetric functions are the generating functions for the Möbius invariants of the maximal intervals of two families of posets. We leave some open questions regarding the cases $r \geq 3$.

3. Binary trees

A tree is a connected graph that has no loops or cycles. We say that a tree is rooted if one of its nodes is specially marked and called the *root*. For two nodes x and y on a rooted tree T, x is said to be the *parent* of y (and y the *child* of x) if x is the node that follows y in the unique path from y to the root. A node is called a *leaf* if it has no children, otherwise is said to be *internal*. A rooted tree T is said to be *planar* if for every internal node x of T the set of children of x is totally ordered. A *(leaf-)labeled (planar) tree* (T, σ) is defined as a planar tree T whose jth leaf from left to right has been labeled $\sigma(j)$, where σ is a permutation. A rooted tree is said to be *complete* if all internal nodes have the same number of children. A *(complete planar) binary tree* is a planar rooted tree in which every internal node has exactly two children, a left and a right child. We denote $\mathcal{BT}_n$ the set of leaf-labeled binary trees with n leaves. These will play a relevant role in the following. See Figure 1 for some examples of labeled binary trees.

3.1. Drake's interpretation of compositional inverse

In [5] B. Drake proposes an interesting interpretation of the compositional inverse of an exponential generating function in terms of trees with allowed and forbidden links. This interpretation was also rediscovered by Dotsenko in [4].

Consider a set of rooted trees (either planar or not) whose leaves are all labeled with distinct positive integers. Two leaf-labeled rooted trees T_1 and T_2, with label sets $A_1, A_2 \subset \mathbb{P}$ such that $|A_1| = |A_2|$, are said to be *equivalent* (and we write $T_1 \sim T_2$) if we can obtain T_2 from T_1 by replacing the labels in T_1 according to the unique order-preserving bijection between A_1 and A_2. We also consider that the rooted trees in this set come together with a local labeling function, that allows us to extend consistently the labels from the children to the parents. Hence, every internal node has been also assigned a label coming from the set of labels of the leaves. Formally a *local labeling* is a recursively defined rule that assigns to each node x (internal or leaf) of a leaf-labeled rooted tree T a unique element $v(x)$ such that $v(x) \in \{v(y) \mid y \text{ a child of } x\}$ if x is an internal node, or $v(x) = l$ if x is a leaf with label l. In addition, the rule $v(x)$ should be consistent along the

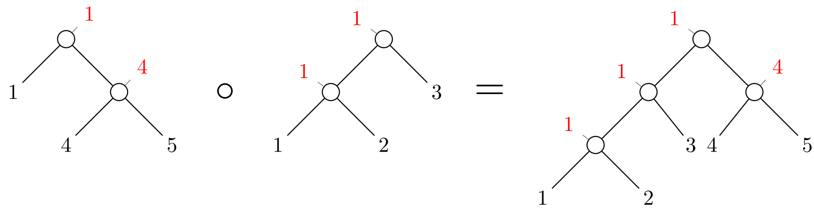


Figure 1: Example of composition of leaf-labeled rooted trees.

unique order-preserving bijection on equivalent trees. The *label of a rooted tree* $v(T)$ is defined as the label of its root.

Example 3.1. We can consider as an example the set of binary trees with distinct leaf-labels. A possible local labeling function is the one that assigns to a parent the smallest label among the labels of its children. In Figure 1 we illustrate some of these trees with the labels indicated near each node.

For two rooted trees T_1 and T_2 with label sets that only intersect in one label that is both a leaf of T_1 and the label of T_2, the *composition* $T_1 \circ T_2$, is defined to be the tree obtained by deleting the common label from the leaf of T_1 and attaching the root of T_2 instead in its position (see Figure 1). If the above condition for T_1 and T_2 is not satisfied $T_1 \circ T_2$ is undefined. Note that composition is associative and so expressions like $T_1 \circ T_2 \circ \cdots \circ T_k$ are well defined.

If $\mathcal{A}$ is a set of leaf-labeled trees, $\mathcal{A}$ is said to have the *label substitution property* if whenever $T_1 \sim T_2$ then $T_1 \in \mathcal{A}$ if and only if $T_2 \in \mathcal{A}$. $\mathcal{A}$ is said to have the *unique decomposition property* if for every $T \in \mathcal{A}$ then $T \neq T_1 \circ T_2 \circ \cdots \circ T_k$ for trees $T_j \in \mathcal{A}$, i.e., T cannot be written as a nontrivial composition of other trees in $\mathcal{A}$. A set $\mathcal{A}$ with these two properties is called an *alphabet* and any tree in $\mathcal{A}$ is called a *letter*. We can also consider alphabets $\mathcal{A}_S$ that are formed by colored letters, that is, pairs (T, s) where $T \in \mathcal{A}$ and $s \in S$ for some set S. A *link* is the composition of two (colored) letters when defined.

Assume that $\mathcal{A}_S$ is partitioned into equivalence classes of colored letters and let K be the set of equivalence classes. For a leaf-labeled tree T constructed composing letters from $\mathcal{A}_S$ we denote $m_j(T)$ the number of letters of the equivalence class j that are in T. We also denote $|T| = \sum_{j\in K} m_j(T)$ the total number of letters in T. Consider now a partition of the set of links into two parts, that we will call from now on *allowed links* $\mathcal{L}(\mathcal{A}_S)$ and *forbidden links* $\overline{\mathcal{L}(\mathcal{A}_S)}$. Let $\mathcal{T}_S^n$ and $\overline{\mathcal{T}_S}^n$ for $n \geq 1$ be the families of trees constructed exclusively with allowed links or exclusively with forbidden links respectively, and whose labels are the elements of the set $[n]$, each label occuring exactly once. Define $\mathcal{T}_S = \cup_{n\geq 1}\mathcal{T}_S^n$ and $\overline{\mathcal{T}_S} = \cup_{n\geq 1}\overline{\mathcal{T}_S}^n$. In particular, $\mathcal{T}_S^1 = \overline{\mathcal{T}_S}^1 = \{\bullet_1\}$ is the tree with a single node labeled 1 and we consider letters in $\mathcal{A}_S$ as if they are both in $\mathcal{T}_S$ and $\overline{\mathcal{T}_S}$.

Define the monomials

$$\mathbf{x}^{m(T)} = \prod_{j\in K} x_j^{m_j(T)}, \tag{4}$$

and the generating functions

$$\begin{aligned} F(y) &= \sum_{n\geq 1}\sum_{T\in\mathcal{T}_S^n} \mathbf{x}^{m(T)}\frac{y^n}{n!}, \\ \overline{F}(y) &= \sum_{n\geq 1}\sum_{T\in\overline{\mathcal{T}}_S^n} (-1)^{|T|}\mathbf{x}^{m(T)}\frac{y^n}{n!}, \end{aligned} \tag{5}$$

where y and x_j for $j\in K$ are indeterminates.

The following theorem of Drake [5] reveals a beautiful algebraic relation between the exponential generating function for the trees constructed using only allowed links and the exponential generating function for the trees constructed using only forbidden links. Its proof is a consequence of the combinatorial interpretation of the composition of exponential generating functions given in [28].

Theorem 3.2 ([5] Theorem 1.3.3)**.** *We have*

$$F^{\langle -1\rangle}(y) = \overline{F}(y). \tag{6}$$

There is a gap in the argument in the original proof of Theorem 3.2 in [5]. For the sake of completeness we provide a proof of this Theorem fixing this gap.

We begin by considering the set of leaf-labeled rooted trees T constructed as follows: Starting from an ordered partition $\pi = (\pi_1,\dots,\pi_\ell)$ of $[n]$ (that is, where the blocks are linearly ordered), T is of the form $T^a\circ T_1^f\circ T_2^f\circ\cdots\circ T_\ell^f$ where $T^a\in\mathcal{T}_S$ and $T_i^f\in\overline{\mathcal{T}_S}$ for all i, with the condition that T_i^f has label set π_i and T^a has label set $\{v(T_i^f)\}_{i=1}^{\ell}$. Note that different factorizations (different partitions and set of subtrees) of the form above can create the same underlying tree T. Indeed, the links between the tree T^a and the trees T_i^f can be either in $\mathcal{L}(\mathcal{A}_S)$ or in $\overline{\mathcal{L}(\mathcal{A}_S)}$ and hence there could be multiple choices as to where T^a finishes and T_i^f starts. Here we want to consider

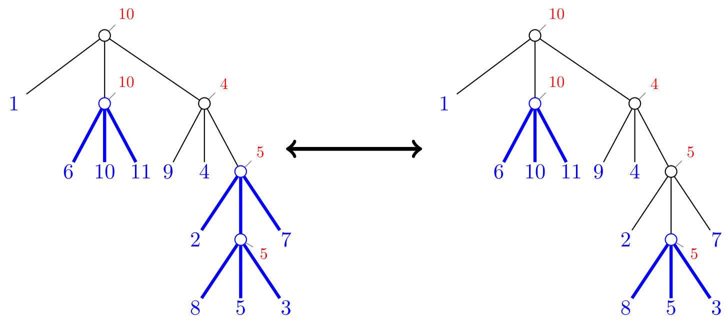


Figure 2: Two $(\mathcal{T}_S, \overline{\mathcal{T}_S})$-composite trees.

different factorizations of the same tree T as different objects. A tree T together with a factorization as above is called a $(\mathcal{T}_S, \overline{\mathcal{T}_S})$-*composite tree.*

Example 3.3. We can consider the alphabet formed by planar rooted trees with exactly one internal node and three leaves with distinct labels. We choose the set of forbidden links to be formed by connecting one of the trees in our alphabet as the middle child of another tree. Hence the set of allowed links are the ones where trees are not connected in a middle child. In Figure 2 we illustrate two examples of $(\mathcal{T}_S, \overline{\mathcal{T}_S})$-composite trees under these conditions. Note that even though both of these composite trees have the same underlying tree they correspond to two different factorizations. The underlying ordered partition of the composite tree on the left is $1|6\ 10\ 11|9|4|2\ 3\ 5\ 7\ 8$ while the one of the tree on the right is $1|6\ 10\ 11|9|4|2|3\ 5\ 8|7$.

Lemma 3.4 ([5] Lemma 1.3.2)**.** *The composition $F(\overline{F}(y))$ is the exponential generating function for $(\mathcal{T}_S, \overline{\mathcal{T}_S})$-composite trees T weighted by $(-1)^{m_f}\mathbf{x}^{m(T)}$ where m_f is the number of letters in the forbidden trees.*

Proof. This follows from a classical combinatorial interpretation of composition of exponential generating functions (see [28, Theorem 5.1.4]). □

Proof of Theorem 3.2. Using Lemma 3.4 we only need to show that the weighted exponential generating function for $(\mathcal{T}_S, \overline{\mathcal{T}_S})$-composite trees is y. We define a sign-reversing involution ι on the set of $(\mathcal{T}_S, \overline{\mathcal{T}_S})$-composite trees where the only fixed point is the tree with a single node (whose factorization is unique). Let $\mathfrak{T} = T^a \circ T_1^f \circ T_2^f \circ \cdots \circ T_\ell^f$ be a $(\mathcal{T}_S, \overline{\mathcal{T}_S})$-composite tree. Now, recursively and starting at the letter R that contains the root of $\mathfrak{T}$, we move to the child of R with smallest label among the ones that have an allowed link with R. Note that the choice of this child is unique since the children of R have disjoint sets of descendants and each leaf of $\mathfrak{T}$ has a unique label. The fact that the local labeling is a recursive assignment implies that every child of R has a different label. We continue recursively repeating the same process until we find a letter R_0 whose children are either leaves or form forbidden links with R_0. The map works as follows: If R_0 is a letter of T^a then $\iota(\mathfrak{T})$ is the factorization where the tree starting at R_0 with all its descendants is part of the forest of trees with forbidden links. If R_0 is not a letter of T^a then $\iota(\mathfrak{T})$ is the factorization where $T^a \circ R_0$ is the new tree with allowed links. Note that the map is well-defined since every child of R_0 is either a leaf or forms a forbidden link with R_0. Note also that the choice of R_0 does not depend on the factorization of the underlying tree T and every time we apply the process the role of R_0 changes between being part of the tree with allowed links or being part of a tree in the forest of trees with forbidden links. Therefore this process is an involution that reverses the sign as defined in Lemma 3.4. □

Example 3.5. In Figure 2 we illustrate two trees (under the same conditions as in Example 3.3) that are related by the involution ι described in the proof of Theorem 3.2. In this example R_0 is the tree whose children have labels 2, 5 and 7. Note that the number of letters in the set of forbidden trees changes by one and hence also the weight $(-1)^{m_f}\mathbf{x}^{m(T)}$ alternates sign between these two composite trees.

Remark 3.6. The proof of Theorem 1.3.3 in [5] defines the map ι as follows: First select a letter R_0 of T^a traveling always from the root to the smallest label that has a letter of T^a substituted in (it is assumed that the trees are planar and the local labeling chooses the leftmost label in the subtree). For this letter R_0 either all children are leaves or it has at least one child forming a link with a tree in $\mathfrak{T}$. If every child is a leaf then $\iota(\mathfrak{T})$ is the factorization where R_0 is considered as a tree in $\overline{\mathcal{T}_S}$. Otherwise let R_1 be the letter substituted into the child with smallest label. If $R_0 \circ R_1$ is an allowed link then $\iota(\mathfrak{T})$ is the factorization where $R_0 \circ R_1$ is part of the tree in $\mathcal{T}_S$. Otherwise make R_0 part of the forest of trees with forbidden links.

The issue here is that ι is not a well-defined map. For example, assume that we are considering planar letters with the conditions in Example 3.3. Then the $(\mathcal{T}_S, \overline{\mathcal{T}_S})$-composite tree in the left of Figure 3 does not have a well defined image. In the figure, the factorization suggested for the tree in the left has underlying ordered partition $\pi = 1|234|5|6|7$. The reader can check that the process described in the previous paragraph, will assign the new factorization of the tree in the right of Figure 3 that is not a $(\mathcal{T}_S, \overline{\mathcal{T}_S})$-composite tree (recall that forbidden trees can only have links involving middle children).

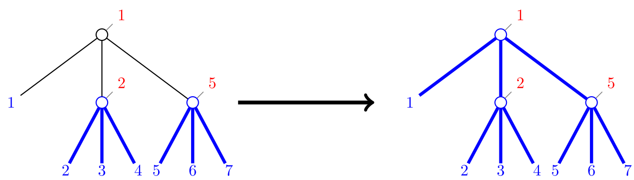
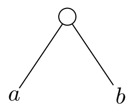
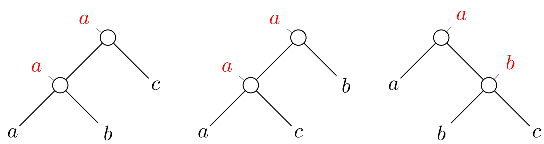


Figure 3: Counterexample to the map defined in [5].

In the following we will be using an alphabet of colored planar binary letters of the form:



where $a < b$ and with colors from $\mathbb{P}$. The possible links are of the form:



where $a < b < c$.

3.2. Normalized binary trees

For each internal node x of a labeled binary tree, let $L(x)$ denote the left child of x and $R(x)$ denote its right child. For each node x of a labeled binary tree (T, σ) define its *label* $v(x)$ to be the smallest leaf label of the subtree rooted at x. Figure 4 illustrates the labels of the internal nodes of a labeled binary tree.

We say that a labeled binary tree is *normalized* if the leftmost leaf of each subtree has the smallest label in the subtree. This is equivalent to requiring that for every internal node x,

$$v(x) = v(L(x)).$$

Note that a normalized tree can be regarded as a labeled nonplanar binary tree (or binary phylogenetic tree) that has been drawn in the plane following the convention above. We denote Nor_n the set of normalized labeled binary trees on label set $[n]$. It is well-known that there are $(2n-3)!! := 1 \cdot 3 \cdots (2n-3)$ binary phylogenetic trees on $[n]$ and so $|\mathsf{Nor}_n| = (2n-3)!!$.

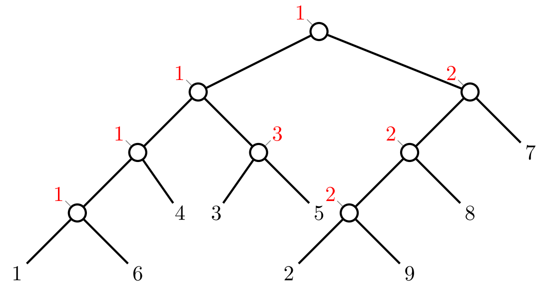


Figure 4: Example of a Lyndon tree. The numbers above the lines correspond to the labels of the internal nodes.

A *Lyndon tree* is a normalized tree (T, σ) such that for every internal node x of T, whose left child $L(x)$ is also internal, we have

$$v(R(L(x)) > v(R(x)). \tag{7}$$

We will say that an internal node x of a labeled binary tree (T, σ) is a *Lyndon node* if (7) holds. Hence (T, σ) is a Lyndon tree if and only if it is normalized and all its internal nodes are Lyndon nodes. A Lyndon tree is illustrated in Figure 4. It is known that the set of Lyndon trees with n leaves gives a basis for the multilinear component of the free Lie algebra on n generators (see for example [30]).

3.3. Colored normalized trees

We will also be considering labeled binary trees with colored internal nodes. A *colored labeled binary tree* is a labeled binary tree such that every internal node x has been assigned a *color* $\mathbf{color}(x) \in \mathbb{P}$. For a weak composition $\mu \in \mathrm{wcomp}_{n-1}$ we denote $\mathcal{BT}_\mu$ the set of colored labeled binary trees that contain exactly $\mu(j)$ internal nodes colored j for each j.

A *colored Lyndon tree* is a normalized binary tree such that for any node x that is not a Lyndon node the following condition must be satisfied:

$$\mathbf{color}(L(x)) > \mathbf{color}(x). \tag{8}$$

For $\mu \in \mathrm{wcomp}_{n-1}$, let Lyn_μ be the set of colored Lyndon trees in $\mathcal{BT}_\mu$ and $\mathsf{Lyn}_n = \cup_{\mu \in \mathrm{wcomp}_{n-1}} \mathsf{Lyn}_\mu$. Note that equation (8) implies that the monochromatic Lyndon trees are just the classical Lyndon trees. Figure 5 shows an example of a colored Lyndon tree.

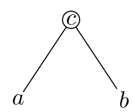
A *colored comb* is a normalized colored binary tree that satisfies the following coloring restriction: for each internal node x whose right child $R(x)$ is not a leaf,

$$\mathbf{color}(x) > \mathbf{color}(R(x)). \tag{9}$$

Let Comb_μ be the set of colored combs in $\mathcal{BT}_\mu$ and Comb_n the set of all colored combs. Figure 6 shows an example of a colored comb. Note that in a monochromatic comb every right child has to be a leaf and hence they are the classical left combs that are known to give a basis for the multilinear component of the free Lie algebra on n generators $\mathcal{L}ie(n)$ (see [30, Proposition 2.3]). The μ-colored Lyndon trees and combs generalize the classical Lyndon trees and combs and both give bases for the $\mathfrak{S}_n$-module $\mathcal{L}ie(\mu)$ in [12] (see also [14]).

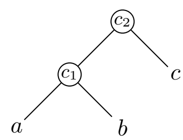
Using Drake's approach we have another perspective to define these types of trees.

Consider the alphabet $\mathcal{A}_\mathbb{P}$ with letters:

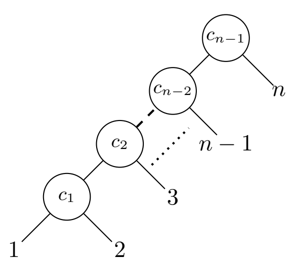


where $c \in \mathbb{P}$ is any color and $a < b$.

To define the colored Lyndon trees we consider the following forbidden links:



with $a < b < c$ and $c_1 \leq c_2$, i.e., the colors weakly increase towards the root. Then the allowed trees are colored Lyndon trees since they satisfy condition (8) and the forbidden trees are of the form:



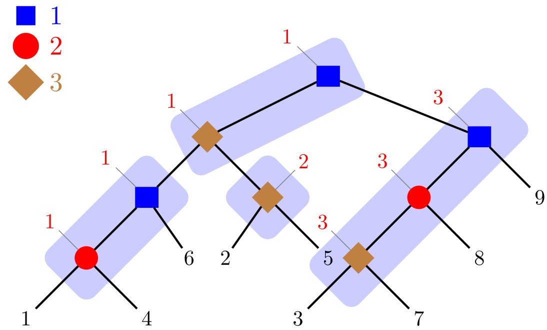


Figure 5: Example of a colored Lyndon tree of type (3,2,2,1). The numbers above the lines correspond to the labels of the internal nodes.

with $c_1 \leq c_2 \leq \cdots \leq c_{n-1}$. Since we can completely characterize any such tree by defining how many times the color i appears among the $n-1$ internal nodes for each $i \in \mathbb{P}$, and considering that

$$h_n(\mathbf{x}) := \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

we obtain the following expression for the exponential generating series $\overline{F_{\mathsf{Lyn}}}(y)$ for the forbidden trees.

Lemma 3.7. *We have*

$$\overline{F_{\mathsf{Lyn}}}(y) = \sum_{n \geq 1} (-1)^{n-1} h_{n-1}(\mathbf{x}) \frac{y^n}{n!}.$$

3.3.1. Lyndon type of a normalized tree With a normalized tree $\Upsilon = (T, \sigma) \in \mathsf{Nor}_n$ we can associate a set partition $\pi^{\mathsf{Lyn}}(\Upsilon)$ of the set of internal nodes of Υ, defined to be the finest set partition satisfying the condition:

- for every internal node x that is not Lyndon, x and $L(x)$ belong to the same block of $\pi^{\mathsf{Lyn}}(\Upsilon)$.

For the tree in Figure 5, the shaded rectangles indicate the blocks of $\pi^{\mathsf{Lyn}}(\Upsilon)$.

Note that the coloring condition (8) implies that in a colored Lyndon tree Υ there are no repeated colors in each block B of the partition $\pi^{\mathsf{Lyn}}(\Upsilon)$ associated with Υ. Hence after choosing a set of $|B|$ colors for the internal nodes in B there is a unique way to assign the different colors such that the colored tree Υ is a colored Lyndon tree (the colors must decrease towards the root in each block of $\pi^{\mathsf{Lyn}}(\Upsilon)$).

Define the *Lyndon type* $\lambda^{\mathsf{Lyn}}(\Upsilon)$ of a normalized tree (colored or uncolored) Υ to be the (integer) partition whose parts are the block sizes of $\pi^{\mathsf{Lyn}}(\Upsilon)$. For the tree Υ in Figure 5, we have $\lambda^{\mathsf{Lyn}}(\Upsilon) = (3, 2, 2, 1)$.

Proposition 3.8. *We have*

$$F_{\mathsf{Lyn}}(y) = \sum_{n\geq 1} \sum_{\Upsilon\in \mathsf{Nor}_n} e_{\lambda^{\mathsf{Lyn}}(\Upsilon)}(\mathbf{x}) \frac{y^n}{n!}.$$

Proof. For a colored labeled binary tree Ψ we define its *content* $\mu(\Psi)$ to be the weak composition μ where $\mu(i)$ is the number of internal nodes of Ψ that have color i. Let $\widetilde{\Psi}$ denote the underlying uncolored labeled binary tree of Ψ. Note that the comments above, together with the fact that

$$e_n(\mathbf{x}) := \sum_{1\leq i_1<i_2<\cdots<i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

imply that for $\Upsilon \in \mathsf{Nor}_n$, the generating function of colored Lyndon trees associated with Υ is

$$\sum_{\substack{\Psi\in \mathsf{Lyn}_n \\ \widetilde{\Psi}=\Upsilon}} \mathbf{x}^{\mu(\Psi)} = e_{\lambda^{\mathsf{Lyn}}(\Upsilon)}(\mathbf{x}). \tag{10}$$

Indeed, the internal nodes in a block of size i in the partition $\pi^{\mathsf{Lyn}}(\Upsilon)$ can be colored uniquely with any set of i different colors and so the contribution from this block of $\pi^{\mathsf{Lyn}}(\Upsilon)$ to the generating function in (10) is $e_i(\mathbf{x})$. Then

$$\begin{aligned}\sum_{\Psi\in \mathsf{Lyn}_n} \mathbf{x}^{\mu(\Psi)} &= \sum_{\Upsilon\in \mathsf{Nor}_n} \sum_{\substack{\Psi\in \mathsf{Lyn}_n \\ \widetilde{\Psi}=\Upsilon}} \mathbf{x}^{\mu(\Psi)} \\ &= \sum_{\Upsilon\in \mathsf{Nor}_n} e_{\lambda^{\mathsf{Lyn}}(\Upsilon)}(\mathbf{x}),\end{aligned}$$

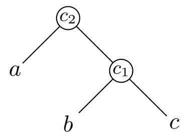
with the last equality following from (10). □

We obtain the following theorem as a corollary of Theorem 3.2.

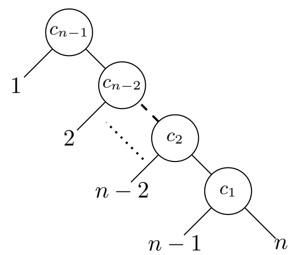
Theorem 3.9 ([12, Theorems 1.5 and 4.3])**.** *We have*

$$\left(\sum_{n\geq 1}\sum_{\Upsilon\in \mathit{Nor}_n} e_{\lambda^{\mathit{Lyn}}(\Upsilon)}(\mathbf{x})\frac{y^n}{n!}\right)^{\langle -1\rangle} = \sum_{n\geq 1}(-1)^{n-1}h_{n-1}(\mathbf{x})\frac{y^n}{n!}. \tag{11}$$

To define the colored combs now we consider forbidden links of the form:



with $a < b < c$ and $c_1 \geq c_2$, i.e., the colors weakly increase towards the root. Then the allowed trees are colored combs and the forbidden trees look like



with $c_1 \geq c_2 \geq \cdots \geq c_{n-1}$. Then following the same argument as the one before Lemma 3.7 we obtain the following expression for the exponential generating series $\overline{F_{\mathsf{Comb}}}(y)$ of the forbidden trees.

Lemma 3.10. *We have*

$$\overline{F_{\mathit{Comb}}}(y) = \sum_{n\geq 1}(-1)^{n-1}h_{n-1}(\mathbf{x})\frac{y^n}{n!}.$$

3.3.2. Comb type of a normalized tree We can associate a new type to each $\Upsilon \in \mathsf{Nor}_n$ in the following way: Let $\pi^{\mathsf{Comb}}(\Upsilon)$ be the finest set partition of the set of internal nodes of Υ satisfying

- for every pair of internal nodes x and y such that y is a right child of x, x and y belong to the same block of $\pi^{\mathsf{Comb}}(\Upsilon)$.

We define the *comb type* $\lambda^{\mathsf{Comb}}(\Upsilon)$ of Υ to be the (integer) partition whose parts are the sizes of the blocks of $\pi^{\mathsf{Comb}}(\Upsilon)$.

Note that the coloring condition (9) is closely related to the comb type of a normalized tree. The coloring condition implies that in a colored comb

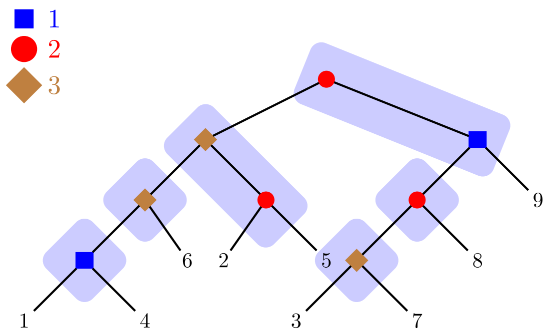


Figure 6: Example of a colored comb of comb type $(2, 2, 1, 1, 1, 1)$.

Υ there are no repeated colors in each block B of the partition $\pi^{\mathsf{Comb}}(\Upsilon)$ associated to Υ. So after choosing $|B|$ different colors for the internal nodes of Υ in B, there is a unique way to assign the colors such that Υ is a colored comb (the colors must decrease towards the right in each block of $\pi^{\mathsf{Comb}}(\Upsilon)$). In Figure 6 this relation is illustrated.

In the same manner as for Proposition 3.8 and Theorem 3.9 we derive the corresponding results for colored combs.

Proposition 3.11. *We have*

$$F_{\mathit{Comb}}(y) = \sum_{n\geq 1} \sum_{\Upsilon \in \mathit{Nor}_n} e_{\lambda^{\mathit{Comb}}(\Upsilon)}(\mathbf{x}) \frac{y^n}{n!}.$$

Theorem 3.12. *We have*

$$\left(\sum_{n\geq 1} \sum_{\Upsilon \in \mathit{Nor}_n} e_{\lambda^{\mathit{Comb}}(\Upsilon)}(\mathbf{x}) \frac{y^n}{n!} \right)^{\langle -1 \rangle} = \sum_{n\geq 1} (-1)^{n-1} h_{n-1}(\mathbf{x}) \frac{y^n}{n!}.$$

Remark 3.13. In [12] Theorem 3.9 is proved using a different technique. The proof involves the recursive definition of the Möbius invariant and an EL-labeling of a poset of partitions weighted by weak compositions where the ascent-free (or falling) chains coming from the EL-labeling are naturally described by colored Lyndon trees. The proof of Theorem 3.12 in [12] is a corollary of Theorems 3.9 and 3.14.

Theorems 3.9 and 3.12 together provide a new proof of the following theorem that was proved bijectively in [12].

Theorem 3.14 ([12, Theorem 5.4]). *For every* $\mu \in \mathrm{wcomp}$,

$$|\mathsf{Lyn}_\mu| = |\mathsf{Comb}_\mu|.$$

Remark 3.15. It is interesting to note that under Drake's interpretation Theorem 3.14 becomes somewhat more transparent: The two sets Lyn_n and Comb_n are constructed using the same alphabet avoiding two different sets of forbidden links that are in bijection with each other. It is also interesting that both types of forbidden trees have a kind of "shape duality", colored Lyndon trees have forbidden trees that look like "left-combs" and colored combs have forbidden trees that look like "right-combs".

4. Stirling permutations

Now we consider permutations of the multiset $\{1, 1, 2, 2, \cdots, n, n\}$ satisfying the condition that all the numbers between the two occurrences of any fixed number m are larger than m, that is, multiset permutations θ satisfying the following condition:

$$\text{if } i < j < k \text{ and } \theta_i = \theta_k = m \text{ then } \theta_j \geq m. \tag{12}$$

For example the permutation 12234431 satisfies condition (12) but the permutation 11322344 does not ($2 < 3$ and 2 is between the two occurrences of 3). The permutations satisfying condition (12) were introduced by Gessel and Stanley in [10] and are known as *Stirling permutations*. For $n \geq 0$ we will denote the set of Stirling permutations of $\{1, 1, 2, 2, \cdots, n, n\}$ by $\mathcal{Q}_n$.

For $\theta \in \mathcal{Q}_n$, assuming always that $\theta_0 = \theta_{2n+1} = 0$, consider the sets

$$\begin{aligned} \mathrm{DES}(\theta) &= \{i \,|\, \theta_i > \theta_{i+1}\}, \\ \mathrm{ASC}(\theta) &= \{i \,|\, \theta_i < \theta_{i+1}\} \text{ and} \\ \mathrm{PLA}(\theta) &= \{i \,|\, \theta_i = \theta_{i+1}\}. \end{aligned} \tag{13}$$

These are respectively the sets of *descents*, *ascents* and *plateaux* defined in [10] and [2]. Let $\mathrm{des}(\theta) = |\mathrm{DES}(\theta)|$, $\mathrm{asc}(\theta) = |\mathrm{ASC}(\theta)|$ and $\mathrm{pla}(\theta) = |\mathrm{PLA}(\theta)|$ be their cardinalities. It is an immediate observation that the statistics des and asc are equidistributed. For this it is enought to consider the function $\rho : \mathcal{Q}_n \to \mathcal{Q}_n$ that reverses a permutation $\rho(\theta)_i := \theta_{2n+1-i}$. Bóna [2] proved that these two statistics are also equidistributed with pla by showing that they satisfy the same recurrence relation. Janson, Kuba and Panholzer [18] gave a simple combinatorial proof using a bijection of Gessel between Stirling permutations and increasing ternary trees.

The triangle of numbers given by any one of these equidistributed statistics are known as the *second-order Eulerian numbers* (see [15]). This terminology emphasizes that the Stirling permutations are the second case ($r = 2$) of a more general family $\mathcal{Q}_n(r)$ of permutations of the multiset $\{1^r, 2^r, \cdots, n^r\}$ satisfying condition (12). Note that $\mathcal{Q}(1) = \mathfrak{S}_n$ and $\mathcal{Q}_n(2) = \mathcal{Q}_n$. These more general multiset permutations have been also studied (see [24, 23, 22, 18, 20, 16] and [25]) with the name of *r-Stirling permutations* or *r-multipermutations*. We borrow the terminology in [15] and call in general any statistic that is equidistributed with the descent statistic in $\mathcal{Q}_n(r)$ an *rth-order Eulerian statistic*.

From a permutation in $\mathcal{Q}_{n-1}$ we can obtain a permutation in $\mathcal{Q}_n$ by inserting the consecutive labels nn in $2n - 1$ possible positions. We have that $\mathcal{Q}_1 = \{11\}$ and so $|\mathcal{Q}_1| = 1$. By induction we obtain that $|\mathcal{Q}_n| = 1 \cdot 3 \cdots (2n-1) = (2n-1)!!$.

4.1. Type of a Stirling permutation

In this section we define several types associated to a Stirling permutation. These types were introduced in [12].

A *segment* u of a Stirling permutation $\theta = \theta_1\theta_2\cdots\theta_{2n}$ is a subword of θ of the form $u = \theta_i\theta_{i+1}\cdots\theta_{i+\ell}$, i.e., all the letters of u are adjacent in θ. A *block* in a Stirling permutation θ is a segment of θ that starts and ends with the same letter. For example, 455774 is a block of 12245577413366. We define $B_\theta(a)$ to be the block of θ that starts and ends with the letter a, and define $\mathring{B}_\theta(a)$ to be the segment obtained from $B_\theta(a)$ after removing the two occurrences of the letter a. For example, $B_\theta(1) = 1224557741$ in $\theta = 12245577413366$ and $\mathring{B}_\theta(1) = 22455774$.

We call (a, b) an *ascending adjacent pair* of $\theta \in \mathcal{Q}_n$ if $a < b$ and the blocks $B_\theta(a)$ and $B_\theta(b)$ are adjacent in θ, i.e., $\theta = \theta' B_\theta(a) B_\theta(b)\theta''$. An *ascending adjacent sequence* of θ of length k is a subsequence $a_1 < a_2 < \cdots < a_k$ such that (a_j, a_{j+1}) is an ascending adjacent pair for $j = 1, \ldots, k-1$. Similarly, we call (a, b) a *terminally nested pair* if $a < b$ and the block $B_\theta(b)$ is the last block in $\mathring{B}_\theta(a)$, i.e., $\mathring{B}_\theta(a) = \theta' B_\theta(b)$ for some Stirling permutation θ' on a subset of the letters. A *terminally nested sequence* of θ of length k is a subsequence $a_1 < a_2 < \cdots < a_k$ such that (a_j, a_{j+1}) is a terminally nested pair for $j = 1, \ldots, k-1$. If we apply the map ρ that reverses the permutation to the two definitions above we obtain the notions of *descending adjacent* and *initially nested* pairs and sequences.

We can associate a *type* to a Stirling permutation $\theta \in \mathcal{Q}_n$ using the different types of sequences defined above. We define the *ascending adjacent type*

$\lambda^{\mathsf{AA}}(\theta)$, to be the partition whose parts are the lengths of maximal ascending adjacent sequences; the *terminally nested type* $\lambda^{\mathsf{TN}}(\theta)$, to be the partition whose parts are the lengths of maximal terminally nested sequences; the *descending adjacent type* $\lambda^{\mathsf{DA}}(\theta)$, to be the partition whose parts are the lengths of maximal descending adjacent sequences; and the *initially nested type* $\lambda^{\mathsf{IN}}(\theta)$, to be the partition whose parts are the lengths of maximal initially nested sequences.

Example 4.1. If $\theta = 158851244667729933$, the maximal ascending adjacent sequences are 129, 467, 3, 5 and 8 and $\lambda^{\mathsf{AA}}(\theta) = (3, 3, 1, 1, 1)$. The maximal descending adjacent sequences are 1, 2, 93, 4, 6, 7, 5 and 8 and $\lambda^{\mathsf{DA}}(\theta) = (2, 1, 1, 1, 1, 1, 1, 1)$. The maximal terminally nested sequences are 158, 27, 3, 4, 6 and 9 and $\lambda^{\mathsf{TN}}(\theta) = (3, 2, 1, 1, 1, 1)$. The maximal initially nested sequences are 158, 24, 6, 7, 9 and 3 and $\lambda^{\mathsf{IN}}(\theta) = (3, 2, 1, 1, 1, 1)$. All of them are partitions of $n = 9$.

Note that since every descent in θ occurs at the end of a maximal ascending adjacent sequence then $\ell(\lambda^{\mathsf{AA}}(\theta)) = \mathrm{des}(\theta)$ where $\ell(\lambda)$ indicates the number of parts of a partition λ. So λ^{AA} is a refinement of the des statistic. In the same manner since each plateau can be considered as occurring either at the end of a maximal terminally nested sequence or at the end of a maximal initially nested sequence then $\ell(\lambda^{\mathsf{TN}}(\theta)) = \ell(\lambda^{\mathsf{IN}}(\theta)) = \mathrm{pla}(\theta)$. By a similar argument we have that $\ell(\lambda^{\mathsf{DA}}(\theta)) = \mathrm{asc}(\theta)$. Table 1 gives the values of des, asc, pla, λ^{AA}, λ^{DA}, λ^{TN} and λ^{IN} for $n = 3$. For $n = 3$ it happens that λ^{TN} and λ^{IN} are equal but this is not true in general.

The following proposition was proved in [12].

Proposition 4.2 ([12, Proposition 4.6])**.** *There is a bijection* $\xi : \mathcal{Q}_n \to \mathcal{Q}_n$ *that satisfies:*

1. (i, j) *is an ascending adjacent pair in* θ *if and only if* (i, j) *is a terminally nested pair in* $\xi(\theta)$*,*
2. $\lambda^{\mathsf{TN}}(\xi(\theta)) = \lambda^{\mathsf{AA}}(\theta)$*.*

From Proposition 4.2 we see that λ^{AA} and λ^{TN} are equidistributed on $\mathcal{Q}_n$. Since the map ρ that reverses θ (defined above) also implies $\lambda^{\mathsf{AA}} \cong \lambda^{\mathsf{DA}}$ and $\lambda^{\mathsf{TN}} \cong \lambda^{\mathsf{IN}}$ we have as corollaries the following two results.

Theorem 4.3. *The types* λ^{AA}*,* λ^{DA}*,* λ^{TN} *and* λ^{IN} *are equidistributed.*

Corollary 4.4 ([2])**.** *The statistics* des*,* asc *and* pla *are equidistributed.*

Remark 4.5. Theorem 4.3 can be proved in a different way using a bijection of Gessel between Stirling permutations and increasing planar ternary trees (see [18]). The idea is that from the perspective of increasing planar ternary

Table 1: Stirling permutations and statistics for $n = 3$

θ	des(θ)	asc(θ)	pla(θ)	$\lambda^{\mathsf{AA}}(\theta)$	$\lambda^{\mathsf{DA}}(\theta)$	$\lambda^{\mathsf{TN}}(\theta)$	$\lambda^{\mathsf{IN}}(\theta)$
112233	1	3	3	(3)	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)
113322	2	2	3	(2, 1)	(2, 1)	(1, 1, 1)	(1, 1, 1)
221133	2	2	3	(2, 1)	(2, 1)	(1, 1, 1)	(1, 1, 1)
223311	2	2	3	(2, 1)	(2, 1)	(1, 1, 1)	(1, 1, 1)
331122	2	2	3	(2, 1)	(2, 1)	(1, 1, 1)	(1, 1, 1)
122331	2	3	2	(2, 1)	(1, 1, 1)	(2, 1)	(2, 1)
112332	2	3	2	(2, 1)	(1, 1, 1)	(2, 1)	(2, 1)
133122	2	3	2	(2, 1)	(1, 1, 1)	(2, 1)	(2, 1)
122133	2	3	2	(2, 1)	(1, 1, 1)	(2, 1)	(2, 1)
133221	3	2	2	(1, 1, 1)	(2, 1)	(2, 1)	(2, 1)
221331	3	2	2	(1, 1, 1)	(2, 1)	(2, 1)	(2, 1)
233211	3	2	2	(1, 1, 1)	(2, 1)	(2, 1)	(2, 1)
331221	3	2	2	(1, 1, 1)	(2, 1)	(2, 1)	(2, 1)
123321	3	3	1	(1, 1, 1)	(1, 1, 1)	(3)	(3)
332211	3	1	3	(1, 1, 1)	(3)	(1, 1, 1)	(1, 1, 1)

trees (reading the types after the bijection), the equidistributivity of the types λ^{AA}, λ^{DA}, λ^{TN} and λ^{IN} is a consequence of the bijection on the set of increasing planar ternary trees defined by reordering simultaneously the 3 children of every internal node of a ternary tree using a fixed permutation. Since there are 6 permutations in $\mathfrak{S}_3$, proving Theorem 4.3 in this way also reveals that there are two other different types equidistributed with the ones discussed here. We leave the details of this proof to the reader.

The link between normalized trees and Stirling permutations is given by the following proposition in [12].

Proposition 4.6 ([12, Proposition 4.8])**.** *There is a bijection*[1] $\gamma : \mathsf{Nor}_n \to \mathcal{Q}_{n-1}$ *that satisfies for each* $\Upsilon \in \mathsf{Nor}_n$

1. $\lambda^{\mathsf{AA}}(\gamma(\Upsilon)) = \lambda^{\mathsf{Lyn}}(\Upsilon)$,
2. $\lambda^{\mathsf{TN}}(\gamma(\Upsilon)) = \lambda^{\mathsf{Comb}}(\Upsilon)$.

Proof of Theorem 2.5. This is a corollary of the results in Propositions 4.2 and 4.6; and Theorems 3.9 and 3.12. ☐

4.2. r-Stirling permutations or r-multipermutations

An r- *Stirling permutation* or r-*multipermutation* is a permutation of the multiset $\{1^r, 2^r, \cdots, n^r\}$ satisfying condition (12). We denote the set of r-

[1]This bijection appeared first in [4].

Stirling permutations by $\mathcal{Q}_n(r)$. For example, 123332215551466644 is in $\mathcal{Q}_n(3)$. In particular $\mathcal{Q}_n(2) = \mathcal{Q}_n$ and $\mathcal{Q}_n(1) = \mathfrak{S}_n$. We want to extend the right-hand side of equation (2) to the generality of r-Stirling permutations.

For a permutation $\theta \in \mathcal{Q}_n(r)$, assuming always that $\theta_0 = \theta_{rn+1} = 0$, we define the sets DES and ASC and the statistics des and asc as before. Let $n(\theta, i)$ denote the number of occurrences of the label θ_i in the subword $\theta_1\theta_2 \cdots \theta_i$. We use a refinement of PLA defined in [18], the set

$$\mathrm{PLA}_j(\theta) = \{i \,|\, \theta_i = \theta_{i+1},\, n(\theta, i) = j\}$$

of *j-plateaux* (plateaux between the occurrences j and $j+1$ of a label) and the statistic $\mathrm{pla}_j(\theta) = |\mathrm{PLA}_j(\theta)|$ its cardinality. In [18] it is shown that des, asc and pla_j are equidistributed in $\mathcal{Q}_n(r)$.

4.2.1. Types on the set of r-Stirling permutations A *block* $B_\theta(a)$ in an r-Stirling permutation θ is a segment of θ that starts and ends with a and contains all the occurrences of a in θ. For example, $B_\theta(1) = 122214555441$ is a block of $\theta = 122214555441333$. Removing all occurrences of a in $B_\theta(a)$ gives a sequence $\mathring{B}_\theta(a)$ of (possibly empty) r-Stirling permutations $\mathring{B}_\theta(a)_j$ for $j = 1, \dots, r-1$. For example $\mathring{B}_\theta(1) = (222, 455544)$.

We call (a, b) an *ascending adjacent pair* in $\theta \in \mathcal{Q}_n(r)$ if $a < b$ and the blocks $B_\theta(a)$ and $B_\theta(b)$ are adjacent in θ, i.e., $\theta = \theta' B_\theta(a) B_\theta(b) \theta''$. An *ascending adjacent sequence* of θ of length k is a subsequence $a_1 < a_2 < \cdots < a_k$ such that (a_j, a_{j+1}) is an ascending adjacent pair for $j = 1, \dots, k-1$. Similarly, for $j \in [r-1]$ we call (a, b) a *j-terminally nested pair* if $a < b$ and the block $B_\theta(b)$ is the last block in $\mathring{B}_\theta(a)_j$, i.e., $\mathring{B}_\theta(a)_j = \theta' B_\theta(b)$ for some r-Stirling permutation θ'. A *j-terminally nested sequence* of θ of length k is a subsequence $a_1 < a_2 < \cdots < a_k$ such that (a_s, a_{s+1}) is a j-terminally nested pair for $s = 1, \dots, k-1$. If we apply the map ρ that reverses the permutation to the two definitions above we obtain the notions of *descending adjacent* and *j-initially nested* pairs and sequences.

We then associate a *type* to an r-Stirling permutation $\theta \in \mathcal{Q}_n(r)$ in different ways according to the lengths of maximal sequences of a given type as before. We define in this way the *ascending adjacent type* $\lambda^{\mathsf{AA}}(\theta)$, the *$j$-terminally nested type* $\lambda^{\mathsf{TN}}_j(\theta)$, the *descending adjacent type* $\lambda^{\mathsf{DA}}(\theta)$ and the *j-initially nested type* $\lambda^{\mathsf{IN}}_j(\theta)$. Note that similar to the case $r = 2$, in the general case λ^{AA} refines des, λ^{DA} refines asc and both λ^{TN}_j and λ^{IN}_j refine pla_j.

The proof of the following theorem is similar to the proof of Theorem 4.3 in [12].

Theorem 4.7. *The types λ^{AA}, λ^{DA}, λ^{TN}_j and λ^{IN}_j for all $j = 1, \dots, r-1$ are equidistributed.*

Remark 4.8. We can also prove Theorem 4.7 following the same idea discussed in Remark 4.5. This time we use instead a more general bijection of Gessel between r-Stirling permutations and increasing planar $(r+1)$-ary trees.

With the more general definitions in place we can consider the family of symmetric functions

$$\mathrm{SP}_n^{(r)}(\mathbf{x}) = \sum_{\theta \in \mathcal{Q}_n(r)} e_{\lambda(\theta)}(\mathbf{x}), \tag{14}$$

where $\lambda(\theta)$ is any of the types of θ defined above.

The question is to determine if this more general definition provides interesting results and has any combinatorial applications for some $r \neq 2$. We will show in Section 5 that for $r = 1$ there is a positive answer with a very similar story to the one for $r = 2$.

5. The case $r = 1$ of standard permutations of $[n]$

In this section we prove Theorem 2.4. To prove this theorem we first introduce a combinatorial interpretation of the multiplicative inverse of an exponential generating function in terms of words with allowed and forbidden links that follows from a theorem discovered by Fröberg [6], Carlitz-Scoville-Vaughan [3] and Gessel [11]. The theory outlined in [3] and [11] is more general and applies to a larger family of counting algebras (as defined in [11]) and not only to exponential generating functions. Here we give a simplified description that applies to the exponential generating function result.

5.1. Combinatorial interpretation of the multiplicative inverse

We consider permutations of any finite label set $A \subset \mathbb{P}$. In particular any label by itself and the empty word $\emptyset$ are examples of permutations. For any two permutations w_1 and w_2 that have disjoint label sets $A_1, A_2 \subset \mathbb{P}$ we define the *product* $w_1 w_2$ to be the permutation with label set $A_1 \cup A_2$ constructed by concatenation. If these conditions are not satisfied the product is not defined. For example if $w_1 = 1345$ and $w_2 = 276$ then $w_1 w_2 = 1345276$. Note that the product is associative and so expressions like $w_1 w_2 \cdots w_k$ are well defined. Two permutations w_1 and w_2 with label sets $A_1, A_2 \subset \mathbb{P}$ such that $|A_1| = |A_2|$ are said to be *equivalent*, and we write $w_1 \sim w_2$, if we can obtain w_2 from w_1 by replacing the labels in w_1 according to the unique order preserving bijection between A_1 and A_2. If $\mathcal{A}$ is a set of permutations, $\mathcal{A}$ is said to have the *label substitution property* if whenever $w_1 \sim w_2$

then $w_1 \in \mathcal{A}$ if and only if $w_2 \in \mathcal{A}$. In a set $\mathcal{A}$ of permutations, we call $w \in \mathcal{A}$ *irreducible* if w is a nonempty permutation such that $w = w_1 w_2$ and $w_1, w_2 \in \mathcal{A}$ imply either $w_1 = w$ or $w_2 = w$, i.e., a permutation that cannot be decomposed as the concatenation of other permutations in $\mathcal{A}$. A set $\mathcal{A}$ of permutations is said to have the *unique decomposition property* if all its permutations are irreducible. We call an *alphabet* a set $\mathcal{A}$ of permutations that has the label substitution property and the unique decomposition property. The elements of $\mathcal{A}$ are called the *letters* of the alphabet. The set of permutations (including the empty permutation) that can be constructed as the product of letters in $\mathcal{A}$ is denoted $\mathcal{A}^*$. We can also consider alphabets $\mathcal{A}_S$ that are formed by colored letters, i.e., pairs (a, s) where $a \in \mathcal{A}$ and $s \in S$ for some set S. We call a *link* the product of two (colored) letters. Assume that the set $\mathcal{A}_S$ of colored letters is partitioned into equivalence classes and let K be the set of equivalence classes. For a permutation $w \in \mathcal{A}_S^*$ we denote $m_j(w)$ the number of letters from the class $j \in K$ that are present in w and $|w| := \sum_{j \in K} m_j(w)$ the *length* of w. Consider a partition of the set of links into two parts that we call the *allowed links* $\mathcal{L}(\mathcal{A}_S)$ and *forbidden links* $\overline{\mathcal{L}(\mathcal{A}_S)}$. Let $\mathcal{W}_S^n$ be the set of permutations with underlying label set $[n]$ for $n \geq 0$ constructed with only allowed links and let $\overline{\mathcal{W}_S}^n$ the ones constructed using only forbidden links. Define $\mathcal{W}_S = \cup_{n \geq 0} \mathcal{W}_S^n$ and $\overline{\mathcal{W}_S} = \cup_{n \geq 0} \overline{\mathcal{W}_S}^n$. In particular, $\mathcal{W}_S^0 = \overline{\mathcal{W}_S}^0 = \emptyset$ and we consider letters in $\mathcal{A}_S$ as if they are both in $\mathcal{W}_S$ and $\overline{\mathcal{W}_S}$.

Define the monomials

$$X^{m(w)} = \prod_{j \in K} x_j^{m_j(w)},$$

and the generating functions

$$F(y) = \sum_{n \geq 0} \sum_{w \in \mathcal{W}_S^n} X^{m(w)} \frac{y^n}{n!},$$

$$\overline{F}(y) = \sum_{n \geq 0} \sum_{w \in \overline{\mathcal{W}_S}^n} (-1)^{|w|} X^{m(w)} \frac{y^n}{n!}.$$

Theorem 5.1 (cf. [11])**.** *We have*

$$F^{-1}(y) = \overline{F}(y).$$

For the sake of completeness we provide a proof of Theorem 5.1. The idea of the proof is the one that appears in [11] where a more general version of this theorem is proved.

To prove Theorem 5.1 we consider the set of permutations of the form $w = w_1 w_2$ where $w_1 \in \mathcal{W}_S$ and $w_2 \in \overline{\mathcal{W}_S}$. Note that if $w \neq \emptyset$ then w has exactly two different factorizations. Indeed, if $w = a_1 a_2 \ldots a_n$ then either there is a number $1 \leq k < n$ such that $a_k a_{k+1} \in \mathcal{L}(\mathcal{A}_S)$ but $a_{k+1} a_{k+2} \in \overline{\mathcal{L}(\mathcal{A}_S)}$, or $a_i a_{i+1} \in \mathcal{L}(\mathcal{A}_S)$ for all i (in which case we let $k = n$) or $a_i a_{i+1} \in \overline{\mathcal{L}(\mathcal{A}_S)}$ for all i (in which case we let $k = 0$). Assuming that $a_0 = a_{n+1} = \emptyset$ the two valid factorizations of w are

$$(a_0 \ldots a_{k+1})(a_{k+2} \ldots a_{n+1}) \text{ and } (a_0 \ldots a_k)(a_{k+1} \ldots a_{n+1}).$$

Here we want to consider the two different factorizations of w as different objects. We call these objects (a permutation w together with its factorization) $(\mathcal{W}_S, \overline{\mathcal{W}_S})$*-composite permutations.*

Lemma 5.2. *The multiplication $F(y)\overline{F}(y)$ is the exponential generating function for $(\mathcal{W}_S, \overline{\mathcal{W}_S})$-composite permutations w weighted by $(-1)^{m_f} \mathbf{x}^{m(w)}$ where m_f is the number of letters in the forbidden permutation.*

Proof. This follows from the combinatorial interpretation of multiplication of exponential generating functions in [28, Proposition 5.1.1]. ☐

Proof of Theorem 5.1. Using Lemma 5.2 we only need to show that the weighted exponential generating function for $(\mathcal{W}_S, \overline{\mathcal{W}_S})$-composite permutations is equal to 1. We define a sign-reversing involution ι on the set of $(\mathcal{W}_S, \overline{\mathcal{W}_S})$-composite permutations where the only fixed point is the empty permutation (whose factorization is unique). Let w be a $(\mathcal{W}_S, \overline{\mathcal{W}_S})$-composite permutation that is nonempty. By the comments above we know that the underlying permutation of w can be associated with two different factorizations w and w'. We then define $\iota(w) = w'$ if $w \neq \emptyset$ and $\iota(\emptyset) = \emptyset$. This process is an involution that reverses the sign as defined in Lemma 5.2 since w and w' differ by one in the number of letters in the forbidden permutation. ☐

5.2. Colored permutations

Let S be any subset of $\mathbb{P}$. A *colored permutation* is a permutation $\sigma \in \mathfrak{S}_n$ in which each letter $j \in [n]$ has been asigned a color $\mathbf{color}(j) \in S$ with the condition that for every occurrence of an ascending adjacent pair $\sigma(j) < \sigma(j+1)$ in σ it must happen that $\mathbf{color}(\sigma(j)) > \mathbf{color}(\sigma(j+1))$. For example for $S = [3]$, $\sigma = 1^3 2^2 4^1 3^3 5^2$ is a colored permutation with the colored letters $(i, \mathbf{color}(i))$ represented as $i^{\mathbf{color}(i)}$. Since in any ascending adjacent sequence of $\sigma \in \mathfrak{S}_n$ the colors need to strictly decrease, $e_{\lambda^{\mathsf{AA}}(\sigma)}(\mathbf{x})$ enumerates the colored permutations with colors in $S = \mathbb{P}$ and underlying uncolored permutation σ.

Proof of Theorem 2.4. Let $\mathcal{A}_S$ be the alphabet with colored letters a^c where $a \in [n]$ and $c \in \mathbb{P}$. Consider the set of forbidden links $\overline{\mathcal{L}(\mathcal{A}_s)}$ to be of the form $a^{c_1}b^{c_2}$ with $a < b$ and $c_1 \leq c_2$. The forbidden permutations are of the form $1^{c_1}2^{c_2}\cdots n^{c_n}$ with $c_1 \leq c_2 \leq \cdots \leq c_n$. Since each of the forbidden colored permutations is completely determined after selecting a multiset of colors, the generating polynomial of the forbidden permutations is $h_n(\mathbf{x})$ and

$$\overline{F}(y) = \sum_{n\geq 0}(-1)^n h_n(\mathbf{x})\frac{y^n}{n!}.$$

The allowed permutations are colored permutations whose generating function corresponds by the comments above to the right-hand side of equation (1). Applying Theorem 5.1 completes the proof of this theorem. □

6. Specializations

Since the $e_i(\mathbf{x})$ are algebraically independent generators for Λ, we can define an algebra homomorphism or *specialization* $E : \Lambda \to \mathbb{Q}[t]$ by $E(e_i(\mathbf{x})) = t$ for all $i \geq 1$ and $E(1) = 1$. It is not hard to verify that the specialization E extends to a specialization $\tilde{E} : \Lambda[[y]] \to \mathbb{Q}[t][[y]]$ in the algebra of power series in y with symmetric function coefficients in Λ (with variables $\mathbf{x}$) defined by applying E coefficientwise. Moreover, it is also true and easy to verify that $\tilde{E}$ is a monoid homomorphism $(\Lambda[[y]], \circ) \to (\mathbb{Q}[[y]], \circ)$, where $\circ$ indicates composition of power series, i.e., for power series $f, g \in \Lambda[[y]]$ $\tilde{E}(f[g(y)]) = \tilde{E}(f)[\tilde{E}(g)(y)]$.

Lemma 6.1. *For every $n \geq 1$,*

$$E(h_n(\mathbf{x})) = t(t-1)^{n-1}. \tag{15}$$

Proof. Using Proposition 2.1 and the definition of E we have

$$\begin{aligned} E\left(\sum_{n\geq 0} h_n(\mathbf{x})y^n\right) &= E\left(\left(\sum_{n\geq 0}(-1)^n e_n(\mathbf{x})y^n\right)^{-1}\right) \\ &= \left(1 + \sum_{n\geq 1}(-1)^n t y^n\right)^{-1} \\ &= \left(1 - \frac{yt}{1+y}\right)^{-1} \end{aligned}$$

$$= \frac{1+y}{1-y(t-1)}$$
$$= 1 + \frac{yt}{1-y(t-1)}$$
$$= 1 + \sum_{n\geq 1} t(t-1)^{n-1}y^n. \qquad \square$$

Recall from Section 4 that $\ell(\lambda^{\mathsf{AA}}(\theta)) = \operatorname{des}(\theta)$ and note that for any $\lambda \vdash n$ we have

$$E(e_\lambda(\mathbf{x})) = t^{\ell(\lambda)}. \tag{16}$$

Using these observations we can apply the specialization E to definition (14) to obtain

$$\begin{aligned} E(\mathrm{SP}_n^{(r)}(\mathbf{x})) &= E\left(\sum_{\theta\in\mathcal{Q}_n(r)} e_{\lambda^{\mathsf{AA}}(\theta)}(\mathbf{x})\right) \\ &= \sum_{\theta\in\mathcal{Q}_n(r)} t^{\ell(\lambda^{\mathsf{AA}}(\theta))} \\ &= \sum_{\theta\in\mathcal{Q}_n(r)} t^{\operatorname{des}(\theta)} \\ &:= A_n^{(r)}(t), \end{aligned}$$

the *r-th order Eulerian polynomial*.

Remark 6.2. Applying the specialization $\tilde{E}$ to equation (2) and using equations (16) and (15) we obtain as a corollary Theorem 2.7. In the same manner, applying $\tilde{E}$ to equation (1) we obtain as a corollary Riordan's result (Theorem 2.6).

7. Connections and future directions

In this section we discuss some instances where the functions $\mathrm{SP}_n^{(r)}(\mathbf{x})$ appear for the cases $r = 1$ and $r = 2$.

7.1. Multiplicative inverse and Lagrange inversion

Propositions 2.1 and 2.2, and Theorems 2.4 and 2.5 are closely related to the problem of finding multiplicative and compositional inverses of general exponential generating functions.

We denote $\widetilde{\mathrm{SP}}_n^{(r)}(h_1, h_2, \dots)$ the symmetric function $\mathrm{SP}_n^{(r)}(\mathbf{x})$ written as a polynomial in the generators $h_i(\mathbf{x})$. Let

$$F(y) = \sum_{n\geq 0} f_n \frac{y^n}{n!}, \tag{17}$$

where the f_n are in some commutative ring A.

Recall that the multiplicative inverse F^{-1} exists if and only if f_0 is a unit in A. The compositional inverse $F^{\langle -1\rangle}$ exists if and only if $f_0 = 0$ and f_1 is a unit in A. The reader can check that in the former case, when we apply the specialization $h_n \mapsto f_n/f_0$ for $n \geq 0$ to equation (1) we obtain that the n-th coefficient in the power series F^{-1} is

$$n![x^n]F^{-1}(y) = (-1)^n f_0^{-1}\widetilde{\mathrm{SP}}_n^{(1)}(f_1/f_0, f_2/f_0, \dots). \tag{18}$$

In the latter case, after applying the specialization $h_n \mapsto f_{n+1}/f_1$ for $n \geq 1$ to equation (2) we obtain that the n-th coefficient of $F^{\langle -1\rangle}$ is

$$n![x^n]F^{\langle -1\rangle}(y) = (-1)^{n-1} f_1^{-n}\widetilde{\mathrm{SP}}_{n-1}^{(2)}(f_2/f_1, f_3/f_1, \dots). \tag{19}$$

Equations (18) and (19) contain another interesting piece of information. Let $f_n = \mathrm{SP}_n^{(1)}(\mathbf{x})$ in (18) and $f_n = \mathrm{SP}_{n-1}^{(2)}(\mathbf{x})$ in (19) for all n. Then Theorems 2.4 and 2.5 imply that the left-hand side of (18) becomes $h_n(\mathbf{x})$ and the left-hand side of (19) becomes $h_{n-1}(\mathbf{x})$ respectively. Since the symmetric functions $h_n(\mathbf{x})$ for $n \geq 1$ form a set of generators of the ring $\Lambda_{\mathbb{Q}}$, we get as a corollary that the symmetric functions $\mathrm{SP}_n^{(1)}(\mathbf{x})$ and $\mathrm{SP}_n^{(2)}(\mathbf{x})$ also generate $\Lambda_{\mathbb{Q}}$. For a partition $\lambda \vdash n$ define

$$\mathrm{SP}_\lambda^{(r)}(\mathbf{x}) := \mathrm{SP}_{\lambda_1}^{(r)}(\mathbf{x})\,\mathrm{SP}_{\lambda_2}^{(r)}(\mathbf{x}) \cdots \mathrm{SP}_{\lambda_{\ell(\lambda)}}^{(r)}(\mathbf{x}).$$

Theorem 7.1. *For $n \geq 0$, the sets $\{\mathrm{SP}_\lambda^{(1)}(\mathbf{x}) \mid \lambda \vdash n\}$ and $\{\mathrm{SP}_\lambda^{(2)}(\mathbf{x}) \mid \lambda \vdash n\}$ are bases for the n-th homogeneous graded component of $\Lambda_{\mathbb{Q}}$.*

7.2. Poset (co)homology

The symmetric functions $\mathrm{SP}_n^{(1)}(\mathbf{x})$ and $\mathrm{SP}_n^{(2)}(\mathbf{x})$ also appear in the context of poset topology.

A *weighted subset* of $[n]$ is a pair (A, μ) where $A \subseteq [n]$ and $\mu \in \mathrm{wcomp}$. We will use A^μ to denote (A, μ). A *weighted partition* of $[n]$ is a collection $\{B_1^{\nu_1}, B_2^{\nu_2}, ..., B_t^{\nu_t}\}$ where $\{B_1, B_2, ..., B_t\}$ is a set partition of $[n]$ and $\nu_i \in$

$\text{wcomp}_{|B_i|-1}$ for all i. For $\nu, \mu \in \text{wcomp}$ we say that $\nu \leq \mu$ if $\nu(i) \leq \mu(i)$ for every i. The *poset of weighted partitions* Π_n^w is the set of weighted partitions of $[n]$ with order relation given by $\{A_1^{\mu_1}, A_2^{\mu_2}, ..., A_s^{\mu_t}\} \leq \{B_1^{\nu_1}, B_2^{\nu_2}, ..., B_t^{\nu_t}\}$ if the following conditions hold:

- $\{A_1, A_2, ..., A_s\}$ is a refinement of $\{B_1, B_2, ..., B_t\}$ and,
- If $B_j = A_{i_1} \cup A_{i_2} \cup ... \cup A_{i_l}$ then $\mu_{i_1} + \mu_{i_2} + ... + \mu_{i_l} \leq \nu_j$.

The poset Π_n^w has one minimal element $\hat{0} := \{1^{\mathbf{0}}, 2^{\mathbf{0}}, \ldots, n^{\mathbf{0}}\}$, where $\mathbf{0} = (0, 0, \ldots)$, and maximal elements $[n]^\mu := \{[n]^\mu\}$ indexed by weak compositions $\mu \in \text{wcomp}_{n-1}$. The following result was found in [12] using poset topology techniques (see [12] for all the definitions).

Theorem 7.2 ([12]). *For all $n \geq 1$,*

$$\sum_{\mu \in \text{wcomp}_{n-1}} \mu_{\Pi_n^w}((\hat{0}, [n]^\mu))\mathbf{x}^\mu = (-1)^{n-1}\text{SP}_{n-1}^{(2)}(\mathbf{x}),$$

where $\mu_{\Pi_n^w}((\hat{0}, [n]^\mu))$ is the Möbius invariant of the maximal interval $(\hat{0}, [n]^\mu)$.

It was also proved in [12] that there is an $\mathfrak{S}_n$-module isomorphism

$$\mathcal{L}ie(\mu) \simeq_{\mathfrak{S}_n} \widetilde{H}^{n-3}((\hat{0}, [n]^\mu)) \otimes \text{sgn}_n$$

for all $\mu \in \text{wcomp}_{n-1}$, where $\mathcal{L}ie(\mu)$ is the multilinear component determined by μ of the free Lie algebra with multiple compatible brackets, the vector space $\widetilde{H}^{n-3}((\hat{0}, [n]^\mu))$ is the reduced cohomology of the interval $(\hat{0}, [n]^\mu)$ and sgn_n is the sign representation of the symmetric group $\mathfrak{S}_n$. The following theorem is a corollary of this isomorphism, Philip Hall's theorem, Theorem 7.2 and the fact that Π_n^w is Cohen-Macaulay (see [12]).

Theorem 7.3 ([12]). *For all $n \geq 1$,*

$$\sum_{\mu \in \text{wcomp}_{n-1}} \dim \mathcal{L}ie(\mu)\mathbf{x}^\mu = \text{SP}_{n-1}^{(2)}(\mathbf{x}),$$

where $\dim V$ is the dimension of the vector space V.

It turns out that $\text{SP}_{n-1}^{(1)}(\mathbf{x})$ also makes an appearance in the context of poset topology in subsequent work of the author [13].

Let $\mathbb{B}_n$ be the boolean algebra (the poset of subsets of $[n]$ ordered by inclusion) and $\mathbb{WCOMP}_n$ be the poset formed by weak compositions μ such that $|\mu| \leq n$ together with the order relation defined before. Both posets are ranked and hence have well-defined poset maps $rk : \mathbb{B}_n \to C_{n+1}$ and

$rk : \mathbb{WCOMP}_n \to C_{n+1}$ to the $n+1$ chain C_{n+1}. Recall that the *Segre or fiber product* $A \overset{\times}{_{f,g}} B$ of two poset maps $f : A \to C$ and $g : B \to C$ is the induced subposet of the product $A \times B$ with elements $\{(a,b) | \, f(a) = g(b)\}$. Denote $\mathbb{B}_n^w = \mathbb{B}_n \overset{\times}{_{rk,rk}} \mathbb{WCOMP}_n$ the *poset of weighted subsets* of $[n]$. The elements of $\mathbb{B}_n^w$ are of the form A^μ with $A \subseteq [n]$ and $\mu \in \mathrm{wcomp}_{|A|}$. This poset has one minimal element $\hat{0} := \varnothing^{\mathbf{0}}$ and maximal elements $[n]^\mu$ indexed by weak compositions $\mu \in \mathrm{wcomp}_n$. The following theorem is proved in [13].

Theorem 7.4 ([13]). *For all $n \geq 0$,*

$$\sum_{\mu \in \mathrm{wcomp}_n} \mu_{\mathbb{B}_n^w}((\hat{0}, [n]^\mu)) \mathbf{x}^\mu = (-1)^n \mathrm{SP}_n^{(1)}(\mathbf{x}).$$

There is a natural colored extension of the exterior algebra $\Lambda(V)$ over a vector space V that provides an analogous version of Theorem 7.3. In this case we have that the multilinear components $\Lambda(\mu)$ of the colored exterior algebra satisfy the $\mathfrak{S}_n$-isomorphism

$$\Lambda(\mu) \simeq_{\mathfrak{S}_n} \widetilde{H}^{n-2}((\hat{0}, [n]^\mu))$$

for all $\mu \in \mathrm{wcomp}_n$, where $[\hat{0}, [n]^\mu]$ is the closed maximal interval in $\mathbb{B}_n^w$ determined by μ.

Theorem 7.5 ([13]). *For all $n \geq 1$,*

$$\sum_{\mu \in \mathrm{wcomp}_n} \dim \Lambda(\mu) \mathbf{x}^\mu = \mathrm{SP}_n^{(1)}(\mathbf{x}).$$

Remark 7.6. Both Theorem 2.5 and Theorem 2.4 can also be derived from Theorem 7.2 and 7.4 using the recursive definition of the Möbius invariant, see [12, 13].

7.3. Stable n-pointed curves

In [19] another surprising connection with the symmetric functions $\mathrm{SP}_n^{(2)}(\mathbf{x})$ appeared in the context of moduli spaces $\overline{M}_{0,n}$ of stable n-pointed curves of genus 0. See [19] for the proper definitions and notation. Let $\omega_n(i) \in H^{2i}(\overline{M}_{0,n}, \mathbb{Q})$ denote the Mumford classes. For a partition λ denote $m_i(\lambda)$ the number of times the part i appears in λ and denote $\omega_n^\lambda = \prod_i \omega_n(i)^{m_i(\lambda)}$. The *higher Weil-Petersson volumes* are defined as

$$WP(\lambda) = \int_{\overline{M}_{0,n}} \omega_n^\lambda.$$

The following theorem follows directly from Theorem 2.5 and [19, Theorem 2.4] after making the identifications $s_k \mapsto -\frac{p_k(\mathbf{x})}{k}$ and $y \mapsto -y$ (s_k a formal variable and $p_k(\mathbf{x})$ the power sum symmetric function) and writing down the proper definitions.

Theorem 7.7. *For $n \geq 0$*

$$\mathrm{SP}_n^{(2)}(\mathbf{x}) = \sum_{\lambda \vdash n} (-1)^{n-1-l(\lambda)} WP(\lambda) \frac{p_\lambda(\mathbf{x})}{z_\lambda}$$

In [19] the authors also provide a recursive definition, a differential equation and the following closed formula for the coefficients $WP(\lambda)$. For partitions $\nu_1, \nu_2 \dots, \nu_k$ denote $\nu_1 + \nu_2 + \cdots + \nu_k$ the partition obtained by taking the union of all the parts of all the ν_i.

Theorem 7.8 ([19] Corollary 2.3)**.** *For $n \geq 0$ and $\lambda \vdash n$*

$$WP(\lambda) = n! \sum_{k=0}^{l(\lambda)} (-1)^{l(\lambda)-k} \binom{n+k}{k} \sum_{\substack{\nu_1,\dots,\nu_k \\ \nu_1+\cdots+\nu_k=\lambda \\ \nu_i \neq 0}} \frac{\prod_{j=1}^{l(\lambda)} \binom{m_j(\lambda)}{m_j(\nu_1), m_j(\nu_2) \dots, m_j(\nu_k)}}{\prod_{i=1}^{k} (|\nu_i|+1)!}.$$

7.4. Open questions

Theorems 2.4 and 2.5 say that $\mathrm{SP}_n^{(1)}(\mathbf{x})$ and $\mathrm{SP}_n^{(2)}(\mathbf{x})$ are involved in the computation of multiplicative and compositional inverses of power series. According to Theorem 7.1 these functions are generators of $\Lambda_{\mathbb{Q}}$. They are also the generating functions for the Möbius invariants of the maximal intervals of the posets B_n^w (Theorem 7.4) and Π_n^w (Theorem 7.2), respectively. They are the dimension generating functions of the multilinear components of the free colored exterior algebra (Theorem 7.5) and the free multibracketed Lie algebra (Theorem 7.3). Finally, $\mathrm{SP}_n^{(2)}(\mathbf{x})$ is the generating function (viewed in the p basis expansion) of the generalized Weil-Petersson volumes of the moduli space $\overline{M}_{0,n}$ (Theorem 7.7). The question now is to understand whether some or all of these results extend to the more general family of symmetric functions $\mathrm{SP}_n^{(r)}(\mathbf{x})$ for $r \geq 3$.

Question 7.9. *Is there a more general family of posets $P(n,r)$ such that the weighted generating function for the Möbius invariant of maximal intervals is given up to sign by $\mathrm{SP}_n^{(r)}(\mathbf{x})$ as in Theorems 7.2 and 7.4?*

Question 7.10. *Is there any combinatorial context where the functions* $\mathrm{SP}_n^{(r)}(\mathbf{x})$ *are meaningful for any* $r \geq 3$*? Are there formulas similar to equations (1) and (2) for the* $\mathrm{SP}_n^{(r)}(\mathbf{x})$ *when* $r \geq 3$*?*

Question 7.11. *Are the sets* $\{\mathrm{SP}_\lambda^{(r)}(\mathbf{x}) \mid \lambda \vdash n\}$ *bases for the* n*-th homogeneous graded component of* $\Lambda_{\mathbb{Q}}$ *for every* $r \geq 3$*?*

As of the time of the construction of this article we do not know of any partial result in any of these directions. One central question is whether the family of r-Stirling permutations for $r \geq 3$ is the right family of multipermutations to extend the results in this work or if another family of multipermutations generalizing both $\mathfrak{S}_n$ and $\mathcal{Q}_n$ is needed.

7.5. Further work

Theorem 2.5 can be generalized in a different direction by considering families of normalized k-ary trees that satisfy a certain coloring condition. This generalization however does not include Theorem 2.4 as a special case. We present and explore this generalization in a future article.

Acknowledgments

The author is grateful to Michelle Wachs for her guidance during this project and all the valuable discussions. The author also would like to thank Ira Gessel and François Bergeron for the valuable discussions and useful comments to improve both the context and the presentation of the results. The author also appreciates the very helpful comments of the anonymous referees. In particular, one of the referees suggested the simpler proof of Lemma 6.1 that was included in the final version. A great part of this work was done while the author was supported by NSF Grant DMS 1202755 and by Colciencias (Departamento Administrativo de Ciencia, Tecnología e Innovación).

References

[1] F. Bergeron, G. Labelle, and P. Leroux. *Combinatorial species and tree-like structures*, volume 67 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1998. Translated from the 1994 French original by Margaret Readdy, With a foreword by Gian-Carlo Rota. MR1629341

[2] M. Bóna. Real zeros and normal distribution for statistics on Stirling permutations defined by Gessel and Stanley. *SIAM J. Discrete Math.*, 23(1):401–406, 2008/09.

[3] L. Carlitz, R. Scoville, and T. Vaughan. Enumeration of pairs of sequences by rises, falls and levels. *Manuscripta Math.*, 19(3):211–243, 1976. MR0432472

[4] V. Dotsenko. Pattern avoidance in labelled trees. *Sém. Lothar. Combin.*, 67:Art. B67b, 27, 2011/12.

[5] B. Drake. *An inversion theorem for labeled trees and some limits of areas under lattice paths*. ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–Brandeis University. MR2712031

[6] R. Fröberg. Determination of a class of Poincaré series. *Math. Scand.*, 37(1):29–39, 1975.

[7] A. M. Garsia. A q-analogue of the Lagrange inversion formula. *Houston J. Math.*, 7(2):205–237, 1981.

[8] A. M. Garsia and M. Haiman. A remarkable q,t-Catalan sequence and q-Lagrange inversion. *J. Algebraic Combin.*, 5(3):191–244, 1996.

[9] I. Gessel. A noncommutative generalization and q-analog of the Lagrange inversion formula. *Trans. Amer. Math. Soc.*, 257(2):455–482, 1980. MR0552269

[10] I. Gessel and R. P. Stanley. Stirling polynomials. *J. Combinatorial Theory Ser. A*, 24(1):24–33, 1978.

[11] I. M. Gessel. *Generating functions and enumeration of sequences*. ProQuest LLC, Ann Arbor, MI, 1977. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[12] R. S. González D'León. On the free Lie algebra with multiple brackets. *Adv. in Appl. Math.*, 79:37–97, 2016.

[13] R. S. González D'León. The colored symmetric and exterior algebras. *J. Algebra*, 496C:187–215, 2018. MR3737838

[14] R. S. González D'León and M. L. Wachs. On the (co)homology of the poset of weighted partitions. *Trans. Amer. Math. Soc.*, 368(10):6779–6818, 2016.

[15] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete mathematics: a foundation for computer science*, volume 2. Addison-Wesley Reading, MA, 1994.

[16] J. Haglund and M. Visontai. Stable multivariate Eulerian polynomials and generalized Stirling permutations. *European J. Combin.*, 33(4):477–487, 2012.

[17] M. D. Haiman. Conjectures on the quotient ring by diagonal invariants. *J. Algebraic Combin.*, 3(1):17–76, 1994.

[18] S. Janson, M. Kuba, and A. Panholzer. Generalized Stirling permutations, families of increasing trees and urn models. *J. Combin. Theory Ser. A*, 118(1):94–114, 2011.

[19] R. Kaufmann, Y. Manin, and D. Zagier. Higher Weil-Petersson volumes of moduli spaces of stable n-pointed curves. *Comm. Math. Phys.*, 181(3):763–787, 1996.

[20] M. Kuba and A. Panholzer. Analysis of statistics for generalized Stirling permutations. *Combin. Probab. Comput.*, 20(6):875–910, 2011.

[21] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[22] S. Park. Inverse descents of r-multipermutations. *Discrete Math.*, 132(1–3):215–229, 1994.

[23] S. Park. P-partitions and q-Stirling numbers. *J. Combin. Theory Ser. A*, 68(1):33–52, 1994.

[24] S. Park. The r-multipermutations. *J. Combin. Theory Ser. A*, 67(1):44–71, 1994.

[25] J. B. Remmel and A. T. Wilson. Block patterns in Stirling permutations. *J. Comb.*, 6(1–2):179–204, 2015. MR3338849

[26] J. Riordan. Triangular permutation numbers. *Proc. Amer. Math. Soc.*, 2:429–432, 1951.

[27] R. P. Stanley. Parking functions and noncrossing partitions. *Electron. J. Combin.*, 4(2): Research Paper 20, approx. 14 pp. (electronic), 1997. The Wilf Festschrift (Philadelphia, PA, 1996).

[28] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[29] R. P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.

[30] M. L. Wachs. On the (co)homology of the partition lattice and the free Lie algebra. *Discrete Math.*, 193(1–3):287–319, 1998. Selected papers in honor of Adriano Garsia (Taormina, 1994).

Rafael S. González D'León
Escuela de Ciencias Exactas e Ingeniería
Universidad Sergio Arboleda
Bogotá
Colombia
E-mail address: rafael.gonzalezl@usa.edu.co
URL: http://dleon.combinatoria.co

Received 6 February 2018