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Combinatorics in ZFC limbo

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In their paper, Large-scale regularities of lattice embeddings of posets, Remmel and Williamson study posets and their incomparability graphs on N^k. Properties (1) through (3) of their main result, Theorem 1.5, are proved using Ramsey theory. The proof of Theorem 1.5 (4), however, uses Friedman's Jump Free Theorem, a powerful ZFC independent extension of Ramsey theory. Attempts to prove Theorem 1.5 (4) within the ZFC axioms have thus far failed. This leaves the main result of the Remmel-Williamson paper in what we informally call "ZFC limbo." In this paper we explore other results of this type. In particular, Theorem 6.2 of this paper, which we prove to be independent of ZFC, directly implies our very similar Theorem 6.3 for which we have no ZFC proof. On the basis of the close structural similarity between these two theorems, we conjecture that Theorem 6.3 is also independent of ZFC. However, Theorem 6.3 also follows directly from "subset sum is solvable in polynomial time." Of course, if our conjecture is true, "subset sum is solvable in polynomial time" cannot be proved in ZFC.

1. Introduction

Basic references are Friedman [Fri97], *Applications of large cardinals to graph theory*, and the expository article, *Lattice exit models*, Williamson [Wil17a]. In Sections 2 and 4 we develop background material and intuition related to certain recursively constructed families of functions on finite subsets of N^k, N the nonnegative integers. In Section 5, we extend a technique of Friedman [Fri97], Theorem 3.4 plus Theorem 4.4 through Theorem 4.15, for creating new independent combinatorial results related to his ZFC independent jump free theorem. In Section 6 we use these results to relate the classical subset sum problem to the techniques developed in Section 5.

2. Elementary background

Let N be the set of nonnegative integers and $k \geq 2$. For $z = (n_1, \ldots, n_k) \in N^k$, $\max\{n_i \mid i = 1, \ldots, k\}$ will be denoted by $\max(z)$. Define $\min(z)$ similarly.

Definition 1 (**Downward directed graph**)**.** Let $G = (N^k, \Theta)$ (vertex set N^k, edge set Θ) be a directed graph. If every (x, y) of Θ satisfies $\max(x) > \max(y)$ then we call G a *downward directed lattice graph.* For $z \in N^k$, let $G^z = \{x : (z, x) \in \Theta\}$ denote the vertices of G *adjacent* to z. **All lattice graphs that we consider will be *downward directed.***

Definition 2 (**Vertex induced subgraph** G_D)**.** For $D \subset N^k$ let $G_D = (D, \Theta_D)$ be the subgraph of G with vertex set D and edge set $\Theta_D = \{(x, y) \mid (x, y) \in \Theta,\ x, y \in D\}$. We call G_D the *subgraph of G induced by D.*

Definition 3 (**Cubes and Cartesian powers in** N^k)**.** The set $E_1 \times \cdots \times E_k$, where $E_i \subset N$, $|E_i| = p$, $i = 1, \ldots, k$, are k-cubes of length p. If $E_i = E, i = 1, \ldots, k$, then this cube is $E^k = \times^k E$, the kth Cartesian power of E.

Definition 4 (**Equivalent ordered** k**-tuples**)**.** Two k-tuples in N^k, $x = (n_1, \ldots, n_k)$ and $y = (m_1, \ldots, m_k)$, are *order equivalent tuples* ($x\, ot\, y$) if $\{(i, j) \mid n_i < n_j\} = \{(i, j) \mid m_i < m_j\}$ and $\{(i, j) \mid n_i = n_j\} = \{(i, j) \mid m_i = m_j\}$.

Note that *ot* is an equivalence relation on N^k. The standard SDR (system of distinct representatives) for the *ot* equivalence relation is gotten by replacing $x = (n_1, \ldots, n_k)$ by $\mathbf{r}(x) = (\mathbf{r}_{S_x}(n_1), \ldots, \mathbf{r}_{S_x}(n_k))$ where $\mathbf{r}_{S_x}(n_j)$ is the rank of n_j in $S_x = \{n_1, \ldots, n_k\}$ (e.g, $x = (3, 8, 5, 3, 8)$, $S_x = \{x\} = \{3, 8, 5, 3, 8\} = \{3, 5, 8\}$, $\mathbf{r}(x) = (0, 2, 1, 0, 2)$). The number of equivalence classes is $\sum_{j=1}^{k} \sigma(k, j) < k^k$, $k \geq 2$, where $\sigma(k, j)$ is the number of surjections from a k set to a j set. We use "$x\, ot\, y$" and "x, y of order type *ot*" to mean x and y belong to the same order type equivalence class.

3. Basic definitions and theorems

We present some basic definitions due to Friedman [Fri97], [Fri98].

Definition 5 (**regressive value**)**.** Let $X \subseteq N^k$ and $f : X \to Y \subseteq N$. An integer n is a *regressive value* of f on X if there exist x such that $f(x) = n < \min(x)$.

Definition 6 (**field of a function and reflexive functions**)**.** For $A \subseteq N^k$ define field(A) to be the set of all coordinates of elements of A. A function f is reflexive in N^k if domain$(f) \subseteq N^k$ and range$(f) \subseteq$ field(domain(f)).

Definition 7 (**the set of functions** $T(k)$)**.** $T(k)$ denotes all reflexive functions with finite domain: $|\text{domain}(f)| < \infty$.

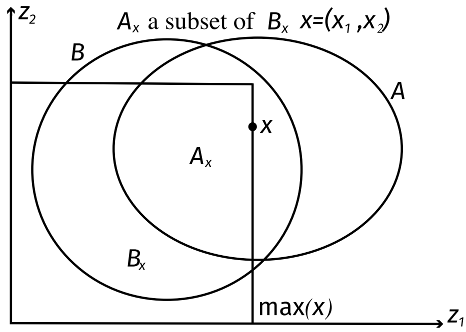

Figure 1: Basic jump free condition 8.

Definition 8 (**full and jump free).** Let $Q \subset T(k)$.

1. **full:** Q is a *full* family of functions on N^k if for every finite subset $D \subset N^k$ there is at least one function f in Q whose domain is D.
2. **jump free:** For $D \subset N^k$ and $x \in D$ define $D_x = \{z \mid z \in D, \max(z) < \max(x)\}$. Suppose that for all f_A and f_B in Q, where f_A has domain A and f_B has domain B, the conditions $x \in A \cap B$, $A_x \subseteq B_x$, and $f_A(y) = f_B(y)$ for all $y \in A_x$ imply that $f_A(x) \geq f_B(x)$. Then Q will be called a *jump free* family of functions on N^k (see figure 1).

Definition 9 (**Regressively regular over** E**).** Let $k \geq 2$, $D \subset N^k$, D finite, $f : D \to N$. We say f is *regressively regular* over E, $E^k \subset D$, if for each order type equivalence class *ot* of k-tuples of E^k either (1) or (2) occurs:

1. **constant less than min E:** For all $x, y \in E^k$ of order type *ot*, $f(x) = f(y) < \min(E)$
2. **greater than min:** For all $x \in E^k$ of order type *ot* $f(x) \geq \min(x)$.

Theorem 3.1 (**Jump free theorem ([Fri97], [Fri98])).** *Let* $p, k \geq 2$ *and* $S \subseteq T(k)$ *be a full and jump free family. Then some* $f \in S$ *has at most* k^k *regressive values on some* $E^k \subseteq \mathrm{domain}(f)$, $|E| = p$. *In fact, some* $f \in S$ *is regressively regular over some* E *of cardinality* p.

Intuitively, referring to Figure 1, suppose that the region A_x is to be searched for the smallest of some quantity and the result recorded at x. Next, the search region is expanded to a superset B_x with the search results for A_x still valid (i.e., $f_A(y) = f_B(y)$ for all $y \in A_x$). Then, clearly $f_A(x) \geq f_B(x)$.

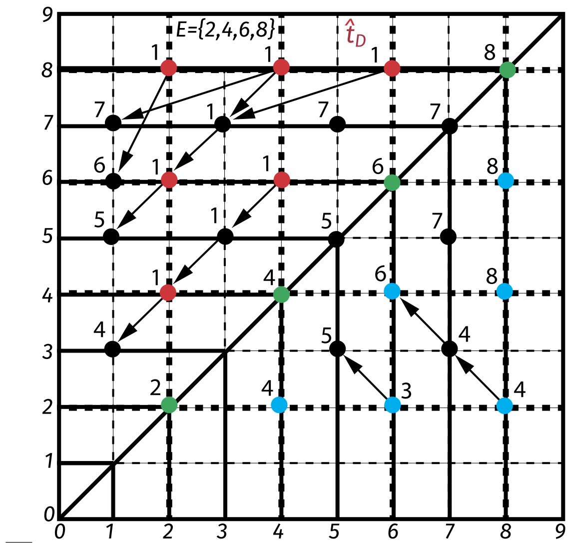

Figure 2: $\hat{t}_D$ regressively regular over $E = \{2, 4, 6, 8\}$.

This expansion property of search algorithms occurs, perhaps somewhat disguised, in many examples.

We use ZFC for the axioms of set theory, Zermelo-Frankel plus the axiom of choice. The jump free theorem can be proved in ZFC + $(\forall n)(\exists$ n-subtle cardinal) but not in ($\exists$ n-subtle cardinal) for any fixed n (assuming this theory is consistent). A proof is in Section 2 of [Fri97], "Applications of Large Cardinals to Graph Theory," October 23, 1997, No. 11 of Downloadable Manuscripts.

We next discuss a class of geometrically natural problems that give rise to applications of the jump free theorem. Using standard terminology, we use $(x_1, \ldots, x_s)$ to denote a directed path of length s in G_D. If $z \in D$, (z) denotes a path of length one. A path $(x_1, \ldots, x_s)$ is terminal if $G_D^{x_s} = \varnothing$.

Definition 10 (**$\hat{t}_D$ terminal path label function**)**.** For finite $D \subset N^k$, let $G_D = (D, \Theta_D)$. Let $T_D(z)$ be the set of all last vertices of terminal paths $(x_1, x_2, \ldots, x_t)$ where $z = x_1$. Define $\hat{t}_D$ (domain D, range field(D)) by

$$\hat{t}_D(z) = \max(z) \text{ if } (z) \text{ terminal, else}$$
$$\hat{t}_D(z) = \min(\{\min(x) \mid x \in T_D(z)\})$$

We call $\hat{t}_D$ the *terminal path label function*.

The choice $\hat{t}_D(z) = \max(z)$ if (z) terminal is used instead of the more natural $\hat{t}_D(z) = \min(z)$ to make possible the following application of the jump free theorem (due to Friedman [Fri97]).

Lemma 11 ($\{\hat{t}_D\}$ **full, reflexive, jump free).** *Take*

$$S = \{\hat{t}_D \mid D \subset N^k, |D| < \infty\}.$$

Then S is full, reflexive, and jump free.

Proof. Full and reflexive is immediate. By the downward condition, $\hat{t}_D = \max(z)$ if and only if (z) is terminal (i.e., $G^z_D = \varnothing$). Let $\hat{t}_A$ and $\hat{t}_B$ satisfy the conditions of f_A and f_B in definition 8 (2). Note that by definition, $x \notin A_x$ or B_x. If (x) is terminal in A then $\hat{t}_A(x) = \max(x) \geq \hat{t}_B(x)$ by the downward condition on G. Else, let $(x, \ldots, y)$ be a terminal path in G_A. Then $\hat{t}_B(y) = \hat{t}_A(y) = \max(y)$ implies $(x, \ldots, y)$ is a terminal path in G_B. Thus, $\hat{t}_A(x) \geq \hat{t}_B(x)$ as was to be shown. □

Theorem 3.2 (Jump free theorem for $\hat{t}_D$). *Let $S = \{\hat{t}_D \mid D \subset N^k,\ |D| < \infty\}$ and let $p, k \geq 2$. Then some $f \in S$ has at most k^k regressive values on some $E^k \subseteq \operatorname{domain}(f)$, $|E| = p$. In fact, some $f \in S$ is regressively regular over some E of cardinality p.*

Proof. Follows from Lemma 11 and the jump free Theorem 3.1. □

Figure 2 shows an example of $\hat{t}_D$ regressively regular over a set $E = \{2, 4, 6, 8\}$, where $D \subset N^2$, $|D| = 28$. Theorem 3.2 is one of the most structurally simple combinatorial results in ZFC limbo. [1] It is the result used to prove the main theorem in [RW99].

We discuss more complex generalizations in the next section.

4. More general recursive constructions

Definition 12 (Partial selection). A function F with domain a subset of X and range a subset of Y will be called a *partial function* from X to Y (denoted by $F : X \to Y$). If $z \in X$ but z is not in the domain of F, we say F is *not defined* at z. Let $r \geq 1$. A partial function $F : N^k \times (N^k \times N)^r \to N$ is *partial selection* function [Fri97] if when $F[x, (y_1, n_1), (y_2, n_2), \ldots (y_r, n_r)]$ is defined $F[x, (y_1, n_1), (y_2, n_2), \ldots (y_r, n_r)] = n_i$ for some $1 \leq i \leq r$.

[1] Harvey Friedman (personal communication) has conjectured that Theorem 3.2 is itself independent of ZFC, but "…it would take 50 years to prove it."

Definition 13 (**max constant sets** D_a). Let $N^k \supset D$, D finite. Let $D_a = \{x \mid x \in D, \max(x) = a\}$. Let $m_0 < m_1 < \cdots < m_q$ be the integers n such that $D_n \neq \varnothing$.

Definition 14 (**Committee model** $\hat{s}_D$ [**Fri97**], [**Wil17a**]). Let $r \geq 1$, $k \geq 2$, $G = (N^k, \Theta)$, $G_D = (D, \Theta_D)$, D finite, $G_D^z = \{x \mid (z,x) \in \Theta_D\}$. Let $F : N^k \times (N^k \times N)^r \to N$ be a partial selection function. If $G_D^z = \varnothing$ define $\Phi_z^D = \varnothing$. Thus, $\Phi_z^D = \varnothing$ if $z \in D_{m_0}$. We define Φ_z^D and $\hat{s}_D(z)$ (domain D, range field(D)) recursively (on the m_t, $t = 0, \ldots, q$) as follows. Let

$$\Phi_z^D = \{F[z, (y_1, n_1), (y_2, n_2), \ldots, (y_r, n_r)],\ y_i \in G_D^z\}$$

be the set of defined values of F where $n_i = \hat{s}_D(y_i)$ if $\Phi_{y_i}^D \neq \varnothing$ and $n_i = \min(y_i)$ if $\Phi_{y_i}^D = \varnothing$. If $\Phi_z^D \neq \varnothing$, define $\hat{s}_D(z)$ to be the minimum over Φ_z^D. If $\Phi_z^D = \varnothing$, define $\hat{s}_D(z) = \max(z)$.

NOTE: An easy induction on $\max(z)$ shows $\hat{s}_D(z) \leq \max(z)$ with equality if and only if $\Phi_z^D = \varnothing$. We give a proof and introduce some terminology.

Lemma 15 ($\hat{s}_D(z)$ **structure**). *$\hat{s}_D(z) \leq \max(z)$ with $\hat{s}_D(z) = \max(z)$ if and only if $\Phi_z^D = \varnothing$.*

Proof. We use induction on $\max(z)$ to construct both $\hat{s}_D(z)$ and Φ_z^D. Let $D_a = \{x \mid x \in D, \max(x) = a\}$. Let $m_0 < m_1 < \cdots < m_q$ be the integers n such that $D_n \neq \varnothing$. If $z \in D_{m_0}$ then the set of adjacent vertices $G_D^z = \varnothing$. Thus, $\Phi_z^D = \varnothing$ and $\hat{s}_D(z) = \max(z)$, $z \in D_{m_0}$. The result holds for $z \in D_{m_0}$. In general, assume that for $t < j$, $z \in D_{m_t}$, $\hat{s}_D(z) \leq \max(z)$ with $\hat{s}_D(z) = \max(z)$ if and only if $\Phi_z^D = \varnothing$. Consider $z \in D_{m_j}$. If (1) $\Phi_z^D = \varnothing$ then $\hat{s}_D(z) = \max(z)$. If (2) $\Phi_z^D \neq \varnothing$ let $n = F[z, (y_1, n_1), (y_2, n_2), \ldots, (y_r, n_r)] \in \Phi_z^D$, $y_i \in G_D^z$ thus $y_i \in D_t$, $t < j$.

First, if $\Phi_{y_i}^D = \varnothing$. then $n_i = \min(y_i) < \max(z)$.

Second, if $\Phi_{y_i}^D \neq \varnothing$ then, by the induction hypothesis, $n_i = \hat{s}_D(y_i) < \max(y_i) < \max(z)$. Thus, $\hat{s}_D(z) \leq \max(z)$ with $\hat{s}_D(z) = \max(z)$ if and only if $\Phi_z^D = \varnothing$. $\square$

The following result is due to Friedman [Fri97].

Theorem 4.1 (**Large scale regularities for** $\hat{s}_D$ ([**Fri97**]). *Let $r \geq 1$, $p, k \geq 2$. $S = \{\hat{s}_D \mid D \subset N^k,\ |D| < \infty\}$. Then some $f \in S$ has at most k^k regressive values over some $E^k \subseteq$ domain(f), $|E| = p$. In fact, some $f \in S$ is regressively regular over some E of cardinality p.*

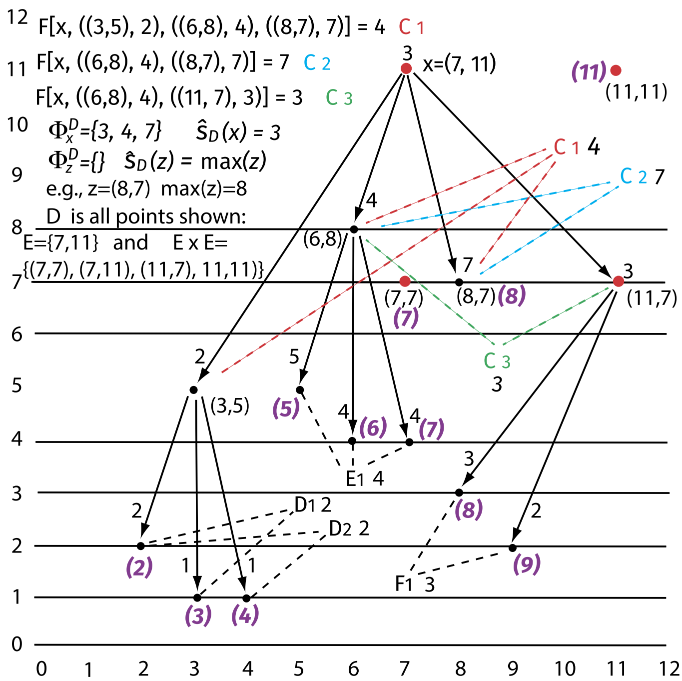

Figure 3: An example of $\hat{s}_D$.

Proof. Recall 3.1. Let $S = \{\hat{s}_D \mid D \subset N^k,\ |D| < \infty\}$. S is obviously full and reflexive. We show S is jump free. We show for all $\hat{s}_A$ and $\hat{s}_B$ in S, the conditions $x \in A \cap B$, $A_x \subseteq B_x$, and $\hat{s}_A(y) = \hat{s}_B(y)$ for all $y \in A_x$ imply that $\hat{s}_A(x) \geq \hat{s}_B(x)$ (i.e., S is *jump free*). If $\Phi_x^A = \varnothing$ then $\hat{s}_A(x) = \max(x) \geq \hat{s}_B(x)$. Assume $\Phi_x^A \neq \varnothing$.

Let $n = F[x, (y_1, n_1), (y_2, n_2), \dots (y_r, n_r)] \in \Phi_x^A$ (note that $y_i \in G_A^x \subseteq G_B^x$) where $n_i = \hat{s}_A(y_i)$ if $\hat{s}_A(y_i) < \max(y_i)$ (i.e., $\Phi_{y_i}^A \neq \varnothing$) and $n_i = \min(y_i)$ if $\hat{s}_A(y_i) = \max(y_i)$ (i.e., $\Phi_{y_i}^A = \varnothing$). But $\hat{s}_A(y_i) = \hat{s}_B(y_i)$, $i = 1, \dots, r$, implies $n \in \Phi_x^B$ and thus $\Phi_x^A \subseteq \Phi_x^B$ and $\hat{s}_A(x) = \min(\Phi_x^A) \geq \min(\Phi_x^B) = \hat{s}_B(x)$. □

Next we give an example of $\hat{s}_D$.

As an example of computing $\hat{s}_D$, consider figure 3. The computation is recursive on the max norm (and doesn't illustrate all of the subtleties). The values of the terminal vertices where $\Phi_x^A = \varnothing$ are shown in parentheses, left to right: (2), (3), (4), (5), (6), (7), (8), (8), (9). These numbers are $\max((a, b))$ for each terminal vertex (a, b). Partial selection functions are of the form $F : N^2 \times (N^2 \times N)^r \to N$ ($r = 2, 3$ here). In particular we have $F[x, ((3, 5), 2), ((6, 8), 4), ((8, 7), 7)] = 4$, $F[x, ((6, 8), 4), ((8, 7), 7)] = 7$, and

$F[x,((6,8),4),((11,7),3)] = 3$. Intuitively, we think of these as (ordered) committees reporting values to the boss, $x = (7,11)$. The first committee, C1, consists of subordinates, $(3,5),(6,8),(8,7)$ reporting respectively $2,4,7$. The committee decides to report 4 (indicated by C1 4 in figure 3). The recursive construction starts with terminal vertices reporting their minimal coordinates. But, the value reported by each committee is not, in general, the actual minimum of the reports of the individual members. Nevertheless, the boss, $x = (7,11)$, *always* takes the minimum of the values reported by the committees. In this case the values reported by the committees are $4,7,3$ the boss takes 3 (i.e., $\hat{s}_D(x) = 3$ for the boss, $x = (7,11)$). Note that a function like $F((7,11),((6,8),4),((8,7),7)$ where $r = 2$, can be padded to the case $r = 3$ (e.g., $F((7,11),((6,8),4),((8,7),7),((8,7),7)))$.

Observe in figure 3 that the values in parentheses, (2), (3), (4), (5), (6), (7), (8), (8), (9), don't figure into the recursive construction of $\hat{s}_D$. They immediately pass their minimum values on to the computation: 2, 1, 1, 5, 4, 4, 7, 3, 2. We discuss some generalizations.

5. Combinatorial generalizations

In this section we present some results that are based on results of Friedman [Fri97] (specifically, Theorem 4.4 and the ideas of Theorem 4.1 and the earlier Theorem 3.3). Friedman removes any mention of the graph G and works with an equivalent streamlined version. We stick with the graph model in this discussion.

We extend Friedman's results slightly by introducing a class of functions $\{\rho_D \mid N^k \supset D \text{ finite}, \rho_D : D \to N, \min(x) \le \rho_D(x), x \in D\}$. These "min dominant" functions allow us to relax the reflexive condition and will be of use for certain combinatorial applications.

Definition 16 (h^{ρ_D} **for** G_D)**.** Let $r \ge 1$, $k \ge 2$, $G = (N^k, \Theta)$, $G_D = (D, \Theta_D)$, D finite, $G_D^z = \{x \mid (z,x) \in \Theta_D\}$. Let $F : N^k \times (N^k \times N)^r \to N$ be a partial selection function. An *initial min dominant family of functions* is specified as follows where D ranges over all finite subsets of N^k:

$$\mathbb{R} = \{\rho_D \mid N^k \supset D \text{ finite}, \rho_D : D \to N, \min(x) \le \rho_D(x), x \in D\}.$$

We define $\Phi_z^{\rho_D}$, h^{ρ_D} recursively on $\max(z)$. If $G_D^z = \varnothing$ define $\Phi_z^{\rho_D} = \varnothing$. Thus, $\Phi_z^{\rho_D} = \varnothing$, $z \in D_{m_0}$. We define $\Phi_z^{\rho_D}$ and $h^{\rho_D}(z)$ recursively (on the m_t, $t = 1,\ldots q$ of definition 13) as follows. Let

$$\Phi_z^{\rho_D} = \{F[z,(y_1,n_1),(y_2,n_2),\ldots,(y_r,n_r)],\ y_i \in G_D^z\}$$

be the set of defined values of F where $n_i = h^{\rho_D}(y_i)$ if $\Phi^{\rho_D}_{y_i} \neq \varnothing$ and $n_i = \min(y_i)$ if $\Phi^{\rho_D}_{y_i} = \varnothing$. If $\Phi^{\rho_D}_z = \varnothing$, define $h^{\rho_D}(z) = \rho_D(z)$. If $\Phi^{\rho_D}_z \neq \varnothing$, define $h^{\rho_D}(z)$ to be the minimum over $\Phi^{\rho_D}_z$.

Note that ρ_D need not be reflexive on D.

Recall definition 14 and the recursive construction of $\hat{s}_D$ and Φ^D_z.

Lemma 17 (**Compare $\hat{s}_D$, h^{ρ_D}**). *For all $z \in D$, either (1) $\Phi^D_z = \Phi^{\rho_D}_z = \varnothing$ and $h^{\rho_D}(z) = \rho_D(z)$, $\hat{s}_D(z) = \max(z)$ or (2) $\Phi^D_z = \Phi^{\rho_D}_z \neq \varnothing$ and $h^{\rho_D}(z) = \hat{s}_D(z)$.*

Proof. Let $D_a = \{x \mid x \in D, \max(x) = a\}$. Let $m_0 < m_1 < \cdots < m_q$ be the integers n such that $D_n \neq \varnothing$. If $z \in D_{m_0}$ then $G^z_D = \varnothing$. Thus, $\Phi^D_z = \Phi^{\rho_D}_z = \varnothing$ and $h^{\rho_D}(z) = \rho_D(z)$, $\hat{s}_D(z) = \max(z)$, $z \in D_{m_0}$.

Assume, for all $z \in D_{m_t}$, $0 \leq t < j$, either $\Phi^D_z = \Phi^{\rho_D}_z = \varnothing$ and $h^{\rho_D}(z) = \rho_D(z)$, $\hat{s}_D(z) = \max(z)$ or $\Phi^D_z = \Phi^{\rho_D}_z \neq \varnothing$ and $h^{\rho_D}(z) = \hat{s}_D(z)$.

Let $z \in D_{m_j}$. If $\Phi^D_z = \Phi^{\rho_D}_z = \varnothing$, then $h^{\rho_D}(z) = \rho_D(z)$ and $\hat{s}_D(z) = \max(z)$. Otherwise, either $\Phi^D_z \neq \varnothing$ or $\Phi^{\rho_D}_z \neq \varnothing$. Assume WLOG that $\Phi^{\rho_D}_z \neq \varnothing$. Let

$$\Phi^{\rho_D}_z = \{F[z, (y_1, n_1), (y_2, n_2), \ldots, (y_r, n_r)],\ y_i \in G^z_D\}.$$

Choose $n = F[z, (y_1, n_1), (y_2, n_2), \ldots, (y_r, n_r)] \in \Phi^{\rho_D}_z$. Thus $y_i \in D_{m_t}$ for some $t < j$. By the induction hypothesis, either (1) $\Phi^D_{y_i} = \Phi^{\rho_D}_{y_i} = \varnothing$, $h^{\rho_D}(y_i) = \rho_D(y_i)$ and $\hat{s}_D(y_i) = \max(y_i)$, in which case $n_i = \min(y_i)$, or (2) $\Phi^D_{y_i} = \Phi^{\rho_D}_{y_i} \neq \varnothing$ and $h^{\rho_D}(y_i) = \hat{s}_D(y_i) = n_i$. In either case, $n \in \Phi^D_z$ and thus $\Phi^{\rho_D}_z \subseteq \Phi^D_z$ In the same manner we conclude that $\Phi^D_z \subseteq \Phi^{\rho_D}_z$. Thus, in fact, $\Phi^D_z = \Phi^{\rho_D}_z \neq \varnothing$ and $h^{\rho_D}(z) = \hat{s}_D(z)$. This completes the proof of the first part of this lemma. □

Next, we consider *regressive regularity.*

Lemma 18 (**Compare regressive regularity $\hat{s}_D$, h^{ρ_D}**). *Let E be of cardinality $p \geq 2$. Then $\hat{s}_D$ is regressively regular over E iff h^{ρ_D} regressively regular over E.*

Proof. For $z \in D$ we have shown (Lemma 17) there are two cases:

$$(1)\ \ \Phi^D_z = \Phi^{\rho_D}_z = \varnothing,\ h^{\rho_D}(z) = \rho_D(z) \text{ and } \hat{s}_D(z) = \max(z)$$

or

$$(2)\ \ \Phi^D_z = \Phi^{\rho_D}_z \neq \varnothing \text{ and } h^{\rho_D}(z) = \hat{s}_D(z).$$

First we show for all $x, y \in E^k$ of order type ot, $\hat{s}_D(x) = \hat{s}_D(y) < \min(E)$ if and only if $h^{\rho_D}(x) = h^{\rho_D}(y) < \min(E)$. Case (1) above is ruled out because $h^{\rho_D}(z) = \rho_D(z) \geq \min(z) \geq \min(E)$ and $\hat{s}_D(z) = \max(z) \geq \min(E)$. Thus we have case (2) $\Phi_z^D = \Phi_z^{\rho_D} \neq \varnothing$ and $h^{\rho_D}(z) = \hat{s}_D(z)$ for $z = x, y$. Thus, $\hat{s}_D(x) = \hat{s}_D(y) < \min(E)$ if and only if $h^{\rho_D}(x) = h^{\rho_D}(y) < \min(E)$.

Second suppose for all $x \in E^k$ of order type *ot*, $h^{\rho_D}(x) \geq \min(x)$. This set of order type *ot* can be partitioned into two sets, $\{x \mid \Phi_x^D \neq \varnothing\}$ and $\{x \mid \Phi_x^D = \varnothing\}$. On the first set, $\min(x) \leq h^{\rho_D}(x) = \hat{s}_D(x)$ and on the second set $h^{\rho_D}(x) = \rho_D(x) \geq \min(x)$ and $\hat{s}_D(x) = \max(x) \geq \min(x)$. Thus, $\hat{s}_D(x) \geq \min(x)$. The same argument works if we assume for $x \in E^k$ of order type *ot* $\hat{s}_D(x) \geq \min(x)$. Thus, for $x \in E^k$ of order type *ot*, $h^{\rho_D}(x) \geq \min(x)$ if and only if $\hat{s}_D(x) \geq \min(x)$. $\square$

Theorem 5.1 (**Regressive regularity** h^{ρ_D}). *Let $G = (N^k, \Theta)$, $r \geq 1$, $p, k \geq 2$. Let $S = \{h^{\rho_D} \mid D \subset N^k,\ |D| < \infty\}$. Then some $f \in S$ has at most k^k regressive values on some $E^k \subseteq \mathrm{domain}(f) = D$, $|E| = p$. In fact, some $f \in S$ is regressively regular over some E of cardinality p.*

Proof. Follows from Theorem 4.1 and Lemma 17, 18. We claim that the set $S = \{h^{\rho_D} \mid D \subset N^k,\ |D| < \infty\}$ is a full family of functions such that for any $p \geq 2$ there is a function h^{ρ_D} which is regressively regular over some E, $|E| = p$. Lemmas 17, 18 show that to find such an E for h^{ρ_D} we can invoke Theorem 4.1 and find such an E for $\hat{s}_D$. $\square$

Remark: Independence of the families of Theorem 5.1. From Theorem 5.1 the regressive regularity of the families of functions $\{h^{\rho_D} \mid D \subset N^k,\ |D| < \infty\}$ is in ZFC limbo as the only proof we have at this point is using the ZFC independent jump free theorem. However, Friedman [Fri97], has liberated these families en masse. In particular, it has been shown by Friedman [Fri97], Theorem 4.4 through Theorem 4.15 that a special case of Theorem 5.1 ($\rho_D = \min$) requires the same large cardinals to prove as the jump free theorem. Thus, Theorem 5.1 provides a family of ZFC independent theorems parameterized by a choice of an initial min dominant family of functions:

$$\mathbb{R} = \{\rho_D \mid N^k \supset D \text{ finite}, \rho_D : D \to N,\ \min(x) \leq \rho_D(x),\ x \in D\}.$$

6. Using the ρ_D and the subset sum problem

Definition 19 (D **capped by** $E^k \subset D$). For $k \geq 2$, $E^k \subseteq D \subset N^k$, let $\max(D)$ be the maximum over $\max(z)$, $z \in D$. Let $\mathrm{setmax}(D) = \{z \mid z \in D, \max(z) = \max(D)\}$. If $\mathrm{setmax}(D) = \mathrm{setmax}(E^k)$, we say that D is *capped by* $E^k \subseteq D$ with the *cap* defined to be $\mathrm{setmax}(E^k)$.

Note that if D is *capped by* $E^k \subseteq D$ then D determines E^k uniquely in the obvious way. An example is shown in figure 2.

The following theorem is analogous to Theorem 5.1.

Theorem 6.1 (**Regressively regular h^{ρ_D}, capped version).** *Let $G = (N^k, \Theta)$, $r \geq 1$, $p, k \geq 2$. Let $S = \{h^{\rho_D} \mid D \subset N^k,\ |D| < \infty\}$. Then some $f \in S$ has at most k^k regressive values on some $E^k \subseteq \mathrm{domain}(f) = D$, $|E| = p$. In fact, some $f \in S$ is regressively regular over some such E, $E^k \subseteq D = \mathrm{domain}(f)$, D capped by E^k.*

Proof. From Theorem 5.1 there is an $h^{\rho_D} \in S$ that is regressively regular over some E, $|E| = p$, $E^k \subseteq D$. Let $E = \{e_0, \ldots, e_{p-1}\}$. Let $D_x = \{z \mid z \in D, \max(z) < \max(x)\}$. Let $\widehat{D} = D_{e_{p-1}} \cup \mathrm{setmax}(E^k)$ so $\widehat{D}$ is capped by E^k. Using the downward condition on G_D and hence $G_{\widehat{D}}$ we have the restriction $h^{\rho_D}|\widehat{D}$ is regressively regular over E. Note that $h^{\rho_D}|\widehat{D}$ may or may not equal the function $h^{\rho_{\widehat{D}}} \in S$. But Lemmas 17 and 18 apply in either case. Thus the function $h^{\rho_{\widehat{D}}} \in S$ is also regressively regular over E. □

Definition 20 (**t-log bounded).** Let $p, k \geq 2$, $t \geq 1$. The function ρ_D is t-log bounded over $E^k \subset D$ where $E = \{e_0, \ldots, e_{p-1}\}$, if the cardinality

$$|\{\rho_D(x) - \min(x) : 0 < \rho_D(x) - \min(x) < e_0 k^k,\ x \in E^k\}| \leq t \log_2(p^k).$$

We write $\rho_D \in \mathrm{LOG(k, E, p, D, t)}$. The set

$$\mathbb{R} = \{\rho_D \mid \rho_D : D \to N,\ \min(x) \leq \rho_D(x),\ x \in D\}$$

is t-log bounded if $\rho_D \in \mathrm{LOG(k, E, p, D, t)}$ when D is capped by E^k. In this case we write $\mathbb{R}_t$ for $\mathbb{R}$.

Remarks on definition 20. Conventions on $\rho_D \in \mathrm{LOG(k, E, p, D, t)}$. Recalling that $\rho_D(x) \geq \min(x)$ and $\rho_D(x)$ can be arbitrarily large, we can choose the cardinality $|\{x : \rho_D(x) - \min(x) \geq e_0 k^k\}|$ large enough to make $\rho_D \in \mathrm{LOG(k, E, p, D, t)}$. We can also choose the $\rho_D(x) - \min(x) \geq e_0 k^k$ distinct. We make that general assumption in what follows.

Theorem 6.2 (**Regressive regularity t-log bounded case).** *Let $G = (N^k, \Theta)$, $r, t \geq 1$, $p, k \geq 2$. Let $S = \{h^{\rho_D} \mid D \subset N^k,\ |D| < \infty\}$ where the set $\mathbb{R}_t = \{\rho_D \mid \rho_D : D \to N,\ \min(x) \leq \rho_D(x),\ x \in D\}$ is t-log bounded. Then some $f \in S$ has at most k^k regressive values on some $E^k \subseteq \mathrm{domain}(f) = D$, $|E| = p$. In fact, some $f \in S$ is regressively regular over some such E, $E^k \subseteq D = \mathrm{domain}(f)$, D capped by E^k and $\rho_D \in \mathrm{LOG(k, E, p, D, t)}$.*

Proof. Follows from Theorem 6.1 which states that some $f \in S$ has at most k^k regressive values on some $E^k \subseteq \text{domain}(f) = D$, $|E| = p$. In fact, some $f \in S$ is regressively regular over some such E, $E^k \subseteq D = \text{domain}(f)$, D capped by E^k. From Definition 20, for each such capped pair D and E^k, ρ_D has already been defined so that $\rho_D \in \mathrm{LOG(k, E, p, D, t)}$. ☐

Theorem 6.2 is independent of ZFC as is Theorem 6.1.

Given any h^{ρ_D}, we have a natural partition of $E^k \subseteq D$ into three sets

$$E_0^k = \{x \in E^k : h^{\rho_D}(x) < \min(E)\}$$
$$E_1^k = \{x \in E^k : \min(E) \leq h^{\rho_D}(x) < \min(x)\}$$
$$E_2^k = \{x \in E^k : \min(x) \leq h^{\rho_D}(x)\}.$$

In Definition 21 we associate sets of integers with each of the three blocks of this partition. This choice can be done in many ways. Our associated sets are chosen because of their natural, generic, relationship to regressive regularity.

We use the terminology of theorem 6.2.

Definition 21 **($Q^{k,t}_{F,G}(E, p, D)$ family of sets).** Let $G = (N^k, \Theta)$, $r, t \geq 1$, $p, k \geq 2$. Let $S = \{h^{\rho_D} \mid D \subset N^k,\ |D| < \infty\}$ where the set $\mathbb{R}_t = \{\rho_D \mid \rho_D : D \to N,\ \min(x) \leq \rho_D(x),\ x \in D\}$ is t-log bounded. Let

$$Q^{k,t}_{F,G}(E, p, D) = \{\cup_{i=0}^2 \Delta h^{\rho_D} E_i^k\}$$

be the family of sets ranging over the indicated parameters and defined by

$$\Delta h^{\rho_D} E_0^k = \{h^{\rho_D}(x) - \min(E) : x \in E^k, h^{\rho_D}(x) < \min(E)\}$$
$$\Delta h^{\rho_D} E_1^k = \{h^{\rho_D}(x) - \min(x) : x \in E^k, \min(E) \leq h^{\rho_D}(x) < \min(x)\}$$
$$\Delta h^{\rho_D} E_2^k = \{\rho_D(x) - \min(x) : x \in E^k, \min(x) \leq h^{\rho_D}(x)\}$$

The sets of Definition 21 are constructed to be sensitive to the case where h^{ρ_D} is regressively regular over E (to be used in the proof of Theorem 6.3). Note that $|\cup_{i=0}^2 \Delta h^{\rho_D} E_i^k| \leq p^k$.

We summarize some terminology involved in the h^{ρ_D}. (1) N the nonnegative integers. (2) N^k the nonnegative integral lattice of dimension $k \geq 2$. (3) $\mathbb{R}$ a collection of functions ρ_D, $\rho_D(x) \geq \min(x)$, one for each finite $D \subset N^k$. (4) $F : N^k \times (N^k \times N)^r \to N$, $r \geq 1$, partial selection functions. (5) $G = (N^k, \Theta)$ a downward directed graph on N^k. (6) $G_D = (D, \Theta_D)$ restriction of G to D. (7) h^{ρ_D} functions defined recursively on $D \subset N^k$. (8)

$E = \{e_0, \dots, e_{p-1}\} \subset N$, $|E| = p \geq 2$. (9) $E^k \subseteq D$, D capped by E^k. (10) $\mathbb{R}_t$ a subclass of $\mathbb{R}$ that are t-log bounded, $t \geq 1$.

Theorem 6.3 (Subset sum connection). *Regard the sets in $Q^{k,t}_{F,G}(E, p, D)$ as instances to the subset sum problem, target 0, size measured (approximately) by $p = |E|$, $E = \{e_0, \dots, e_{p-1}\}$. For fixed F, G, k, t consider sets of instances $\{H^{k,t}_{F,G}(E, p, D) : E, p, D\}$ where*

$$H^{k,t}_{F,G}(E, p, D) = \cup_{i=0}^{2} \Delta h^{\rho_D} E_i^k.$$

For each p there exists $\hat{E}$ and $\hat{D}$, $|\hat{E}| = p$, such that the subset sum problem for

$$\{H^{k,t}_{F,G}(\hat{E}, p, \hat{D}) : p = 2, 3, \dots\}$$

is solvable in time $O(p^{kt})$.

Proof. From the definition of $H^{k,t}_{F,G}(E, p, D)$ the set $\mathbb{R}_t$ is t-log bounded. From Theorem 6.2, for any p, we can choose $\hat{D}$ capped by $\hat{E}^k$ such that $h^{\rho_{\hat{D}}}$ is regressively regular over $\hat{E}$. For notational simplicity we set $\hat{E} = \{e_0, \dots, e_{p-1}\}$.

By regressive regularity, The set $\{x \in \hat{E}^k : \min(\hat{E}) \leq h^{\rho_{\hat{D}}}(x) < \min(x)\}$ is empty, thus $\Delta h^{\rho_{\hat{D}}} \hat{E}_1^k = \varnothing$.

For $\Delta h^{\rho_{\hat{D}}} \hat{E}_0^k$ we have $h^{\rho_{\hat{D}}}(x) - e_0 < 0$. Note $|h^{\rho_{\hat{D}}}(x) - e_0| < e_0$ and, by regressive regularity, the cardinality $|\Delta h^{\rho_{\hat{D}}} \hat{E}_0^k| < k^k$ so

$$\sum_{\Delta h^{\rho_{\hat{D}}} \hat{E}_0^k} |h^{\rho_{\hat{D}}}(x) - e_0| < e_0 k^k. \tag{22}$$

From t-log bounded, we have

$$|\{\rho_{\widehat{D}}(x) - \min(x) : 0 < \rho_{\widehat{D}}(x) - \min(x) < e_0 k^k,\ x \in \hat{E}^k\}| \leq t \log_2(p^k).$$

The negative terms in the instance come from

$$\Delta h^{\rho_{\hat{D}}} \hat{E}_0^k = \{h^{\rho_{\hat{D}}}(x) - \min(\hat{E}) : x \in \hat{E}^k,\ h^{\rho_{\hat{D}}}(x) < \min(\hat{E})\}$$

The cardinality $|\Delta h^{\rho_{\hat{D}}} \hat{E}_0^k| < k^k$.

The positive terms in the instance come from

$$\Delta h^{\rho_{\hat{D}}} \hat{E}_2^k = \{\rho_{\widehat{D}}(x) - \min(x) : x \in \hat{E}^k, \min(x) \leq h^{\rho_{\hat{D}}}(x)\}$$

If $0 \in \Delta h^{\rho_{\hat{D}}} \hat{E}_2^k$ then the solution is trivial as 0 is the target. We use Equation 22 to rule out having to consider positive values of $\rho_D(x) - \min(x) \geq e_0 k^k$. Otherwise, from t-log bounded, we have $|\Delta h^{\rho_{\hat{D}}} \hat{E}_2^k| \leq t \log_2(p^k)$. We can check all possible solutions by comparing the sums of less than 2^{k^k} subsets of negative terms with less than $2^{t \log_2(p^k)}$ subsets of positive terms. Thus we can check all possible solutions in $O(p^{kt})$ comparisons. □

We have proved Theorem 6.3 for each $t \geq 1$ from Theorem 6.2. Theorem 6.2 is independent of ZFC for each fixed t. We know of no other proof. Thus, Theorem 6.3 for each fixed $t \geq 1$ is in ZFC limbo. If a ZFC proof could be found that the subset sum problem is solvable in polynomial time $O(n^\gamma)$ where n is the length of the instance (p^k for fixed k, $p \geq 2$ here), then that result would prove Theorem 6.3 for $t = \gamma$ and thus remove that case from limbo by showing that it is provable within ZFC. We conjecture, however, that Theorem 6.3 for each fixed $t \geq 1$ is itself independent of ZFC. The basis for this conjecture is that the subset sum problem arises from Theorem 6.2 in a very natural, generic way. Of course, if our conjecture is true, "subset sum is solvable in polynomial time" cannot be proved in ZFC.

There are other natural possibilities for the sets of instances 21 as well (e.g., [Wil17b]). The challenge is to find a family for which the independence of the analog of Theorem 6.3 can be proved.

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