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Patterns in words of ordered set partitions

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An ordered set partition of $\{1, 2, \ldots, n\}$ is a partition with an ordering on the parts. Let $\mathcal{OP}_{n,k}$ be the set of ordered set partitions of $[n]$ with k blocks. Godbole, Goyt, Herdan and Pudwell defined $\mathcal{OP}_{n,k}(\sigma)$ to be the set of ordered set partitions in $\mathcal{OP}_{n,k}$ avoiding a permutation pattern σ and obtained the formula for $|\mathcal{OP}_{n,k}(\sigma)|$ when the pattern σ is of length 2. Later, Chen, Dai and Zhou found a formula algebraically for $|\mathcal{OP}_{n,k}(\sigma)|$ when the pattern σ is of length 3.

In this paper, we define a new pattern avoidance for the set $\mathcal{OP}_{n,k}$, called $\mathcal{WOP}_{n,k}(\sigma)$, which includes the questions proposed by Godbole, Goyt, Herdan and Pudwell. We obtain formulas for $|\mathcal{WOP}_{n,k}(\sigma)|$ combinatorially for any σ of length 3. We also define 3 kinds of descent statistics on ordered set partitions and study the distribution of the descent statistics on $\mathcal{WOP}_{n,k}(\sigma)$ for σ of length 3.

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1. Introduction

In [4], Godbole, Goyt, Herdan and Pudwell initiated the study of patterns in ordered set partitions. In particular, they studied the number of ordered set partitions which avoid certain types of permutations of length 2 and 3. A *partition* π of $[n] = \{1,\ldots,n\}$ is a family of nonempty, pairwise disjoint subsets $B_1, B_2, \ldots, B_k$ of $[n]$ called *parts* (*blocks*) such that $\bigcup_{i=1}^k B_i = [n]$. We let $\ell(\pi)$ denote the number of parts in π and $|\pi| = n$ denote the size of π. We let $\min(B_i)$ and $\max(B_i)$ denote the minimal and maximal elements of B_i and we use the convention that we order the parts so that $\min(B_1) < \cdots < \min(B_k)$. To simplify notation, we shall write π as $B_1/\cdots/B_k$. Thus we would write $\pi = 134/268/57$ for the set partition π of $[8]$ with parts $B_1 = \{1,3,4\}$, $B_2 = \{2,6,8\}$ and $B_3 = \{5,7\}$. Pattern avoidance problems in set partitions was studied by Sagan [17]; Jelínek and Mansour [8]; Jelínek, Mansour and Shattuck [9]. See Mansour [14] for a comprehensive introduction to set partitions.

An *ordered set partition* with underlying set partition π is just a permutation of the parts of π, i.e. $\delta = B_{\sigma_1}/\cdots/B_{\sigma_k}$ for some permutation σ in the symmetric group S_k. For example, $\delta = 57/134/268$ is an ordered set partition of the set $[8]$ with underlying set partition $\pi = 134/268/57$. Given an ordered set partition $\delta = B_{\sigma_1}/\cdots/B_{\sigma_k}$, we let the *word* of δ, $w(\delta)$, be the word obtained from δ by removing all the slashes. For example, if

$\delta = 57/134/268$, then $w(\delta) = 57134268$. We let $\mathcal{OP}_n$ denote the set of ordered set partitions of $[n]$ and $\mathcal{OP}_{n,k}$ denote the set of ordered set partitions of $[n]$ with k parts.

If $b_1, \ldots, b_k$ are positive integers, then we let

1. $\mathcal{OP}_{[b_1,\ldots,b_k]}$ denote the set of ordered set partitions $B_1/\cdots/B_k$ of $[b_1 + \cdots + b_k]$ such that $|B_i| = b_i$ for $i = 1, \ldots, b_k$,
2. $\mathcal{OP}_{n,\{b_1,\ldots,b_k\}}$ denote the set of ordered set partitions $\pi \in \mathcal{OP}_n$ such that the size of any part in π is an element of $\{b_1, \ldots, b_k\}$, and
3. $\mathcal{OP}_{\langle b_1^{\beta_1},\ldots,b_k^{\beta_k}\rangle}$ denote the set of ordered set partitions π of $[\sum_{i=1}^k \beta_i b_i]$ which has β_i parts of size b_i for $i = 1, \ldots, k$.

Note that

$$\bigcup_{n\geq 0} \mathcal{OP}_{n,\{b_1,\ldots,b_k\}} = \bigcup_{\beta_1\geq 0,\ldots,\beta_k\geq 0} \mathcal{OP}_{\langle b_1^{\beta_1},\ldots,b_k^{\beta_k}\rangle}.$$

Clearly, $|\mathcal{OP}_{[b_1,\ldots,b_k]}| = \binom{n}{b_1,\ldots,b_k}$ if $b_1 + \cdots + b_k = n$.

Given a sequence of distinct positive integers $w = w_1 \cdots w_n$, we let $\mathrm{red}(w)$ denote the permutation in S_n obtained from w by replacing the i^{th} smallest letter in w by i. For example, $\mathrm{red}(4592) = 2341$. Following [4], we say that a permutation $\sigma = \sigma_1 \cdots \sigma_j$ **occurs** in an ordered set partition $\delta = B_1/\cdots/B_k$ if and only if there exists $1 \leq i_1 < \cdots < i_j \leq k$ and $b_{i_m} \in B_{i_m}$ such that $\mathrm{red}(b_{i_1} \cdots b_{i_j}) = \sigma$, and δ **avoids** σ if σ does not occur in δ. For example, if $\delta = 57/134/268$, then 213 occurs in δ since $\mathrm{red}(518) = 213$, but δ avoids 123 because every element in the first part $\{5, 7\}$ of δ is bigger than every element in the second part $\{1, 3, 4\}$ of δ. If α is a permutation in S_j, then we let $\mathcal{OP}_n(\alpha)$ denote the set of ordered set partitions of $[n]$ that avoid α. We can then define $\mathcal{OP}_{n,k}(\alpha)$, $\mathcal{OP}_{[b_1,\ldots,b_k]}(\alpha)$, $\mathcal{OP}_{n,\{b_1,\ldots,b_k\}}(\alpha)$ and $\mathcal{OP}_{\langle b_1^{\beta_1},\ldots,b_k^{\beta_k}\rangle}(\alpha)$ in a similar manner. We let

$$\begin{aligned} op_n(\alpha) &:= |\mathcal{OP}_n(\alpha)|, \\ op_{n,k}(\alpha) &:= |\mathcal{OP}_{n,k}(\alpha)|, \\ op_{[b_1,\ldots,b_k]}(\alpha) &:= |\mathcal{OP}_{[b_1,\ldots,b_k]}(\alpha)|, \quad \text{and} \\ op_{\langle b_1^{\beta_1},\ldots,b_k^{\beta_k}\rangle}(\alpha) &:= |\mathcal{OP}_{\langle b_1^{\beta_1},\ldots,b_k^{\beta_k}\rangle}(\alpha)|. \end{aligned}$$

Godbole, Goyt, Herdan and Pudwell [4] proved a number of interesting results about these quantities. For example, they showed that

$$op_{n,k}(\sigma) = op_{n,k}(123)$$

for all permutations σ of length 3. They also proved that

$$op_{n,3}(123) = op_{n,3}(132) = \left(\frac{n^2}{8} + \frac{3n}{8} - 2\right) 2^n + 3$$

and

$$op_{n,n-1}(123) = \frac{3(n-1)^2\binom{2n-2}{n-1}}{n(n+1)}.$$

Later, Chen, Dai and Zhou [2] proved that

$$(1.1)\quad 1 + \sum_{n\geq 1} t^n \sum_{k=1}^{n} op_{n,k}(123)x^k = \frac{-x + 2xt - 2t + 2t^2x + 2t^2 + x\sqrt{1 - 4xt - 4t + 4t^2x + 4t^2}}{2t(x+1)^2(t-1)}.$$

The goal of this paper is to study an alternative notion of pattern avoidance in ordered set partitions. Given an ordered set partition $\delta = B_1/\cdots/B_k$ of $[n]$, let $w(\delta) = w_1 \cdots w_n$ denote the word of δ. Then we say that a permutation $\alpha = \alpha_1 \cdots \alpha_j \in S_j$ **occurs in the word of** δ if there exists $1 \leq i_1 < \cdots < i_j \leq n$ such that $\mathrm{red}(w_{i_1} \cdots w_{i_j}) = \alpha$. Thus α occurs in the word of δ if α classically occurs in $w(\delta)$. We say that an ordered set partition δ **word-avoids** α if α does not occur in the word of δ. For example, if $\delta = 57/134/268$, we saw that δ avoids 123 in the sense of [4], but clearly 123 occurs in the word of δ since $\mathrm{red}(134) = 123$. Then we let $\mathcal{WOP}_n(\alpha)$ denote the set of ordered set partitions which word-avoid α. Similarly, we can define $\mathcal{WOP}_{n,k}(\alpha)$, $\mathcal{WOP}_{[b_1,\ldots,b_k]}(\alpha)$, $\mathcal{WOP}_{n,\{b_1,\ldots,b_k\}}(\alpha)$ and $\mathcal{WOP}_{\langle b_1^{\beta_1},\ldots,b_k^{\beta_k}\rangle}(\alpha)$. Then we let

$$\begin{aligned} wop_n(\alpha) &:= |\mathcal{WOP}_n(\alpha)|, \\ wop_{n,k}(\alpha) &:= |\mathcal{WOP}_{n,k}(\alpha)|, \\ wop_{[b_1,\ldots,b_k]}(\alpha) &:= |\mathcal{WOP}_{[b_1,\ldots,b_k]}(\alpha)|, \quad \text{and} \\ wop_{\langle b_1^{\alpha_1},\ldots,b_k^{\alpha_k}\rangle}(\alpha) &:= |\mathcal{WOP}_{\langle b_1^{\alpha_1},\ldots,b_k^{\alpha_k}\rangle}(\alpha)|. \end{aligned}$$

We also study the corresponding generating functions

$$\begin{aligned} \mathbb{WOP}_\alpha(t) &:= 1 + \sum_{n\geq 1} wop_n(\alpha)\ t^n, \\ \mathbb{WOP}_\alpha(x,t) &:= 1 + \sum_{n\geq 1} t^n \sum_{k=1}^{n} wop_{n,k}(\alpha)\ x^k, \quad \text{and} \end{aligned}$$

$$\mathbb{WOP}_{\alpha,\{b_1,\dots,b_k\}}(x,t,q_1,\dots q_k) := \sum_{\beta_1\geq 0}\cdots\sum_{\beta_k\geq 0} wop_{\langle b_1^{\beta_1},\dots,b_k^{\beta_k}\rangle}(\alpha)\ t^{\sum_{i=1}^k b_i\beta_i} x^{\sum_{i=1}^k \beta_i}\ q_1^{\beta_1}\cdots q_k^{\beta_k}.$$

Note that $wop_{n,k}(321) = op_{n,k}(321)$. That is, if 321 occurs in the word of an ordered set partition δ, then the occurrences of 3, 2 and 1 must have been in different parts of the partition δ so that 321 would occur in δ in the sense of Godbole, Goyt, Herdan and Pudwell [4]. However, for other $\sigma \in S_3$, it is not the case that $wop_{n,k}(\sigma) = op_{n,k}(\sigma)$. In fact, it follows from the results of the this paper that we have 3 Wilf-equivalence where $wop_n(\sigma)$ for $\sigma \in S_3$, namely $wop_n(123)$, $wop_n(132) = wop_n(231) = wop_n(312) = wop_n(213)$ and $wop_n(321)$.

We shall also study refinements of these generating functions by descents. Recall that for a permutation $\sigma = \sigma_1\cdots\sigma_n \in S_n$, the *descent set* of σ is defined as $\mathrm{Des}(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$, and the number of descents of σ is $des(\sigma) = |\mathrm{Des}(\sigma)|$. In fact, there are four natural notions of descents in an ordered set partition $\pi = B_1/\cdots/B_k \in \mathcal{OP}_n$. That is, we let $des(\pi)$ be the number of descents in the word of π, $w(\pi) = w_1\cdots w_n$. Thus $des(\pi) := |\{i : w_i > w_{i+1}\}|$. Given two consecutive parts B_i and B_{i+1}, we write $B_i >_p B_{i+1}$ if every element of B_i is greater than every element of B_{i+1} and we write $B_i >_{min} B_{i+1}$ if the minimal element of B_i is greater than the minimal element of B_{i+1}. We shall call elements i such that $B_i >_p B_{i+1}$ *part-descents* and elements i such that $B_i >_{min} B_{i+1}$ *min-descents*. We also let i such that $\max(B_i) > \max(B_{i+1})$ be a *max-descent*. Then we define

$$\begin{aligned}
des(\pi) &:= |\{i : w(\pi)_i > w(\pi)_{i+1}\}| = |\{i : \max(B_i) > \min(B_{i+1})\}|,\\
pdes(\pi) &:= |\{i : B_i >_p B_{i+1}\}| = |\{i : \min(B_i) > \max(B_{i+1})\}|,\\
mindes(\pi) &:= |\{i : B_i >_{min} B_{i+1}\}| = |\{i : \min(B_i) > \min(B_{i+1})\}| \quad \text{and}\\
maxdes(\pi) &:= |\{i : \max(B_i) > \max(B_{i+1})\}|.
\end{aligned}$$

The statistics des, *pdes* and *mindes* are not equi-distributed on $\mathcal{OP}_n$ (as can be seen when $n = 3$). We shall show in Section 2 that the statistics *maxdes* and *mindes* are equi-distributed on $\mathcal{OP}_n$. A number of other Euler-Mahonian statistics of ordered set partitions were studied in [6, 7, 10, 16, 18]. Wilson [19] also studied Mahonian statistics of ordered multiset partitions.

For each type of generating function above, we consider the refined generating function where we keep track of the number of descents of each type.

In particular, we shall study the following generating functions,

$$\begin{aligned}
\mathbb{WOP}_{\alpha}^{des}(x,y,t) &:= 1+\sum_{n\geq 1} t^n \sum_{\pi\in\mathcal{WOP}_n(\alpha)} x^{\ell(\pi)} y^{des(\pi)},\\
\mathbb{WOP}_{\alpha}^{pdes}(x,y,t) &:= 1+\sum_{n\geq 1} t^n \sum_{\pi\in\mathcal{WOP}_n(\alpha)} x^{\ell(\pi)} y^{pdes(\pi)}, \text{ and}\\
\mathbb{WOP}_{\alpha}^{mindes}(x,y,t) &:= 1+\sum_{n\geq 1} t^n \sum_{\pi\in\mathcal{WOP}_n(\alpha)} x^{\ell(\pi)} y^{mindes(\pi)}.
\end{aligned}$$

Similarly, we shall study

$$\begin{aligned}
&\mathbb{WOP}_{\alpha,\{b_1,\ldots,b_k\}}^{des}(x,y,t,q_1,\ldots,q_n)\\
&\qquad := \sum_{\beta_1\geq 0,\ldots,\beta_k\geq 0,} \sum_{\pi\in\mathcal{WOP}_{\langle b_1^{\beta_1},\ldots,b_k^{\beta_k}\rangle}(\alpha)} t^{|\pi|} x^{\ell(\pi)} y^{des(\pi)} q_1^{\beta_1}\cdots q_k^{\beta_k},
\end{aligned}$$

$$\begin{aligned}
&\mathbb{WOP}_{\alpha,\{b_1,\ldots,b_k\}}^{pdes}(x,y,t,q_1,\ldots,q_n)\\
&\qquad := \sum_{\beta_1\geq 0,\ldots,\beta_k\geq 0,} \sum_{\pi\in\mathcal{WOP}_{\langle b_1^{\beta_1},\ldots,b_k^{\beta_k}\rangle}(\alpha)} t^{|\pi|} x^{\ell(\pi)} y^{pdes(\pi)} q_1^{\beta_1}\cdots q_k^{\beta_k},
\end{aligned}$$

$$\begin{aligned}
&\mathbb{WOP}_{\alpha,\{b_1,\ldots,b_k\}}^{mindes}(x,y,t,q_1,\ldots,q_n)\\
&\qquad := \sum_{\beta_1\geq 0,\ldots,\beta_k\geq 0,} \sum_{\pi\in\mathcal{WOP}_{\langle b_1^{\beta_1},\ldots,b_k^{\beta_k}\rangle}(\alpha)} t^{|\pi|} x^{\ell(\pi)} y^{mindes(\pi)} q_1^{\beta_1}\cdots q_k^{\beta_k}.
\end{aligned}$$

The main focus of this paper is studying the generating functions described above where α is in S_2 or S_3. One advantage of our notion of word-avoidance in ordered set partitions is that we can employ standard techniques from the theory of generating functions such as the Lagrange Inversion Theorem to give us nice answers. For example, we will show that

$$\begin{aligned}
&\mathbb{WOP}_{132}(x,t) = \frac{t+1-\sqrt{(t+1)^2-4t(x+1)}}{2t(1+x)},\\
&wop_{n,k}(132) = \frac{1}{k}\binom{n-1}{k-1}\binom{n+k}{k-1},
\end{aligned}$$

and

$$wop_{\langle b_1^{\beta_1},\ldots,b_k^{\beta_k}\rangle}(132) = \frac{1}{n}\binom{k}{\beta_1,\ldots,\beta_k}\binom{n+k}{n-1},$$

where $n = \sum_{i=1}^k b_i\beta_i$ and $k = \sum_{i=1}^k \beta_i$.

Similarly, we will show that

$$\mathbb{WOP}_{132}^{des}(x,y,t) = \frac{(1+2yt+xyt-t-xt)}{2t(y+xy)} - \frac{\sqrt{((1+2yt+xyt-t-xt))^2 - 4t(1-t+ty)(x+yx)}}{2t(y+xy)}$$

and

$$\sum_{\pi\in\mathcal{WOP}_{n,k}(132)} y^{des(\pi)} = \frac{1}{k}\binom{n-1}{k-1}\sum_{j-0}^{k-1}\binom{k}{j}\binom{n-1}{k-1-j}y^{k-1-j}.$$

The outline of this paper is as follows. In Section 2, we will compute generating functions for ordered set partitions word-avoiding patterns of length 2 and prove some symmetries in the generating functions $\mathbb{WOP}_\alpha^{des}(x,y,t)$, $\mathbb{WOP}_\alpha^{pdes}(x,y,t)$ and $\mathbb{WOP}_\alpha^{mindes}(x,y,t)$ for $\alpha \in S_j$ and $j \geq 3$. In Section 3, we will show how to compute generating functions $\mathbb{WOP}_\alpha^{des}(x,y,t)$ for all $\alpha \in S_3$. In Sections 4 and 5, we will study generating functions $\mathbb{WOP}_\alpha^{pdes}(x,y,t)$ and $\mathbb{WOP}_\alpha^{mindes}(x,y,t)$ for α in S_3. In Section 6, we will summarize open problems about our research.

2. Preliminaries

The structures of elements in $\mathcal{WOP}_n(12)$ and $\mathcal{WOP}_n(21)$ are quite easy to describe. For example, if $\pi \in \mathcal{WOP}_n(12)$, then the word of π must be $n(n-1)\cdots 21$ and hence $\pi = n/n-1/\cdots/1$. Similarly, if $\pi \in \mathcal{WOP}_n(21)$, then the word of π must be $12\cdots(n-1)n$ and hence π must be of the form $B_1/B_2/\cdots/B_k$ where for each $i = 1,\ldots,k-1$, all the elements of B_i are smaller than all the elements of B_{i+1}. It follows that $wop_{n,k}(21) = \binom{n-1}{k-1}$ because to specify an ordered set partition $\pi \in \mathcal{WOP}_{n,k}(21)$ with k parts, we only need to specify where we place the $k-1$ slashes in the $n-1$ spaces between the letters $1,\ldots,n$.

Thus,

$$\mathbb{WOP}_{12}^{des}(x,y,t) = 1 + \sum_{n\geq 1} y^{n-1}x^n t^n = 1 + \frac{xt}{1-xyt}$$

and $\mathbb{WOP}_{12}^{des}(x,y,t) = \mathbb{WOP}_{12}^{pdes}(x,y,t) = \mathbb{WOP}_{12}^{mindes}(x,y,t)$. Similarly,

$$\begin{aligned}
\mathbb{WOP}_{21}^{des}(x,y,t) &= 1 + \sum_{n\geq 1} t^n \sum_{k=1}^{n} \binom{n-1}{k-1} x^k \\
&= 1 + xt\sum_{n\geq 1} t^{n-1} \sum_{k=1}^{n} \binom{n-1}{k-1} x^{k-1} \\
&= 1 + xt\sum_{n\geq 1} t^{n-1}(1+x)^{n-1} \\
&= 1 + \frac{xt}{1-t(1+x)},
\end{aligned}$$

and $\mathbb{WOP}_{21}^{des}(x,y,t) = \mathbb{WOP}_{21}^{pdes}(x,y,t) = \mathbb{WOP}_{21}^{mindes}(x,y,t)$.

Next consider the generating functions $\mathbb{WOP}_{\alpha}^{des}(x,y,t)$, $\mathbb{WOP}_{\alpha}^{pdes}(x,y,t)$, and $\mathbb{WOP}_{\alpha}^{mindes}(x,y,t)$ when $\alpha \in S_j$ for $j \geq 3$. There are some obvious symmetries in our situation. Recall that for a permutation $\sigma = \sigma_1 \cdots \sigma_n$, the *reverse* of σ is defined by $\sigma^r = \sigma_n \cdots \sigma_1$ and the *complement* of σ is defined by $\sigma^c = (n+1-\sigma_1)\cdots(n+1-\sigma_n)$. It is easy to see that $des(\sigma) = des((\sigma^r)^c)$.

We can define reverse and complement on ordered set partitions as well. That is, suppose that $\pi = B_1/\cdots/B_k$ is an ordered set partition of $[n]$. Then if $B_i = \{a_1^i < a_2^i < \cdots < a_j^i\}$, we let the complement of B_i be $B_i^c = \{(n+1-a_j^i) < \cdots < (n+1-a_2^i) < (n+1-a_1^i)\}$. Then we let the reverse of π be $\pi^r = B_k/\cdots/B_1$ and the complement of π be $\pi^c = B_1^c/\cdots/B_k^c$. Thus $(\pi^r)^c = B_k^c/\cdots/B_1^c$.

It is easy to see that if $w(\pi) = w_1\cdots w_n$, then the word of $(\pi^r)^c$ is $(n+1-w_n)\cdots(n+1-w_1) = (w(\pi)^r)^c$. Similarly it is easy to see that if $B_i >_p B_{i+1}$, then $B_{i+1}^c >_p B_i^c$; and if $\min(B_i) > \min(B_{i+1})$, then $\max(B_{i+1}^c) > \max(B_i^c)$. Thus the operation of reverse-complement shows that *maxdes* and *mindes* are equi-distributed on $\mathcal{OP}_n$, and

$$\begin{aligned}
\sum_{\pi\in\mathcal{WOP}_{n,k}(\alpha)} x^{\ell(\pi)} y^{des(\pi)} &= \sum_{\pi\in\mathcal{WOP}_{n,k}((\alpha^r)^c)} x^{\ell(\pi)} y^{des(\pi)}, \\
\sum_{\pi\in\mathcal{WOP}_{n,k}(\alpha)} x^{\ell(\pi)} y^{pdes(\pi)} &= \sum_{\pi\in\mathcal{WOP}_{n,k}((\alpha^r)^c)} x^{\ell(\pi)} y^{pdes(\pi)}, \\
\sum_{\pi\in\mathcal{WOP}_{n,k}(\alpha)} x^{\ell(\pi)} y^{maxdes(\pi)} &= \sum_{\pi\in\mathcal{WOP}_{n,k}((\alpha^r)^c)} x^{\ell(\pi)} y^{mindes(\pi)}.
\end{aligned}$$

This allows us to skip the computation of *maxdes* distribution on $\mathcal{WOP}_n(\alpha)$.

It follows that for all $1 \leq b_1 < \cdots < b_s$,

$$\begin{aligned} \mathbb{WOP}^*_{132}(x,y,t) &= \mathbb{WOP}^*_{213}(x,y,t), \\ \mathbb{WOP}^*_{231}(x,y,t) &= \mathbb{WOP}^*_{312}(x,y,t), \\ \mathbb{WOP}^*_{132,\{b_1,\ldots,b_s\}}(x,y,t,q_1,\ldots,q_s) &= \mathbb{WOP}^*_{213,\{b_1,\ldots,b_s\}}(x,y,t,q_1,\ldots,q_s), \end{aligned}$$

and

$$\mathbb{WOP}^*_{231,\{b_1,\ldots,b_s\}}(x,y,t,q_1,\ldots,q_s) = \mathbb{WOP}^*_{312,\{b_1,\ldots,b_s\}}(x,y,t,q_1,\ldots,q_s),$$

where $*$ is either *des* or *pdes*.

Reverse-complement does not always preserve *mindes*. For example,

$$\sum_{\pi \in \mathcal{WOP}_3(132)} x^{\ell(\pi)} y^{mindes(\pi)} \neq \sum_{\pi \in \mathcal{WOP}_3(213)} x^{\ell(\pi)} y^{mindes(\pi)}.$$

In general, reverse and complement by themselves do not preserve these generating functions. For example, since $|\mathcal{WOP}_n(123)| \neq |\mathcal{WOP}_n(321)|$ for any $n \geq 3$, it follows that

$$\mathbb{WOP}^*_{123}(x,y,t) \neq \mathbb{WOP}^*_{321}(x,y,t),$$

where $*$ is *des*, *pdes* or *mindes*.

Our next theorem will show that

$$\mathbb{WOP}^{des}_{312}(x,y,t) = \mathbb{WOP}^{des}_{213}(x,y,t)$$

and

$$\mathbb{WOP}^{mindes}_{312}(x,y,t) = \mathbb{WOP}^{mindes}_{213}(x,y,t).$$

Thus, there are only three different generating functions of the form $\mathbb{WOP}^{des}_{\alpha}(x,y,t)$ for $\alpha \in S_3$. Similarly, our next theorem will show that for all $1 \leq b_1 < \cdots < b_s$,

$$\mathbb{WOP}^{des}_{213,\{b_1,\ldots,b_s\}}(x,y,t,q_1,\ldots,q_s) = \mathbb{WOP}^{des}_{312,\{b_1,\ldots,b_s\}}(x,y,t,q_1,\ldots,q_s)$$

and

$$\mathbb{WOP}^{mindes}_{213,\{b_1,\ldots,b_s\}}(x,y,t,q_1,\ldots,q_s) = \mathbb{WOP}^{mindes}_{312,\{b_1,\ldots,b_s\}}(x,y,t,q_1,\ldots,q_s).$$

Theorem 2.1. *There is a bijection $\phi_n : \mathcal{WOP}_n(312) \to \mathcal{WOP}_n(213)$ such that for all $\pi = B_1/\cdots/B_k \in \mathcal{WOP}_n(312)$, $\phi_n(\pi) = C_1/\cdots/C_k \in \mathcal{WOP}_n(213)$ where $|B_i| = |C_i|$ for $i = 1,\ldots,k$. The number 1 is in position k in $w(\pi)$ if and only if 1 is in position k in $w(\phi_n(\pi))$, and $des(\pi) = des(\phi_n(\pi))$, $\mathrm{Des}(w(\pi)) = \mathrm{Des}(w(\phi_n(\pi)))$, $mindes(\pi) = mindes(\phi_n(\pi))$.*

Proof. We shall define $\phi_n : \mathcal{WOP}_n(312) \rightarrow \mathcal{WOP}_n(213)$ by induction on n. For $1 \leq n \leq 2$, we let ϕ_n be the identity map. Now assume that we have defined $\phi_k : \mathcal{WOP}_k(312) \rightarrow \mathcal{WOP}_k(213)$ for $k \leq n-1$. We classify the ordered set partitions π in $\mathcal{WOP}_n(312)$ by the position of 1 in $w(\pi)$.

First suppose that 1 occurs in position 1 in $w(\pi)$. If 1 is in a part by itself, then π is of the form $1/B_2/\cdots/B_k$ for some $k \geq 2$. In this case, we can subtract 1 from each element in $B_2/\cdots/B_k$ to obtain an ordered set partition $\pi^* = B_2^*/\cdots/B_k^*$ in $\mathcal{WOP}_{n-1}(312)$. Then let $\phi_{n-1}(B_2^*/\cdots/B_k^*) = C_2^*/\cdots/C_k^*$ and let $C_2/\cdots/C_k$ be result of adding 1 to each element of $C_2^*/\cdots/C_k^*$. It is easy to see that if we let $\phi_n(1/B_2/\cdots/B_k) = 1/C_2/\cdots/C_k$, then $1/C_2/\cdots/C_k \in \mathcal{WOP}_n(213)$, $|B_i| = |C_i|$ for $i = 2,\ldots,k$, $des(1/B_2/\cdots/B_k) = des(1/C_2/\cdots/C_k)$, $\mathrm{Des}(w(1/B_2/\cdots/B_k)) = \mathrm{Des}(w(1/C_2/\cdots/C_k))$, and $mindes(1/B_2/\cdots/B_k) = mindes(1/C_2/\cdots/C_k)$. If 1 is not in a part by itself, then π is of the form $B_1/\cdots/B_k$ where $1 \in B_1$ and $|B_1| \geq 2$. In this case, we can remove 1 from B_1 and subtract 1 from each of the remaining element to obtain an ordered set partition $\pi^* = B_1^*/\cdots/B_k^*$ in $\mathcal{WOP}_{n-1}(312)$. Then let $\phi_{n-1}(B_1^*/\cdots/B_k^*) = C_1^*/\cdots/C_k^*$ and let $C_1/\cdots/C_k$ be result of adding 1 to each element of $C_1^*/\cdots/C_k^*$ and then adding 1 to the first part. Again it is easy to see that if we let $\phi_n(B_1/\cdots/B_k) = C_1/\cdots/C_k$, then $C_1/\cdots/C_k \in \mathcal{WOP}_n(213)$, $|B_i| = |C_i|$ for $i = 1,\ldots,k$, $des(B_1/\cdots/B_k) = des(C_1/\cdots/C_k)$, $\mathrm{Des}(w(B_1/\cdots/B_k)) = \mathrm{Des}(w(C_1/\cdots/C_k))$, and $mindes(B_1/\cdots/B_k) = mindes(C_1/\cdots/C_k)$.

Next suppose that 1 occurs in position r in $w(\pi)$ where $r \geq 2$. Then π must be of the form $B_1/\cdots/B_j/B_{j+1}/\cdots/B_k$ where $j \geq 1$ and 1 is the first element of part B_{j+1}. Since $w(\pi)$ is 312-avoiding, it must be the case all the elements of $B_1,\ldots,B_j$ are less than all the elements of $B_{j+1} - \{1\}, B_{j+2},\ldots,B_k$. It follows that $B_1/\cdots/B_j$ is a set partition of $\{2,\ldots,r\}$ such that $w(B_1/\cdots/B_j)$ reduces to a 312-avoiding permutation and $B_{j+1} - \{1\}/\cdots/B_k$ is a set partition of $\{r+1,\ldots,n\}$ such that the reduction of $w(B_{j+1}/\cdots/B_k)$ is 312-avoiding. Moreover, $r-1$ is a descent in $w(\pi)$ and $B_j >_{min} B_{j+1}$. In this case, we let $B_{j+1}^*/\cdots/B_k^*$ be the result of subtracting $r-1$ from each element of $B_{j+1},\ldots,B_k$ except the element 1 so that $B_{j+1}^*/\cdots/B_k^*$ is an ordered set partition in $\mathcal{WOP}_{n-r+1}(312)$ whose word starts with 1. We let $B_1^*/\cdots/B_j^*$ be the result of subtracting 1 from each element of $B_1/\cdots/B_j$ so that $B_1^*/\cdots/B_j^*$ is an element of $\mathcal{WOP}_{r-1}(312)$. Now let $\phi_{r-1}(B_1^*/\cdots/B_j^*) = C_1/\cdots/C_j$ and $\phi_{n-r+1}(B_{j+1}^*/\cdots/B_k^*) = D_1/\cdots/D_{k-j}$. We can then add $n-r+1$ to each element of $C_1/\cdots/C_j$ to produce an ordered set partition $C_1^*/\cdots/C_j^*$ of $\{n-r+2,\ldots,n\}$ whose word reduces to a 213-avoiding permutation such that $des(\mathrm{red}(w(C_1^*/\cdots/C_j^*))) =$

$des(w(B_1/\cdots/B_j))$, $\mathrm{Des}(\mathrm{red}(w(C_1^*/\cdots/C_j^*))) = \mathrm{Des}(w(B_1/\cdots/B_j))$, and $mindes(C_1^*/\cdots/C_j^*) = mindes(B_1/\cdots/B_j)$. Then we let

$$\phi_n(\pi) = C_1^*/\cdots/C_j^*/D_1/\cdots/D_{k-j}.$$

It is easy to see by induction that $des(w(\pi)) = des(w(\phi_n(\pi)))$, $\mathrm{Des}(w(\pi)) = \mathrm{Des}(w(\phi_n(\pi)))$ and $mindes(\pi) = mindes(\phi_n(\pi))$. Moreover, by construction 1 is in position r in both $w(\pi)$ and $w(\phi_n(\pi))$. The only thing we have to check is that $w(\phi_n(\pi))$ is 213-avoiding, but this follows from the fact that all the elements in $C_1^*/\cdots/C_j^*$ are bigger than all the elements in $D_1/\cdots/D_{k-j}$, and the permutations $\mathrm{red}(w(C_1^*/\cdots/C_j^*))$ and $w(D_1/\cdots/D_{k-j})$ are both 213-avoiding. □

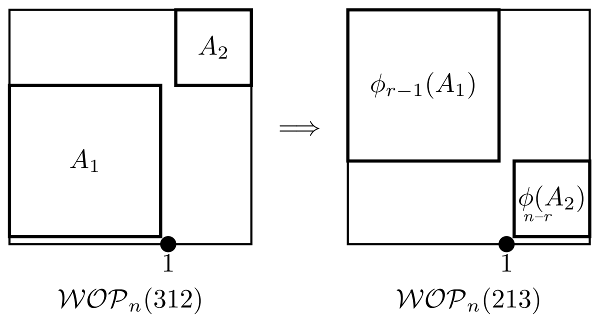

Figure 1: Bijection $\phi_n : \mathcal{WOP}_n(312) \to \mathcal{WOP}_n(213)$.

Figure 2 shows that $\phi_5(3/24/15) = 5/34/12$. Observe that the number of descents, word descent set, and the number of min-descents are preserved, while the number of part-descents is *not* preserved.

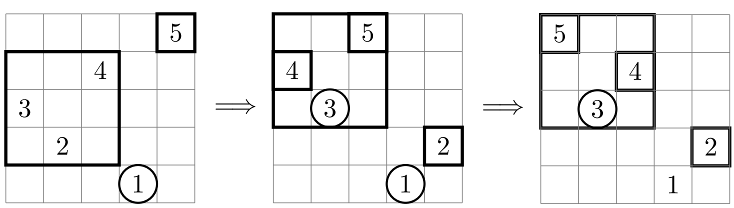

Figure 2: $\pi = 3/24/15 \in \mathcal{WOP}_{5,3}(312)$ and $\phi_5(\pi) = 5/34/12 \in \mathcal{WOP}_{5,3}(213)$.

We end this section with two observations. Suppose that $\pi = B_1/\cdots/B_k \in \mathcal{WOP}_{n,k}(132)$. First, we notice that if the last element $\max(B_i)$ of B_i

is greater than the first element $\min(B_{i+1})$ of B_{i+1} so that there is a descent in $w(\pi)$ at position $\sum_{j=1}^{i}|B_j|$, then it must be the case that $\min(B_i) > \min(B_{i+1})$. That is, if $\min(B_i) < \min(B_{i+1})$, then $\min(B_i) \neq \max(B_i)$ and hence $(\min(B_i), \max(B_i), \min(B_{i+1}))$ would reduce to 132. It follows that for all $\pi \in \mathcal{WOP}_n(132)$, $des(\pi) = mindes(\pi)$, and hence,

$$\mathbb{WOP}_{132}^{des}(x,y,t) = \mathbb{WOP}_{132}^{mindes}(x,y,t). \tag{2.1}$$

Second, for any $\pi = B_1/\cdots/B_k \in \mathcal{WOP}_{n,k}(132)$, i is a max-descent if and only if i is a part-descent. Otherwise if $\min(B_i) < \max(B_{i+1})$, then the triple $(\min(B_i), \max(B_i), \max(B_{i+1}))$ matches the pattern 132. Let

$$\mathbb{WOP}_{132}^{maxdes}(x,y,t) := 1 + \sum_{n\geq 1} t^n \sum_{\pi \in \mathcal{WOP}_n(132)} x^{\ell(\pi)} y^{maxdes(\pi)},$$

then we have $\mathbb{WOP}_{132}^{pdes}(x,y,t) = \mathbb{WOP}_{132}^{maxdes}(x,y,t)$. Note that the set $\mathcal{WOP}_n(213)$ is in bijection with $\mathcal{WOP}_n(132)$ by the action of reverse-complement, and the *maxdes* statistic on $\mathcal{WOP}_n(132)$ corresponds to the *mindes* statistic on $\mathcal{WOP}_n(213)$. By Theorem 2.1, we have

$$\begin{aligned}\mathbb{WOP}_{132}^{pdes}(x,y,t) &= \mathbb{WOP}_{213}^{pdes}(x,y,t) = \mathbb{WOP}_{132}^{maxdes}(x,y,t) \\ &= \mathbb{WOP}_{213}^{mindes}(x,y,t) = \mathbb{WOP}_{312}^{mindes}(x,y,t).\end{aligned} \tag{2.2}$$

3. Computing $\mathbb{WOP}_{\alpha}^{des}(x,y,t)$ for $\alpha \in S_3$

In this section, we shall derive generating functions $\mathbb{WOP}_{\alpha}^{des}(x,y,t)$ for all $\alpha \in S_3$.

3.1. The functions $\mathbb{WOP}_{132}^{des}(x,y,t) = \mathbb{WOP}_{213}^{des}(x,y,t) = \mathbb{WOP}_{231}^{des}(x,y,t) = \mathbb{WOP}_{312}^{des}(x,y,t)$

In Section 2, we have showed the equality of the four generating functions. We shall compute $\mathbb{WOP}_{132}^{des}(x,y,t)$. In this case, we shall classify the ordered set partitions π in $\mathcal{WOP}_n(132)$ by the size of the last part. That is, suppose that $\pi = B_1/\cdots/B_k$ where $B_k = \{a_1 < \cdots < a_r\}$. Then we let A_{r+1} denote the set of elements in $B_1/\cdots/B_{k-1}$ that are greater that a_r, A_1 denote the set of elements in $B_1/\cdots/B_{k-1}$ that are less that a_1, and A_i denote the set of elements j in $B_1/\cdots/B_{k-1}$ such that $a_i > j > a_{i-1}$ for $i = 2,\ldots,r$. Since $w(\pi)$ is 132-avoiding, for any $i \geq 2$, every element y in A_i must appear to the left of every element x in A_{i-1} since otherwise xya_i would be an occurrence of 132 in $w(\pi)$. It follows that the word of π has the structure pictured in Figure 3. Note that it is possible that any given A_i is empty. However, this structure ensures that no part of π can contain elements from two different A_i's so that if A_i is non-empty, then A_i is a union of consecutive parts of π, say $A_i = B_a/\cdots/B_b$ for some $a < b$. Moreover, if $i \geq 2$ and $A_i \neq \emptyset$, then the last element of B_b is a descent in $w(\pi)$. That is, either $A_1, \ldots, A_{i-1}$ are empty and there is a descent from the last element of B_b to a_1 which is the first element of B_k or one of $A_1, \ldots, A_{i-1}$ is non-empty. Let p be the largest integer r such that $1 \leq r \leq i-1$ and A_r is non-empty, then there is a descent from the last element of B_b to the first element of the first part of A_p.

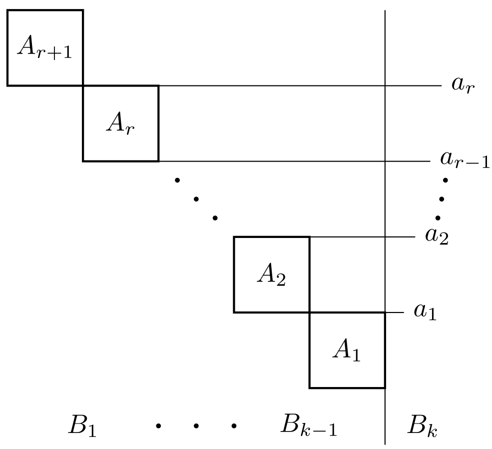

Figure 3: The structure of $\pi \in \mathcal{WOP}_n(132)$.

Let $B(x,y,t) = \mathbb{WOP}_{132}^{des}(x,y,t)$. This structure implies that $B(x,y,t)$ satisfies the following recursion:

$$B(x,y,t) = 1 + \sum_{r\geq 1} xt^r(1 + y(B(x,y,t) - 1))^r B(x,y,t). \tag{3.1}$$

In (3.1) the factor xt^r accounts for those ordered set partitions π whose last part is of size r. We get a factor $1+y(B(x,y,t)-1)$ for A_i for $i = 2, \ldots, r+1$ where the 1 accounts for the possibility that A_i is empty and the term $y(B(x,y,t)-1)$ accounts for the fact that there is descent starting at the last element of A_i if A_i is non-empty. Finally the last factor $B(x,y,t)$ corresponds to the contribution over all possible A_1.

It follows that

$$B(x,y,t) = 1 + \frac{xtB(x,y,t)(1+y(B(x,y,t)-1))}{1-t(1+y(B(x,y,t)-1))}. \tag{3.2}$$

Multiplying both sides of (3.2) by $1-t(1+y(B(x,y,t)-1))$ leads to the quadratic equation

$$0 = (1-t+ty) - B(x,y,t)(1+2yt+xyt-t-tx) + t(xy+y)B(x,y,t)^2,$$

and solving for $B(x,y,t)$ gives that

$$B(x,y,t) = \frac{(1+2yt+xyt-t-tx)}{2t(xy+y)} - \frac{\sqrt{(1+2yt+xyt-t-tx)^2-4(1-t+ty)(t(y+xy))}}{2t(xy+y)}.$$

If we let $f(x,y,t) = B(x,y,t)-1$, then (3.2) gives that

$$f(x,y,t) = x\frac{t(f(x,y,t)+1)(1+y(f(x,y,t))}{1-t(1+yf(x,y,t))}.$$

The Lagrange Inversion Theorem implies that the coefficient of x^k in $f(x,y,t)$ is given by

$$f(x,y,t)|_{x^k} = \frac{1}{k}\delta(x)^k\Big|_{x^{k-1}},$$

where $\delta(x) = \frac{t(x+1)(1+yx)}{1-t(1+yx)}$. Using Newton's binomial theorem, we have

$$\begin{aligned}
f(x,y,t)|_{x^kt^n} &= \frac{1}{k}\frac{t^k(1+x)^k(1+yx)^k}{(1-t(1+yx))^k}\Bigg|_{x^{k-1}t^n} \\
&= \frac{1}{k}(1+x)^k(1+yx)^k \\
&\quad \cdot \left(\sum_{s\geq 0}\binom{k+s-1}{k-1}t^s(1+xy)^s\right)\Bigg|_{x^{k-1}t^{n-k}} \\
&= \frac{1}{k}(1+x)^k(1+yx)^n\binom{k+n-k-1}{k-1}\Bigg|_{x^{k-1}} \\
&= \frac{1}{k}\binom{n-1}{k-1}\sum_{j=0}^{k-1}\binom{k}{j}\binom{n}{k-1-j}y^{k-1-j}.
\end{aligned}$$

Thus we have the following theorem.

Theorem 3.1. *The generating function*

$$\mathbb{WOP}_{132}^{des}(x,y,t) = \frac{(1+2yt+xyt-t-tx)}{2t(y+yx)} - \frac{\sqrt{(1+2yt+xyt-t-tx)^2-4(1-t+ty)(t(y+xy))}}{2t(y+yx)},$$

and

$$\sum_{\pi\in\mathcal{WOP}_{n,k}(132)} y^{des(\pi)} = \frac{1}{k}\binom{n-1}{k-1}\sum_{j=0}^{k-1}\binom{k}{j}\binom{n}{k-1-j}y^{k-1-j}.$$

Setting $y=1$ in Theorem 3.1 and observing that $\sum_{j=0}^{k-1}\binom{k}{j}\binom{n}{k-1-j} = \binom{n+k}{k-1}$, we have the following corollary.

Corollary 3.2. *The generating function*

$$\mathbb{WOP}_{132}(x,t) = \frac{(1+t)-\sqrt{(1+t)^2-4t(1+x)}}{2t(1+x)},$$

and

$$wop_{n,k}(132) = \frac{1}{k}\binom{n-1}{k-1}\binom{n+k}{k-1}.$$

It follows from Theorem 3.1 that $wop_n(132)$ is the number of rooted planar trees with $n+1$ leaves that have no vertices of outdegree 1 because their generating functions both satisfy the recurrence

$$F(t) = 1+\sum_{r\geq 1} t^r F(t)^{r+1}.$$

A bijection follows naturally from the generating function: let $\pi = B_1/\cdots/B_k \in \mathcal{WOP}_n(132)$ where $B_k = \{a_1 < \cdots < a_r\}$, and $A_1,\ldots,A_{r+1}$ be the sub-ordered-partitions of π defined by the previous construction. Then the last part B_k is mapped into a root with outdegree $r+1$, and each A_i is a subgraph connected to the root. Figure 4 shows an example of the bijection. Based on the recursion, the number of non-leaves is equal to the number of blocks of the ordered set partition, and the out-degree of the root is one more than the size of the last block.

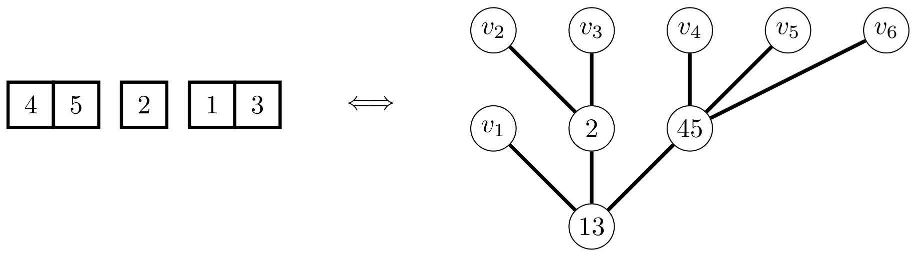

Figure 4: Bijection between $\mathcal{WOP}_n(132)$ and rooted planar trees with no vertices of outdegree 1.

Given any sequence of positive numbers $1 \leq b_1 < b_2 < \cdots < b_s$, we let

$$A = A(x,y,t,q_1,\ldots,q_s) = \mathbb{WOP}^{des}_{132,\{b_1,\ldots,b_s\}}(x,y,t,q_1,\ldots,q_s).$$

It follows from the block structure pictured in Figure 3 that

$$A = 1 + \sum_{i=1}^{s} x q_i t^{b_i} (1 + y(A-1))^{b_i} A.$$

If we set $F = F(x,y,t,q_1,\ldots,q_s) = A(x,y,t,q_1,\ldots,q_s) - 1$, then

$$F = x(F+1) \sum_{i=1}^{s} q_i t^{b_i} (1+yF)^{b_i}.$$

It follows from Lagrange Inversion that

$$F|_{x^k} = \frac{1}{k} \delta^k(x)\Big|_{x^{k-1}},$$

where $\delta(x) = (x+1)\sum_{i=1}^{s} q_i t^{b_i}(1+yx)^{b_i}$. Thus

$$(3.3) \quad F|_{x^k t^n}$$
$$= \frac{1}{k}(x+1)^k \sum_{\substack{\alpha_i \geq 0 \\ \alpha_1+\cdots+\alpha_s=k}} \binom{k}{\alpha_1,\ldots,\alpha_s} t^{\sum_{i=1}^{k}\alpha_i b_i}(1+yx)^{(\sum_{i=1}^{k}\alpha_i b_i)} \prod_{i=1}^{s} q_i^{\alpha_i}\Big|_{x^{k-1}t^n}$$

$$= \frac{1}{k}(x+1)^k(1+yx)^n \sum_{\substack{\alpha_1+\cdots+\alpha_s=k \\ \alpha_1 b_1+\cdots\alpha_k b_k=n}} \binom{k}{\alpha_1,\ldots,\alpha_s} \prod_{i=1}^{s} q_i^{\alpha_i}\Big|_{x^{k-1}}$$

$$= \frac{1}{k}\left(\sum_{j=0}^{k-1}\binom{k}{j}\binom{n}{k-1-j}y^{k-1-j}\right) \sum_{\substack{\alpha_1+\cdots+\alpha_s=k \\ \alpha_1 b_1+\cdots\alpha_k b_k=n}} \binom{k}{\alpha_1,\ldots,\alpha_s} \prod_{i=1}^{s} q_i^{\alpha_i}.$$

If $\sum \alpha_i b_i = n$, then taking the coefficient of $q_1^{\alpha_1}\cdots q_s^{\alpha_s}$ on both sides of equation (3.3) yields the following theorem.

Theorem 3.3. *Suppose that* $0 < b_1 < \cdots < b_s$, $\sum_{i=1}^{s}\alpha_i = k$, *and* $\sum_{i=1}^{s}\alpha_i b_i = n$. *Then*

$$\sum_{\pi\in\mathcal{WOP}_{\langle b_1^{\alpha_1},\ldots,b_s^{\alpha_s}\rangle}(132)} y^{des(\pi)} = \frac{1}{k}\binom{k}{\alpha_1,\ldots,\alpha_s}\left(\sum_{j=0}^{k-1}\binom{k}{j}\binom{n}{k-1-j}y^{k-1-j}\right).$$

Setting $y = 1$ in Theorem 3.3 and observing that $\sum_{j=0}^{k-1}\binom{k}{j}\binom{n}{k-1-j} = \binom{n+k}{k-1}$ yield the following corollary.

Corollary 3.4. *Suppose that* $0 < b_1 < \cdots < b_s$, $\sum_{i=1}^{s}\alpha_i = k$, *and* $\sum_{i=1}^{s}\alpha_i b_i = n$. *Then*

$$wop_{\langle b_1^{\alpha_1},\ldots,b_s^{\alpha_k}\rangle}(132) = \frac{1}{k}\binom{n+k}{k-1}\binom{k}{\alpha_1,\ldots,\alpha_s}.$$

3.2. The function $\mathbb{WOP}^{des}_{123,\{1,2\}}(x,y,t,q_1,q_2)$

Next we turn our attention to ordered set partitions π such that $w(\pi)$ avoids 123. In this case, all parts of π are of size 1 or 2 since any part B_i of size greater than 2 immediately yields a consecutive increasing sequence of size 3 in $w(\pi)$.

Thus we will compute the generating function

$$\mathbb{WOP}^{des}_{123,\{1,2\}}(x,y,t,q_1,q_2) := \sum_{k,\ell\geq 0}\sum_{\pi\in\mathcal{WOP}_{\langle 1^k,2^\ell\rangle}} y^{des(\pi)}t^{k+2\ell}x^{k+\ell}q_1^k q_2^\ell.$$

To compute $\mathbb{WOP}^{des}_{123,\{1,2\}}(x, y, t, q_1, q_2)$, we must first review a bijection Ψ of Deutsch and Elizalde [3] between 123-avoiding permutations and Dyck paths.

Given an $n\times n$ chessboard, we set the origin $(0, 0)$ at the lower left corner, and label the coordinates of the columns from left to right with $0, 1, \ldots, n$ and the coordinates of the rows from bottom to top with $0, 1 \ldots, n$. A *Dyck path* is a path made up of unit down-steps D and unit right-steps R which starts at $(0, n)$, which is at the top left-hand corner, and ends at $(n, 0)$, which is at the bottom right-hand corner, and stays weakly below the diagonal $y = n - x$. We let $\mathcal{D}_n$ denote the set of Dyck paths on the $n \times n$ board.

Given a Dyck path P, we let

$$Return(P) := \{1 \leq i \leq n - 1 : P \text{ goes through the point } (i, n - i)\}$$

denote the set of *return positions* and let $\mathrm{return}(P) = \min(Return(P))$ be the *smallest (first) return position*. For example, for the Dyck path

$$P = DDRDDRRRDDRDRDRRDR$$

shown on the right in Figure 5, $Return(P) = \{4, 8\}$ and $\mathrm{return}(P) = 4$.

Given any permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n(123)$, we write it on our $n \times n$ chessboard by placing σ_i in the i^{th} column and σ_i^{th} row, reading from bottom to top. Then, we shade the cells to the north-east of the cell that contains σ_i. $\Psi(\sigma)$ is the path that goes along the south-west boundary of the shaded cells. For example, this process is pictured in Figure 5 for the permutation $\sigma = 869743251 \in S_9(123)$ which is mapped into the Dyck path $DDRDDRRRDDRDRDRRDR$.

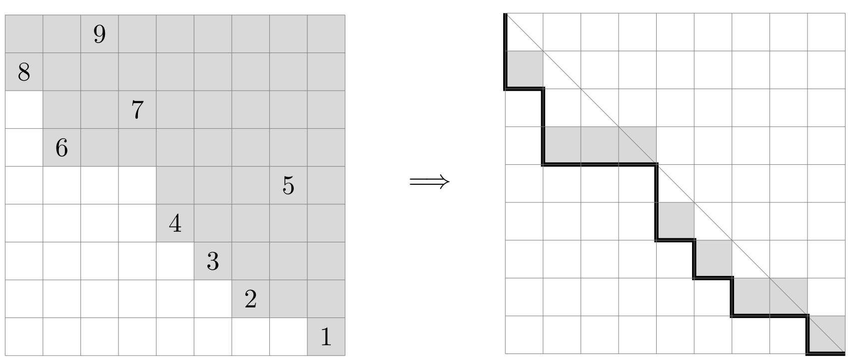

Figure 5: $\Psi(\sigma) = DDRDDRRRDDRDRDRRDR$ for $\sigma = 869743251$.

Given any Dyck path P, we construct the permutation $\Psi^{-1}(P)$ as follows. First we place a $\times$ in every outer corner of P. Then we consider the rows and columns which do not have a $\times$. Processing the columns from top to bottom and the rows from left to right, we place a $\times$ in the i^{th} empty row and i^{th} empty column. Finally we replace the $\times$s with numbers $\{1, \ldots, n\}$ from bottom to top. This process is pictured in Figure 6. The details that Ψ is bijection between $S_n(123)$ and $\mathcal{D}_n$ can be found in [3].

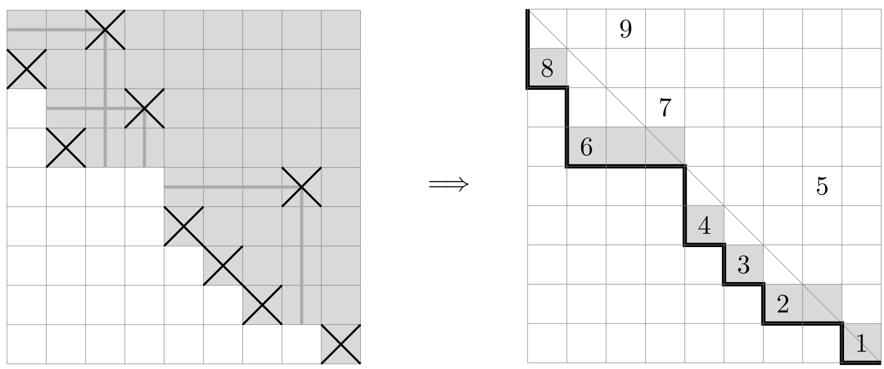

Figure 6: $\Psi^{-1}(P) = 869743251$ for $P = DDRDDRRRDDRDRDRRDR$.

We shall classify the ordered set partitions $\pi \in \mathcal{WOP}_n(123)$ by the first return (from left to right) of the path $P = \Psi(w(\pi))$. Suppose that the first return of the path P is at the point $(n-k, k)$, then the path P is divided by the first return into 2 paths, path DAR and path B, as shown in Figure 7 (a). The numbers in the outer corners above the point $(n-k, k)$ must come from $\{k+1, \ldots n\}$. Because we place the $\times$s in the columns which are not occupied by the $\times$s in the outer corners of P, in a decreasing manner, reading from left to right, it follows that by the time we have reached column $n-k$, we must have used all of the numbers in $\{k+1, \ldots, n\}$. This means that there is no $\times$s in the shaded area in 7 (a) so that all the $\times$s in the last k columns must lie in the lower k rows. In particular, this implies that in $w(\pi)$, all the elements in $\{k+1, \ldots, n\}$ proceed all the elements in $\{1, \ldots, k\}$. The elements in $\{k+1, \ldots, n\}$ are determined by the path DAR and the elements in $\{1, \ldots, k\}$ are determined by the path B, and there is a descent at the $n-k^{\text{th}}$ position in $w(\pi)$ if $k > 0$. Hence we can break any ordered set partition $\pi = B_1/\cdots/B_j$ such that $\Psi(w(\pi)) = P$ into two parts, $B_1/\cdots/B_i$ that contains all the elements in $\{k+1, \ldots, n\}$ and $B_{i+1}/\cdots/B_j$ that contains all the elements in $\{1, \ldots, k\}$.

Let $A(x,y,t,q_1,q_2) = \mathbb{WOP}^{des}_{123,\{1,2\}}(x,y,t,q_1,q_2)$. It is easy to see that the contribution to $A(x,y,t,q_1,q_2)$ by summing over the weights of all possible choices of $B_{i+1}/\cdots/B_j$ as k varies over all choices of $k>0$ is $y(A(x,y,t,q_1,q_2)-1)$ and is equal to 1 if $k=0$.

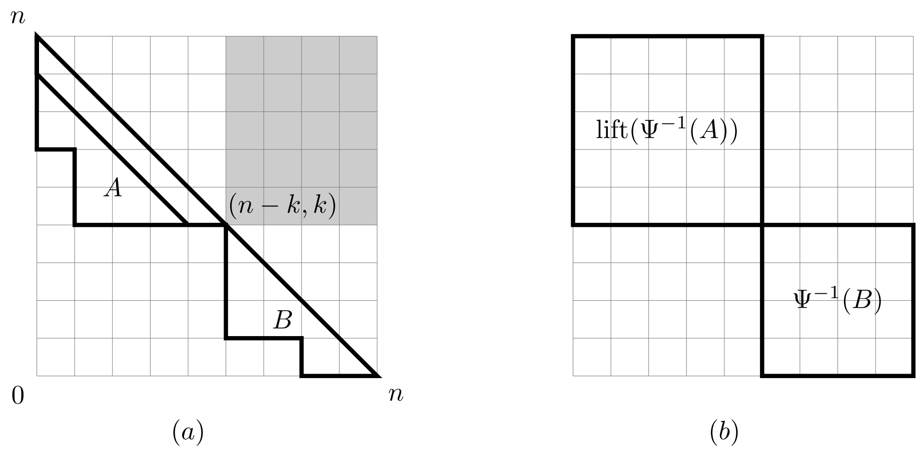

Figure 7: Breaking the Dyck path P at the first return.

To analyze the contribution from parts $B_1/\cdots/B_i$, we need to work on the path DAR, which can be seen as lifting the path A one unit in the south-west direction. We let lift(P) be the path DPR. For $\sigma \in S_n(123)$ and $P = \Psi(\sigma)$, we write lift(σ) for the permutation $\Psi^{-1}(\text{lift}(P)) = \Psi^{-1}(DPR) \in S_{n+1}$ corresponding to path lift(P).

We say that a pair of consecutive DR steps is a *peak* (*outer corner*) of a Dyck path, and in the corresponding 123-avoiding permutation, the numbers in the rows that contain peaks are called *peaks* of a permutation. A number is called a *non-peak* if it is not a peak. It is easy to see that the peaks of a permutation $\sigma \in S_n(123)$ and lift(σ) are the same. Since we label the rows and columns that do not contain peaks from left to right with the non-peak numbers in decreasing order under the map Ψ^{-1}, in lift(σ), $n+1$ is in the column of the first non-peak and each remaining non-peak shifts to the next column that does not contain a peak. Figure 8 illustrates the lift action of $\sigma = (8,6,9,7,4,3,2,5,1) \in S_9(123)$.

Following the construction, σ and lift(σ) have the same descent set in the first $n-1$ positions, and there is a descent in the n^{th} position if and only if σ_n is a non-peak. Since the word $w(\pi)$ of an ordered set partition $\pi \in \mathcal{WOP}_n(123)$ is determined by the Dyck path $DARB$, we can study smaller Dyck paths A and B instead of π when computing the generating function.

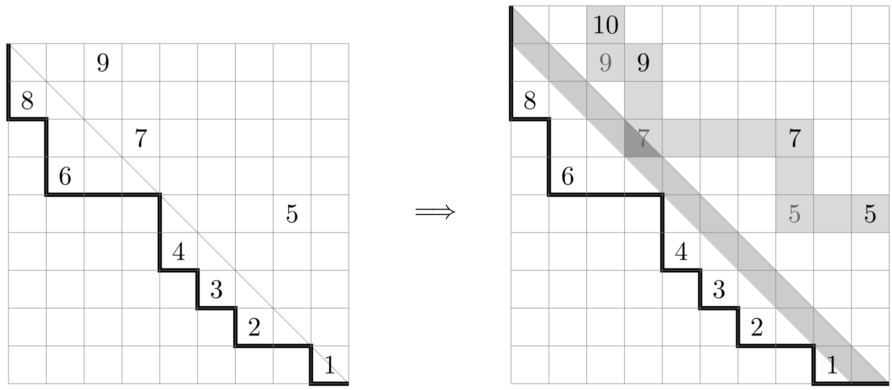

Figure 8: $\sigma = (8,6,9,7,4,3,2,5,1)$ and $\text{lift}(\sigma) = (8,6,10,9,4,3,2,7,1,5)$.

Let $\pi = B_1/\cdots/B_j \in \mathcal{WOP}_n(123)$ such that the first return is $n-k$ and the numbers $\{k+1,\ldots,n\}$ are contained in parts $B_1/\cdots/B_i$. We have the following four cases when computing the function $A(x,y,t,q_1,q_2)$.

Case 1. The first return of P is at the point $(1, n-1)$.

In this case, P starts of DR and n is the first outer corner of path P. This means that $w(\pi)$ starts with n, $i = 1$, and $B_1 = \{n\}$. It is easy to see that in this case the contribution to $A(x,y,t,q_1,q_2)$ is $xtq_1(1 + y(A(x,y,t,q_1,q_2) - 1))$. That is, if $n = 1$, then we get a contribution of xtq_1 and otherwise, n will cause a descent in $w(\pi)$ which gives a contribution of $xtq_1y(A(x,y,t,q_1,q_2) - 1)$.

Case 2. The first return of P is at the point $(2, n-2)$.

In this case, P starts of $DDRR$, $n-1$ is the first outer corner of P, n is in the square $(2,n)$ and $w(\pi)$ starts out with $(n-1)n$. Then it is either the case that $i = 2$, $B_1 = \{n-1\}$, and $B_2 = \{n\}$ or $i = 1$ and $B_1 = \{n-1, n\}$. It is easy to see that in the first case, the contribution to $A(x,y,t,q_1,q_2)$ is $x^2t^2q_1^2(1 + y(A(x,y,t,q_1,q_2) - 1))$. That is, if $n = 2$, then we get a contribution of $x^2t^2q_1^2$ and otherwise, n will cause a descent in $w(\pi)$ which gives a contribution of $x^2t^2q_1^2y(A(x,y,t,q_1,q_2) - 1)$. Similarly, in the second case the contribution to $A(x,y,t,q_1,q_2)$ is $xt^2q_2(1+y(A(x,y,t,q_1,q_2)-1))$. Thus the total contribution to $A(x,y,t,q_1,q_2)$ from Case 2 is

$$(x^2t^2q_1^2 + xt^2q_2)(1 + y(A(x,y,t,q_1,q_2) - 1)).$$

Case 3. The first return of P is at the point $(n-k,k)$ where $k < n-2$, and the last three steps before the first return are DRR.

In this case, we have the situation pictured in Figure 9. Thus $w(\pi) = w_1 \cdots w_n$ where $w_{n-k-1} = k+1$ and $w_{n-k} = p$ where $p > k+1$. It follows that either $B_i = \{k+1,p\}$ or $B_{i-1} = \{k+1\}$ and $B_i = \{p\}$. We claim that the contribution to $A(x,y,t,q_1,q_2)$ in the first case where $B_i = \{k+1,p\}$ is

$$y(A(x,y,t,q_1,q_2) - 1)xt^2q_2(1 + y(A(x,y,t,q_1,q_2) - 1)).$$

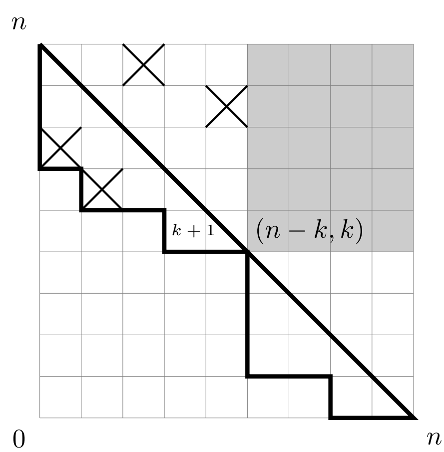

Figure 9: The situation in Case 3.

That is, the first factor y comes from the fact that there is a descent caused by the last element of B_{i-1} and the first element of B_i which is $k+1$. The next factor $(A(x,y,t,q_1,q_2) - 1)$ comes from summing over all possible choices of $B_1/\cdots/B_{i-1}$. The factor xt^2q_2 comes from B_i. If $B_{i+1}/\cdots/B_j$ is empty then we get a factor of 1, and if $B_{i+1}/\cdots/B_j$ is not empty, then we get a factor of y coming from the descents between the last element of B_i and the first element of B_{i+1} and a factor of $(A(x,y,t,q_1,q_2) - 1)$ coming summing the weights over all possible choices of $B_{i+1}/\cdots/B_j$.

Similar reasoning shows that the contribution to $A(x, y, t, q_1, q_2)$ in the second case where $B_{i-1} = \{k + 1\}$ and $B_i = \{p\}$ is

$$y(A(x, y, t, q_1, q_2) - 1)x^2t^2q_1^2(1 + y(A(x, y, t, q_1, q_2) - 1)).$$

Thus the total contribution to $A(x, y, t, q_1, q_2)$ in Case 3 is

$$y(A(x, y, t, q_1, q_2) - 1)(xt^2q_2 + x^2t^2q_1^2)(1 + y(A(x, y, t, q_1, q_2) - 1)).$$

Case 4. The first return of P is at the point $(n - k, k)$ where $k < n - 2$, and the last three steps before the first return are RRR.

In this case, we have the situation pictured in Figure 10. Thus $w(\pi) = w_1 \cdots w_n$ where $w_r = k + 1$ and $w_{r+1} \cdots w_{n-k}$ is a decreasing sequence of length at least 2. In this situation, B_i must be a singleton part $\{w_{n-k}\}$. We claim that the contribution to $A(x, y, t, q_1, q_2)$ from the ordered set partitions in Case 4 is

$$xytq_1(A(x, y, t, q_1, q_2) - 1 - xtq_1 - xytq_1(A(x, y, t, q_1, q_2) - 1)) \cdot (1 + y(A(x, y, t, q_1, q_2) - 1)).$$

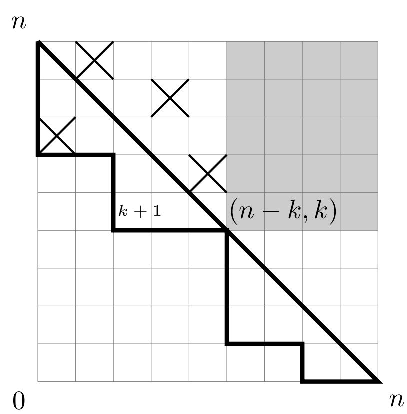

Figure 10: The situation in Case 4.

That is, the first factor $xytq_1$ is the weight of part B_i, where y comes from the fact that there is a descent caused by the last element of B_{i-1} and the first element of B_i. The next factor comes summing over all possible choices of $B_1/\cdots/B_{i-1}$. It is not difficult to see that this corresponds to the sum of the weights over all non-empty ordered set partitions π where 1 is not the last element of the word of π. Let

$$A_n(x,y,q_1,q_2) := \sum_{\pi \in \mathcal{WOP}_n(123)} x^{\ell(\pi)} y^{des(\pi)} q_1^{\text{one}(\pi)} q_2^{\text{two}(\pi)},$$

where $\text{one}(\pi)$ is the number of parts of size 1 and $\text{two}(\pi)$ is the number of parts of size 2 in π. Then $A_n(x,y,q_1,q_2) - xytq_1A_{n-1}(x,y,q_1,q_2)$ is the weight over all ordered set partitions π of size n such that 1 is not the last element of $w(\pi)$. Thus the sum of the weights over all non-empty ordered set partitions π where 1 is not the last element of $w(\pi)$ equals

$$\sum_{n\geq 2} t^n(A_n(x,y,q_1,q_2) - xytq_1A_{n-1}(x,y,q_1,q_2)) = \\ (A(x,y,t,q_1,q_2) - 1 - xtq_1) - xytq_1(A(x,y,t,q_1,q_2) - 1).$$

Finally we get a factor of 1 if $B_{i+1}/\cdots/B_j$ is empty and a factor of $y(A(x,y,t,q_1,q_2)-1)$ over all possible choices of $B_{i+1}/\cdots/B_j$ if $B_{i+1}/\cdots/B_j$ is non-empty.

Summing the contributions from Cases 1–4, we have

$$\begin{aligned} A(x,y,t,q_1,q_2) &= 1 + (y-1)^2(q_1xt + q_2xt^2 - q_1^2x^2t^2(y-1)) \\ &\quad - 2A(x,y,t,q_1,q_2)(y(y-1)(q_1xt + q_2xt^2 - q_1^2x^2t^2(y-1)) \\ &\quad + A(x,y,t,q_1,q_2)^2y^2(q_1xt + q_2xt^2 - q_1^2x^2t^2(y-1)). \end{aligned} \tag{3.4}$$

Because (3.4) involves both linear and quadratic terms in x, we can not apply the Lagrange Inversion Theorem to get an explicit formula for $\mathbb{WOP}^{des}_{123,\{1,2\}}(x,y,t,q_1,q_2)|_{x^k}$. Nevertheless, (3.4) gives us a quadratic equation which we can solve for $A(x,y,t,q_1,q_2)$ to prove the following theorem.

Theorem 3.5. *The generating function*

$$\mathbb{WOP}^{des}_{123,\{1,2\}}(x,y,t,q_1,q_2) = \frac{P(x,y,t,q_1,q_2) - \sqrt{Q(x,y,t,q_1,q_2)}}{R(x,y,t,q_1,q_2)},$$

where

$$\begin{aligned} P(x,y,t,q_1,q_2) &= 1 + 2y(y-1)q_1xt + 2y(y-1)q_2xt^2 - 2y(y-1)^2q_1^2x^2t^2, \\ Q(x,y,t,q_1,q_2) &= 1 - 4yq_1xt - 4yq_2xt^2 + 4(y(y-1)q_1^2x^2t^2, \ \textit{and} \\ R(x,y,t,q_1,q_2) &= 2y^2q_1xt + 2y^2q_2xt^2 - 2y^2(y-1)q_1^2x^2t^2. \end{aligned}$$

Setting $y = 1$ in $\mathbb{WOP}^{des}_{123,\{1,2\}}(x, y, t, q_1, q_2)$ gives us the following corollary.

Corollary 3.6. *We have*

$$\mathbb{WOP}_{123,\{1,2\}}(x, t, q_1, q_2) = \frac{1 - \sqrt{1 - 4tx(q_1 + xq_2)}}{2tx(q_1 + xq_2)}, \tag{3.5}$$

and

$$wop_{\langle 1^{\alpha_1}, 2^{\alpha_2}\rangle}(123) = \frac{1}{\alpha_1 + \alpha_2 + 1}\binom{2\alpha_1 + 2\alpha_2}{\alpha_1 + \alpha_2}\binom{\alpha_1 + \alpha_2}{\alpha_1}, \tag{3.6}$$

$$wop_{n,k}(123) = wop_{\langle 1^{2k-n}, 2^{n-k}\rangle}(123) = \frac{1}{k+1}\binom{2k}{k}\binom{k}{n-k}. \tag{3.7}$$

Proof. Let $A_{123}(x, t, q_1, q_2) = \mathbb{WOP}^{des}_{123,\{1,2\}}(x, 1, t, q_1, q_2) = \mathbb{WOP}_{123,\{1,2\}}(x, t, q_1, q_2)$, then the recursion becomes

$$A_{123}(x, t, q_1, q_2) = 1 + tq_1xA^2_{123}(x, t, q_1, q_2) + tq_2x^2A^2_{123}(x, t, q_1, q_2).$$

Equation (3.5) is obtained by solving the quadratic equation. Since

$$wop_{n,k}(123) = wop_{\langle 1^{2k-n}, 2^{n-k}\rangle}(123) = A_{123}(x, t, q_1, q_2)|_{t^n x^k q_1^{2k-n} q_2^{n-k}},$$

we can get equation (3.6) and equation (3.7) by applying Lagrange Inversion. □

Thus, we have enumerated the number of ordered set partitions in $\mathcal{WOP}_n(123)$ with certain numbers of blocks of size 1 and size 2. Now we give a formula for the number of ordered set partitions in $\mathcal{WOP}_n(123)$ with a certain block size composition. In [4], Godbole, *et al.* showed that

$$\mathrm{op}_{[b_1,\dots,b_i,b_{i+1},\dots,b_k]}(321) = \mathrm{op}_{[b_1,\dots,b_{i+1},b_i\dots,b_k]}(321)$$

by constructing a bijective map between $\mathcal{OP}_{[b_1,\dots,b_i,b_{i+1},\dots,b_k]}(321)$ and $\mathcal{OP}_{[b_1,\dots,b_{i+1},b_i\dots,b_k]}(321)$.

For our new definition of pattern avoidance, we prove a similar result that the order of block sizes in block size composition does not affect $wop_{[b_1,\dots,b_k]}(123)$, and we have the following theorem.

Theorem 3.7. *We have*

$$wop_{[b_1,\ldots,b_i,b_{i+1},\ldots,b_k]}(123) = wop_{[b_1,\ldots,b_{i+1},b_i,\ldots,b_k]}(123)$$

and

$$wop_{[b_1,\ldots,b_i,b_{i+1},\ldots,b_k]}(321) = wop_{[b_1,\ldots,b_{i+1},b_i,\ldots,b_k]}(321).$$

Proof. The second equation is included in the bijection constructed by Godbole, *et al.* that

$$\begin{aligned} wop_{[b_1,\ldots,b_i,b_{i+1},\ldots,b_k]}(321) &= \mathrm{op}_{[b_1,\ldots,b_i,b_{i+1},\ldots,b_k]}(321) \\ &= \mathrm{op}_{[b_1,\ldots,b_{i+1},b_i,\ldots,b_k]}(321) \\ &= wop_{[b_1,\ldots,b_{i+1},b_i,\ldots,b_k]}(321). \end{aligned}$$

For the first equation, we prove by a bijection.

For a block size composition $B = [b_1, \ldots, b_i, b_{i+1}, \ldots, b_k]$, since we are considering the 123-avoiding ordered set partitions, all the blocks are of size 1 or 2. We have the following 2 cases.

(1) If $b_i = b_{i+1} = 1$ or 2, then $wop_{[b_1,\ldots,b_i,b_{i+1},\ldots,b_k]}(123)$ and $wop_{[b_1,\ldots,b_{i+1},b_i,\ldots,b_k]}(123)$ are exactly the same enumeration.

(2) If $b_i \neq b_{i+1}$, then without loss of generality, we suppose $b_i = 1$ and $b_{i+1} = 2$. We show that there is a bijective map between $\mathcal{WOP}_{[b_1,\ldots,1,2,\ldots,b_k]}(123)$ and $\mathcal{WOP}_{[b_1,\ldots,2,1,\ldots,b_k]}(123)$. We suppose the 3 integers filled in blocks b_i and b_{i+1} are $a_1 < a_2 < a_3$. Since there is no 123 pattern-occurrence, there are only 2 possible fillings for both $[\ldots, 1, 2, \ldots]$ and $[\ldots, 2, 1, \ldots]$ cases. They are a_2/a_1a_3 and a_3/a_1a_2 for $[\ldots, 1, 2, \ldots]$, a_2a_3/a_1 and a_1a_3/a_2 for $[\ldots, 2, 1, \ldots]$. We construct a map, as showed in Figure 11, sending a_3/a_1a_2 to a_1a_3/a_2 and a_2/a_1a_3 to a_2a_3/a_1.

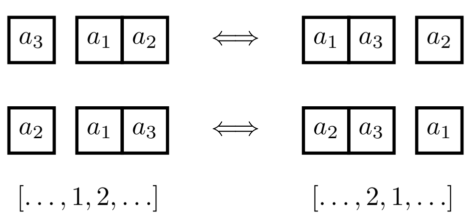

Figure 11: Bijection between $\mathcal{WOP}_{[b_1,\ldots,1,2,\ldots,b_k]}(123)$ and $\mathcal{WOP}_{[b_1,\ldots,2,1,\ldots,b_k]}(123)$.

It is not difficult to check that the map is bijective and preserves the 123-avoiding condition. Thus $wop_{[b_1,\ldots,b_i,b_{i+1},\ldots,b_k]}(123) = wop_{[b_1,\ldots,b_{i+1},b_i\ldots,b_k]}(123)$. $\square$

The formula for $wop_{[b_1,\ldots,b_k]}(123)$ follows the bijection.

Corollary 3.8. *For any composition $[b_1,\ldots,b_k]$ such that $b_i \in \{1,2\}$, we have*

$$wop_{[b_1,\ldots,b_k]}(123) = C_k,$$

here $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k^{th} Catalan number.

Proof. Let α_1 be the number of 1's and α_2 be the number of 2's in $[b_1,\ldots,b_k]$. By Corollary 3.6, we have

$$wop_{\langle 1^{\alpha_1},2^{\alpha_2}\rangle}(123) = \frac{1}{\alpha_1+\alpha_2+1}\binom{2\alpha_1+2\alpha_2}{\alpha_1+\alpha_2}\binom{\alpha_1+\alpha_2}{\alpha_1}.$$

Since the order of block sizes does not affect $wop_{[b_1,\ldots,b_k]}(123)$ and there are $\binom{\alpha_1+\alpha_2}{\alpha_1}$ ways to permute the block sizes, we have

$$\begin{aligned} wop_{[b_1,\ldots,b_k]}(123) = \frac{\frac{1}{\alpha_1+\alpha_2+1}\binom{2\alpha_1+2\alpha_2}{\alpha_1+\alpha_2}\binom{\alpha_1+\alpha_2}{\alpha_1}}{\binom{\alpha_1+\alpha_2}{\alpha_1}} &= \frac{1}{\alpha_1+\alpha_2+1}\binom{2\alpha_1+2\alpha_2}{\alpha_1+\alpha_2} \\ &= \frac{1}{k+1}\binom{2k}{k} = C_k. \end{aligned}$$ $\square$

Setting $y = q_1 = q_2 = 1$ in $\mathbb{WOP}^{des}_{123,\{1,2\}}(x,y,t,q_1,q_2)$ gives us the following corollary.

Corollary 3.9. *We have*

$$\mathbb{WOP}_{123}(x,t) = \frac{1-\sqrt{1-4tx-4t^2x}}{2(xt+xt^2)}.$$

We pause to make some observations about some special cases of elements of $\mathcal{WOP}_n(123)$. First consider the case of ordered set partitions in $\mathcal{WOP}_n(123)$ where every part has size 1. In this case, we are just considering the generating function of $y^{des(\sigma)}$ over all 123-avoiding permutations. We can obtain this generating function from $\mathbb{WOP}^{des}_{123,\{1,2\}}(x,y,t,q_1,q_2)$ by setting x equal to $1/x$, t equal to tx, and then setting $x = 0$. We carried out these steps in Mathematica and obtained the following corollary which was first proved by Barnabei, Bonetti and Silimbani [1].

Corollary 3.10. *We have*

$$1+\sum_{n\geq 1} t^n \sum_{\sigma\in S_n(123)} y^{des(\sigma)} = \frac{-1-2ty(y-1)+2t^2y(y-1)^2+\sqrt{1-4ty-4t^2y(y-1)}}{2ty^2(-1+t(y-1))}.$$

We can do a similar computation starting with the generating function $\mathbb{WOP}^{des}_{132}(x,y,t)$ to obtain the following corollary.

Corollary 3.11. *For any* $\alpha \in \{132, 231, 312, 213\}$,

$$1+\sum_{n\geq 1} t^n \sum_{\sigma\in S_n(\alpha)} y^{des(\sigma)} = \frac{1+t(y-1)-\sqrt{1+t^2(y-1)^2-2t(y+1)}}{2yt}.$$

In this case, the coefficients are the coefficients of the triangle of the Narayana numbers $T(n,k) = \frac{1}{k}\binom{n}{k-1}\binom{n-1}{k-1}$ which is entry A001263 in the OEIS [15].

3.3. The function $\mathbb{WOP}^{des}_{321}(x,y,t)$

The final generating function that we shall consider in this section is $\mathbb{WOP}^{des}_{321}(x,y,t)$. Since a permutation σ is 321-avoiding if and only if its reverse σ^r is 123-avoiding, we shall again appeal to the bijection Ψ of Deutsch and Elizalde between 123-avoiding permutations and Dyck paths and classify the ordered set partitions δ which word-avoid 321 by $\Psi(w(\delta))$. The main difference in this case is that we obtain the permutation $w(\delta)$ by reading the elements in the diagram from right to left, rather from left to right, and we classify the ordered set partitions by the last return of $\Psi(w(\delta))$. In this situation, we have two cases for any $\delta \in \mathcal{WOP}_n(321)$.

Case 1. The last return of $\Psi(w(\delta))$ is at position $(n-1,1)$ in which case $w(\delta)$ starts with 1.

In this case, 1 can not be part of an occurrence of 321 in the word of the ordered set partition. Thus either 1 is in a part by itself in which case we get a contribution of $xt\mathbb{WOP}^{des}_{321}(x,y,t)$ to $\mathbb{WOP}^{des}_{321}(x,y,t)$, or 1 is part of the first part of the ordered set partition arising from the part of the ordered set partition above and to the left of 1 which gives a contribution of $t(\mathbb{WOP}^{des}_{321}(x,y,t)-1)$ to $\mathbb{WOP}^{des}_{321}(x,y,t)$. Thus the total contribution to

$\mathbb{WOP}_{321}^{des}(x,y,t)$ of the ordered set partitions that word-avoid 321 and start with 1 is

$$xt\mathbb{WOP}_{321}^{des}(x,y,t) + t(\mathbb{WOP}_{321}^{des}(x,y,t) - 1).$$

Case 2. Either $\Psi(w(\delta))$ has no return or the last return is at position $(n-k,k)$ where $k>1$.

Let us first consider the cases of ordered set partitions $\delta \in \mathcal{WOP}_n(321)$ such that $\Psi(w(\delta))$ hits the diagonal only at $(0,n)$ and $(n,0)$ and $n \geq 2$. For such ordered set partitions, we have two subcases.

Subcase 2.1 The second element of $w(\delta)$ equals 1.

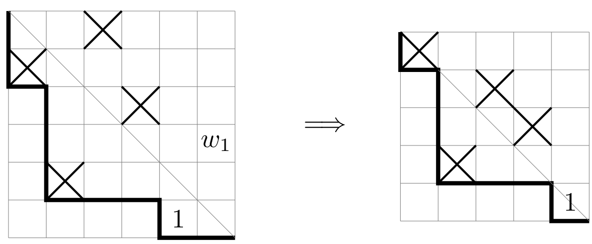

Figure 12: Ordered set partitions in Subcase 2.1.

In this case, suppose that $w(\delta) = w_1 \cdots w_n$ where $w_2 = 1$. Then we have the situation pictured in Figure 12. Since $w_1 > w_2 = 1$, it must be the case that w_1 is in a part by itself so that it contributes a factor of xyt to the weight of δ. If we remove the row and column containing w_1 and keep the same outer corner squares, and possibly relabel the $\times$s in the columns with no outer corner squares by having the $\times$s in those columns decreasingly, reading from left to right, we will obtain an arbitrary ordered set partition $\pi \in \mathcal{WOP}_{n-1}(321)$ such that $w(\pi)$ starts with 1. Hence the ordered set partitions in this subcase contribute to $\mathbb{WOP}_{321}^{des}(x,y,t)$ a factor of

$$xyt(xt\mathbb{WOP}_{321}^{des}(x,y,t) + t(\mathbb{WOP}_{321}^{des}(x,y,t) - 1)).$$

Subcase 2.2 The second element of $w(\delta)$ does not equal 1.

In this case, suppose that $w(\delta) = w_1 \cdots w_n$ where $w_i = 1$ for $i > 2$. Then we have the situation pictured in Figure 13. In this case, since $w_1 < w_2 <$

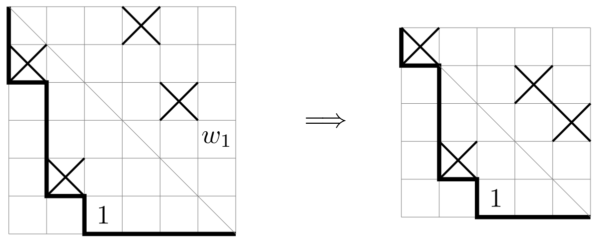

Figure 13: Ordered set partitions in Subcase 2.2.

$\cdots < w_{i-1} > w_i = 1$, it must be the case that w_i starts a new part in δ. If we remove the row and column containing w_1 and keep the same outer corner squares, and possibly relabel the $\times$s in the columns with no outer corner squares by having the $\times$s in those columns decreasingly, reading from left to right, we will obtain an arbitrary ordered set partition $\pi \in \mathcal{WOP}_{n-1}(321)$ such that $w(\pi)$ does not start with 1. The sum of the weights of the ordered set partitions π such that $w(\pi)$ does not start with 1 is

$$\mathbb{WOP}_{321}^{des}(x,y,t) - 1 - xt\mathbb{WOP}_{321}^{des}(x,y,t) - t(\mathbb{WOP}_{321}^{des}(x,y,t) - 1).$$

Then w_1 is either in a part by itself in which case it contributes a factor of xt or is in the same part with w_2 in which case it contributes a factor of t. Hence the ordered set partitions in this subcase contribute a factor of

$$(xt+t)(\mathbb{WOP}_{321}^{des}(x,y,t) - 1 - xt\mathbb{WOP}_{321}^{des}(x,y,t) - t(\mathbb{WOP}_{321}^{des}(x,y,t) - 1))$$

to $\mathbb{WOP}_{321}^{des}(x,y,t)$.

Let

$$NR(x,y,t) := \sum_{n\geq 2} t^n \sum_{\substack{\pi \in \mathcal{WOP}_n(321), \\ Return(\Psi(w(\delta)))=\emptyset}} x^{\ell(\pi)} y^{des(\pi)}$$

be the contribution of ordered set partitions in Subcases 2.1 and 2.2 to $\mathbb{WOP}_{321}^{des}(x,y,t)$, then

$$NR(x,y,t) = xyt(xt\mathbb{WOP}_{321}^{des}(x,y,t) + t(\mathbb{WOP}_{321}^{des}(x,y,t) - 1)) + \\ (xt+t)(\mathbb{WOP}_{321}^{des}(x,y,t) - 1 - xt\mathbb{WOP}_{321}^{des}(x,y,t) - t(\mathbb{WOP}_{321}^{des}(x,y,t) - 1)).$$

Now consider in the general case in Case 2 when the last return is at $(n-k,k)$ where $1 < k \leq n-1$. This situation is pictured in Figure 14. Because we fill the columns which do not have outer corner squares in a

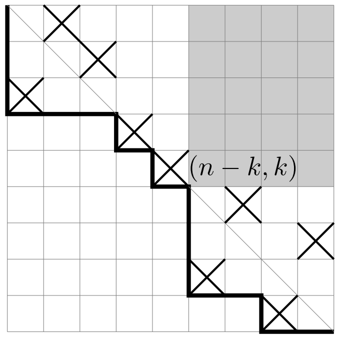

Figure 14: Ordered set partitions in Case 2 when the last return is at $(n-k,k)$.

decreasing manner, reading from left to right, it is easy to see that there is no $\times$ in the squares of the shaded area in Figure 14. This means that the $\times$s corresponding to $1,\ldots,k$ must be all in the bottom $k\times k$ squares. What we do not know is how the final increasing sequence of the elements $1,\ldots,k$ in $w(\delta)$ union of the initial increasing sequence of the remaining elements break up into parts in δ. For example, in Figure 14, $k=4$ and the last increasing sequence of the elements $1,\ldots,4$ in $w(\delta)$ is the single digit 2 and the initial increasing sequence of the remaining elements is $6,7,9,10$. Then we have two cases. The first case is when there is no overlap between the parts containing $1,\ldots,k$ and the remaining parts. In this case, we get a contribution of $NR(x,y,t)(\mathbb{WOP}_{321}^{des}(x,y,t)-1)$ to $\mathbb{WOP}_{321}^{des}(x,y,t)$. If there is an overlap, then we need to remove the x corresponding to the last part in the generating function $NR(x,y,t)$ so that we would get a contribution of $\frac{1}{x}NR(x,y,t)(\mathbb{WOP}_{321}^{des}(x,y,t)-1)$.

It follows that the total contribution to $\mathbb{WOP}_{321}^{des}(x,y,t)$ from the ordered set partitions $\delta\in\mathcal{WOP}_n(321)$ in Case 2 is

$$NR(x,y,t)+(1+\frac{1}{x})NR(x,y,t)(\mathbb{WOP}_{321}^{des}(x,y,t)-1).$$

Hence we have

$$\begin{aligned}\mathbb{WOP}_{321}^{des}(x,y,t) \quad &= \quad 1+xt\mathbb{WOP}_{321}^{des}(x,y,t)+t(\mathbb{WOP}_{321}^{des}(x,y,t)-1)+\\ &\quad NR(x,y,t)+(1+\frac{1}{x})NR(x,y,t)(\mathbb{WOP}_{321}^{des}(x,y,t)-1).\end{aligned}$$

This is a quadratic equation in $\mathbb{WOP}_{321}^{des}(x,y,t)$ which we can solve to obtain the following theorem.

Theorem 3.12. *The generating function*

$$\mathbb{WOP}_{321}^{des}(x,y,t) = \frac{x+2t(x+1)+2t^2(x+1)(xy-x-1)}{2t(x+1)^2(t(x(y-1)-1)+1)} - \frac{x\sqrt{1-4t(x+1)(t(x(y-1)-1)+1)}}{2t(x+1)^2(t(x(y-1)-1)+1)}. \quad (3.8)$$

Setting $y=1$ in (3.8), we obtain the following corollary which recovers the result of Chen, Dai and Zhou [2].

Corollary 3.13. *The generating function*

$$\mathbb{WOP}_{321}(x,t) = \frac{x+2t(1+x)-2t^2(1+x)-x\sqrt{1-4(1-t)t(1+x)}}{2(1-t)t(1+x)^2}.$$

The recursion that we used to compute $\mathbb{WOP}_{321}^{des}(x,y,t)$ does not allow us to control the size of the parts of the ordered set partitions $\pi \in \mathcal{WOP}_n(321)$ so that we have not been able to compute generating functions of the form $\mathbb{WOP}_{321,\{b_1,\ldots,b_k\}}^{des}(x,y,t,q_1,\ldots,q_k)$ in general.

4. Generating functions for min-descents

Based on the analysis in Section 2, we need to study the following 5 kinds of generating functions,

$$\begin{array}{lcl} \mathbb{WOP}_{213}^{mindes}(x,y,t) & = & \mathbb{WOP}_{312}^{mindes}(x,y,t), \\ \mathbb{WOP}_{132}^{mindes}(x,y,t), & & \mathbb{WOP}_{231}^{mindes}(x,y,t), \\ \mathbb{WOP}_{123}^{mindes}(x,y,t), & & \mathbb{WOP}_{321}^{mindes}(x,y,t). \end{array}$$

We are able to explicitly determine the functions $\mathbb{WOP}_{132}^{mindes}(x,y,t)$, $\mathbb{WOP}_{231}^{mindes}(x,y,t)$ and $\mathbb{WOP}_{213}^{mindes}(x,y,t) = \mathbb{WOP}_{312}^{mindes}(x,y,t)$, and write the functions $\mathbb{WOP}_{123}^{mindes}(x,y,t)$ and $\mathbb{WOP}_{321}^{mindes}(x,y,t)$ as roots of polynomial equations.

4.1. The function $\mathbb{WOP}_{132}^{mindes}(x,y,t)$

As we observed in Section 2,

$$\mathbb{WOP}_{132}^{des}(x,y,t) = \mathbb{WOP}_{132}^{mindes}(x,y,t),$$

thus we have the following theorem.

Theorem 4.1. *The generating function*

$$\mathbb{WOP}_{132}^{mindes}(x,y,t) = \mathbb{WOP}_{132}^{des}(x,y,t) = \frac{1+2yt+xyt-t-tx}{2t(y+yx)} - \frac{\sqrt{(1+2yt+xyt-t-tx)^2-4(1-t+ty)(t(y+xy))}}{2t(y+yx)},$$

and

$$\sum_{\pi\in\mathcal{WOP}_{n,k}(132)} y^{mindes(\pi)} = \frac{1}{k}\binom{n-1}{k-1}\sum_{j=0}^{k-1}\binom{k}{j}\binom{n}{k-1-j}y^{k-1-j}.$$

4.2. The function $\mathbb{WOP}_{231}^{mindes}(x,y,t)$

Next consider $\mathbb{WOP}_{231}^{mindes}(x,y,t)$. Let

$$C_n(x,y) := \sum_{\pi\in\mathcal{WOP}_n(231)} x^{\ell(\pi)}y^{mindes(\pi)}.$$

We can classify ordered set partitions $\pi = B_1/\cdots/B_k \in \mathcal{WOP}_n(231)$ by the position i of n in the word of π. Assume $n \geq 2$.

Case 1. $i = 1$.
In this case $w(\pi)$ starts with n which means that n must be in a part by itself so that $B_1 = \{n\}$. Then B_1 contributes a factor of xy since it automatically causes a min-descent with B_2. Thus the ordered set partitions $\pi \in \mathcal{WOP}_n(231)$ in Case 1 contribute $xyC_{n-1}(x,y)$ to $C_n(x,y)$.

Case 2. $i = n$.
In this case $w(\pi)$ ends with n. If n is in a part by itself, then $B_k = \{n\}$ and there is no min-descent between B_{k-1} and B_k. Hence we get a contribution of $xC_{n-1}(x,y)$ in this case. If $n \in B_k$ where $|B_k| \geq 2$, then we can simply remove n from B_k and obtain an ordered set partition in $\mathcal{WOP}_n(231)$ with the same number of parts and the same number of min-descents, and we will get a contribution of $C_{n-1}(x,y)$. Thus the ordered set partitions $\pi \in \mathcal{WOP}_n(231)$ in Case 2 contribute $(1+x)C_{n-1}(x,y)$ to $C_n(x,y)$.

Case 3. $2 \leq i \leq n-1$.
In this case, n must be the last element in some part B_j. Because $w(\pi)$ is 231-avoiding, it must be the case that all the elements in $B_1/\cdots/B_j - \{n\}$

are less than all the elements in $B_{j+1}/\cdots/B_k$. If $B_j = \{n\}$, then B_j contributes a factor of xy since B_j will cause a min-descent with B_{j+1}. Our choices over all possibilities of $B_1/\cdots/B_{j-1}$ contribute a factor of $C_{i-1}(x,y)$ and our choices over all possibilities of $B_{j+1}/\cdots/B_k$ contribute a factor of $C_{n-i}(x,y)$. Thus we get a contribution of $xyC_{i-1}(x,y)C_{n-i}(x,y)$ in this case. If $|B_j| \geq 2$, then we can eliminate n from B_j. Our choices over all possibilities of $B_1/\cdots/B_j - \{n\}$ contribute a factor of $C_{i-1}(x,y)$ and our choices over all possibilities of $B_{j+1}/\cdots/B_k$ contribute a factor of $C_{n-i}(x,y)$. Hence we get a contribution of $C_{i-1}(x,y)C_{n-i}(x,y)$ in this situation. Thus the ordered set partitions $\pi \in \mathcal{WOP}_n(231)$ in Case 3 contribute $(1+xy)C_{i-1}(x,y)C_{n-i}(x,y)$ to $C_n(x,y)$.

It follows that for $n \geq 2$,

$$C_n(x,y) = (1+x+xy)C_{n-1}(x,y) + \sum_{i=2}^{n-1}(1+xy)C_{i-1}(x,y)C_{n-i}(x,y).$$

Hence,

$$\begin{aligned}
\mathbb{WOP}_{231}^{mindes}(x,y,t) &= 1 + xt + \sum_{n\geq 2} C_n(x,y)t^n \\
&= 1 + xt + (1+x+xy)t\sum_{n\geq 2} C_{n-1}(x,y)t^{n-1} \\
&\quad +(1+xy)t\sum_{n\geq 2}\sum_{k=2}^{n-1} C_{k-1}(x,y)C_{n-k}(x,y) \\
&= 1 + xt + (1+x+xy)t(\mathbb{WOP}_{231}^{mindes}(x,y,t)-1) \\
&\quad +(1+xy)t(\mathbb{WOP}_{231}^{mindes}(x,y,t)-1)^2.
\end{aligned}$$

This gives us a quadratic equation in which we can solve to prove the following theorem.

Theorem 4.2. *The generating function*

$$\mathbb{WOP}_{231}^{mindes}(x,y,t) = \frac{1+t-tx+txy-\sqrt{(1+t-tx+txy)^2-4(t+txy)}}{2(t+txy)}.$$

4.3. The functions $\mathbb{WOP}_{213}^{mindes}(x,y,t) = \mathbb{WOP}_{312}^{mindes}(x,y,t)$

As we observed in Section 2,

$$\mathbb{WOP}_{213}^{mindes}(x,y,t) = \mathbb{WOP}_{132}^{maxdes}(x,y,t).$$

Then we can work on the set $\mathcal{WOP}_n(132)$ and track the *maxdes* statistic to compute the function $\mathbb{WOP}_{132}^{maxdes}(x, y, t)$ instead of $\mathbb{WOP}_{213}^{mindes}(x, y, t)$.

We shall again classify the ordered set partitions $\pi \in \mathcal{WOP}_n(132)$ by the size of the last part and we will use the structure in Figure 3. Now suppose that $C(x, y, t) = \mathbb{WOP}_{132}^{maxdes}(x, y, t)$. In this case, we get a factor of xt^r from the last part $\{a_1, \ldots, a_r\}$. Next we shall analyze when the last part from any A_i will cause a max-descent in π. Let s be the smallest index i such that A_i is non-empty. If $s = r + 1$, then there is a max-descent from the last part of A_{r+1} to $\{a_1, \ldots, a_r\}$ so that we would get a factor of $y(C(x, y, t) - 1)$. If $s \leq r$, then the last part of A_s does not create a max-descent with $\{a_1, \ldots, a_r\}$ so it contributes a factor of $(C(x, y, t) - 1)$. However, each non-empty A_j with $j > s$ creates a max-descent between the last part of A_j and the first part of the next non-empty A_i, so each such A_j contributes a factor of $1 + y(C(x, y, t) - 1)$. Thus $C = C(x, y, t)$ satisfies the following recursive relation:

$$
\begin{aligned}
(4.1)\ C(x,y,t) &= 1 \\
&\quad + \sum_{r\geq 1} xt^r \left((1 + y(C-1) + \sum_{s=1}^{r} (C-1)(1+y(C-1))^{r+1-s} \right) \\
&= 1 + x(1+y(C-1)) \sum_{r\geq 1} t^r \left(1 + (C-1) \sum_{s=1}^{r} (1+y(C-1))^{r-s} \right) \\
&= 1 + x(1+y(C-1)) \sum_{r\geq 1} t^r \left(1 + (C-1) \frac{(1+y(C-1))^r - 1}{(1+y(C-1)) - 1} \right) \\
&= 1 + x(1+y(C-1)) \sum_{r\geq 1} t^r \left(1 + \frac{(1+y(C-1))^r - 1}{y} \right) \\
&= 1 + x\frac{(1+y(C-1))}{y} \sum_{r\geq 1} t^r \left(y - 1 + (1+y(C-1))^r \right) \\
&= 1 + x\frac{(1+y(C-1))}{y} \left(\frac{t(y-1)}{1-t} + \frac{t(1+y(C-1))}{1-t(1+y(C-1))} \right) \\
&= 1 + \frac{tx}{y}(1+y(C-1)) \left(\frac{(y-1)}{1-t} + \frac{(1+y(C-1))}{1-t(1+y(C-1))} \right).
\end{aligned}
$$

Clearing the fractions gives a quadratic equation in C which we can solve to show that

$$C(x, y, t) = \frac{P(x, y, t) - \sqrt{Q(x, y, t)}}{R(x, y, t)},$$

where

$$\begin{aligned} P(x,y,t) &= 1-2t+t^2-tx+2ty-2t^2y+txy+2t^2xy-2t^2xy^2 \\ Q(x,y,t) &= 1-4t+6t^2-4t^3+t^4-2tx+4t^2x-2t^3x+t^2x^2-2txy+ \\ &\quad 4t^2xy-2t^3xy-2t^2x^2y+t^2x^2y^2, \text{ and} \\ R(x,y,t) &= 2(ty-t^2y+txy-t^2xy^2). \end{aligned}$$

If we let $f(x,y,t) = C(x,y,t)-1$, then (4.1) gives that

$$f(x,y,t) = \frac{tx}{y}(1+yf)\left(\frac{y-1}{1-t} + \frac{1+yf}{1-t(1+yf)}\right).$$

The Lagrange Inversion Theorem implies that the coefficient of x^k in $f(x,y,t)$ is given by

$$f(x,y,t)|_{x^k} = \frac{1}{k}\delta(x)^k\Big|_{x^{k-1}},$$

where

$$\delta(x) = \frac{t}{y}(1+yx)\left(\frac{y-1}{1-t} + \frac{1+yx}{1-t(1+yx)}\right).$$

Thus,

$$\begin{aligned} & f(x,y,t)|_{x^kt^n} \\ &= \frac{1}{k}\frac{t^k}{y^k}(1+yx)^k \sum_{a=0}^{k} \binom{k}{a} \frac{(y-1)^{k-a}}{(1-t)^{k-a}} \frac{(1+xy)^a}{(1-t(1+xy))^a}\Bigg|_{x^{k-1}t^n} \\ &= \frac{1}{k}\frac{1}{y^k} \sum_{a=0}^{k} \binom{k}{a} \frac{(y-1)^{k-a}}{(1-t)^{k-a}} \frac{(1+xy)^{k+a}}{(1-t(1+xy))^a}\Bigg|_{x^{k-1}t^{n-k}}. \end{aligned}$$

By Newton's Binomial Theorem, we have

$$\begin{aligned} \frac{1}{(1-t)^{k-a}} &= \sum_{u\geq 0} \binom{k-a+u-1}{u} t^u \quad \text{and} \\ \frac{1}{(1-t(1+xy))^a} &= \sum_{v\geq 0} \binom{a+v-1}{v} t^v (1+xy)^v. \end{aligned}$$

It follows that

$$\begin{aligned} & f(x,y,t)|_{x^k t^n} \\ = \; & \frac{1}{k}\frac{1}{y^k}\sum_{a=0}^{k}\sum_{v=0}^{n-k}\binom{k}{a}\binom{a+v-1}{v}\binom{k-a+(n-k-v)-1}{n-k-v} \\ & \qquad \cdot (y-1)^{k-a}(1+xy)^{k+a+v}|_{x^{k-1}} \\ = \; & \frac{1}{k}\frac{1}{y^k}\sum_{a=0}^{k}\sum_{v=0}^{n-k}\binom{k}{a}\binom{a+v-1}{v}\binom{k-a+(n-k-v)-1}{n-k-v} \\ & \qquad \cdot \binom{k+a+v}{k-1}(y-1)^{k-a}y^{k-1} \\ = \; & \frac{1}{ky}\sum_{a=0}^{k}\sum_{v=0}^{n-k}\binom{k}{a}\binom{a+v-1}{v}\binom{k-a+(n-k-v)-1}{n-k-v} \\ & \qquad \cdot \binom{k+a+v}{k-1}(y-1)^{k-a}. \end{aligned}$$

Thus we have the following theorem.

Theorem 4.3. *The generating functions*

$$\begin{aligned} \mathbb{WOP}_{213}^{mindes}(x,y,t) = \mathbb{WOP}_{312}^{mindes}(x,y,t) &= \mathbb{WOP}_{132}^{maxdes}(x,y,t) \\ &= \frac{P(x,y,t)-\sqrt{(Q(x,y,t)}}{R(x,y,t)}, \end{aligned}$$

where

$$\begin{aligned} P(x,y,t) &= 1-2t+t^2-tx+2ty-2t^2y+txy+2t^2xy-2t^2xy^2, \\ Q(x,y,t) &= 1-4t+6t^2-4t^3+t^4-2tx+4t^2x-2t^3x+t^2x^2-2txy+ \\ & \quad\; 4t^2xy-2t^3xy-2t^2x^2y+t^2x^2y^2, \text{ and} \\ R(x,y,t) &= 2(ty-t^2y+txy-t^2xy^2), \end{aligned}$$

and the generating function

$$\begin{aligned} \sum_{\pi\in\mathcal{WOP}_{n,k}(213)} y^{mindes(\pi)} &= \frac{1}{ky}\sum_{a=0}^{k}\sum_{v=0}^{n-k}\binom{k}{a}\binom{a+v-1}{v} \\ & \cdot \binom{k-a+(n-k-v)-1}{n-k-v}\binom{k+a+v}{k-1}(y-1)^{k-a}. \end{aligned}$$

We can compute the limit as y approaches 0 of $\mathbb{WOP}_{213}^{mindes}(x,y,t)$ to obtain the generating function of ordered set partitions in $\mathcal{WOP}_n(213)$ which have no min-descents. In this case, we obtain the following corollary.

Corollary 4.4. *The generating function*

$$1+\sum_{n\geq 1} t^n \sum_{\pi\in\mathcal{WOP}_n(213), mindes(\pi)=0} x^{\ell(\pi)} = \frac{1+t(-2+t-tx)}{1+t^2-t(2+x)}.$$

Setting $x=1$, the coefficient list $\{a_n\}_{n\geq 0}$ in the Taylor series expansion is

$$1,1,2,5,13,34,89,233,610,1597,4181,10946,28657,\ldots$$

which is a bisection of the Fibonacci numbers appears as sequence A001519 in the OEIS [15] which has a large number of combinatorial interpretations. In fact, we can prove this combinatorially by showing the recurrence: $a_n = 3a_{n-1}-a_{n-2}$ for all $n\geq 3$. Note that a_n is the number of ordered set partitions that word-avoid 213 and have no min-descents. The number 1 must be in the first position in the word of each such ordered set partition. There are a_{n-1} such ordered set partitions when 1 is in a block of size 1. When the number 1 is in a block of size larger than 1, we suppose that 2 is in the k^{th} position in the word. It is easy to show that we have a_{n-k+1} such ordered set partitions. Thus,

$$\begin{aligned} a_n = a_{n-1}+\sum_{k=2}^{n} a_{n-k+1} = a_{n-1}+\sum_{k=1}^{n-1} a_k = 2a_{n-1}+\sum_{k=1}^{n-2} a_k \\ = 2a_{n-1}+(a_{n-1}-a_{n-2}), \end{aligned}$$

which proves the recurrence relation.

Given any sequence of positive numbers $1\leq b_1<b_2<\cdots<b_s$, we let

$$A = A(x,y,t,q_1,\ldots,q_s) = \mathbb{WOP}_{213,\{b_1,\ldots,b_s\}}^{mindes}(x,y,t,q_1,\ldots,q_s).$$

It follows from the structure pictured in Figure 3 and our analysis above that

$$\begin{aligned} A &= 1+\sum_{i=1}^{s} xq_i t^{b_i}(1+y(A-1)+\sum_{a=1}^{b_i}(A-1)(1+y(A-1))^{b_i+1-a}) \\ &= 1+\sum_{i=1}^{s} xq_i t^{b_i}(1+y(A-1))\left(1+\frac{(1+y(A-1))^{b_i}-1}{y}\right). \end{aligned}$$

If we set $F = F(x, y, t, q_1, \dots, q_s) = A(x, y, t, q_1, \dots, q_s) - 1$, then we have

$$F = x \sum_{i=1}^{s} q_i t^{b_i} (1 + yF) \left(1 + \frac{(1 + yF)^{b_i} - 1}{y}\right).$$

It follows from the Lagrange Inversion Theorem that

$$F|_{x^k} = \frac{1}{k} \delta^k(x)|_{x^{k-1}}$$

where $\delta(x) = \sum_{i=1}^{s} q_i t^{b_i} (1 + yx) \left(1 + \frac{(1+yx)^{b_i} - 1}{y}\right)$.

One can use this expression to show that if $\alpha_1, \dots, \alpha_s$ are non-negative integers such that $\sum_{i=1}^{s} \alpha_i = k$ and $\sum_{i=1}^{s} \alpha_i b_i = n$, then

$$F|_{x^k t^n q_1^{\alpha_1} \cdots q_s^{\alpha_s}} = \frac{1}{k} \binom{k}{\alpha_1, \dots, \alpha_s} \frac{(1 + xy)^k}{y^k} \prod_{i=1}^{s} \left((1 + xy)^{b_i} - 1\right)^{\alpha_i} \Bigg|_{x^{k-1}}.$$

Hence it is possible to get a closed expression for $F|_{x^k t^n q_1^{\alpha_1} \cdots q_s^{\alpha_s}}$, and we shall omit the messy details.

4.4. The function $\mathbb{WOP}^{mindes}_{123,\{1,2\}}(x, y, t, q_1, q_2)$

Next let us consider the computation of the generating function

$$A(x, y, t, q_1, q_2) = \mathbb{WOP}^{mindes}_{123,\{1,2\}}(x, y, t, q_1, q_2).$$

We will again consider the case analysis of $\pi = B_1/\cdots/B_j \in \mathcal{WOP}_{n,\{1,2\}}(123)$ by looking at the first return of the path $P = \Psi(w(\pi))$ and we will keep the same notation. That is, we shall assume the first return is at $(n - k, k)$, $B_1/\cdots/B_i$ are the parts containing the numbers $\{k+1, \dots, n\}$ and $B_{i+1}/\cdots/B_j$ are the parts containing the number $\{1, \dots, k\}$.

Case 1. The first return of P is at the point $(1, n-1)$.

In this case, we showed that $B_1 = \{n\}$. If $n = 1$, then we get a contribution of xtq_1. Otherwise, n will cause a min-descent between B_1 and B_2 which gives a contribution of $xtq_1 y(A(x, y, t, q_1, q_2) - 1)$. Thus, the contribution in this case is

$$xtq_1(1 + y(A(x, y, t, q_1, q_2) - 1)).$$

Case 2. The first return of P is at the point $(2, n-2)$.

In this case, we showed that either $B_1 = \{n-1\}$ and $B_2 = \{n\}$ or $B_1 = \{n-1, n\}$. It is easy to see that in the first case, the contribution to $A(x, y, t, q_1, q_2)$ is $x^2t^2q_1^2(1 + y(A(x, y, t, q_1, q_2) - 1))$. That is, if $n = 2$, then we get a contribution of $x^2t^2q_1^2$. Otherwise, B_2 will cause a min-descent between B_2 and B_3 which gives a contribution of $x^2t^2q_1^2y(A(x, y, t, q_1, q_2) - 1)$. Similarly, in the second case the contribution to $A(x, y, t, q_1, q_2)$ is $xt^2q_2(1 + y(A(x, y, t, q_1, q_2) - 1))$ as there is a min-descent between B_1 and B_2 if B_2 exists. Thus the total contribution to $A(x, y, t, q_1, q_2)$ from Case 2 is

$$(x^2t^2q_1^2 + xt^2q_2)(1 + y(A(x, y, t, q_1, q_2) - 1)).$$

Case 3. The first return of P is at the point $(n-k, k)$ where $k < n-2$, and $k+1$ is in column $n-k-1$.

In this case, we have the situation pictured in Figure 9. Thus $w(\pi) = w_1 \cdots w_n$ where $w_{n-k-1} = k+1$ and $w_{n-k} = p$ where $k+1 < p$. It follows that either $B_i = \{k+1, p\}$ or $B_{i-1} = \{k+1\}$ and $B_i = \{p\}$. We claim that the contribution to $A(x, y, t, q_1, q_2)$ in the first case where $B_i = \{k+1, p\}$ is

$$y(A(x, y, t, q_1, q_2) - 1)xt^2q_2(1 + y(A(x, y, t, q_1, q_2) - 1)).$$

That is, the first factor y comes from the fact that there is a min-descent between B_{i-1} and B_i since $\min(B_i) = k+1$ which is the smallest element in $B_1/\cdots/B_i$. The next factor $(A(x, y, t, q_1, q_2) - 1)$ comes from summing the weights of the reductions of $B_1/\cdots/B_{i-1}$ over all possible choices of $B_1/\cdots/B_{i-1}$. The factor xt^2q_2 comes from B_i. If $B_{i+1}/\cdots/B_j$ is empty then we get a factor of 1, and if $B_{i+1}/\cdots/B_j$ is not empty, then we get a factor of y, coming from the fact that the minimal element of B_i, $k+1$, is greater than the minimal element of B_{i+1} which is some element in $\{1, \ldots, k\}$, and a factor of $(A(x, y, t, q_1, q_2) - 1)$ comes from summing the weights over all possible choices of $B_{i+1}/\cdots/B_j$.

A similar reasoning will show that the contribution to $A(x, y, t, q_1, q_2)$ in the second case where $B_{i-1} = \{k+1\}$ and $B_i = \{p\}$ is

$$y(A(x, y, t, q_1, q_2) - 1)x^2t^2q_1^2(1 + y(A(x, y, t, q_1, q_2) - 1)).$$

Thus the total contribution to $A(x, y, t, q_1, q_2)$ in Case 3 is

$$y(A(x, y, t, q_1, q_2) - 1)(xt^2q_2 + x^2t^2q_1^2)(1 + y(A(x, y, t, q_1, q_2) - 1)).$$

At this point, our analysis differs from that of $\mathbb{WOP}^{des}_{123,\{1,2\}}(x,y,t,q_1,q_2)$.

Case 4. The first return of P is at the point $(n-k,k)$ where $k<n-2$, $k+1$ is in column $r=n-k-2$, and B_{i-1} has size 2.
Referring to Figure 10, in the word $w(\pi)=w_1\cdots w_n$, we have $w_{n-k-2}=k+1$, $w_{n-k-1}=p_1$, and $w_{n-k}=p_2$, where $k+1<p_2<p_1$. It follows that $B_i=\{p_2\}$, $B_{i-1}=\{k+1,p_1\}$, and there is no min-descent between B_{i-1} and B_i. Referring to the Dyck path structure in Figure 15 that if the path ends with 3 right steps RRR and it does not have a return, then there are two sub-Dyck-path components denoted B in the picture – the part tracking back from last step before the last down step to the step that it first reaches the first diagonal, and the part from the next step back to the start point. The corresponding parts of the two sub-Dyck-paths in the ordered set partition side are $B_1,\ldots,B_{i-2}$ that can be seen as 2 ordered set partitions that word-avoid 123, whose contribution is $(1+y(A(x,y,t,q_1,q_2)-1))^2$. The contribution of parts B_{i-1} and B_i is $x^2t^3q_1q_2$ and the contribution of blocks $B_{i+1}/\cdots/B_j$ is $(1+y(A(x,y,t,q_1,q_2)-1))$ for the same reason as Case 3. Thus the contribution of this case is

$$(1+y(A(x,y,t,q_1,q_2)-1))^2x^2t^3q_1q_2(1+y(A(x,y,t,q_1,q_2)-1)).$$

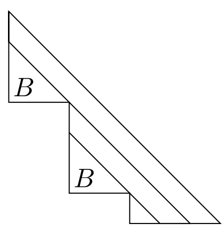

Figure 15: The situation in Case 4.

Case 5. The first return of P is at the point $(n-k,k)$ where $k<n-2$ and the size of B_{i-1} is not 2 (π does not satisfy Case 4).
This case is similar to Case 4 of $\mathbb{WOP}^{des}_{123,\{1,2\}}(x,y,t,q_1,q_2)$ in Section 3.2. In this case, B_i must be a singleton, and we claim that the contribution of this case is

$$\begin{aligned}&y(A(x,y,t,q_1,q_2)-1-xtq_1(1+y(A(x,y,t,q_1,q_2)-1))\\&-xt^2q_2(1+y(A(x,y,t,q_1,q_2)-1))^2)\cdot xtq_1(1+y(A(x,y,t,q_1,q_2)-1)).\end{aligned}$$

That is, the first factor y comes from the fact that there is a min-descent caused by parts B_{i-1} and B_i. The next factor comes summing the weights of all possible choices of $B_1/\cdots/B_{i-1}$. The contribution of part B_i is txq_1 and the last factor $(1+y(A(x,y,t,q_1,q_2)-1))$ is the contribution of blocks $B_{i+1}/\cdots/B_j$.

Adding up the contributions leads to the following theorem.

Theorem 4.5. *The function* $\mathbb{WOP}^{mindes}_{123,\{1,2\}}(x,y,t,q_1,q_2)$ *is the root of the following degree* 3 *polynomial equation about* A*:*

$$A = 1+txq_1(1+y(A-1))+(t^2xq_2+t^2x^2q_1^2)(1+y(A-1))^2+t^3x^2q_1q_2(1+y(A-1))^3+txyq_1(1+y(A-1))(A-1-txq_1(1+y(A-1))-t^2xq_2(1+y(A-1))^2).$$

One can use Mathematica to compute the generating function:

$$\begin{aligned}\mathbb{WOP}^{mindes}_{123,\{1,2\}}(x,y,t,q_1,q_2) &= 1+txq_1+t^2\left(q_2x+q_1^2x^2+q_1^2x^2y\right)+t^3\left(q_1q_2x^2\right.\\ &\left.+3q_1q_2x^2y+4q_1^3x^3y+q_1^3x^3y^2\right)+t^4\left(2q_2^2x^2y+9q_1^2q_2x^3y+2q_1^4x^4y\right.\\ &\left.+6q_1^2q_2x^3y^2+11q_1^4x^4y^2+q_1^4x^4y^3\right)+t^5\left(5q_1q_2^2x^3y+5q_1^3q_2x^4y+10q_1q_2^2x^3y^2\right.\\ &\left.+41q_1^3q_2x^4y^2+15q_1^5x^5y^2+10q_1^3q_2x^4y^3+26q_1^5x^5y^3+q_1^5x^5y^4\right)+\cdots.\end{aligned}$$

4.5. The function $\mathbb{WOP}^{mindes}_{321}(x,y,t)$

We write $C(x,y,t)=\mathbb{WOP}^{mindes}_{321}(x,y,t)$. To study the function $C(x,y,t)$, we use the fact that the reverse of the word of any $\pi\in\mathcal{WOP}_n(321)$ is 123-avoiding. In other words, if we let $\overline{\mathcal{WOP}}_n(123)$ be the set of ordered set partitions whose numbers are organized in decreasing order inside each part and the word is 123-avoiding, then each $\pi\in\mathcal{WOP}_n(321)$ corresponds to a $\bar{\pi}\in\overline{\mathcal{WOP}}_n(123)$. The *mindes* of π is then equal to the rise of the minimal elements of consecutive blocks (or *minrise*) of $\bar{\pi}$. We shall work on $\overline{\mathcal{WOP}}_n(123)$ and the statistic *minrise* to compute the function $C(x,y,t)$.

We also need to define another generating function

$$C_\ell(x,y,t) := 1+\sum_{n\geq 1}t^n\sum_{\pi\in\overline{\mathcal{WOP}}_n(123)}x^{\ell(\pi)}y^{|\{i:i<\ell(\pi)-1,B_i<_{min}B_{i+1}\}|}$$

that tracks the number of *minrise*'s that are not caused by the last two parts over all ordered set partitions in $\overline{\mathcal{WOP}}_n(123)$.

We will always use the shorthand C and C_ℓ for $C(x,y,t)$ and $C_\ell(x,y,t)$.

We start by studying the function $C(x,y,t)$. Note that the action *lift* defined in Section 3 preserves the *minrise* of any ordered set partitions in

$\overline{\mathcal{WOP}}_n(123)$, which makes it possible to find a recursion for $\overline{\mathcal{WOP}}_n(123)$ using the Dyck path bijection. For any $\pi = B_1/\cdots/B_m \in \overline{\mathcal{WOP}}_n(123)$, we let $w(\pi) = w_1\cdots w_n \in S_n(123)$. Let the first return of the corresponding Dyck path be at the $n-k^{\text{th}}$ column and let B_i be the part containing the number w_{n-k}.

Then there are 5 cases.

Case 1. B_i has size 1 and $w_{n-k-1} = k+1$.
In this case, there is a *minrise* between parts B_{i-1} and B_i. The numbers before $k+1$ reduce to an ordered set partition in $\overline{\mathcal{WOP}}_{n-k-2}(123)$. Either B_{i-1} only has the number $k+1$ or contains other numbers, and in the later case the *minrise* caused by last two parts in the previous numbers is not counted. Thus the contribution of the numbers before w_{n-k} to $C(x,y,t)$ is $tx(C + \frac{C_\ell - 1}{x})$. Since the numbers after w_{n-k} can form any ordered set partition in $\overline{\mathcal{WOP}}_k(123)$ and the *minrise* is not affected, the contribution to the function $C(x,y,t)$ of this case is

$$t^2x^2y\left(C + \frac{C_\ell - 1}{x}\right)C.$$

Case 2. B_i has size larger than 1 and $w_{n-k-1} = k+1$.
In this case, B_i contains no number in $\{w_1,\ldots,w_{n-k-1}\}$ and there is no *minrise* between parts B_{i-1} and B_i. The contribution of the numbers before w_{n-k} is $tx(C + \frac{C_\ell - 1}{x})$, and the contribution of the numbers from w_{n-k} is $tx\left(\frac{C-1}{x}\right)$. The contribution to $C(x,y,t)$ of this case is

$$t^2x^2\left(C + \frac{C_\ell - 1}{x}\right)\left(\frac{C-1}{x}\right).$$

Case 3. B_i has size 1 and $w_{n-k-1} \neq k+1$.
In this case, there is no *minrise* between parts B_{i-1} and B_i. The contribution of the numbers before w_{n-k} is $\left(C - tx\left(C + \frac{C_\ell - 1}{x}\right)\right)$. Since the numbers after w_{n-k} form an ordered set partition in $\overline{\mathcal{WOP}}_k(123)$ and the first part can either contain the number w_{n-k} or not, without changing the *minrise*, the contribution of the numbers from w_{n-k} is $tx\left(C + \frac{C-1}{x}\right)$, and the contribution to the function $C(x,y,t)$ of this case is

$$tx\left(C + \frac{C-1}{x}\right)\left(C - tx\left(C + \frac{C_\ell - 1}{x}\right)\right).$$

Case 4. $w_{n-k-1} \in B_i$ and $w_{n-k+1} \notin B_i$.
In this case, there is no *minrise* between parts B_{i-1} and B_i. We have $w_{n-k} \neq$

$k+1$ and $w_{n-k-1} \neq k+1$ in order to satisfy that $w_{n-k-1} \in B_i$. $w_{n-k+1} \notin B_i$ implies that the first part of the ordered set partition after w_{n-k} does not contain the number w_{n-k}. Thus the numbers up to w_{n-k} contribute $t\left(C-1-tx\left(C+\frac{C_\ell-1}{x}\right)\right)$ and the numbers after w_{n-k} contribute C to the function $C(x,y,t)$. Thus the total contribution of this case is

$$tC\left(C-1-tx\left(C+\frac{C_\ell-1}{x}\right)\right).$$

Case 5. $w_{n-k-1} \in B_i$ and $w_{n-k+1} \in B_i$.
In this case, there is still no *minrise* between parts B_{i-1} and B_i. We have $w_{n-k} \neq k+1$ and $w_{n-k-1} \neq k+1$ in order to satisfy that $w_{n-k-1} \in B_i$. $w_{n-k+1} \in B_i$ implies that the first part of the ordered set partition after w_{n-k} contains the number w_{n-k}. As part B_i connects the numbers before w_{n-k} and the numbers after w_{n-k}, the *minrise* caused by the last two parts before w_{n-k} is not counted. Thus the numbers up to w_{n-k} contribute $t\left(C_\ell-1-tx\left(C+\frac{C_\ell-1}{x}\right)\right)$ and the numbers after w_{n-k} contribute $\frac{C-1}{x}$ to the function $C(x,y,t)$. The total contribution of this case is

$$t\left(\frac{C-1}{x}\right)\left(C_\ell-1-tx\left(C+\frac{C_\ell-1}{x}\right)\right).$$

Summing the contribution of all the five cases, we have

(4.2)
$$C(x,y,t)=1+(y-1)t^2x^2C\left(C+\frac{C_\ell-1}{x}\right)+txC\left(C+\frac{C-1}{x}\right)+tC$$
$$\cdot\left(C-1-tx\left(C+\frac{C_\ell-1}{x}\right)\right)+t\left(\frac{C-1}{x}\right)\left(C_\ell-1-tx\left(C+\frac{C_\ell-1}{x}\right)\right).$$

We can do similar analysis for $C_\ell(x,y,t)$. We have the following 7 cases, of which the first 5 cases are similar to that of $C(x,y,t)$.
Case 1. B_i has size 1, $w_{n-k-1}=k+1$ and $k>0$.
The argument is same as Case 1 of $C(x,y,t)$ except that the contribution of the numbers after w_{n-k} is $C_\ell-1$ instead of C, since $k>0$ implies that B_{i+1} is not empty, and we do not count the *minrise* between the last two parts of π. Thus the contribution to $C_\ell(x,y,t)$ of this case is

$$t^2x^2y\left(C+\frac{C_\ell-1}{x}\right)(C_\ell-1).$$

Case 2. B_i has size larger than 1 and $w_{n-k-1} = k+1$.
Similar to Case 2 of $C(x,y,t)$, the contribution is $t^2x^2\left(C + \frac{C_\ell - 1}{x}\right)\left(\frac{C_\ell-1}{x}\right)$. The only difference is that the contribution of numbers after w_{n-k} is $\frac{C_\ell-1}{x}$ instead of $\frac{C-1}{x}$ as we do not count the *minrise* between the last two parts.
Case 3. B_i has size 1, $w_{n-k-1} \neq k+1$ and $k > 0$.
Similar to Case 3 of $C(x,y,t)$, the contribution is $tx\left(C_\ell - 1 + \frac{C_\ell-1}{x}\right)\left(C - tx\left(C + \frac{C_\ell-1}{x}\right)\right)$. The difference is that the contribution of numbers after w_{n-k} is $\left(C_\ell - 1 + \frac{C_\ell-1}{x}\right)$ as we do not count the *minrise* between the last two parts and the collection of numbers after w_{n-k} is not empty.
Case 4. $w_{n-k-1} \in B_i$, $w_{n-k+1} \notin B_i$ and $k > 0$.
Similar to Case 4 of $C(x,y,t)$, the contribution is $t(C_\ell - 1)\left(C - 1 - tx\left(C + \frac{C_\ell-1}{x}\right)\right)$. The contribution of numbers after w_{n-k} is $(C_\ell - 1)$ since $k > 0$ implies that the collection of numbers after w_{n-k} is not empty.
Case 5. $w_{n-k-1} \in B_i$ and $w_{n-k+1} \in B_i$.
Similar to Case 5 of $C(x,y,t)$, the contribution is $t\left(\frac{C_\ell-1}{x}\right)\left(C_\ell - 1 - tx\left(C + \frac{C_\ell-1}{x}\right)\right)$. The the contribution of numbers after w_{n-k} is $\frac{C_\ell-1}{x}$ as we do not count the *minrise* between the last two parts.
Case 6. $k = 0$ and $w_{n-k-1} \notin B_i$.
In this case, $B_i = \{w_{n-k}\}$. Since we do not count the descents of the last two parts, we do not care whether w_{n-k} is bigger or smaller than the minimum of the previous part. The contribution of this case is txC.
Case 7. $k = 0$ and $w_{n-k-1} \in B_i$.
In this case, B_i can be seen as including w_{n-k} in the last part before w_{n-k}. The last *minrise* before w_{n-k} is not counted, and $w_{n-k}, w_{n-k-1} \neq k+1$. The contribution of this case is $t\left(C_\ell - 1 - tx\left(C + \frac{C_\ell-1}{x}\right)\right)$.

Summing the contribution of all the 7 cases, we have

(4.3)
$$\begin{aligned} C_\ell(x,y,t) = {} & 1 + (y-1)t^2x^2(C_\ell - 1)\left(C + \frac{C_\ell-1}{x}\right) + txC\left(C_\ell - 1 + \frac{C_\ell-1}{x}\right) \\ & + txC + t\left(C_\ell - 1 - tx\left(C + \frac{C_\ell-1}{x}\right)\right) + tx\left(C_\ell - 1 + \frac{C_\ell-1}{x}\right) \\ & \cdot \left(C - tx\left(C + \frac{C_\ell-1}{x}\right)\right) + t(C_\ell-1)\left(C - 1 - tx\left(C + \frac{C_\ell-1}{x}\right)\right). \end{aligned}$$

Using equations (4.2) and (4.3) about $C(x,y,t)$ and $C_\ell(x,y,t)$, we can compute the Groebner basis of the functions to find an equation that $C(x, y,t)$ satisfies, and we have the following theorem.

Theorem 4.6. *The function* $\mathbb{WOP}_{321}^{mindes}(x,y,t)$ *is the root of the following degree* 4 *polynomial equation about* C*:*

$$1+t(-1+2x+2x^2-x^2y)+t^2-t^3+C(-2+t(-3-5x-3x^2+x^2y)+t^2(3-4x-6x^2 +3x^2y)+t^3(2+3x+3x^2-2x^2y))+C^2(1+t(9+6x+x^2)+t^2(-3+5x+8x^2+2x^3) +t^3(-10-6x-3x^2-4x^3-2x^4+x^2y+3x^3y+3x^4y-x^4y^2)+t^4(3-x^2)) +C^3(t(-5-3x)+t^2(-7-8x-3x^2-x^3-3x^2y-x^3y)+t^3(18+17x+6x^2+2x^3 +x^4-6x^2y-4x^3y-x^4y)+t^4(-6-6x-3x^2+x^3+x^4+5x^2y+x^3y-x^4y)) +C^4t^2(2-t+x-tx-tx^2+tx^2y)(3-3t+2x-3tx-tx^2+2tx^2y+tx^3y)=0.$$

One can use Mathematica to compute the generating function:

$$\mathbb{WOP}_{321}^{mindes}(x,y,t) = 1+tx+t^2\left(x^2y+x^2+x\right)+t^3\left(4x^3y+x^3+2x^2y +5x^2+2x\right)+t^4\left(2x^4y^2+11x^4y+x^4+17x^3y+17x^3+4x^2y +22x^2+6x\right)+t^5\left(15x^5y^2+26x^5y+x^5+10x^4y^2+90x^4y +49x^4+65x^3y+123x^3+10x^2y+88x^2+18x\right)+\cdots.$$

5. Generating functions for part-descents

In this section, we shall study the generating function $\mathbb{WOP}_{\alpha}^{pdes}(x,y,t)$ where $\alpha \in S_3$. Based on the analysis in Section 2, we need to study the following 4 kinds of generating functions,

$$\begin{array}{rcl} \mathbb{WOP}_{132}^{pdes}(x,y,t) &=& \mathbb{WOP}_{213}^{pdes}(x,y,t), \\ \mathbb{WOP}_{231}^{pdes}(x,y,t) &=& \mathbb{WOP}_{312}^{pdes}(x,y,t), \\ \mathbb{WOP}_{123}^{pdes}(x,y,t), && \mathbb{WOP}_{321}^{pdes}(x,y,t). \end{array}$$

We are able to explicitly determine the functions $\mathbb{WOP}_{132}^{pdes}(x,y,t) = \mathbb{WOP}_{213}^{pdes}(x,y,t)$, and write the functions $\mathbb{WOP}_{231}^{pdes}(x,y,t) = \mathbb{WOP}_{312}^{pdes}(x,y,t)$ and $\mathbb{WOP}_{321}^{pdes}(x,y,t)$ as roots of polynomial equations. We fail to obtain a recursive formula for $\mathbb{WOP}_{123}^{pdes}(x,y,t)$ since it is hard to get the *pdes* statistic under the the *lift* action of a 123-avoiding permutation.

5.1. The functions $\mathbb{WOP}_{132}^{pdes}(x,y,t) = \mathbb{WOP}_{213}^{pdes}(x,y,t)$

As we observed in Section 2,

$$\mathbb{WOP}_{132}^{pdes}(x,y,t) = \mathbb{WOP}_{213}^{mindes}(x,y,t).$$

Thus we have the following theorem.

Theorem 5.1. *The generating functions*

$$\begin{aligned}\mathbb{WOP}_{132}^{pdes}(x,y,t) &= \mathbb{WOP}_{213}^{pdes}(x,y,t) = \mathbb{WOP}_{132}^{maxdes}(x,y,t) \\ &= \mathbb{WOP}_{213}^{mindes}(x,y,t) = \mathbb{WOP}_{312}^{mindes}(x,y,t) \\ &= \frac{P(x,y,t)-\sqrt{Q(x,y,t)}}{R(x,y,t)},\end{aligned}$$

where

$$\begin{aligned}P(x,y,t) &= 1-2t+t^2-tx+2ty-2t^2y+txy+2t^2xy-2t^2xy^2, \\ Q(x,y,t) &= 1-4t+6t^2-4t^3+t^4-2tx+4t^2x-2t^3x+t^2x^2-2txy \\ &\quad +4t^2xy-2t^3xy-2t^2x^2y+t^2x^2y^2, \text{ and} \\ R(x,y,t) &= 2(ty-t^2y+txy-t^2xy^2),\end{aligned}$$

and

$$\begin{aligned}\sum_{\pi\in\mathcal{WOP}_{n,k}(132)} y^{pdes(\pi)} = \frac{1}{k}\frac{1}{y}\sum_{a=0}^{k}\sum_{v=0}^{n-k}\binom{k}{a}\binom{a+v-1}{v} \\ \binom{k-a+(n-k-v)-1}{n-k-v}\binom{k+a+v}{k-1}(y-1)^{k-a}.\end{aligned}$$

5.2. The functions $\mathbb{WOP}_{231}^{pdes}(x,y,t) = \mathbb{WOP}_{312}^{pdes}(x,y,t)$

We compute the function $\mathbb{WOP}_{312}^{pdes}(x,y,t)$ and write $D(x,y,t) = \mathbb{WOP}_{312}^{pdes}(x,y,t)$. As this is different from the 132-avoiding case, we will consider a new structure for the set $\mathcal{WOP}_n(312)$.

Given any ordered set partition $\pi = B_1/\cdots/B_k \in \mathcal{WOP}_n(312)$. If the size $n=0$, then it contributes 1 to the function $D(x,y,t)$. Otherwise, π has at least one part and we suppose the last part is $B_k = \{a_1,a_2,\ldots,a_r\}$ with $r \geq 1$ numbers. Note that there is no number $a > a_2$ in the previous blocks $B_1,\ldots,B_{k-1}$, otherwise the subsequence (a,a_1,a_2) of $w(\pi)$ is a 312-occurrence. Thus, the subsequence $a_2,\ldots,a_r$ must be a consecutive integer sequence.

Now, we divide the numbers in the previous blocks $B_1,\ldots,B_{k-1}$ into 2 sets: let $A_1 = \{1,\ldots,a_1-1\}$ be the numbers smaller than a_1 and $A_2 = \{a_1+1,\ldots,a_2-1\}$ be the numbers bigger than a_1. The numbers in the set A_1 must appear before the numbers in A_2 as otherwise there is a 312-occurrence in the word. Thus, an ordered set partition $\pi = B_1/\cdots/B_k \in \mathcal{WOP}_n(312)$ has the structure pictured in Figure 16.

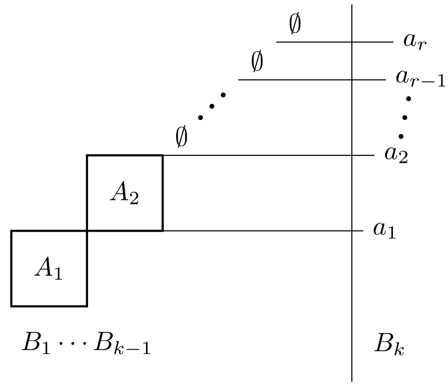

Figure 16: Structure of an ordered set partition in $\mathcal{WOP}_n(312)$.

We let $A_i(\pi)$ be the restriction of π to the set A_i. Then each $A_i(\pi)$ is also an ordered set partition in $\mathcal{WOP}_n(312)$. However, if both A_i's are not empty, then it is possible that the last block of A_1 and the first block of A_2 are contained in the same block in π. In that case, the *pdes* caused by the last two blocks of A_1 (if any) and the *pdes* caused by the first two blocks in A_2 (if any) will not contribute to *pdes*(π). We let $D_\ell(x,y,t)$, $D_f(x,y,t)$ and $D_{\ell f}(x,y,t)$ be the generating functions tracking the number of *pdes* without tracking the *pdes* caused by the last two parts, the first two parts, and both last and first two parts that

$$
\begin{aligned}
D_\ell(x,y,t) &:= 1+\sum_{n\geq 1} t^n \sum_{\pi\in\mathcal{WOP}_n(312)} x^{\ell(\pi)} y^{|\{i:i<\ell(\pi)-1, B_i>_p B_{i+1}\}|},\\
D_f(x,y,t) &:= 1+\sum_{n\geq 1} t^n \sum_{\pi\in\mathcal{WOP}_n(312)} x^{\ell(\pi)} y^{|\{i:i>1, B_i>_p B_{i+1}\}|},\\
D_{\ell f}(x,y,t) &:= 1+\sum_{n\geq 1} t^n \sum_{\pi\in\mathcal{WOP}_n(312)} x^{\ell(\pi)} y^{|\{i:1<i<\ell(\pi)-1, B_i>_p B_{i+1}\}|},
\end{aligned}
$$

then we can compute the recursive equations of functions $D(x,y,t)$, $D_\ell(x,y,t)$, $D_f(x,y,t)$ and $D_{\ell f}(x,y,t)$ respectively.

We first consider the function $D(x,y,t)$.

Case 1. The last part B_k has size bigger than 1.

Then there is always no *pdes* involving the part B_k as the last part contains the number a_2 which is greater than any numbers in $B_1,\ldots,B_{k-1}$. The last part has contribution $tx^2+tx^3+\cdots=\frac{tx^2}{1-t}$, and the contribution of $B_1,\ldots,B_{k-1}$ is $D^2(x,y,t)$ when the last block of A_1 and the first block of

A_2 are in different blocks in π, and $\frac{(D_\ell(x,y,t)-1)(D_f(x,y,t)-1)}{x}$ when the last block of A_1 and the first block of A_2 are in the same block in π. Thus, the contribution of this case to the function $D(x,y,t)$ is

$$\frac{tx^2}{1-t}\left(D^2(x,y,t)+\frac{(D_\ell(x,y,t)-1)(D_f(x,y,t)-1)}{x}\right).$$

Case 2. B_k has size 1, A_2 only contains 1 block which is in the same block as the last block of A_1 in π.
In this case, the set A_1 cannot be empty and there is still no *pdes* caused by the last two parts of π. The contribution is

$$tx\left((D_\ell(x,y,t)-1)\frac{t}{1-t}\right).$$

Case 3. B_k has size 1, A_2 is empty.
In this case, there is no *pdes* caused by the last two parts of π and the contribution is

$$txD(x,y,t).$$

Case 4. B_k has size 1, and π does not satisfy Case 2 or 3.
In this case, there is a *pdes* caused by the last two parts of π. Since it is possible that the last block of A_1 and the first block of A_2 are in the same block in π, the contribution of this case is

$$txy\left(D(x,y,t)(D(x,y,t)-1)+\frac{(D_\ell(x,y,t)-1)(D_f(x,y,t)-\frac{tx}{1-t}-1)}{x}\right).$$

Summing the contribution of all the 4 cases, and we write $D, D_\ell, D_f, D_{\ell f}$ on the right hand side to abbreviate $D(x,y,t), D_\ell(x,y,t), D_f(x,y,t), D_{\ell f}(x, y,t)$, then we have

$$D(x,y,t)=1+\frac{tx}{1-t}\left(D^2+\frac{(D_\ell-1)(D_f-1)}{x}\right) + (y-1)tx\left(D(D-1)+\frac{(D_\ell-1)(D_f-\frac{tx}{1-t}-1)}{x}\right). \tag{5.1}$$

For the function $D_\ell(x,y,t)$, we do not need to consider the contribution to part-descent involving part B_k, thus the analysis is like Case 1 of $D(x,y,t)$

and we have

(5.2)
$$D_\ell(x,y,t) = 1 + \frac{tx}{1-t}\left(D^2 + \frac{(D_\ell - 1)(D_f - 1)}{x}\right).$$

For the function $D_f(x,y,t)$, we have similar cases to $D(x,y,t)$, but one more case when last part is of size 1.

Case 1. B_k has size larger than 1.

In this case, there is always no *pdes* involving part B_k. The last part has contribution $\frac{tx^2}{1-t}$. The contribution of $B_1, \ldots, B_{k-1}$ is $(D_f(x,y,t)-1)D(x,y,t)$ when A_1 is not empty and the last block of A_1 and the first block of A_2 are in different blocks in π, $D_f(x,y,t)$ when A_1 is empty, and $\frac{(D_{\ell f}(x,y,t)-1)(D_f(x,y,t)-1)}{x}$ when the last block of A_1 and the first block of A_2 are in the same block in π. Thus, the contribution of this case to the function $D(x,y,t)$ is

$$\frac{tx^2}{1-t}\Big((D_f(x,y,t)-1)D(x,y,t) + D_f(x,y,t) + \frac{(D_{\ell f}(x,y,t)-1)(D_f(x,y,t)-1)}{x}\Big).$$

Case 2. B_k has size 1, A_2 only contains 1 block and it is in the same block as the last block of A_1.

In this case, the set A_1 cannot be empty and there is still no *pdes* caused by the last two parts of π. The contribution is

$$tx\left((D_{\ell f}(x,y,t)-1)\frac{t}{1-t}\right).$$

Case 3. B_k has size 1, A_2 is empty.

In this case, there is no *pdes* caused by the last two parts of π and the contribution is

$$txD_f(x,y,t).$$

Case 4. B_k has size 1, A_1 is empty, and A_2 only has one block.

In this case, the *pdes* caused by the only two parts of π is not counted as we do not count the first *pdes*, and the contribution is

$$tx\frac{tx}{1-t}.$$

Case 5. B_k has size 1, and the numbers in sets A_1, A_2 does not satisfy Case 2, 3 or 4.

In this case, there is a *pdes* caused by the last two parts of π. Since it is possible that the last block of A_1 and the first block of A_2 are in the same block, the contribution of this case is

$$txy\left((D_f(x,y,t)-1)(D(x,y,t)-1)+(D_f(x,y,t)-1-\frac{tx}{1-t})\right.$$
$$\left.+\frac{(D_{\ell f}(x,y,t)-1)(D_f(x,y,t)-\frac{tx}{1-t}-1)}{x}\right).$$

Summing the contribution of all the 5 cases, we have

$$D_f(x,y,t)=1+\frac{tx}{1-t}\left((D_f-1)D+D_f+\frac{(D_{\ell f}-1)(D_f-1)}{x}\right)$$
$$+(y-1)tx\left((D_f-1)D-\frac{tx}{1-t}+\frac{(D_{\ell f}-1)(D_f-\frac{tx}{1-t}-1)}{x}\right). \tag{5.3}$$

For the function $D_{\ell f}(x,y,t)$, we do not need to consider the contribution to part-descent involving part B_k, thus the contribution is like Case 1 of $D_f(x,y,t)$ and we have

$$D_{\ell f}(x,y,t)=1+\frac{tx}{1-t}\left((D_f-1)D+D_f+\frac{(D_{\ell f}-1)(D_f-1)}{x}\right). \tag{5.4}$$

Using equations (5.1), (5.2), (5.3) and (5.4) about $D(x,y,t)$, $D_\ell(x,y,t)$, $D_f(x,y,t)$ and $D_{\ell f}(x,y,t)$, we can compute the Groebner basis of the functions to find an equation that $D(x,y,t)$ satisfies, and we have the following theorem.

Theorem 5.2. *We have*

$$D(x,y,t)=1+\frac{tx}{1-t}\left(D^2+\frac{(D_\ell-1)(D_f-1)}{x}\right)$$
$$+(y-1)tx\left(D(D-1)+\frac{(D_\ell-1)(D_f-\frac{tx}{1-t}-1)}{x}\right),$$

$$D_\ell(x,y,t) = 1+\frac{tx}{1-t}\left(D^2+\frac{(D_\ell-1)(D_f-1)}{x}\right),$$

$$D_f(x,y,t) = 1 + \frac{tx}{1-t}\left((D_f-1)D + D_f + \frac{(D_{\ell f}-1)(D_f-1)}{x}\right) + (y-1)tx\left((D_f-1)D - \frac{tx}{1-t} + \frac{(D_{\ell f}-1)(D_f - \frac{tx}{1-t} - 1)}{x}\right),$$

$$D_{\ell f}(x,y,t) \quad = \quad 1 \ + \ \frac{tx}{1-t}\left((D_f-1)D + D_f + \frac{(D_{\ell f}-1)(D_f-1)}{x}\right),$$

and the function $\mathbb{WOP}_{312}^{pdes}(x,y,t)$ *is the root of the following degree* 3 *polynomial equation about* D*:*

$$1-t+D(-1+t)(1+t(1+2x(-1+y)))+D^2(1-t)t\left(1+tx^2(-1+y)^2+x(-1+t(-1+y)+2y)\right)+D^3t^2x(-1+y)(-1+t(1+x(-1+y))-xy)=0.$$

We can use Mathematica to compute the generating function:

$$\mathbb{WOP}_{312}^{pdes}(x,y,t) = 1+tx+t^2\left(x^2y+x^2+x\right)+t^3\left(x^3y^2+3x^3y+x^3+x^2y+4x^2+x\right)+t^4\left(x^4y^3+6x^4y^2+6x^4y+x^4+x^3y^2+11x^3y+9x^3+x^2y+8x^2+x\right)+t^5\left(x^5y^4+10x^5y^3+20x^5y^2+10x^5y+x^5+x^4y^3+21x^4y^2+46x^4y+16x^4+x^3y^2+23x^3y+32x^3+x^2y+13x^2+x\right)+\cdots.$$

5.3. The function $\mathbb{WOP}_{321}^{pdes}(x,y,t)$

We write $D(x,y,t) = \mathbb{WOP}_{321}^{pdes}(x,y,t)$. As we defined in Section 4.5, $\overline{\mathcal{WOP}}_n(123)$ is the set of ordered set partitions whose numbers are organized in decreasing order inside each part and the word is 123-avoiding. Each $\pi \in \mathcal{WOP}_n(321)$ corresponds to a $\bar{\pi} \in \overline{\mathcal{WOP}}_n(123)$, and the *pdes* of π is equal to the part-rise (or *prise*) of $\bar{\pi}$. We want to work on $\overline{\mathcal{WOP}}_n(123)$ and the statistic *prise* to compute the function $D(x,y,t)$.

We also need to define $D_\ell(x,y,t)$, $D_f(x,y,t)$ and $D_{\ell f}(x,y,t)$ as the generating functions tracking the number of *prise* without tracking the *prise* caused by the last two parts, the first two parts, and both last and first two parts of ordered set partitions in $\overline{\mathcal{WOP}}_n(123)$ that

$$D_\ell(x,y,t) \quad := \quad 1 + \sum_{n\geq 1} t^n \sum_{\pi \in \overline{\mathcal{WOP}}_n(123)} x^{\ell(\pi)} y^{|\{i: i<\ell(\pi)-1, B_i <_p B_{i+1}\}|},$$

$$D_f(x,y,t) := 1+\sum_{n\geq 1} t^n \sum_{\pi\in\overline{\mathcal{WOP}}_n(123)} x^{\ell(\pi)} y^{|\{i:i>1, B_i<_p B_{i+1}\}|},$$
$$D_{\ell f}(x,y,t) := 1+\sum_{n\geq 1} t^n \sum_{\pi\in\overline{\mathcal{WOP}}_n(123)} x^{\ell(\pi)} y^{|\{i:1<i<\ell(\pi)-1, B_i<_p B_{i+1}\}|}.$$

We will always use D, D_ℓ, D_f and $D_{\ell f}$ to abbreviate $D(x,y,t)$, $D_\ell(x, y,t)$, $D_f(x,y,t)$ and $D_{\ell f}(x,y,t)$. As we are generally looking at the same cases as Section 4.5, we shall briefly describe the classification of cases and give the contribution of each case.

For any $\pi = B_1/\cdots/B_j \in \overline{\mathcal{WOP}}_n(123)$, we let $w(\pi) = w_1\cdots w_n \in S_n(123)$. Let the first return of the corresponding Dyck path be at the $n-k^{\text{th}}$ column and let B_i be the block containing the number w_{n-k}.

For the function $D(x,y,t)$, there are 4 cases.
Case 1. Both B_{i-1} and B_i are of size 1.
The contribution to $D(x,y,t)$ is $t^2x^2yD^2$.
Case 2. $w_{n-k-1}\notin B_i$ and π does not satisfy Case 1.
The contribution to $D(x,y,t)$ is $txD\left(D+\frac{D_f-1}{x}\right)-t^2x^2D^2$.
Case 3. $w_{n-k-1}\in B_i$ and $w_{n-k+1}\notin B_i$.
The contribution to $D(x,y,t)$ is $\left(D-1-xt\left(D+\frac{D_\ell-1}{x}\right)\right)\cdot tD$.
Case 4. $w_{n-k-1}\in B_i$ and $w_{n-k+1}\in B_i$.
The contribution to $D(x,y,t)$ is $\left(D_\ell-1-xt\left(D+\frac{D_\ell-1}{x}\right)\right)\cdot t\frac{D_f-1}{x}$.

Summing the contribution of all the 4 cases, we have

$$D(x,y,t) = 1+t^2x^2(y-1)D^2+txD\left(D+\frac{D_f-1}{x}\right) + tD\left(D-1-xt\left(D+\frac{D_\ell-1}{x}\right)\right) + t\left(\frac{D_f-1}{x}\right)\left(D_\ell-1-xt\left(D+\frac{D_\ell-1}{x}\right)\right). \tag{5.5}$$

For the function $D_\ell(x,y,t)$, there are 6 cases.
Case 1. Both B_{i-1} and B_i are of size 1, and $k>0$.
The contribution to $D_\ell(x,y,t)$ is $t^2x^2yD(D_\ell-1)$.
Case 2. $w_{n-k-1}\notin B_i$, $k>0$, and π does not satisfy Case 1.
The contribution to $D_\ell(x,y,t)$ is $txD\left(D_\ell-1+\frac{D_{\ell f}-1}{x}\right)-t^2x^2D(D_\ell-1)$.
Case 3. $w_{n-k-1}\in B_i$, $k>0$, and $w_{n-k+1}\notin B_i$.
The contribution to $D_\ell(x,y,t)$ is $\left(D-1-xt\left(D+\frac{D_\ell-1}{x}\right)\right)\cdot t(D_\ell-1)$.
Case 4. $w_{n-k-1}\in B_i$ and $w_{n-k+1}\in B_i$.
The contribution to $D_\ell(x,y,t)$ is $\left(D_\ell-1-xt\left(D+\frac{D_\ell-1}{x}\right)\right)\cdot t\frac{D_{\ell f}-1}{x}$.

Case 5. $k = 0$ and $w_{n-k-1} \notin B_i$.
The contribution to $D_\ell(x, y, t)$ is txD.
Case 6. $k = 0$ and $w_{n-k-1} \in B_i$.
The contribution to $D_\ell(x, y, t)$ is $t\left(D_\ell - 1 - tx\left(D + \frac{D_\ell - 1}{x}\right)\right)$.

Summing the contribution of all the 6 cases, we have

$$
\begin{aligned}
(5.6) \quad D_\ell(x,y,t) &= 1 + txD + t^2x^2(y-1)D(D_\ell - 1) \\
&+ txD\left(D_\ell - 1 + \frac{D_{\ell f} - 1}{x}\right) + t(D_\ell - 1)\left(D - 1 - xt\left(D + \frac{D_\ell - 1}{x}\right)\right) \\
&+ t\left(\frac{D_{\ell f} - 1}{x}\right)\left(D_\ell - 1 - xt\left(D + \frac{D_\ell - 1}{x}\right)\right) \\
&+ t\left(D_\ell - 1 - tx\left(D + \frac{D_\ell - 1}{x}\right)\right).
\end{aligned}
$$

The functions $D_f(x, y, t)$ and $D_{\ell f}(x, y, t)$ have exactly the same 4 cases and 6 cases as $D(x, y, t)$ and $D_\ell(x, y, t)$. The main difference on the right hand side expansion is that some D and D_ℓ become D_f and $D_{\ell f}$. We omit the classification of cases and organize the terms of the expressions of $D_f(x, y, t)$ and $D_{\ell f}(x, y, t)$ in the same way as $D(x, y, t)$ and $D_\ell(x, y, t)$, and we have

$$
\begin{aligned}
(5.7) \quad D_f(x,y,t) &= 1 + t^2x^2(y-1)(D_f - 1)D + txD_f\left(D + \frac{D_f - 1}{x}\right) \\
&+ tD\left(D_f - 1 - xt\left(D_f + \frac{D_{\ell f} - 1}{x}\right)\right) \\
&+ t\left(\frac{D_f - 1}{x}\right)\left(D_{\ell f} - 1 - xt\left(D_f + \frac{D_{\ell f} - 1}{x}\right)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
(5.8) \quad D_{\ell f}(x,y,t) &= 1 + txD_f + t^2x^2(y-1)(D_f - 1)(D_\ell - 1) \\
&+ txD_f\left(D_\ell - 1 + \frac{D_{\ell f} - 1}{x}\right) + t(D_\ell - 1)\left(D_f - 1 - xt\left(D_f + \frac{D_{\ell f} - 1}{x}\right)\right) \\
&+ t\left(\frac{D_{\ell f} - 1}{x}\right)\left(D_{\ell f} - 1 - xt\left(D_f + \frac{D_{\ell f} - 1}{x}\right)\right) \\
&+ t\left(D_{\ell f} - 1 - tx\left(D_f + \frac{D_{\ell f} - 1}{x}\right)\right).
\end{aligned}
$$

Using equations (5.5), (5.6), (5.7) and (5.8), one can compute the Groebner basis of the functions to find an equation that $D(x, y, t)$ satisfies, and we have the following theorem.

Theorem 5.3. *The function* $\mathbb{WOP}_{321}^{pdes}(x,y,t)$ *is the root of the following degree* 6 *polynomial equation about* D*:*

$$\begin{aligned}&\big((-1+D)x+t\left(-1-D^2(1+x)^2+2D\left(1+x+x^2(-1+y)\right)\right)\\&-2x^2(-1+y)\big)+D^3t^5x^5(-1+y)^3+D^2t^4x^3(-1+y)^2(-2+2D(1+x)\\&+x(1+x-xy))+t^2\left(1+x+D(-2+x(2(-2+y)+x(4+x(-1+y))(-1\right.\\&\left.+y)))-x\left(x^2(-1+y)^2+y\right)-D^2(1+x)(-1+x(-2+3x(-1+y)+y))\right)\\&+Dt^3x(-1+y)\left(1+D^2(1+x)^2+2x(-1+x(-1+y))+D\left(-2+x^2(4\right.\right.\\&\left.\left.+3x-2y-3xy)\right)\right)\left(1+D\left(-2+D\left(1+t\left(1+x-t\left(1+x+x^2\right)\right.\right.\right.\right.\\&\left.\left.\left.\left.+D(-1+t)\left(1+x+tx^2(-1+y)\right)+tx^2y\right)\right)\right)\right)=0.\end{aligned}$$

We can use Mathematica to compute the following generating function:

$$\begin{aligned}\mathbb{WOP}_{321}^{pdes}(x,y,t)&=1+tx+t^2\left(x^2y+x^2+x\right)+t^3\left(4x^3y+x^3+x^2y+5x^2\right.\\&\left.+x\right)+t^4\left(2x^4y^2+11x^4y+x^4+11x^3y+16x^3+x^2y+13x^2+x\right)\\&+t^5\left(15x^5y^2+26x^5y+x^5+5x^4y^2+65x^4y+42x^4\right.\\&\left.+23x^3y+76x^3+x^2y+29x^2+x\right)+\cdots.\end{aligned}$$

6. Open problems

In this paper, we mainly use the classical recursion of 132-avoiding permutations and the Dyck path bijection of 123-avoiding permutations to prove results on the generating functions of ordered set partitions that word-avoid some patterns of length 3 tracking several statistics. Our definition of word-avoidance of an ordered set partition differs from the pattern avoidance defined by Godbole, Goyt, Herdan and Pudwell [4]. Notwithstanding, our definition of 321-word-avoiding ordered set partition coincides α-avoiding ordered set partition in the sense of [4] for any pattern $\alpha \in S_3$.

Due to this coincidence, we spent much of this paper dealing with the set $\mathcal{WOP}_n(321)$ of ordered set partitions word-avoiding 321. In Section 3, we solved all the generating functions tracking the statistic *descent* about $\mathcal{WOP}_n(\alpha)$ for any pattern α of length 3, and obtained many beautiful symmetries and formulas with multinomial coefficients. However, the enumeration for $wop_{[b_1,\ldots,b_k]}(321)=op_{[b_1,\ldots,b_k]}(321)$ and $wop_{\langle b_1^{\alpha_1},\ldots,b_k^{\alpha_k}\rangle}(321)=op_{\langle b_1^{\alpha_1},\ldots,b_k^{\alpha_k}\rangle}(321)$ are still open. As a first question, an explicit formula for $wop_{\langle b_1^{\alpha_1},\ldots,b_k^{\alpha_k}\rangle}(321)$ is desired.

In Section 4 and Section 5, we got nice results for all the generating functions tracking the statistics *mindes* and *pdes*, except that we did not have any result about $\mathbb{WOP}_{123}^{pdes}(x,y,t)$. In particular, we had polynomial equations about the generating functions $\mathbb{WOP}_{321}^{mindes}(x,y,t)$ and $\mathbb{WOP}_{321}^{pdes}(x,y,t)$ stated in Section 4.5 and Section 5.3, which would still make sense when using pattern avoidance definition in the sense of [4]. The polynomial equations have all the information of the generating functions, and one can come up with efficient recursions easily with the equations. The open problem in this part is the function $\mathbb{WOP}_{123}^{pdes}(x,y,t)$. We have not been able to get recursions about $\mathbb{WOP}_{123}^{pdes}(x,y,t)$ since the *pdes* statistic changes abnormally at the action *lift*.

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