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On the approximate shape of degree sequences that are not potentially H-graphic

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A sequence of nonnegative integers π is *graphic* if it is the degree sequence of some graph G. In this case we say that G is a *realization* of π, and we write $\pi = \pi(G)$. A graphic sequence π is *potentially H-graphic* if there is a realization of π that contains H as a subgraph.

Given nonincreasing graphic sequences $\pi_1 = (d_1, \dots, d_n)$ and $\pi_2 = (s_1, \dots, s_n)$, we say that π_1 *majorizes* π_2 if $d_i \geq s_i$ for all i, $1 \leq i \leq n$. In 1970, Erdős showed that for any K_{r+1}-free graph H, there exists an r-partite graph G such that $\pi(G)$ majorizes $\pi(H)$. In 2005, Pikhurko and Taraz generalized this notion and showed that for any graph F with chromatic number $r+1$, the degree sequence of an F-free graph is, in an appropriate sense, nearly majorized by the degree sequence of an r-partite graph.

In this paper, we give similar results for degree sequences that are not potentially H-graphic. In particular, there is a graphic sequence $\pi^*(H)$ such that if π is a graphic sequence that is not potentially H-graphic, then π is close to being majorized by $\pi^*(H)$. Similar to the role played by complete multipartite graphs in the traditional extremal setting, the sequence $\pi^*(H)$ asymptotically gives the maximum possible sum of a graphic sequence π that is not potentially H-graphic.

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1. Introduction

All graphs considered in this paper are finite. A connected component of a graph is *nontrivial* if it has at least one edge. We let $\Delta(G), \delta(G)$ and $\alpha(G)$

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denote the maximum degree, minimum degree and independence number of a graph G, respectively and let $\vee$ denote the standard graph join. Additionally, let $N(v)$ denote the neighborhood of a vertex v in a graph G, and for $X \subseteq V(G)$ let $N_X(v) = N(v) \cap X$. If H is a subgraph of G, let $N_H(v) = N_{V(H)}(v)$ and also let $d(v) = |N(v)|$ and $d_X(v) = |N_X(v)|$.

A sequence of nonnegative integers π is *graphic* if it is the degree sequence of some graph G. Unless otherwise noted, we will assume that all graphic sequences are written in nonincreasing order. In this case we say that G *realizes* π or is a *realization* of π, and we write $\pi = \pi(G)$. A graphic sequence π is *potentially H-graphic* if there is a realization of π that contains H as a subgraph. Let $\pi(H)$ be the degree sequence of the graph H. If $\pi(H) = (s_1, \ldots, s_k)$, then we say $\pi = (d_1, \ldots, d_n)$ is *degree sufficient for H* if $d_i \geq s_i$ for each i, $1 \leq i \leq k$. When the context is clear, we will write $\pi = (d_1^{m_1}, \ldots, d_t^{m_t})$ to indicate that degree d_i has multiplicity m_i in π.

In this paper, we study the structure of degree sequences that are not potentially H-graphic. As the realizations of a graphic sequence may have a great deal of structural variety, it is perhaps more appropriate to say that we examine the "shape" (in the Ferrer's diagram sense) of these sequences. Our inspiration comes from several results on H-free graphs from the extremal literature.

Given (not necessarily graphic) sequences $S_1 = (x_1, \ldots, x_n)$ and $S_2 = (y_1, \ldots, y_n)$, we say S_1 *majorizes* S_2 and write $S_1 \succeq S_2$ if $x_i \geq y_i$ for all i, $1 \leq i \leq n$. In [4], Erdős showed the following.

Theorem 1. *If G is a K_{r+1}-free graph of order n, then there exists an n-vertex r-partite graph F such that $\pi(F) \succeq \pi(G)$.*

Given positive integers m and k, define $D_{k,m}(S_1)$ to be the sequence

$$(\underbrace{x_k - m, \ldots, x_k - m}_{k \text{ times}}, x_{k+1} - m, \ldots, x_n - m).$$

We say that S_2 *(k, m)-majorizes* S_1 if $S_2 \succeq D_{k,m}(S_1)$. In 2005, Pikhurko and Taraz [20] used this notion to examine the shape of the degree sequences of general H-free graphs.

Theorem 2. *Let H be a graph with chromatic number $\chi(H) = r + 1 \geq 2$. For any $\epsilon > 0$ and $n \geq n_0(\epsilon, H)$, the degree sequence π of an H-free graph G of order n is $(\epsilon n, \epsilon n)$-majorized by the degree sequence of some r-partite graph of order n.*

It was noted in [20] that both the operation of "leveling off" the first k terms of $\pi(G)$ and the operation of reducing all of the terms in $\pi(G)$ by m are necessary.

That the degree sequences of r-partite graphs appear as the bounding class in Theorems 1 and 2 is unsurprising given the central role played by the Turán graph $T_{n,r}$, the complete r-partite graph of order n with parts as equal as possible, in the extremal literature.

The *extremal number*, denoted $ex(H, n)$, is the maximum number of edges in a graph of order n that does not contain H as a subgraph. While the exact value of the extremal function is known for very few graphs (cf. [1, 3, 5, 23]), in 1966 Erdős and Simonovits [8] extended previous work of Erdős and Stone [9] and determined $ex(H, n)$ asymptotically for arbitrary H. More precisely, this seminal theorem gives exact asymptotics for $ex(H, n)$ when H is a nonbipartite graph.

Theorem 3 (The Erdős-Stone-Simonovits Theorem)**.** *If H is a graph with chromatic number $\chi(H) = r + 1 \geq 2$, then*

$$ex(H, n) = \max\{|E(G)| \colon |G| = n, H \not\subseteq G\} = |E(T_{n,r})| + o(n^2).$$

It is our goal to examine the structure of degree sequences that are not potentially H-graphic in a manner similar to Theorems 1 and 2. In order to do so, we will next discuss a recent result on potentially H-graphic sequences that will allow us to identify the sequences that form our bounding class. For a graphic sequence π, we let $\sigma(\pi)$ denote the sum of the terms of π.

1.1. The potential function $\sigma(H, n)$

In 1991, Erdős, Jacobson and Lehel [7] proposed the following problem:

> Determine $\sigma(H, n)$, the minimum even integer such that every n-term graphic sequence π with $\sigma(\pi) \geq \sigma(H, n)$ is potentially H-graphic.

We refer to $\sigma(H, n)$ as the *potential number* or *potential function* of H. As $\sigma(\pi)$ is twice the number of edges in any realization of π, the Erdős-Jacobson-Lehel problem can be viewed as a degree sequence relaxation of the Turán problem. While the exact value of $\sigma(H, n)$ has been determined for a number of specific graph classes (c.f. [2, 7, 10, 12, 17, 19]), little was known about the behavior of the potential function for general graphs until recently when Ferrara, LeSaulnier, Moffatt and Wenger [11] determined $\sigma(H, n)$ asymptotically for all H. We describe their result next.

Let H be a graph on k vertices with at least one nontrivial connected component. For each $i \in \{\alpha(H)+1, \ldots, k\}$, define

$$\nabla_i(H) = \min\left\{\Delta(F) : F \leq H, |V(F)| = i\right\},$$

where $F \leq H$ denotes that F is an induced subgraph of H. Let n be sufficiently large, and consider the sequence

$$\widetilde{\pi}_i(H,n) = ((n-1)^{k-i}, (k-i+\nabla_i(H)-1)^{n-k+i}).$$

This sequence is graphic provided that $n-k+i$ and $\nabla_i(H)-1$ are not both odd. If they are both odd, then reduce the last term of the sequence by 1. As given in [11], the resulting sequence is graphic, but not potentially H-graphic. Consequently, $\sigma(H,n) \geq \max_i (\sigma(\widetilde{\pi}_i(H,n))$. This maximum is attained by $\widetilde{\pi}_{i^*}(H,n)$, where we define $i^* = i^*(H)$ to be the smallest index i in $\{\alpha(H)+1, \ldots, k\}$ that minimizes the quantity $2i - \nabla_i(H)$ and therefore maximizes $\sigma(\widetilde{\pi}_i(H,n))$. The main result of [11] states that $\widetilde{\pi}_{i^*}(H,n)$ determines $\sigma(H,n)$ asymptotically for all H, which can be viewed as an Erdős-Stone-Simonovits-type theorem for the Erdős-Jacobson-Lehel problem.

Theorem 4 (Ferrara, LeSaulnier, Moffatt and Wenger [11])**.** *If H is a graph and n is a positive integer, then*

$$\sigma(H,n) = \sigma(\widetilde{\pi}_{i^*}(H,n)) + o(n).$$

1.2. Main results

Through the remainder of this paper, unless otherwise noted we will assume that all sequences have minimum term at least 1. Given two n-term graphic sequences $\pi_1 = (d_1, \ldots, d_n)$ and π_2, and nonnegative integers a_1, a_2, and b with $a_1 \leq a_2$, we say that π_1 is $([a_1,a_2],b)$-*close* to π_2 if there is a (not necessarily graphic) sequence π_1' with $\pi_2 \succeq \pi_1'$ such that π_1' can be obtained from π_1 via the following two steps:

1. Create the sequence

$$S_1 = (d_1, \ldots, d_{a_1-1}, \underbrace{d_{a_2}, \ldots, d_{a_2}}_{a_2-a_1+1 \text{ times}}, d_{a_2+1}, \ldots, d_n).$$

2. Create π_1' from S_1 by subtracting a *total* of at most b from the terms of S_1.

We will refer to step (1) as *leveling off* terms a_1 to a_2 of π_1 and the procedure in step (2) as *editing* the sequence S_1.

In contrast to the idea of (k, m)-majorization, $([a_1, a_2], b)$-closeness leaves the first $a_1 - 1$ terms unchanged, and after the leveling off step allows for variable editing, provided that the total amount of editing in step (2) is at most b.

We show that if a sequence is not potentially H-graphic, then it is close (in the above sense) to being majorized by one of the sequences $\widetilde{\pi}_i(H, n)$. Our main results are as follows.

Theorem 5. *Let H be a graph with degree sequence $\pi(H) = (h_1, \dots, h_k)$, and let $\pi = (d_1, \dots, d_n)$ be a graphic sequence that is not degree sufficient for H. Further, let j be the largest integer for which $d_{k-j+1} < h_{k-j+1}$. If $j \geq \alpha(H) + 1$, then π is majorized by $\widetilde{\pi}_j(H, n)$. If $j < \alpha(H) + 1$, then π is $([k - \alpha(H), k - j + 1], 0)$-close to $\widetilde{\pi}_{\alpha(H)+1}(H, n)$. This result is best possible.*

The case where π is degree sufficient for H seems to be much more technical, and requires both the leveling off and editing operations outlined above. Recall that $i^* = i^*(H)$ is the smallest i in $\{\alpha(H) + 1, \dots, k\}$ that minimizes $2i - \nabla_i(H)$.

Theorem 6. *Let H be a graph of order k with at least one nontrivial component and let π be an n-term graphic sequence that is degree sufficient for H. If π is not potentially H-graphic, then π is $([k - i^* + 1, k], (6\alpha + 3)k^2 + \alpha^3 k)$-close to $\widetilde{\pi}_{i^*}(H, n)$.*

In Section 3, we provide an example of a graph H and a sequence that is not potentially H-graphic and requires $O(\alpha k^2)$ editing, proving that Theorem 6 is in some sense best possible up to the coefficient of αk^2. However, this sharpness remains an open question in those cases when $\alpha \gg k^{1/2}$.

2. Preliminaries

The following results will be used repeatedly throughout our proofs of Theorems 5 and 6. The first is the well-known characterization of graphic sequences due to Erdős and Gallai.

Theorem 7 (Erdős and Gallai [6]). *A sequence $\pi = (d_1, \dots, d_n)$ such that $d_1 \geq \dots \geq d_n$ is graphic if and only if $\sum_{i=1}^{n} d_i$ is even and, for all $p \in \{1, \dots, n-1\}$,*

$$\sum_{i=1}^{p} d_i \leq p(p-1) + \sum_{i=p+1}^{n} \min\{d_i, p\}. \qquad (1)$$

The following Lemma is central to the proof of Theorem 6, and is likely also of independent interest. As the proof of this result is quite technical, we postpone it until Section 4.1.

Lemma 8. *Let r and k be positive integers with $r < k$, and let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence. Suppose that $d_{k-r} - d_k \geq r(k+2)$. If there are at least $r(k+r+1)$ terms among $d_{k+1}, \ldots, d_n$ with values in $\{k-r, \ldots, k-1\}$, then π has a realization containing the graph $K_{k-r,r}$ with vertices of degree $d_1, \ldots, d_{k-r}$ forming the partite set of order $k-r$.*

The following gives a bound on the length of a sequence with fixed maximum term that is not potentially H-graphic.

Lemma 9. *Let H be a graph with $\pi(H) = (h_1, \ldots, h_k)$. Let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence with $d_1 \leq M$ such that there are terms $d_{i_1}, \ldots, d_{i_k}$ of π satisfying $d_{i_j} \geq h_j$ for $1 \leq j \leq k$. If π has at least $2M^2 + k$ positive terms, then there is a realization G of π with a copy of H that lies on vertices of degree $d_{i_1}, \ldots, d_{i_k}$.*

Proof. We may assume that $d_{i_j} = d_j$ for all j and also that $n \geq 2M^2 + k$ and $d_n \geq 1$. First note that if $M = 1$, then H must be a set of disjoint edges and isolated vertices, and π is potentially H-graphic as long as $n \geq k$. We therefore assume $M \geq 2$.

Let $V(H) = \{u_1, \ldots, u_k\}$, with the vertices in nonincreasing order by degree. In a realization G of π, let $S = \{v_1, \ldots, v_k\}$ be the vertices with the k highest degrees (in order) and let H_S be the graph with vertex set S and $v_iv_j \in E(H_S)$ if and only if $u_iu_j \in E(H)$. If all of the edges of H_S are in G, then H_S is a subgraph of G that is isomorphic to H.

Assume now that G is a realization of π that maximizes $|E(H_S) \cap E(G)|$, but this quantity is less than $|E(H_S)|$. Thus, there exist $v_i, v_j \in V(G)$ such that $v_iv_j \notin E(G)$ but $v_iv_j \in E(H_S)$. Since π is degree sufficient for H, it follows that v_i and v_j must each have a neighbor, say a_i and a_j, respectively, such that $v_ia_i, v_ja_j \notin E(H_S)$ but $v_ia_i, v_ja_j \in E(G)$. Note that possibly $a_i = a_j$.

Since the maximum degree in G is M, there are at most M^2+1 vertices at distance at most 2 from a_i, and at most $M^2 + 1$ vertices at distance at most 2 from a_j. Since a_i and a_j have distinct neighbors in S, there are at most $k-2$ vertices in S that are distance at least 3 from both a_i and a_j. Therefore, there is a vertex w in $V(G) \backslash S$ that is distance at least 3 from both a_i and a_j. Let x be a neighbor of w; consequently x is not adjacent to a_i or a_j, and $xw \notin E(H_S)$. Exchanging the edges v_ia_i, v_ja_j, and wx for the non-edges v_iv_j, a_iw, and a_jx yields a realization G' of π such that $|E(H_S) \cap E(G')| > |E(H_S) \cap E(G)|$, contradicting the maximality of G. $\square$

Finally, the following theorem of Li and Yin gives useful sufficient conditions for a degree sequence to be potentially K_k-graphic. We will use this repeatedly in our proofs of Theorems 5 and 6.

Theorem 10 (Li and Yin, [19]). *Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence and let k be a positive integer.*

(a) If $d_k \geq k-1$ and $d_i \geq 2(k-1)-i$ for $1 \leq i \leq k-2$, then π is potentially K_k-graphic.
(b) If $d_k \geq k-1$ and $d_{2k} \geq k-2$, then π is potentially K_k-graphic.

3. Sharpness

Prior to proving Theorems 5 and 6, we will discuss the sharpness of these results.

The number of terms leveled off in Theorem 5 is best possible in light of the following example. Let $H = K_{k-r} \vee \overline{K}_r$, where r is at least 2, and for $1 \leq j < \alpha(H)+1$ let

$$\pi_j = \left(\left(\frac{n}{k+1}\right)^{k-j}, (k-r-1)^{n-k+j}\right)$$

where n is sufficiently large. If the sum of π_j is even, then π_j is graphic; if the sum is odd, then reducing the last term by 1 yields a graphic sequence. Clearly π_j is not degree sufficient for H.

Note that the $(k-j+1)^{\text{st}}$ term of π_j is the first place that degree sufficiency for H fails. Since $k-r-1 \leq k-2$, the last $n-k+j$ terms of π_j are termwise dominated by the last $n-k+j$ terms of $\widetilde{\pi}_{\alpha(H)+1}(H,n)$. However, π_j has $k-j$ large terms, of which only the first $k-\alpha(H)-1$ are dominated by $\widetilde{\pi}_{\alpha(H)+1}(H,n)$. Therefore, we need to reduce terms $d_{k-\alpha(H)}, \ldots, d_{k-j}$ of π_j to $d_{k-j+1} = k-r-1$, and each of these reductions is on the order of n. This yields a sequence that is majorized by $\widetilde{\pi}_{\alpha(H)+1}(H,n)$, but reducing any smaller number of terms would not suffice.

To evaluate the sharpness of Theorem 6, we consider $H = K_k$. Let

$$\pi_k = \left(n-1, (2k-5)^{2k-3}, 1^{n-2k+2}\right)$$

The Erdős-Gallai criteria show that π_k is graphic; clearly π_k is degree sufficient for K_k. Observe that π_k is potentially K_k-graphic if and only if the sequence $\pi'_k = \left((2k-6)^{2k-3}\right)$, obtained by performing the Havel-Hakimi algorithm, is potentially K_{k-1}-graphic. However, the complement of any realization of π'_k is a 2-regular graph, so the maximum size of a clique in any realization of π'_k is at most $k-2$. Hence π_k is not potentially K_k-graphic.

Since $2i - \nabla_i(K_k) = i+1$ for each $i \in \{2,\ldots,k\}$, we have $i^* = 2$. Note that $\widetilde{\pi}_2(K_k, n) = ((n-1)^{k-2}, (k-2)^{n-k+3})$. Leveling off terms $k - i^* + 1 = k-1$ through k of π'_k does not change the sequence. However, each entry from $k-1$ through $2k-2$ is larger than $k-2$, so we need to reduce each of these entries by $k-3$, for a total of $k(k-3)$ editing. Thus we perform a total of $O(\alpha k^2)$ editing.

4. Proofs of Theorems 5 and 6

Proof of Theorem 5. First note that for each $i \geq \alpha(H)+1$, $h_{k-i+1} \leq k - i + \nabla_i(H)$. Otherwise, every i-vertex induced subgraph of H has maximum degree greater than $\nabla_i(H)$, contradicting the definition of $\nabla_i(H)$.

Similarly, if $i < \alpha(H)+1$, then $h_{k-i+1} \leq k - \alpha(H)$. Otherwise, at most $i-1 < \alpha(H)$ vertices have degree at most $k-\alpha(H)$, while there are at least $\alpha(H)$ vertices in H with degree at most $k-\alpha(H)$.

Recall that j is the largest integer for which $d_{k-j+1} < h_{k-j+1}$. First suppose that $j \geq \alpha(H)+1$. In this case, we show that π is majorized by $\widetilde{\pi}_j(H,n)$. Clearly, the first $k-j$ terms of π are majorized by the first $k-j$ terms of $\widetilde{\pi}_j(H,n)$. As $d_{k-j+1} < h_{k-j+1} \leq k - j + \nabla_j(H)$, the remaining terms of π are at most $k-j+\nabla_j(H)-1$. Thus, π is majorized by $\widetilde{\pi}_j(H,n)$.

Now suppose that $j < \alpha(H)+1$. Here we show that π is $([k-\alpha(H), k-j+1], 0)$-close to $\widetilde{\pi}_{\alpha(H)+1}(H,n)$. We know that $d_{k-j+1} < h_{k-j+1} \leq k-\alpha(H)$. Since $k-\alpha(H)+\nabla_{\alpha(H)+1} - 2 \geq k - \alpha(H) - 1$, reducing terms $d_{k-\alpha(H)}$ through d_{k-j} of π to d_{k-j+1} results in a sequence that is majorized by $\widetilde{\pi}_{\alpha(H)+1}(H,n)$. □

For the proof of Theorem 6, we prove a more technical result that follows below. First we define some terminology that is used in the proof.

Given a graphic sequence π that is degree sufficient for $K_r \vee \overline{K}_{k-r}$, we will create a sequence π^w, called the *want sequence of* π *for* $K_r \vee \overline{K}_{k-r}$. Begin by finding a realization G of π on the vertices $\{v_1,\ldots,v_n\}$ with $d(v_i) = d_i$ that maximizes the sum of (a) the number of edges amongst $v_1,\ldots,v_r$ and (b) the number of edges joining $\{v_1,\ldots,v_r\}$ and $\{v_{r+1},\ldots,v_k\}$. Let $G_r = G[v_{r+1},\ldots,v_n]$, and let $\pi_0^w = (w_{r+1},\ldots,w_n)$ be the degree sequence of G_r, indexed so that $w_i = d_{G_r}(v_i)$.

For each v_i with $i \leq r$, we want v_i to be adjacent to each of the vertices in the set $S_i = \{v_1,\ldots,v_k\} \backslash \{v_i\}$. Since π is degree sufficient for $K_r \vee \overline{K}_{k-r}$, we see that for each nonneighbor of v_i in S_i, there is a neighbor of v_i in $\{v_{k+1}, \dots, v_n\}$, and each of these neighbors is distinct. Let W_i be a subset of $N_{G_r}(v_i) \cap \{v_{k+1}, \dots, v_n\}$ that has size $k-1-d_{S_i}(v_i)$. Let W be the multiset $\cup_{i=1}^{k} W_i$.

To create the want sequence from π_0^w, we make the following modifications. Each time the vertex v_y appears in W, add 1 to entry w_y of π_0^w. For each j with $r+1 \leq j \leq k$, subtract $r - d_{\{v_1,\dots,v_r\}}(v_j)$ from w_j. The sequence that results from these modifications is the want sequence, π^w. Note that the largest value that can be subtracted from any entry is r, and since π is degree-sufficient for $K_r \vee \overline{K}_{k-r}$ and the only entries that might be reduced are those with index at most k, no entry of π^w is negative and at most r terms of π^w are 0. The largest value that will be added to any entry of π_0^w is at most r, and only terms with index at least $k+1$ are increased, so the largest entry of π^w is at most the maximum of w_{r+1} and $w_{k+1} + r$.

If π^w is graphic, we can find a realization of π that contains $K_r \vee \overline{K}_{k-r}$. To do this, take the union of the complete split graph $K_r \vee \overline{K}_{k-r}$ on the vertices $\{v_1, \dots, v_k\}$ (with the clique on the vertex set $\{v_1 \dots, v_r\}$) and a realization of π^w on the vertices $\{v_{r+1}, \dots, v_n\}$. Then join each vertex belonging to the clique of the complete split graph (that is, v_i such that $i \leq r$) to the vertices in $N(v_i) \cap \{v_{k+1}, \dots, v_n\} \backslash W_i$. This graph has degree sequence π, so we have a realization of π that contains the desired complete split graph.

We will prove the following, more specific result than that stated in Theorem 6.

Theorem 11. *Let H be a fixed graph of order k with at least one non-trivial component. If π is a graphic sequence of length n that is degree sufficient for H but not potentially H-graphic, then π is*

$$([k-i^*+1, k], k^2+ki^*+2+(6k^2+(i^*-\nabla_{i^*}(H))^2k+\nabla_{i^*}(H))(i^*-\nabla_{i^*}(H)-2))$$

-close to $\widetilde{\pi}_{i^*}(H, n)$.

Since

$$2i^* - \nabla_{i^*}(H) \leq 2(\alpha(H)+1) - \nabla_{\alpha(H)+1}(H) \leq 2\alpha(H) + 1,$$

and $i^* \geq \alpha(H) + 1$, we see that $i^* - \nabla_{i^*}(H) \leq \alpha(H)$. Thus, Theorem 6 follows directly from Theorem 11.

Proof. Let $\pi = (d_1, \dots, d_n)$ and label the vertices of H with $\{v_1, \dots, v_k\}$ such that $d(v_i) \geq d(v_j)$ when $i < j$. To simplify notation, we will write α for $\alpha(H)$ and let $\ell^* = i^* - \nabla_{i^*}(H)$. Let $f(H) = 6k^2 + (\ell^*)^2k + \nabla_{i^*}(H)$. With this notation, we prove that π is $([k-i^*+1,k], k^2+ki^*+2+f(H)(\ell^*-2))$-close to $\widetilde{\pi}_{i^*}(H,n)$.

If either $d_k \geq 2k-3$ or $d_{2k} \geq k-2$, then by Theorem 10, π is potentially K_k-graphic. As this would imply that π is potentially H-graphic, we assume henceforth that $d_k \leq 2k-4$ and, if $n \geq 2k$, that $d_{2k} \leq k-3$.

We will break the proof into several cases. In Cases 1, 2, and 3, we will show that, after reducing the value of terms $d_{k-i^*+1}, \ldots, d_{k-1}$ to d_k, we only require an editing of at most $k^2+ki^*+2+f(H)(\ell^*-2)$. In Case 4, no editing is required, and we show that π is potentially H-graphic.

Case 1: $n < 2k$.

In this case, after leveling off terms d_{k-i^*+1} through d_{k-1}, we need to reduce at most $k+i^*$ terms of the sequence. Each of those terms is reduced by at most $k+\ell^*-3$, so the total amount of editing is at most $k^2+ki^*+(\ell^*-3)(k+i^*)$.

Case 2: $2k \leq n < f(H)$.

Reducing the terms $d_{k-i^*+1}, \ldots, d_{k-1}$ of π to d_k leaves at most $k-i^*$ terms that may be greater than $2k-4$. Now in the resulting sequence, each of the terms from position $k-i^*+1$ to position $2k-1$ needs to be reduced by at most $k+\ell^*-3$ and each term from position $2k$ to the end of the sequence must be reduced by at most ℓ^*-2. Thus the amount of editing required on this sequence is at most

$$(k+i^*-1)(k+\ell^*-3)+(n-2k+1)(\ell^*-2) \leq (\ell^*-2)(f(H)-k+i^*)+(k-1)(k+i^*-1).$$

Case 3: $n \geq f(H)$ and $d_{f(H)} \leq k-\ell^*-1$.

Here, we can stop editing after the term $d_{f(H)}$ since all subsequent terms are at most $k-\ell^*-1$. We perform the same leveling-off step as in Case 2, so the amount of editing required is at most

$$(k+i^*-1)(k+\ell^*-3)+(f(H)-2k+1)(\ell^*-2) = (\ell^*-2)(f(H)-k+i^*)+(k-1)(k+i^*-1).$$

Case 4: $n \geq f(H)$ and $d_{f(H)} > k-\ell^*-1$.

Now we show that π is potentially H-graphic. If $\ell^*=1$, then $d_{f(H)} \geq k-1$, and since $f(H) \geq 2k$, this means $d_{2k} \geq k-1$. Thus by part (b) of Theorem 10, π is potentially K_k-graphic. We assume henceforth that $\ell^* \geq 2$.

Let t be such that $d_t \geq k-1$ but $d_{t+1} < k-1$. We then have two cases.

Case 4a: $t < k - \ell^*$.

In this case, we wish to show that π has a realization that contains the complete split graph $K_t \vee \overline{K}_{k-t}$. First note that π is degree sufficient for such a graph because $d_t \geq k-1$ and $d_k \geq k - \ell^* > t$. Let π_1^w be the want sequence of π for $K_t \vee \overline{K}_{k-t}$. Since every entry of π_0^w is at most $k-2$ (because $d_{t+1} < k-1$), and t is also at most $k-1$, the largest entry of π_1^w is less than $2k$.

Zverovich and Zverovich [26] showed that a sequence with maximum term r and minimum term s is graphic as long as the length of the sequence is at least $\frac{(r+s+1)^2}{4s}$. Since π_1^w has length $n-t$ and up to t terms may be 0, π_1^w is graphic if $n - 2t \geq (k+1)^2$. Since $t < k - \ell^*$, this is true if $n \geq k^2 + 4k + 1 - 2\ell^*$. The observation that $n \geq f(H) > 6k^2$ shows that this is true, and π_1^w is graphic.

Now observe that if π_1^w is graphic, then π is potentially $K_t \vee \overline{K}_{k-t}$-graphic. If $t \geq k - \alpha(H)$, then this complete split graph contains a copy of H, and we are done. So we assume that $t < k - \alpha(H)$. Let F_i denote an i-vertex induced subgraph of H that achieves $\Delta(F_i) = \nabla_i(H)$. If π has a realization containing $K_t \vee \overline{K}_{k-t}$ with a copy of F_{k-t} on the vertices in the independent set, then π is potentially H-graphic.

By Lemma 2.1 of [24], we know that a realization G of π can be found that contains this split graph with the following property: the t vertices of degree $k-1$ are on the t highest-degree vertices of G, and the $k-t$ vertices of degree t are on the next $k-t$ highest degree vertices of G. Delete the t vertices of highest degree in G, and let $\pi' = (d'_1, \ldots, d'_{n-t})$ be the degree sequence of the resulting subgraph of G. It follows that π' satisfies $d'_1 \leq k-2$ and $d'_{f(H)-t} \geq k - \ell^* - t \geq 1$. Since $\Delta(F_{k-t}) = \nabla_{k-t}(H)$ and $d'_{k-t} \geq d'_{f(H)-t} \geq k - \ell^* \geq \nabla_{k-t}(H)$, it follows that π' is degree sufficient for F_{k-t}. Applying Lemma 9 with $H = F_{k-t}$ and $M = k-2$, we see that π' is potentially F_{k-t}-graphic as long as at least $2(k-2)^2 + (k-t)$ terms of π' are positive. Since $f(H) - t \geq 6k^2$, there is a realization of π' that contains a copy of F_{k-t} on the highest degree vertices; overlapping this with the vertices $v_{t+1}, \ldots, v_k$ of G, we get a realization of π containing $K_t \vee F_{k-t}$, which implies that π is potentially H-graphic.

Case 4b: $t \geq k - \ell^*$.

First, suppose $d_{k-\ell^*} - d_k \geq \ell^*(k+2)$. This implies that $d_{k-\ell^*} \geq \ell^*(k+2) + d_k$. Since $\ell^* \geq 2$, it follows that $d_{k-\ell^*} \geq 3k$. We claim that this implies that π is potentially H-graphic.

Since $f(H) > \ell^*(k + \ell^* + 1)$ and $d_{f(H)} \geq k - \ell^*$, Lemma 8 yields a realization G of π containing the complete bipartite graph $K_{k-\ell^*,\ell^*}$, where the vertices $\{v_1, \dots, v_{k-\ell^*}\}$ form the partite set of order $k - \ell^*$. Let S be the set of vertices in the copy of $K_{k-\ell^*,\ell^*}$, let $S' = \{v_1, \dots, v_{k-\ell^*}\}$, and let $R = V(G) \setminus S$. If the vertices of S' induce a complete graph, then G contains $K_{k-\ell^*} \vee \overline{K}_{\ell^*}$; consequently G contains a copy of H since $k - \ell^* \geq k - \alpha$. Suppose there are vertices v_i and v_j in S' such that $v_iv_j \notin E(G)$. Since $d_S(v_i) \leq k-2$ and $d(v_i) \geq 3k-2$, we know that v_i has at least $2k$ neighbors in R. Similarly, v_j has at least $2k$ neighbors in R. If each neighbor of v_i in R is adjacent to each neighbor of v_j in R, then each of these vertices in R has degree at least $2k-1$. Since v_i has at least $2k$ neighbors in R, it follows that G has at least $2k$ vertices with degree at least $2k-1$, contradicting the assumption that $d_k \leq 2k-4$. Thus there are vertices x and y in R such that $v_ix, v_jy \in E(G)$, and $xy \notin E(G)$. Hence we can replace the edges v_ix and v_jy with the non-edges xy and v_iv_j to obtain a new realization of π. Iteratively performing this process for each nonadjacent pair of vertices in S' yields a realization of π in which S' induces a complete graph. Consequently π is potentially H-graphic.

Finally, we must consider the case where $d_{k-\ell^*} < d_k + \ell^*(k+2) < 2k + \ell^*(k+2)$. Observe that π is degree sufficient for $K_{k-\ell^*} \vee \overline{K}_{\ell^*}$ since $t \geq k - \ell^*$ and $f(H) > k$. Let $\pi_2^w = (g_1, \dots, g_{n'})$ be the want sequence of π for $K_{k-\ell^*} \vee \overline{K}_{\ell^*}$ where $n' = n-(k-\ell^*)$. Since constructing the want sequence increases each term by at most $k-\ell^*$, the facts that $d_{k-\ell^*+1} < 2k+\ell^*(k+2)$, $d_k < 2k-3$, and $d_{2k} < k-1$ imply that π_2^w has the following properties:

- $g_i < 3k + (k+1)\ell^*$ for $1 \leq i \leq \ell^*$,
- $g_i < 3k - \ell^*$ for $\ell^* + 1 \leq i \leq \ell^* + k$, and
- $g_i < 2k - \ell^*$ for $\ell^* + k + 1 \leq i \leq n'$.

Claim 1. *The sequence π_2^w is graphic.*

Proof of Claim 1. Note that $n' = n - (k - \ell^*) > 6k^2 + (\ell^*)^2k$.

Tripathi and Vijay [22] showed that the Erdős-Gallai criteria (Theorem 7) need only be checked for certain values of p: it suffices to check all $p \leq s$, where s is the largest integer for which $d_s \geq s-1$, or to check only those values of p for which d_p is strictly greater than d_{p+1}. We will use the Erdős-Gallai criteria and this observation to show that π_2^w is graphic.

Since $g_i < 2k - \ell^*$ for large enough i, we only need to check the inequalities for indices up to $2k$. We can write the right-hand side of (1) as

$$p(p-1) + \sum_{i=p+1}^{r} p + \sum_{i=r+1}^{n'} g_i,$$

where $r \geq p+1$ is the largest index such that $g_r \geq p$ but $g_{r+1} < p$. This then simplifies to

$$\begin{aligned} p(r-1) + \sum_{i=r+1}^{n'} d_i &\geq p(r-1) + n' - r \\ &= r(p-1) - p + n' \\ &\geq (p+1)(p-1) - p + n' \\ &= p^2 - p - 1 + n', \end{aligned}$$

so we need to show that $n' + p^2 - p - 1 \geq \sum_{i=1}^{p} g_i$ for each $p \leq 2k$.

First suppose $p \leq \ell^*$. Then

$$\begin{aligned} \sum_{i=1}^{p} g_i &< p(3k + \ell^*(k+1)) \\ &\leq 3\ell^* k + (\ell^*)^2(k+1). \end{aligned}$$

Since $n' \geq 6k^2 + (\ell^*)^2 k$, the desired inequality holds.

If $\ell^* + 1 \leq p \leq \ell^* + k$, then

$$\begin{aligned} \sum_{i=1}^{p} g_i &\leq 3\ell^* k + (\ell^*)^2(k+1) + (p - \ell^*)(3k - \ell^*) \\ &\leq 2\ell^* k + (\ell^*)^2(k+1) + 3k^2. \end{aligned}$$

For p in this range,

$$\begin{aligned} n' + p^2 - p - 1 &\geq 6k^2 + (\ell^*)^2 k + (\ell^* + 1)^2 - 1 \\ &= 6k^2 + (\ell^*)^2(k+1) + 2\ell^*, \end{aligned}$$

so the inequality holds.

Finally, if $\ell^* + k + 1 \leq p \leq 2k$, then

$$\begin{aligned} \sum_{i=1}^{p} g_i &\leq 2\ell^* k + (\ell^*)^2(k+1) + 3k^2 + (p - \ell^* - k)(2k - \ell^*) \\ &\leq 5k^2 + (\ell^*)^2 k + 2(\ell^*)^2 - \ell^* k. \end{aligned}$$

Now, $n' + p^2 - p - 1 > 6k^2 + (\ell^*)^2 k + 4k^2$, so the Erdős-Gallai inequality is satisfied. Thus Claim 1 is proved. □

We can now use a realization of π_2^w to create a copy of $K_{k-\ell^*} \vee \overline{K}_{\ell^*}$ in a realization of π. Since $H \subseteq K_{k-\ell^*} \vee \overline{K}_{\ell^*}$, this implies that π is potentially H-graphic. □

4.1. Proof of Lemma 8

Kleitman and Wang gave the following generalization of the graphicality criteria due independently to Havel and Hakimi [13, 14].

Theorem 12 (Kleitman and Wang, [16])**.** *Let $\pi = (d_1, \dots, d_n)$ be a nonincreasing sequence of nonnegative integers, and let $i \in [n]$. If π_i is the sequence defined by*

$$\pi_i = \begin{cases} (d_1 - 1, \dots, d_{d_i} - 1, d_{d_i+1}, \dots, d_{i-1}, d_{i+1}, \dots, d_n) & d_i < i \\ (d_1 - 1, \dots, d_{i-1} - 1, d_{i+1} - 1, \dots, d_{d_i+1} - 1, d_{d_i+2}, \dots, d_n) & d_i \geq i, \end{cases}$$

then π is graphic if and only if π_i is graphic.

Let π' be the sequence resulting from sorting π_i in nonincreasing order, and call π' the *residual sequence* obtained by *laying off* d_i. Repeated application of Theorem 12 yields an efficient algorithm to test for graphicality.

Proof of Lemma 8.
Idea of the proof: The proof of Lemma 8 is based on a careful analysis of repeated applications of the Kleitman-Wang algorithm (Theorem 12). Observe that when laying off a term d_i from a graphic sequence, the d_i terms of highest degree, aside from d_i, are each reduced by 1. If there are many terms of the same value that will be reduced, the order in which these reductions occur does not matter. In particular, provided we reduce the correct number of terms, we may reduce any of the terms equal to d_{d_i} and will get the same residual sequence. This fact is the key to constructing a realization of π that contains $K_{k-r,r}$, as referenced in the statement of Lemma 8.

Kleitman-Wang provides a means by which this realization can be constructed on the vertex set $V = \{v_1, \dots, v_n\}$ so that the vertices v_j have degree d_j for $j = 1, \dots, n$. The vertex v_j is associated with the jth term in π. When d_j is laid off, the resulting sequence, π', is a graphic sequence. We can use it to construct a graph on $V - \{v_j\}$ with (the reordered) π' as its degree sequence. The vertex v_j is then added, adjacent to the first d_j members of $\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n\}$. In this way, when we lay off a term d_j of π, we will say that the vertices associated with the terms that are reduced are assigned to the neighborhood of v_j. Repeating this process, we will create the desired realization of $K_{k-r,r}$.

The problem with this procedure is that applying it more than once requires that each of the degree sequences must be reordered, which makes keeping track of the vertices that are assigned to a particular neighborhood difficult.

For clarity, we will often abuse terminology and say we lay off vertex v_j to mean we lay off the term of π whose value corresponds to the degree of v_j. This makes sense when we think about laying off a term d_j of π as assigning a set of vertices to the neighborhood of v_j. We will lay off at most $r(k+r+1)$ vertices with the aim of obtaining just r of them whose neighborhood contains $\{v_1, \ldots, v_{k-r}\}$. Our parameters are chosen just for this purpose. The entries in $\{d_{k+1}, \ldots, d_n\}$ that have value in $\{k-r, \ldots, k-1\}$ will be the candidates for entries to lay off. Because $d_{k-r} - d_k \geq r(k+2)$, we can guarantee that, for each of the degree sequences that result from the laying off, the entries that correspond to $v_1, \ldots, v_{k-r}$ will always stay within the first $k-1$ entries.

Terms and definitions: Now we proceed to prove that the procedure outlined above does indeed produce the graph we want. We will lay off entries of π corresponding to vertices $v_{a_1}, v_{a_2}, \ldots, v_{a_p}, \ldots$, where v_{a_p} will be determined at step p. Let $V_0 = V$ and for $p = 1, 2, \ldots$, let $V_p = V_{p-1} - v_{a_p}$. The neighborhood we assign to v_{a_p}, which we will call N_p, is a subset of V_p. We will call the process of laying off $v_{a_1}, \ldots, v_{a_{r(k+2)}}$ the Laying-off Algorithm.

For $p = 0, 1, 2, \ldots$, we define $\hat{d}_p(v_i)$ to be the *remaining degree* of v_i after $v_{a_1}, \ldots, v_{a_p}$ are laid off. That is, for every vertex v_i, $\hat{d}_0(v_i) = d_i$ and for $p = 1, 2, \ldots, r(k+2)$, we have $\hat{d}_p(v_i) := d_i - |\{j\colon 1 \leq j \leq p \text{ and } v_i \in N_j\}|$. Iteratively,

$$\hat{d}_p(v_i) = \begin{cases} \hat{d}_{p-1}(v_i) & \text{if } v_i \notin N_p \\ \hat{d}_{p-1}(v_i) - 1 & \text{if } v_i \in N_p. \end{cases}$$

To determine which vertex v_{a_p} to lay off, for $p = 1, 2, \ldots$, we define $S_{p-1} \subset V_{p-1}$ to be the set of all vertices $w \in V_{p-1}$ for which $\hat{d}_{p-1}(w) \in \{k-r, \ldots, k-1\}$. Then choose v_{a_p} to be a vertex in S_{p-1} for which $\hat{d}_{p-1}(v_{a_p})$ is minimum. Let $\ell_{p-1} = \hat{d}_{p-1}(v_{a_p})$; this is the number of vertices that will be assigned to N_p. Note that the neighborhood of v_{a_p} may not consist solely of the vertices in N_p. In particular, if $\hat{d}_{p-1}(v_{a_p}) < \hat{d}_0(v_{a_p})$, then v_{a_p} is in $N_{p'}$ for some $p' < p$. Thus, the neighborhood of v_{a_p} in our final graph contains $v_{a_{p'}}$ although $v_{a_{p'}}$ is not in N_p.

The natural ordering on $V = V_0$ is simply $(v_1, \ldots, v_n)$. This corresponds to the nonincreasing order of π. We say that v_i *naturally precedes* v_j if $i < j$, and will write $v_i \propto v_j$. We will define π_p to be the sequence given by each

$\hat{d}_p(v_i)$, for all $v_i \in V_p$, that is nonincreasing and, when equality holds, to obey the natural ordering. That is, $\hat{d}_p(v_i)$ precedes $\hat{d}_p(v_j)$ in π_p if either (a) $\hat{d}_p(v_i) > \hat{d}_p(v_j)$, or (b) $\hat{d}_p(v_i) = \hat{d}_p(v_j)$ and $i < j$. This is simply the degree sequence obtained from π by p iterations of the Kleitman-Wang algorithm; thus, π_p is graphic.

Observe that in defining π_p, we have prescribed the order of the terms based on the vertices with which they are associated. This is because we need to keep track of not only the remaining degree of a vertex but also the position of that vertex in π_p. To make this precise, let τ_p be a function from $\{1, \ldots, |V_p|\} \to V_p$ in which $\tau_p(j)$ is the vertex in the jth position in the order defined by π_p, and let T_p be the sequence $\tau_p(1), \ldots, \tau_p(n-p)$. Thus, T_p is simply the sequence of vertices of V_p, ordered according to the position of their remaining degree in π_p. A subsequence $\tau_p(b_1), \ldots, \tau_p(b_m)$ of T_p is *consistent* if $\tau_p(b_i) \propto \tau_p(b_j)$ for all $b_i < b_j$. In essence, this means that all vertices in the subsequence are in order by index, from lowest to highest. We say that T_p itself is consistent if $\tau_p(1), \ldots, \tau_p(n-p)$ is consistent.

In the Kleitman-Wang algorithm, when the term d_i is laid off it is first removed from the sequence; then the first d_i terms of the resulting sequence are each reduced by one. To incorporate this into the Laying-off Algorithm, we define $\hat{\pi}_{p-1}$ to be π_{p-1} with the term associated with v_{a_p} removed. So, $\hat{\tau}_{p-1}$ and $\hat{T}_{p-1}$ are the corresponding order function and sequence of vertices.

Finding the "neighborhoods" N_p: Now we can describe our modification of the Kleitman-Wang algorithm more precisely. At step p of the Laying-off Algorithm, we choose N_p in the following way:

1. If T_{p-1} is consistent, then simply let N_p be the first ℓ_{p-1} vertices in $\hat{T}_{p-1}$.
2. If T_{p-1} is not consistent but $\hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1})) > \hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1}+1))$, then we again let N_p consist of the first ℓ_{p-1} vertices in T_{p-1}.
3. If T_{p-1} is not consistent but $\hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1})) = \hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1}+1))$, then N_p consists of all vertices $w \in V_{p-1}$ for which $\hat{d}_{p-1}(w) > \hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1}))$, and the vertices x with the highest index for which $\hat{d}_{p-1}(x) = \hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1}))$.

In words, what we do is identify the vertices with largest $\hat{d}_p$ values and reduce their values by 1. If T_{p-1} is not consistent and we cannot reduce all of those with the same value, we reduce those with largest index (i.e., those that come later in the ordering). When T_{p-1} is consistent, we still take the first ℓ_{p-1} vertices, even if all of those with the same value are not reduced.

We will say that N_p is *good* if $\{v_1, \ldots, v_{k-r}\} \subseteq N_p$. The existence of at least r vertices among $\{v_{k+1}, \ldots, v_n\}$ such that laying off each gives a good N_p will yield the $K_{k-r,r}$ we seek.

An example: Let us do a small example to illustrate the way the Laying-off Algorithm works.

Begin with the graphic sequence $\pi = (9, 9, 9, 9, 8, 8, 7, 7, 7, 7, 4, 4, 4, 4) = (d(v_1), \ldots, d(v_{14}))$. For the purposes of this example, we will only identify the neighborhoods of the vertices with degree 4.

Step 1 Since the original ordering of vertices is consistent, we assign the neighborhood of v_{14} to be $N_1 = \{v_1, v_2, v_3, v_4\}$. The new sequence is $\pi_1 = (8^6, 7^4, 4^3)$, and since $\hat{d}_1(v_4) \geq \hat{d}_1(v_5)$, there is no reordering of vertices and T_1 is consistent.

Step 2 Since T_1 is consistent, we can simply assign the set

$$N_2 = \{v_1, v_2, v_3, v_4\}$$

to the neighborhood of v_{13}. Now $\pi_2 = (8, 8, 7^8, 4^2)$. However, the vertices are no longer in their original order; the sequence T_2 is:

$$v_5, v_6, v_1, v_2, v_3, v_4, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}.$$

Step 3 Since T_2 is not consistent, we must consider $\hat{d}_2(\hat{\tau}_2(4))$. Since

$$\hat{d}_2(\hat{\tau}_2(4)) = \hat{d}_2(\hat{\tau}_2(5)),$$

we cannot simply assign the four highest-degree vertices to N_3. We begin with $N_3 = \{v_5, v_6\}$, the two highest-degree vertices. Then we need two more vertices, so we take the two vertices of degree $\hat{d}_2(\hat{\tau}_2(4)) = 7$ that have the highest index, that is v_9 and v_{10}. So $N_3 = \{v_5, v_6, v_9, v_{10}\}$. This leaves $\pi_3 = (7^8, 6^2, 4)$, and T_3 is consistent.

Step 4 Since T_3 is consistent, we let $N_4 = \{v_1, v_2, v_3, v_4\}$. Then $\pi_4 = (7^4, 6^6)$.

Observe that at each step π_i is exactly the sequence we would get after i iterations of the Kleitman-Wang algorithm if a term of value 4 is laid off each time.

Proof that the Laying-off Algorithm gives r good neighborhoods: Now we show that this process does create r vertices among $\{v_{k+1}, \ldots, v_n\}$ that have good neighborhoods. We begin with several claims that develop useful properties of the Laying-off Algorithm, in particular the key observation that $\ell_p \geq \ell_{p-1}$ for all $p \leq rk$. Then, we show that at each iteration of the algorithm, the sequence T_p has a certain structure that allows us to easily count the number of iterations needed to find r good neighborhoods.

Claim 1. *If $p \leq r(k+1)$ and $i < j$, then $\hat{d}_p(v_j) \leq \hat{d}_p(v_i) + 1$.*

Proof of Claim 1. If $\hat{d}_p(v_j) \geq \hat{d}_p(v_i) + 2$ then, since $d_0(v_i) \geq d_0(v_j)$, there exists a p' such that $\hat{d}_{p'-1}(v_j) = \hat{d}_{p'-1}(v_i)$, $v_i \in N_{p'}$ and $v_j \notin N_{p'}$, and there also exists a p'' such that $\hat{d}_{p''-1}(v_j) = \hat{d}_{p''-1}(v_i) + 1$, $v_i \in N_{p''}$ and $v_j \notin N_{p''}$. But such a p'' cannot exist because if $\hat{d}_{p''-1}(v_j) > \hat{d}_{p''-1}(v_i)$, then $v_i \in N_{p''}$ implies v_j is also in $N_{p''}$. This contradiction proves Claim 1. □

Claim 2. *If $p \leq r(k+1)$ and $j \leq k-r$, then $\hat{d}_{p-1}(v_j) \geq k$. In addition, if $S_{p-1} \cap N_p \neq \emptyset$, then N_p is good.*

Proof of Claim 2. If v_j is not laid off, then d_j decreases by at most 1 at each step and so $\hat{d}_{p-1}(v_j) \geq d_j - (p-1)$. Because $d_{k-r} \geq d_k + r(k+2)$, we have the following:

$$\hat{d}_{p-1}(v_j) \geq d_j - (p-1) \geq d_{k-r} - (p-1) \geq d_k + r(k+2) - (p-1) \geq d_k + r.$$

The conditions on the sequence force $d_k \geq k - r$, giving

$$\hat{d}_{p-1}(v_j) \geq d_k + r \geq k.$$

As a result, if $S_{p-1} \cap N_p \neq \emptyset$, then N_p must contain every vertex with remaining degree greater than $\hat{d}_p(v_{a_p}) \leq k-1$. This includes all of $\{v_1, \ldots, v_{k-r}\}$ and so N_p must be good. □

Let g_p denote the number of good neighborhoods $N_{p'}$ with $p' \leq p$. We may assume that $g_p \leq r-1$ for all $p \leq r(k+1)$. Otherwise, we would have r good neighborhoods, hence our copy of $K_{k-r,r}$. In particular, by Claim 2 we can assume that there are at most $r-1$ values of p for which $S_{p-1} \cap N_p \neq \emptyset$.

Claim 3. *If $p \leq r(k+1)$, then $|S_{p-1} \cap N_p| \leq r-1$ and $|S_{p-1}| > r(k+r+1) - g_{p-1}(r-1) - (p-1) \geq 2r$. In addition, every $v \in N_p$ has $\hat{d}_{p-1}(v)$ at least as large as the least value of $\hat{d}_{p-1}$ among members of S_{p-1}.*

Proof of Claim 3. Consider the vertex v_{a_p}. It has degree at most $k-1$ when it is laid off. By Claim 2, there are at least $k-r$ vertices v_j with $\hat{d}_{p-1}(v_j) \geq k$ and so $|S_{p-1} \cap N_p| \leq (k-1) - (k-r) = r-1$. Because a vertex will only leave the set S_p if it has been laid off or assigned to the neighborhoods of enough other vertices that its remaining degree is too low,

$$\begin{aligned}|S_{p-1}| &\geq r(k+r+1) - \left|\bigcup_{j=1}^{p-1}\{S_{j-1}\cap N_j\}\right| - (p-1)\\ &\geq r(k+r+1) - g_{p-1}(r-1) - (p-1).\end{aligned}$$

Since $g_{p-1} \leq r-1$, we have $|S_{p-1}| \geq r(k+r+1)-(r-1)^2-(p-1) \geq 2r$. If we include the vertices $\{v_1, \ldots, v_{k-r}\}$, there are a total of at least k vertices w for which $\hat{d}_{p-1}(w)$ is at least the minimum value of $\hat{d}_{p-1}$ among the members of S_{p-1}. This proves Claim 3. □

Claim 4. *If $\ell_p < \ell_{p-1}$ for some $p \leq rk$, then at most r more iterations of the Laying-off Algorithm will create the desired copy of $K_{k-r,r}$.*

Proof of Claim 4. By definition, $\ell_{p-1} = \hat{d}_{p-1}(v_{a_p})$ and $\ell_p = \hat{d}_p(v_{a_{p+1}})$. Since v_{a_p} was chosen to minimize $\hat{d}_{p-1}$ among S_{p-1}, we know that

$$\hat{d}_{p-1}(v_{a_{p+1}}) \geq \ell_{p-1} \geq k-r.$$

Since $\hat{d}_p(v_{a_{p+1}}) \geq \hat{d}_{p-1}(v_{a_{p+1}})-1$, we get $\ell_p = \hat{d}_p(v_{a_{p+1}}) \geq \ell_{p-1}-1$. Thus, $\ell_p = \ell_{p-1} - 1$. This means that $\hat{d}_{p-1}(v_{a_{p+1}}) = \ell_{p-1}$ and $v_{a_{p+1}} \in N_p$. Since $v_{a_{p+1}} \in S_{p-1}$ as well, Claim 2 gives that N_p is good.

Further, as ℓ_{p-1} is also the minimum remaining degree of any vertex in S_{p-1}, Claim 3 gives that every vertex w in N_p has $\hat{d}_{p-1}(w) \geq \ell_{p-1}$. Since $v_{a_{p+1}} \in N_p$, we conclude that $\hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1})) = \ell_{p-1}$. Since $\hat{d}_{p-1}(v_{k-r}) > \hat{d}_{p-1}(v_{a_{p+1}})$, there are at most $\ell_{p-1}-1$ vertices w with $\hat{d}_{p-1}(w) \geq \hat{d}_{p-1}(v_{k-r})$.

So, if we can show that $\ell_{p'} \geq \ell_{p-1} - 1$ for all p' such that $p \leq p' \leq p+(r-g_p)$, then each $N_{p'}$ is good and we have the desired $K_{k-r,r}$ in at most r more steps. From Claim 3, there are at most $r-2$ vertices in $S_{p-1} \cap N_p$ that have remaining degree larger than $\hat{d}_{p-1}(v_{a_{p+1}}) = \ell_{p-1}$. Also from Claim 3, $|S_{p-1}| > r(k+r+1) - g_{p-1}(r-1) - (p-1)$. Thus, there are at least

$$r(k+r+1) - g_{p-1}(r-1) - (p-1) - (r-2) > r(r-g_{p-1})$$

vertices of remaining degree equal to ℓ_{p-1} in S_{p-1}. Since Claim 3 gives that $|S_{p'-1} \cap N_{p'}| \leq r-1$ for all $p' \geq p$, each of the next $r-g_p$ iterations of the Laying-off Algorithm will remove at most r vertices from S_{p-1}, and each of those vertices has remaining degree equal to ℓ_{p-1}.

Hence there is always a vertex in S_{p-1} with degree equal to ℓ_{p-1}. Thus, no vertex with degree $\ell_{p-1} - 1$ will be placed into $N_{p'}$, and Claim 4 is proved. □

We can thus assume that $\ell_p \geq \ell_{p-1}$ for all $p \leq rk$.

Now we are prepared to examine the structure of the sequence T_p. Claim 5 below is the main observation, that even when the Laying-off Algorithm results in an inconsistent sequence, the sequence that results is of a very specific form. Thus, the Laying-off Algorithm ensures that the number of iterations between consistent sequences is less than k.

To show this, we say the sequence T_p is of *proper form* if there is a partition of V_p into four ordered sets $\tau_p^{(1)}$, $\tau_p^{(2)}$, $\tau_p^{(3)}$ and $\tau_p^{(4)}$ (where the order is inherited from τ_p) such that $\hat{d}_p$ is constant on each of $\tau_p^{(2)}$ and $\tau_p^{(3)}$ and, when $i < j$ and $v_i, v_j \in V_p$, v_j precedes v_i if and only if $v_j \in \tau_p^{(2)}$ and $v_i \in \tau_p^{(3)}$. By Claim 1, we know that in this case $\hat{d}_p(v_j) \leq \hat{d}_p(v_i) + 1$. Note that this allows for $\tau_p^{(2)}$ and $\tau_p^{(3)}$ to be empty, in which case T_p is consistent.

We will abuse notation to let $\cup$ represent the "concatenation" of ordered sets; that is, $\tau_p^{(i)} \cup \tau_p^{(j)}$ is also an ordered set, where the elements of $\tau_p^{(i)}$ precede those of $\tau_p^{(j)}$, and within each set the original order is maintained. Thus, if T_p is of proper form, both $\tau_p^{(1)} \cup \tau_p^{(2)}$ and $\tau_p^{(3)} \cup \tau_p^{(4)}$ are consistent. For a sequence T_p that is of proper form, the *inconsistency* of T_p is $\left|\tau_p^{(2)} \cup \tau_p^{(3)}\right|$. A consistent sequence has inconsistency zero.

Claim 5. *For all $p \in \{0, \ldots, rk - 1\}$, T_p is of proper form. If T_{p-1} is consistent or has inconsistency at least k, then N_p is good. If T_p is inconsistent, then $|\tau_p^{(1)} \cup \tau_p^{(3)}| \leq \ell_{p-1}$. If T_{p-1} has positive inconsistency, then either*

- *T_p is consistent (and N_p is good),*
- *$T_p = \hat{T}_{p-1}$ and N_p is good, or*
- *T_p has inconsistency strictly less than the inconsistency of T_{p-1}.*

Proof of Claim 5. We will prove the claim by induction on p.

If $p = 0$, then $T_p = T_0$ is consistent. Moreover N_{p+1} is good because it is simply the first ℓ_p entries of $\hat{T}_p$, which must contain $v_1, \ldots, v_{k-r}$. In fact, this is true for any consistent T_p and this will be our base case for the induction.

We assume the statement of the claim is true for $T_0, \ldots, T_{p-1}$.

Case 1: T_{p-1} is consistent.

The set N_p is good because it is simply the first ℓ_{p-1} entries of $\hat{T}_{p-1}$, which must contain $v_1, \ldots, v_{k-r}$. If T_p is consistent, then it is, by definition, of proper form.

If T_p is not consistent, then $\hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1})) = \hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1}+1))$. We can partition $\hat{T}_{p-1}$ into $\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(4)L} \cup \hat{\tau}_{p-1}^{(4)R}$, where $\hat{\tau}_{p-1}^{(4)L}$ contains all vertices with remaining degree exactly $\hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1}))$ and $\hat{\tau}_{p-1}^{(4)R}$ contains those with lower remaining degree. We can further partition $\hat{\tau}_{p-1}^{(4)L}$ into $\hat{\tau}_{p-1}^{(4)L_1}$ and $\hat{\tau}_{p-1}^{(4)L_2}$, where $\hat{\tau}_{p-1}^{(4)L_1}$ contains all vertices of $\hat{\tau}_{p-1}^{(4)L}$ that are included in N_p, and $\hat{\tau}_{p-1}^{(4)L_2}$ consists of those that are not.

Now T_p can be partitioned into

$$\begin{aligned}\tau_p^{(1)} &= \hat{\tau}_{p-1}^{(1)}\\ \tau_p^{(2)} &= \hat{\tau}_{p-1}^{(4)L_2}\\ \tau_p^{(3)} &= \hat{\tau}_{p-1}^{(4)L_1}, \text{ and}\\ \tau_p^{(4)} &= \hat{\tau}_{p-1}^{(4)R},\end{aligned}$$

and it is of proper form. Clearly $|\tau_p^{(1)} \cup \tau_p^{(3)}| = \ell_{p-1}$.

Observe that if T_{p-1} is consistent and T_p is not, then $\{v_1, \ldots, v_{k-r}\}$ is contained in $\tau_p^{(1)} \cup \tau_p^{(2)} \cup \tau_p^{(3)}$.

Case 2: T_{p-1} is not consistent.

Recall that $\ell_{p-1} = |N_p|$, the number of vertices in V_p that are reduced by one when a vertex of T_{p-1} is laid off. The effect of the Laying-off Algorithm on T_p depends on the value of ℓ_{p-1}.

Note that $\ell_{p-1} \leq |\hat{\tau}_{p-1}^{(1)}|$ is not possible because Claim 4 allows us to assume that $\ell_{p-1} \geq \ell_{p-2}$. Since $\ell_{p-2} \geq |\tau_{p-1}^{(1)} \cup \tau_{p-1}^{(3)}| > |\hat{\tau}_{p-1}^{(1)}|$, this is a contradiction.

With this information, we can show that if the inconsistency of T_{p-1} is at least k, then N_p is good. The largest k entries of T_{p-1} are in $\tau_{p-1}^{(1)} \cup \tau_{p-1}^{(2)} \cup \tau_{p-1}^{(3)}$ and N_p contains all of $\hat{\tau}_{p-1}^{(1)}$. Because there are $r(k+r+1)$ vertices eligible to be laid off from $\{v_{k+1}, \ldots, v_n\}$, and we've laid off at most rk, the vertices $v_{k-r}, \ldots, v_k$ will not be laid off. If v_k is in $\hat{\tau}_{p-1}^{(3)}$, then its value is at most $d_k - 1$, and if v_k is in $\hat{\tau}_{p-1}^{(2)}$, then its value is at most d_k. But the degree of each of $v_1, \ldots, v_{k-r}$ is at least $d_{k-r} - kr \geq d_k + r(k+2) - kr > d_k$. So, each of $v_1, \ldots, v_{k-r}$ are in $\hat{\tau}_{p-1}^{(1)}$ and will be in N_p as long as the inconsistency is at least k.

Case 2a: $|\hat{\tau}_{p-1}^{(1)}| < \ell_{p-1} < |\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(2)}|$.

In this case, we can partition $\hat{\tau}_{p-1}^{(2)}$ into two pieces: $\hat{\tau}_{p-1}^{(2)L}$ and $\hat{\tau}_{p-1}^{(2)R}$. The members of $\hat{\tau}_{p-1}^{(2)R}$ are reduced when v_{a_p} is laid off, but those of $\hat{\tau}_{p-1}^{(2)L}$ are not.

After reordering, we obtain the following

$$\begin{aligned}\tau_p^{(1)} =&\hat{\tau}_{p-1}^{(1)},\\ \tau_p^{(2)} =&\hat{\tau}_{p-1}^{(2)L},\\ \tau_p^{(3)} =&\hat{\tau}_{p-1}^{(3)},\\ \tau_p^{(4)} =&\hat{\tau}_{p-1}^{(2)R} \cup \hat{\tau}_{p-1}^{(4)}.\end{aligned}$$

Moreover, $\ell_{p-1} \geq \ell_{p-2} \geq |\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(3)}| = |\hat{\tau}_p^{(1)} \cup \hat{\tau}_p^{(3)}|$. In addition, the inconsistency of T_p is $|\tau_p^{(2)} \cup \tau_p^{(3)}| = |\hat{\tau}_{p-1}^{(2)L} \cup \hat{\tau}_{p-1}^{(3)}|$, which is strictly less than the inconsistency of T_{p-1} because $\hat{\tau}_{p-1}^{(2)L}$ is a strict subset of $\hat{\tau}_{p-1}^{(2)}$. This finishes the proof of Case 2a.

To proceed through the next cases, we must partition $\hat{\tau}_{p-1}^{(4)}$ into two pieces: $\hat{\tau}_{p-1}^{(4)L}$ and $\hat{\tau}_{p-1}^{(4)R}$. The members of $\hat{\tau}_{p-1}^{(4)L}$ have the same remaining degree as those in $\hat{\tau}_{p-1}^{(3)}$, and those of $\hat{\tau}_{p-1}^{(4)R}$ have smaller remaining degree (either or both of these may be empty).

Case 2b: $|\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(2)}| \leq \ell_{p-1} \leq |\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(2)}| + |\hat{\tau}_{p-1}^{(4)L}|$.

In this case, the values of $\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(2)}$ as well as some of $\hat{\tau}_{p-1}^{(4)L}$ are reduced. Since the members of $\hat{\tau}_{p-1}^{(2)} \cup \hat{\tau}_{p-1}^{(3)}$ (and the unreduced values of $\hat{\tau}_{p-1}^{(4)L}$) now have the same value, reordering results in T_p being a consistent sequence.

Case 2c: $|\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(2)}| + |\hat{\tau}_{p-1}^{(4)L}| < \ell_{p-1} < |\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(2)} \cup \hat{\tau}_{p-1}^{(3)} \cup \hat{\tau}_{p-1}^{(4)L}|$.

In this case, we can partition $\hat{\tau}_{p-1}^{(3)}$ into two pieces: $\hat{\tau}_{p-1}^{(3)L}$ and $\hat{\tau}_{p-1}^{(3)R}$. The members of $\hat{\tau}_{p-1}^{(3)R}$ are reduced but those of $\hat{\tau}_{p-1}^{(3)L}$ are not.

After reordering, we obtain the following

$$\begin{aligned}\tau_p^{(1)} =&\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(3)L},\\ \tau_p^{(2)} =&\hat{\tau}_{p-1}^{(2)},\\ \tau_p^{(3)} =&\hat{\tau}_{p-1}^{(3)R},\\ \tau_p^{(4)} =&\hat{\tau}_{p-1}^{(4)}.\end{aligned}$$

Moreover, $\ell_{p-1} \geq \ell_{p-2} \geq |\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(3)}| = |\hat{\tau}_p^{(1)} \cup \hat{\tau}_p^{(3)}|$. In addition, the inconsistency of T_p is $|\tau_p^{(2)} \cup \tau_p^{(3)}| = |\hat{\tau}_{p-1}^{(2)} \cup \hat{\tau}_{p-1}^{(3)R}|$, which is strictly less than the inconsistency of T_{p-1} because $\hat{\tau}_{p-1}^{(3)R}$ is a strict subset of $\hat{\tau}_{p-1}^{(3)}$.

Case 2d: $|\hat{\tau}^{(1)}_{p-1} \cup \hat{\tau}^{(2)}_{p-1} \cup \hat{\tau}^{(3)}_{p-1} \cup \hat{\tau}^{(4)L}_{p-1}| \leq \ell_{p-1}$.

In this case, no rearranging is necessary: the order of the vertices in T_p is the same as the order in $\hat{T}_{p-1}$.

Because the only vertices out of order are in $\hat{\tau}^{(2)}_{p-1} \cup \hat{\tau}^{(3)}_{p-1}$, N_p will contain all of the first ℓ_{p-1} vertices. Since $\ell_{p-1} \geq k - r$, the neighborhood N_p must contain $\{v_1, \dots, v_{k-r}\}$ and thus be good.

This concludes the proof of Claim 5. □

Given Claim 5, the proof of Lemma 8 follows easily. There can be at most $k-1$ neighborhoods that are not good between consecutive good neighborhoods. So after $(r-1)k+1$ iterations of the procedure, there will be r good neighborhoods. □

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