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Revisiting the Hamiltonian theme in the square of a block: the general case

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This is the second part of joint research in which we show that every 2-connected graph G has the $\mathcal{F}_4$ property. That is, given distinct $x_i \in V(G)$, $1 \leq i \leq 4$, there is an x_1x_2-hamiltonian path in G^2 containing different edges $x_3y_3, x_4y_4 \in E(G)$ for some $y_3, y_4 \in V(G)$. However, it was shown already in [3, Theorem 2] that 2-connected DT-graphs have the $\mathcal{F}_4$ property; based on this result we generalize it to arbitrary 2-connected graphs. We also show that these results are best possible.

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1. Introduction

This is the second part of joint research in which we establish the most general result for the square of a block (i.e., a 2-connected graph) to be hamiltonian connected. In the first part this was achieved in [3, Theorem 2] for the case of DT-graphs (i.e., graphs in which every edge is incident to a vertex of degree two). In the past, the approach to deal with 2-connected DT-graphs first and then generalize the corresponding results to blocks in

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general, was a logical consequence of the proof methods developed in [6]–[9], say. However, since the 1990's shorter proofs of what has become known as Fleischner's Theorem, were developed first by Říha in [16] and later by Georgakopoulos in [11]. A short proof of an even stronger version of that theorem was proved by Müttel and Rautenbach in [13]. Unfortunately, the methods developed for these shorter proofs do not seem to suffice to prove the main result of this paper (Theorem 4). This is why we had to resort to the concept of EPS-graphs (see, e.g., [6]).

All concepts not defined in this paper, can be found in the cited literature; in cases where contradictions regarding terminology may arise, we prefer the definitions as given in the papers by Fleischner. We also included some additional references to give the interested reader a better insight regarding past developments of the topic. However, to make it easier to read this paper we repeat some definitions. In particular, by a *uv-path* we mean a path from u to v. If a uv-path is hamiltonian, we call it a *uv-hamiltonian path*. Also, we understand an *eulerian* graph to be a not necessarily connected graph all of whose vertices have even degree. Moreover, we let $\delta u = u$ if $d(u) = 1$, and $\delta u = \emptyset$, otherwise.

Next, we repeat some results quoted or proved in [3], using the same numbering as in [3]. Theorems proved in the 1970's and quoted already in [3] are numbered by upper-case letters using the same letters as in [3].

Definition 1. *Let G be a graph and let $A = \{x_1, x_2, \ldots, x_k\}$ be a set of k (≥ 3) distinct vertices in G. An x_1x_2-hamiltonian path in G^2 which contains $k-2$ distinct edges $x_iy_i \in E(G)$, $i = 3, \ldots, k$ is said to be $\mathcal{F}_k$. Hence we speak of an $\mathcal{F}_k$ x_1x_2-hamiltonian path in G^2. If x_i is adjacent to x_j, we insist that x_iy_i and x_jy_j are distinct edges. A graph G is said to have the $\mathcal{F}_k$ property if for any set $A = \{x_1, x_2 \ldots, x_k\} \subseteq V(G)$, there is an $\mathcal{F}_k$ x_1x_2-hamiltonian path in G^2.*

By an *EPS-graph*, *JEPS-graph* of G, denoted $S = E \cup P$, $S = J \cup E \cup P$ respectively, we mean a spanning connected subgraph S of G which is the edge-disjoint union of an eulerian graph E (which may be disconnected) and a linear forest P, respectively a linear forest P together with an open trail J.

Lemma 1. *([3, Lemma 1]) Suppose G is a block chain with a cutvertex, v and w are vertices in different endblocks of G and are not cutvertices. Then*

(i) there exists an EPS-graph $E \cup P \subseteq G$ such that $d_P(v),\ d_P(w) \leq 1$. If the endblock which contains v is 2-connected, then we have $d_P(v) = 0$ and $d_P(w) \leq 1$; and

(ii) there exists a $JEPS$-graph $J \cup E \cup P \subseteq G$ such that $d_P(v) = 0 = d_P(w)$. Moreover, v, w are the only odd vertices of J. Also, we have $d_P(c) = 2$ for at most one cutvertex c of G (and hence $d_P(c') \leq 1$ for all other cutvertices c' of G).

Theorem A. *[3, Theorem 1]) Suppose G is a 2-connected graph and v, w are two distinct vertices in G. Then either*

(i) there exists an EPS-graph $S = E \cup P \subseteq G$ with $d_P(v) = 0 = d_P(w)$; or

(ii) there exists a $JEPS$-graph $S = J \cup E \cup P \subseteq G$ with v, w being the only odd vertices of J, and $d_P(v) = 0 = d_P(w)$.

By a $[v; w_1, \ldots, w_n]$-EPS-graph of G, we mean an EPS-graph $S = E \cup P$ of G such that $d_P(v) = 0$ and $d_P(w_i) \leq 1$ for every $i = 1, \ldots, n$.

Theorem B. *([9, Theorem 3]) Let G be a 2-connected graph and let v, w_1, w_2, w_3 be distinct vertices of G. Suppose K is a cycle in G such that $\{v, w_1, w_2, w_3\} \subseteq K$. Then G has a $[v; w_1, w_2, w_3]$-EPS-graph $S = E \cup P$ such that $K \subseteq E$.*

Suppose G is a 2-connected graph and v, w_1, w_2 are distinct vertices in G. A cycle K in G is a $[v; w_1, w_2]$-*maximal cycle* in G if $\{v, w_1\} \subseteq V(K)$, and $w_2 \in V(K)$ unless G has no cycle containing all of $\{v, w_1, w_2\}$.

Theorem C. *([9, Theorem 2]) Let G be a 2-connected graph and let v, w_1, w_2 be three distinct vertices of G. Suppose K is a $[v; w_1, w_2]$-maximal cycle in G. Then G has a $[v; w_1, w_2]$-EPS graph $S = E \cup P$ such that $K \subseteq E$.*

Theorem D. *([6, Theorem 2]) Let G be a 2-connected graph and let v, w be two distinct vertices of G. Let K be a cycle through v, w. Then G has a $[v; w]$-EPS-graph $S = E \cup P$ with $K \subseteq E$.*

Theorem E. *([8, Theorem 3]). Suppose v and w are two arbitrarily chosen vertices of a 2-connected graph G. Then G^2 contains a hamiltonian cycle C such that the edges of C incident to v are in G and at least one of the edges of C incident to w is in G. Further, if v and w are adjacent in G, then these are three different edges.*

A hamiltonian cycle in G^2 satisfying the conclusion of Theorem E is also called a $[v; w]$-hamiltonian cycle. More generally, a hamiltonian cycle C in G^2 which contains two edges of G incident to v, and at least one edge G incident to each w_i, $i = 1, \ldots, k$, is called a $[v; w_1, \ldots, w_k]$-hamiltonian cycle, provided the edges in question are all different.

Theorem F. *([8, Theorem 4]). Let G be a 2-connected graph. Then the following hold.*

(i) G has the $\mathcal{F}_3$ property.

(ii) For a given $q \in \{x, y\}$, G^2 has an xy-hamiltonian path containing an edge of G incident to q.

By applying Theorems E and F to each block of a block chain B, we have the following.

Corollary 1. *Suppose B is a non-trivial block chain with $|V(B)| \geq 3$ and v and w are vertices in different endblocks of G. Assume further that v, w are not cutvertices of B. Then*

(i) B^2 has a hamiltonian cycle which contains an edge of B incident to v and an edge of B incident to w. In the case that the endblock which contains v is 2-connected, then B^2 has a hamiltonian cycle which contains two edges of B incident to v and an edge of B incident to w. Also,

(ii) B^2 has a vw-hamiltonian path containing an edge of B incident to v and an edge of B incident to w.

Recall that a graph is called a *DT-graph* if every edge is incident to a 2-valent vertex. If G is a graph, we let $V_2(G)$ denote the set of all vertices of degree 2 in G.

The main result of [3] is the following result which is the larger part of the proof of Theorem 4 below.

Theorem 1. *Every 2-connected DT-graph has the $\mathcal{F}_4$ property.*

In proving Theorem 1 we made use of the following Lemma which plays a role also in this paper.

Lemma 2. *Let G be a 2-connected DT-graph and let $G^+ = G \cup \{x_1y, x_2y, y\}$, $y \notin V(G)$ (see [3]), with $N(x_3) \nsubseteq V_2(G)$ and $N(x_4) \nsubseteq V_2(G)$. Suppose $N(x_i) \subseteq V_2(G)$ for some $i \in \{1, 2\}$. Assume further that every proper 2-connected subgraph of G has the $\mathcal{F}_4$ property. Then $(G^+)^2$ has a hamiltonian cycle containing the edges $x_1y, x_2y, x_3z_3, x_4z_4$ where x_3z_3, x_4z_4 are different edges of G.*

Note that in the ensuing discussion and proofs we make use of the fact that in *DT*-graphs G, the existence of an EPS-graph of G yields a hamiltonian cycle of G^2. In order to keep the paper as short as possible the reader is referred to the constructions expounded in [6].

However, before dealing with the main result, Theorem 4 in section 3, we need to prove several preliminary results.

2. Beyond $\mathcal{F}_3$

We now proceed to prove some results needed to shorten the proof of Theorem 4.

Lemma 3. *Let G be a 2-connected DT-graph with at least four vertices, and let v, w_1, w_2 be three distinct vertices in G with $N(v) \subseteq V_2(G)$ and $N(w_1) \subseteq V_2(G)$. Then G^2 has a $[v; w_1, w_2]$-hamiltonian cycle.*

Proof: Since G is a 2-connected graph, G has a cycle K containing v, w_1. Suppose K has been chosen such that it is $[v; w_1, w_2]$-maximal. Then G has a $[v; w_1, w_2]$-EPS-graph $S = E \cup P$ with $K \subseteq E$ by Theorem C.

If $N(w_2) \subseteq V_2(G)$, then it is straightforward to see that S^2 yields a hamiltonian cycle having the required properties. Note that the case $N(w_2) = \{v, w_1\}$ yields $vw_1 \notin E(G)$ in this case (since $|V(G)| \geq 4$) and K contains a path $v_0vw_2w_1w_0$ or $G = K = C_4$ ($v_0 \in N(v), w_0 \in N(w_1)$) all of whose vertices are 2-valent in G and thus the four edges of that path are contained in some hamiltonian cycle of S^2. Hence $N(w_2) \nsubseteq V_2(G)$ and w_2 is a 2-valent vertex.

Depending on the position of w_2 vis-a-vis v and w_1 we now consider the following cases.

Case (A) $N(w_2) = \{v, w_1\}$. It is easy to see that $vw_1 \notin E(G)$.
Next we need to consider two cases separately.

(1) $G - w_2$ is 2-connected. We apply Theorem A and correspondingly consider the following cases.

First we assume that $G - w_2$ has an EPS-graph $S = E \cup P$ with $d_P(v) = d_P(w_1) = 0$. By the construction according to the method developed in [6] we have in $(G - w_2)^2$ a hamiltonian cycle H whose edges in v and in w_1 are in $G - w_2$. Now it is trivial to expand H to a hamiltonian cycle in G^2 as required.

On the other hand, if $G - w_2$ has a $JEPS$-graph $S = J \cup E \cup P$ with v, w_1 being the only odd vertices of J and $d_P(v) = d_P(w_1) = 0$, then $(G - w_2)^2$ has a hamiltonian path $P(v, w_1)$ starting in v with an edge of G and ending in w_1 with an edge of G, then $P(v, w_1) \cup \{w_1w_2, w_2v\}$ defines a hamiltonian cycle of G^2 as claimed by the lemma.

(2) $G - w_2$ is not 2-connected; hence it is a block chain with v and w_1 belonging to different endblocks of $G - w_2$, and they are not cutvertices of $G - w_2$. By Corollary 1*(ii)*, $(G - w_2)^2$ has a hamiltonian path $P(v, w_1)$

starting in v with an edge of G and ending in w_1 with and edge of G. Thus $P(v_1 w_1) \cup \{w_1 w_2, w_2 v\}$ defines hamiltonian cycle of G^2 as claimed by the lemma and thus finishes Case (A).

Because of the cases already treated it follows that there is $t \in V(G)$ satisfying

Case (B) $t \in N(w_2) - V_2(G)$. We assume additionally $|N(w_2) \cap \{v, w_1\}| = 1$.

(i) $t \in \{v, w_1\}$. Let $t' = N(w_2) - t$. Hence $vw_1 \notin E(G)$ and $t' \notin \{v, w_1\}$. Moreover, $w_2 \in V(K)$ and $t' \in V_2(G)$; otherwise we could treat t' like t in (ii) below. In this case we can write

$$K = v, \ldots, t', w_2, w_1, w_1', \ldots, v'', v \qquad \text{if } t = w_1$$

or

$$K = v, w_2, t', \ldots, w_1, w_1', \ldots, v'', v \quad \text{if } t = v.$$

In any case, a $[v; w_1, w_2]$-EPS-graph $S = E \cup E$ with $K \subseteq E$ exists by Theorem C and yields in S^2 a hamiltonian cycle of G^2 as required.

(ii) $t \notin \{v, w_1\}$. Since $\{v, w_1, w_2\} \subset V(K)$ we also have $t \in V(K)$, and by Theorem C, a $[v; t, w_1]$-EPS-graph $S = E \cup P$ with $K \subseteq E$ exists. Also in this case, S^2 has a hamiltonian cycle as claimed by the lemma (in particular, it contains tw_2).

We are thus led to the following case.

Case (C) $t \in N(w_2) - V_2(G)$ and $N(w_2) \cap \{v, w_1\} = \emptyset$.

Further we assume that w_2 is not contained in the cycle K; otherwise, for t as above, K contains v, w_1, w_2, t, and G has a $[v; w_1, w_2, t]$-EPS-graph $S = E \cup P$ with $K \subseteq E$ by Theorem B. Again, S^2 yields a hamiltonian cycle with the required properties.

Partition K into two vw_1-paths, $K = P_1(v, w_1) \cup P_2(v, w_1)$. Since G is 2-connected, there exists a $w_2 u_1$-path $P(w_2, u_1)$ and a $w_2 u_2$-path $P(w_2, u_2)$ in G which are internally disjoint, with $u_1, u_2 \in V(K)$ and such that $(P(w_2, u_i) - u_i) \cap K = \emptyset$, $i = 1, 2$.

Suppose $u_1, u_2 \in P_j(v, w_1)$ for some $j \in \{1, 2\}$. Then there is a cycle K^* in G containing the vertices v, w_1, w_2, t which contradicts the choice of K.

Hence we assume that $u_i \in P_i(v, w_1)$, $i = 1, 2$; it is an internal vertex of $P_i(v, w_1)$.

Now consider

$$\min_{K \supset \{v,w_1\}} \min_{u_1,u_2 \in V(K)} \{ \, l(P(w_2,u_1)) + l(P(w_2,u_2)) \, \} \, ;$$

fix a cycle K and $u_1, u_2 \in V(K)$ together with $P(w_2,u_1), P(w_2,u_2)$ which satisfy this minimality condition.

Set $P(w_2) = P(w_2,u_1) \cup P(w_2,u_2)$ and let $G_2 \subset G$ be induced by $V(P(w_2))$ and by all vertices y lying on a path P_y with endvertices $v_y, w_y \in V(P(w_2))$ such that $\{v_y, w_y\} \neq \{u_1, u_2\}$ and satisfying $(V(P_y) - \{v_y, w_y\}) \cap V(K) = \emptyset$. G_2 is uniquely determined and it is a (trivial or non-trivial) block chain with u_1, u_2 belonging to endblocks of G_2; they are not cutvertices of G_2.

Likewise, define G_K as induced by all vertices z lying on a path P_z with endvertices $v_z, w_z \in V(K)$ and satisfying $(V(P_z) - \{v_z, w_z\}) \cap V(G_2) = \emptyset$. G_K is 2-connected because of $K \subset G_K$.

Observe, that the minimality condition guarantees that there is no path $P(x,y)$ with $x \in V(G_K) - \{u_1,u_2\}$ and $y \in V(G_2) - \{u_1,u_2\}$. Now it is straightforward to see that $G = G_K \cup G_2$, $G_K \cap G_2 = \{u_1,u_2\}$ because of the minimality condition.

Note that the above arguments apply to arbitrary 2-connected graphs. In what follows we restrict ourselves to DT-graphs.

Also, from the choice of K it follows that $\{u_1,u_2\} \cap \{v,w_1\} = \emptyset$. However $K \supset \{v, w_1, u_1, u_2\}$ which is a set of four distinct vertices on K. Hence G_K has a $[v; w_1, u_1, u_2]$-EPS graph $S_K = E_K \cup P_K$ with $K \subseteq E_K$ because of Theorem B.

Now consider the graph G_2.

(a) Suppose w_2 is incident with a bridge of G_2. Then G_2 has an EPS-graph
$S_2 = E_2 \cup P_2$ with $w_2 \notin E_2$, $d_{P_2}(w_2) = 2$ and $d_{P_2}(u_i) \leq 1$, $i = 1,2$ by Lemma 1 (i). It follows that for $E = E_K \cup E_2$ and $P = P_K \cup (P_2 - w_2 t)$, $t \in N(w_2)$, $S = E \cup P$ is an EPS-graph of G with $K \subseteq E$, $d_P(w_2) = 1$ and w_2 is a pendant vertex in S, $d_P(v) = 0$, $d_P(w_1) \leq 1$, and $d_P(u_i) \leq 2$, $i = 1,2$. It now follows that S^2 yields a hamiltonian cycle C in G^2 as required: its edges incident to v are edges of G, and at least one edge of C incident to w_i is in G, $i = 1,2$.

(b) Suppose w_2 lies in a cycle of G_2, i.e., w_2 lies in a 2-connected block $B(w_2)$ of G_2. Let $z_i \in V(B(w_2))$ be such that $z_i = u_i$ if $u_i \in V(B(w_2))$, $i = 1, 2$; otherwise, let z_i be a cutvertex of G_2.

If G_2 is a non-trivial block chain we apply Corollary 1*(i)* to obtain a hamiltonian cycle C_2 of G_2^2. C_2 contains $u_1y_1, u_1v_1, u_2v_2 \in E(C_2) \cap E(G_2)$ provided the endblock $B(u_1)$ containing u_1 is 2-connected; and $\{y_1, v_1\} \subseteq N(u_1)$, $v_2 \in N(u_2)$. However, if $B(u_1)$ is a bridge u_1y_1 then $u_1y_1 \in E(C_2)$, and $u_1v_1 \in E(C_2) - E(G)$. Moreover, in constructing C_2 (which results from applying Theorem E to the 2-connected blocks of G_2) we may apply Lemma 3 by induction to the block $B(w_2)$ containing also $s \in N(w_2)$, to obtain $w_2s \in E(C_2)$ as well.

If however, G_2 is 2-connected, we apply induction to G_2 to obtain a hamiltonian cycle C_2 of G_2^2 where edges incident to u_1 are in G_2 and so is sw_2 and an edge incident to u_2.

To obtain H_2 missing u_1, we make a 'shortcut' by replacing u_1y_1, u_1v_1 with y_1v_1.

Now, S_K yields a hamiltonian cycle $H_K \subseteq (G_K)^2$ with its two edges in v belonging to G_K and in each of w_1, u_1, u_2, H_K traverses at least one edge of G_K (note that $N(w_1) \cup N(u_1) \cup N(u_2) \subset V_2(G)$). Likewise, H_2 contains an edge of G_2 incident with w_2, and one edge of G_2 incident with u_2. Denote $u_2v_K \in H_K \cap G_K$, $u_2v_2 \in H_2 \cap G_2$. Then $H = (H_K - u_2v_K) \cup (H_2 - u_2v_2) \cup \{v_Kv_2\}$ is a hamiltonian cycle C in G^2 as required. □

By an *edge-critical block*, we mean a block which fails to be a block when any edge is deleted from it.

Let G be a graph and let $D(G) = \{uv \in E(G) \mid d(u) > 2, d(v) > 2\}$. Note that G is a DT-graph if and only if $D(G) = \emptyset$.

Theorem G. *([8, Theorem 1]) Suppose G is an edge-critical block which is not a DT-graph. Let x, y be any two distinct vertices in G. Then $D(G)$ contains an edge e such that $G - e$ has a DT-endblock B such that $\{x, y\} \not\subseteq V(B)$, and if $x \in V(B)$, then x is a cutvertex of $G - e$.*

We shall now prove a stronger version of Theorem F*(ii)*.

Theorem 2. *Let G be a 2-connected graph and let x, y be two vertices in G. Then G^2 has an xy-hamiltonian path $P(x, y)$ such that*

(i) $xz \in E(G) \cap E(P(x, y))$ for some $z \in V(G)$, and

(ii) either $yw \in E(G) \cap E(P(x, y))$ for some $w \in V(G)$, or else $P(x, y)$ contains an edge uv for some vertices $u, v \in N(y)$.

Proof: Without loss of generality, assume that G is edge-critical since otherwise we can delete edges of G until we reach an edge-critical block. We consider two cases.

Case (A) $D(G) = \emptyset$.

Let G^* denote the 2-connected graph obtained from G by adding a new vertex z^* and joining z^* to both x and y.

First assume that x and y are not adjacent in G.

(i) Assume that $N_G(x) \cup N_G(y) \subseteq V_2(G)$.

Let C^* denote any cycle containing z^* and let $S^* = E^* \cup P^*$ be an $[x; y]$-EPS-graph of G^* with $C^* \subseteq E^*$ by Theorem D. Let $S = S^* - z^*$. Then $S = J \cup E \cup P$ is a $JEPS$-graph of G with $P = P^*$ and the component of S^* containing C^* becomes the open trail J from x to y in S. By following the construction of an xy-hamiltonian path $P(x, y)$ in S^2 which was used in [6], it is clear that $P(x, y)$ can start with an edge of S incident to x and ends with an edge of S incident to y unless $d_P(y) = 1$. If $d_P(y) = 1$, we jump from a vertex u preceding y in J to the vertex v in P_0 adjacent to y, where P_0 is the component of P containing y.

(ii) Assume that $N_G(x) \not\subseteq V_2(G)$ and $N_G(y) \subseteq V_2(G)$.

Then at least one of the two neighbors of x, say x' has degree greater than 2. Let C^* be a cycle containing z^* and the edge xx'. Note that this is possible because G is 2-connected (so that there is an xy-path in G starting with any given edge). In this case, let $S^* = E^* \cup P^*$ be an $[x; x', y]$-EPS-graph of G^* with $C^* \subset E^*$ by Theorem C because C^* is $[x; x', y]$-maximal. Then proceed as in case (i) and note that x is a pendant vertex in S. A required hamiltonian path in S^2 (with $S = E \cup P \cup J$ as in case (i)) can be constructed starting with the pendant edge incident to x.

(iii) Assume that $N_G(y) \not\subseteq V_2(G)$ and $N_G(x) \subseteq V_2(G)$.

This case can be treated symmetrically to case (ii), starting with an $[x; y', y]$-EPS-graph $S^* = E^* \cup P^*$ of G^* and $y' \in (N_G(y) - V_2(G)) \cap V(C^*)$.

(iv) Assume that $N_G(x) \not\subseteq V_2(G)$ and $N_G(y) \not\subseteq V_2(G)$.

Proceed as in case (ii) with C^* as defined there. Here we operate with an $[x; x', y']$-EPS-graph $S^* = E^* \cup P^*$ of G^* with $C^* \subseteq E^*$, where $y'y \in E(C^* - z^*)$, assuming first that $x' \neq y'$ (i.e., $\ell(C^*) > 4$) and applying Theorem C. Then $d_{P^*}(y) \leq 1$ (because $d_{G^*}(y) = 3$). Again we get a required xy-hamiltonian path HP in G^2.

Note that, if $y'' \in N_G(y) - y'$ and $d_{P^*}(y'') = 2$, $d_{P^*}(y) = 1$ then yy'' is an end-edge of the path in P^* incident to y and $y'y'' \in E(HP)$.

Now assume that $x' = y'$, (i.e. $\ell(C^*) = 4$). Since G is 2-connected, there is an $x'y$-path $P(x', y)$ in $G - x$ not containing $x'y$ ($x'y$ lies in a 2-connected block of $G - x$). Then $\{xx'\} \cup P(x', y)$ is an xy-path in G which together with xz^*y yields a cycle $C' \subset G^* - x'y$ with $\ell(C') > 4$, for which the preceding argument goes through if we operate with an $[x; x', y'']$-EPS-graph of $G^* - x'y$ where y'' is as above ($x'y$ is a chord of C' in G^*).

Next we assume that x and y are adjacent. In this case, we take a longest xy-path in $G - xy$ and combine it with xz^*y to form the cycle C^*; $l(C^*) \geq 5$ follows unless $N(x) \cap N(y) \neq \emptyset$ in which case $G = K_3$ since G is a DT-graph and we are done. If $\ell(C^*) \geq 5$ we proceed as before.

Case (B) $D(G) \neq \emptyset$.

By [7, Theorem 1], $D(G)$ contains an edge $e = st$ such that $G - e$ is a block chain with at least one of its endblocks, say B_e, being a DT-block. Without loss of generality $t \in V(B_e)$.

Suppose $(V(B_e) - c_e) \cap \{x, y\} = \emptyset$, where c_e is the cutvertex of $G - e$ belonging to B_e. Then we replace B_e by a path P^* of length 3 joining t and c_e. The resulting graph H is an edge-critical block and $|D(H)| < |D(G)|$. By induction H^2 has an xy-hamiltonian path with properties *(i)* and *(ii)* as stated by the theorem. Assuming that it contains as many edges of H as possible, any such xy-hamiltonian path in H^2 can be converted into an xy-hamiltonian path in G^2 having properties *(i)* and *(ii)* of the theorem, by the same method used in [7] as long as $c_e \notin \{x, y\}$. The same conclusion can be drawn if said hamiltonian path in H^2 satisfies $c_e \in \{x, y\}$. For, we may proceed as in [7, pp. 32-33], cases 2 and 4: we just look at the xy-hamiltonian path

$$P(x, y) = c_e \dots u^* v^* \dots r$$

($u^* \in V(P^*), v^* \in V(H) - V(P^*)$, $\{r, c_e\} = \{x, y\}$) in H^2 just as we would look at a hamiltonian cycle H_1 in H^2 in [7]

$$H_1 = c_e \dots u^* v^* \dots c_e$$

and using a hamiltonian path in B_e^2 starting in t and ending at c_e with an edge of B_e.

Hence we assume that for every DT-endblock B_e of $G - e$ (where $e \in D(G)$),

$$|(V(B_e) - c_e) \cap \{x, y\}| = 1$$

(note that $D(G) \neq \emptyset$ implies that G has at least two DT-endblocks like B_e). In particular, we assume $x \in V(B_e) - c_e$.

Let B'_e be the other endblock of $G - e$. If B'_e is a DT-block, then it follows from the preceding argument that $|(V(B'_e) - c'_e) \cap \{x, y\}| = 1$ where c'_e is the cutvertex of $G - e$ belonging to B'_e. If B'_e is not a DT-endblock, then B'_e contains a DT-endblock $B_{e'}$ for some $e' \in D(G)$, and we have the same conclusion as in the preceding sentence. Thus we conclude in any case that $y \in V(B'_e) - c'_e$.

Set $G_0 = G - e - (B_e \cup B'_e)$; G_0 is a (trivial or non-trivial) block chain. Possibly $G_0 = \emptyset$ in which case $c_e = c'_e$.

By Theorem F *(ii)*, $(B_e)^2$ has an xc_e hamiltonian path $P(x, e_c)$ starting with an edge xz_1 of B_e; $(B'_e)^2$ has an $c'_e y$-hamiltonian path $P(c'_e, y)$ ending with an edge $z_2 y$ of B'_e. By Corollary 1 *(ii)*, $(G_0)^2$ has a $c_e c'_e$-hamiltonian path $P_0(c_e, c'_e)$, being just a vertex if $c_e = c'_e$. Then

$$P(x, c_e) P_0(c_e, c'_e) P(c'_e, y)$$

is an xy-hamiltonian path in G^2 having properties (i) and (ii) of the theorem. □

Definition 2. *A graph G is said to have the strong $\mathcal{F}_3$ property if for any set of three distinct vertices $\{x_1, x_2, x_3\}$ in G, there is an $x_1 x_2$-hamiltonian path in G^2 containing $x_3 z_3, x_i z_i$ which are distinct edges of G for a given $i \in \{1, 2\}$. Such an $x_1 x_2$-hamiltonian path in G^2 is called a strong $\mathcal{F}_3$ $x_1 x_2$-hamiltonian path.*

Theorem 3. *Every 2-connected graph has the strong $\mathcal{F}_3$ property.*

Proof: Let G be a 2-connected graph. Without loss of generality, assume that G is an edge-critical block; otherwise, we delete edges from G until we reach an edge-critical block. Trivially, the theorem is true if G is a triangle. Thus we assume that $|V(G)| \geq 4$.

(I) Assume that G is a DT-graph.

Proceeding analogously to what we did in proving ([3, Theorem 2]), let G^+ denote the graph obtained from G by adding a new vertex z and join z to x_1, x_2. We shall show that $(G^+)^2$ has a hamiltonian cycle C_i containing $zx_1, zx_2, x_i z_i, x_3 z_3$ which are distinct edges of G^+ for a given $i \in \{1, 2\}$. Then $C_i - z = P_i(x_1, x_2)$ is a required strong $\mathcal{F}_3$ $x_1 x_2$-hamiltonian path in

G^2 containing the edges x_iz_i, x_3z_3 of G. Basically, we apply the construction of a hamiltonian cycle in the square of an EPS-graph in a DT-graph (see [6] and Observation (*) in [3]). In some of the cases, however, we shall proceed by induction, noting that the theorem is trivially true if it is a cycle; and sometimes we proceed by a direct proof.

Let C^+ be a cycle in G^+ containing z, x_1, x_2, x_3.

Case (A): $N(x_j) \subseteq V_2(G)$, $j = 1, 2, 3$.

By Theorem C, let $S = E \cup P$ be an $[x_i; x_{3-i}, x_3]$-EPS-graph of G^+ with $C^+ \subseteq E$. Hence $(G^+)^2$ has an $[x_i; x_{3-i}, x_3]$-hamiltonian cycle C_i for any $i \in \{1, 2\}$ provided $\ell(C^+) > 4$ (see the corresponding argument in the proof of Theorem 2).

However, if $\ell(C^+) = 4$, then $G^- = G - x_3$ is a non-trivial block chain ($x_1x_2 \in E(G)$ yields G being a triangle, contrary to the assumption at the beginning of the proof).

Moreover, x_1 and x_2 are pendant vertices of G^-. By Corollary 1*(ii)*, $(G^-)^2$ has an x_1x_2-hamiltonian path $P^-_{1,2}$ starting with $x_1v_1 \in E(G)$ and ending with $v_2x_2 \in E(G)$. Thus

$$(P^-_{1,2} - x_1v_1) \cup \{x_1x_3, x_3v_1\}$$

and

$$(P^-_{1,2} - v_2x_2) \cup \{x_3x_2, x_3v_2\}$$

yield the hamiltonian paths in G^2 as required by the theorem.

Case (B): $N(x_i) \subseteq V_2(G)$, $i = 1, 2$ and $N(x_3) \not\subseteq V_2(G)$.

Then $d_G(x_3) = 2$. Let $N(x_3) = \{u_3, v_3\}$.

(a) $\{u_3, v_3\} \neq \{x_1, x_2\}$. Without loss of generality assume that $u_3 \notin \{x_1, x_2\}$.

Again, by Theorem C, let $S = E \cup P$ be an $[x_i; x_{3-i}, u_3]$-EPS-graph of G^+ with $C^+ \subseteq E$. A required hamiltonian cycle C_i in $(G^+)^2$ can be constructed using S.

(b) $\{u_3, v_3\} = \{x_1, x_2\}$.

Consider the graph $G' = G - x_3$.

(b1) Suppose G' is 2-connected. We apply Theorem A with x_1, x_2 in place of v, w.

(i) Suppose G' has an EPS-graph $S' = E' \cup P'$ with $d_{P'}(x_i) = 0$, $i = 1, 2$. Let H' be a hamiltonian cycle of $(S')^2$: the edges of H' incident to x_i, $i = 1, 2$ are in G'; denote them by $e_i = x_i u_i$, $f_i = x_i v_i$, $i = 1, 2$. Without loss of generality the notation is chosen in such a way that $P(e_1, e_2)$ is the path in H' starting in x_1 with e_1 and ending in x_2 with e_2; $P(f_2, f_1) \subset H'$ is defined analogously. Then

$$x_1(P(e_1, e_2) - e_2)u_2 v_2(P(f_2, f_1) - \{f_1, f_2\})v_1 x_3 x_2$$

is a hamiltonian path as required for $i = 1$. By a symmetrical argument one obtains a hamiltonian path ending with f_2, say, and containing $x_1 x_3$.

(ii) Suppose G' has a $JEPS$-graph $S' = J' \cup E' \cup P'$ with x_1, x_2 being the only odd vertices of J' and $d_{P'}(x_i) = 0$, $i = 1, 2$. $(S')^2$ contains a hamiltonian path P^* starting with $g_1 = x_1 y_1$ and ending with $g_2 = x_2 y_2$, $\{g_1, g_2\} \subseteq E(G)$. We extend P^* to a hamiltonian path P as required by setting $P = x_1 x_3 y_1 (P^* - g_1)$ or $P = (P^* - g_2) y_2 x_3 x_2$.

(b2) Suppose G' is not 2-connected. Then G' is a block chain. By Lemma 1 *(ii)* with $x_1 = v$ and $x_2 = w$, G' has a $JEPS$-graph $S' = J' \cup E' \cup P'$ with $d_{P'}(x_1) = 0 = d_{P'}(x_2)$; and x_1, x_2 are the odd vertices of J'. Now proceed as in (b1)(ii): $(S')^2$ has an $x_1 x_2$-hamiltonian path P^* starting and ending with edges h_1, h_2 of G; one extends P^* to a corresponding hamiltonian path in G^2 by either traversing $x_1 x_3$ first and ending with h_2 in x_2, or traversing h_1 first and ending in x_2 with $x_3 x_2$.

Case (C): $N(x_1) \subseteq V_2(G)$ and $N(x_2) \not\subseteq V_2(G)$.

Then $d_G(x_2) = 2$; let $N(x_2) = \{u_2, v_2\}$. Without loss of generality assume that u_2 is on the cycle C^+.

(1) $N(x_3) \subseteq V_2(G)$.

(a) Suppose $x_3 \notin N(x_2)$.

By Theorem B, let $S = E \cup P$ be an $[x_i; x_{3-i}, u_2, x_3]$-EPS-graph of G^+ with $C^+ \subseteq E$. Then a required hamiltonian cycle in $(G^+)^2$ can be constructed for each $i \in \{1, 2\}$.

(b) Suppose $x_3 \in N(x_2)$; that is, $x_3 = u_2$.

Let $G' = G^+ - x_2 x_3$ which is a DT-graph.

(i) Suppose G' is 2-connected.

There is a cycle C' in G' containing z, x_1, x_2, v_2, x_3: this follows from the fact that G' contains in this case a path $P(x_1, v_2)$ with $x_3 \in V(P(x_1, v_2))$; it cannot contain z because $d_{G'}(x_2) = d_{G'}(z) = 2$. By Theorem C, let $S = E \cup P$ be an $[x_1; v_2, x_3]$-EPS-graph of G' with $C' \subseteq E$. Hence a required hamiltonian cycle C_i of $(G^+)^2$ can be constructed. A corresponding hamiltonian path in G^2 starts and ends with edges of G.

(ii) Suppose G' is not 2-connected.

Then G' is a block chain with a 2-connected endblock B_z containing z, x_1, x_2, v_2, and a block chain $G_3 = G' - B_z$ containing x_3 (which is not a cutvertex of G_3 and belongs to an endblock of G_3). G_3 is a DT-graph unless $G_3 = K_2$. Denote $V(B_z) \cap V(G_3) = \{c\}$.

By Lemma 1*(i)*, if G_3 has a cutvertex, then it has an EPS-graph $S_3 = E_3 \cup P_3$ such that $d_{P_3}(x_3) \leq 1$ and $d_{P_3}(c) \leq 1$. Moreover, if the endblock B_c in G_3 containing c is 2-connected, then we may achieve $d_{P_3}(c) = 0$; if B_c is a bridge, then $d_{P_3}(c) = 1$ and c is a pendant vertex in S_3. However, if G_3 is 2-connected, then we apply Theorem D to obtain such S_3. If $G_3 = K_2$, then $S_3 = \{cx_3\}$ and $E_3 = \emptyset$, $P_3 = \{cx_3\}$ and $d_{P_3}(c) = d_{P_3}(x_3) = 1$.

Let C_z be a cycle in B_z containing z, x_1, x_2, v_2, c. Such C_z exists because $d(x_2) = 2$.

If $c \notin \{x_1, v_2\}$, then by Theorem B, let $S_z = E_z \cup P_z$ be an $[x_1; v_2, c, z]$-EPS-graph of B_z with $C_z \subseteq E_z$. Set $E = E_z \cup E_3$ and $P = P_z \cup P_3$. Then we have an EPS-graph $S = E \cup P$ in G with $C_z \subseteq E$ and $d_P(x_1) = 0 = d_P(x_2)$, $d_P(v_2) \leq 1$ and $d_P(c) \leq 2$, $d_P(x_3) \leq 1$. Thus a hamiltonian cycle in $(G^+)^2$ can be constructed which contains edges of G incident with x_1, x_2 together with another edge of G incident to x_3,and also containing zx_1, zx_2.

If $c = v_2$, then v_2 is a cutvertex of G: for, $(G_3 \cup \{cx_2, x_2x_3\}) \cap (B_z - \{cx_2\} - z) = c$ and $(G_3 \cup \{cx_2, x_2x_3\}) \cup (B_z - \{cx_2\} - z) = G$. This yields a contradiction.

Hence we are left with the case $c = x_1$.

Suppose $d_{G_3}(x_1) > 1$. Then G_3 has an $[x_1; x_3]$-EPS-graph $S_3 = E_3 \cup P_3$ which we combine with an $[x_1; v_2]$-EPS-graph $S_z = E_z \cup P_z$ of B_z (see Theorem D) to obtain the EPS-graph $S = E \cup P$ by putting $E = E_3 \cup E_z$ and $P = P_3 \cup P_z$. We have $d_P(x_1) = 0 = d_P(x_2)$, $d_P(x_3) \leq 1$, $d_P(v_2) \leq 1$, and since G' is a DT-graph, S^2 has a hamiltonian cycle as required containing $x_1w_1, x_2v_2, x_3w_3 \in E(G)$, and also containing zx_1, zx_2.

Finally, assume $d_{G_3}(x_1) = 1$; i.e., G_3 is a non-trivial block chain or $G_3 = K_2$. Suppose first that $G_3 \neq K_2$. By Corollary 1, $(G_3)^2$ has a hamiltonian cycle $H_3 \supset \{x_1y_1, x_3w_3\}$ with $\{x_1y_1, x_3w_3\} \subset E(G_3)$; and it has a hamiltonian path $P_{1,3}$ starting with x_1y_1 and ending with x_3z_3, say, which are edges of G_3. Likewise, since $G_2 = (G - x_2x_3) - G_3 = B_z - z$ is a non-trivial block chain (x_2 is a pendant vertex of G_2), $(G_2 - \delta x_1)^2$ has a hamiltonian cycle H_2 containing v_2x_2 and if $d_{G_2}(x_1) > 1$, then $x_1s_1, x_1t_1 \in E(G_2)$ unless $G - \delta x_1 = K_2$ in which case $H_2 = x_2v_2$. It also has a hamiltonian path $P_{1,2}$ starting with x_1z_1, say, and ending with v_2x_2 which are edges of G_2.

Setting $H_2' = H_2$ if $\delta x_1 = x_1$ and $H_2' = (H_2 - \{x_1s_1, x_1t_1\}) \cup \{s_1t_1\}$ if $\delta x_1 = \emptyset$, we obtain hamiltonian paths $P_1(x_1,x_2)$, $P_2(x_1,x_2)$ in G^2 as required and defined by

$$E(P_1(x_1,x_2)) = E(P_{1,3} \cup (H_2' - x_2v_2)) \cup \{x_3v_2\}$$

$$E(P_2(x_1,x_2)) = (E(P_{1,2} \cup H_3) - \{x_1z_1, x_1y_1\}) \cup \{y_1z_1\}.$$

If, however, $G_3 = K_2$, then $N_{G^+}(z) = N_{G^+}(x_3) = \{x_1,x_2\}$, i.e., $G^+ - x_3$ is isomorphic to G. Since $d_G(x_2) = 2$ and because of the assumption $d_{G_3}(x_1) = 1$ and because $c = x_1$, it follows that $G - x_3$ is a non-trivial block chain and x_2 is an endvertex of $G - x_3$ and $c = x_1$ is not a cutvertex belonging to the other endblock of $G - x_3$ unless $x_1x_2 \in E(G - x_3)$. However, if $x_1x_2 \in E(G - x_3)$, then we conclude that G is a triangle in this exceptional case, contradicting the assumption $|V(G)| \geq 4$ at the beginning of the proof. Hence $G - x_3$ is a non-trivial block chain.

Now we apply Corollary 1*(ii)* to obtain in $(G - x_3)^2$ a hamiltonian path $P^{(3)}(x_1,x_2)$ starting with $x_1v_1 \in E(G)$ and ending in x_2 with $v_2x_2 \in E(G)$. Now, for $i = 1, 2$,

$$(P^{(3)}(x_1,x_2) - x_iv_i) \cup \{x_ix_3, x_3v_i\}$$

yields a hamiltonian path in G^2 as required.

(2) $N(x_3) \not\subseteq V_2(G)$.

Then $d_G(x_3) = 2$. Let $N(x_3) = \{u_3, v_3\}$. Suppose without loss of generality that C^+ is of the form $zx_1u_1 \ldots u_3x_3v_3 \ldots u_2x_2z$.

(a) $\{u_3, v_3\} \neq \{x_1, x_2\}$.

By Theorem B, let $S = E \cup P$ be an $[x_i; x_{3-i}, x_3^*, u_2]$-$EPS$-graph of G with $C^+ \subseteq E$, where $x_3^* \in \{u_3, v_3\} - \{x_1, x_2\}$. If $\ell(C^+) > 5$, then it is straightforward to see that a required hamiltonian cycle C_i in $(G^+)^2$ can be constructed from S for any $i \in \{1, 2\}$, independent of the size of $N(x_3) \cap (N(x_1) \cup N(x_2))$.

Observe that $\ell(C^+) \geq 4$. However, $\ell(C^+) = 4$ implies $N(x_3) = \{x_1, x_2\}$, contrary to the assumption $\{u_3, v_3\} \neq \{x_1, x_2\}$.

To finish this case (a) we are thus left with the case $\ell(C^+) = 5$ which implies $|\{u_3, v_3\} \cap \{x_1, x_2\}| = 1$. More precisely, we have

$$C^+ = zx_1u_1u_2x_2z,$$

i.e., $x_3 \in \{u_1, u_2\}$.

Suppose $d_G(u_2) = 2$; then $G^- = G - \{u_1, u_2\}$ is a non-trivial block chain (the case $G^- = K_2$ is impossible). By Corollary 1*(ii)*, $(G^-)^2$ has a hamiltonian path P^- starting with $x_1t_1 \in E(G)$ and ending with $v_2x_2 \in E(G)$. Now

$$(P^- - x_1t_1) \cup \{x_1u_2, u_2u_1, u_1t_1\}$$

and

$$(P^- - v_2x_2) \cup \{v_2u_2, u_2u_1, u_1x_2\}$$

yield the required hamiltonian paths.

Finally, suppose $d_G(u_2) > 2$. Then $x_3 = u_1$ since $d_G(x_3) = 2$.

Now $G - x_3$ is either a non-trivial block chain or it is 2-connected. In any case, x_2 and u_2 are not cutvertices of $G - x_3$ and they belong to the same 2-connected block B^* of $G - x_3$. If $G - x_3$ is not 2-connected, let c^* denote the cutvertex of $G - x_3$ in (the endblock) B^*. Set $G^* = (G - x_3) - B^*$. By Corollary 1*(ii)* or if $G^* = K_2$, $(G^*)^2$ has an x_1c^*-hamiltonian path P^* starting with an edge $x_1t_1 \in E(G)$, provided $G^* \neq \emptyset$; if $G^* = \emptyset$ set $P^* = \emptyset$. In any case, however, $(B^*)^2$ has by induction c^*x_2-hamiltonian paths, one starting in c^* with $c^*t^* \in E(B^*)$, whereas the other ends in x_2 with $t_2x_2 \in E(B^*)$, and both containing an edge $u_2w_2 \in E(B^*)$. Denote these paths by P_1^* and P_2^*, respectively. If $G^* = \emptyset$, then set $c^* = x_1$.

It follows that for both $i = 1, 2$,

$$P^* \cup (P_i^* - u_2w_2) \cup \{w_2u_1, u_1u_2\}$$

yield x_1x_2-hamiltonian paths as required. This finishes case (a).

(b) $\{u_3, v_3\} = \{x_1, x_2\}$.

Then $G' = G^+ - x_3$ is 2-connected; it contains a cycle $C' \supseteq \{z, x_1, x_2, v_2\}$. By Theorem D, let $S' = E' \cup P'$ be an $[x_1; v_2]$-EPS-graph of G' with $C' \subseteq E'$. Then $d_{P'}(z) = 0 = d_{P'}(x_1) = d_{P'}(x_2)$ and $d_{P'}(v_2) \leq 1$. Let $E = E'$ and $P = P' \cup \{x_ix_3\}$. Then $S = E \cup P$ is an EPS-graph of G^+ and a required hamiltonian cycle C_i in $(G^+)^2$ containing $x_ix_3, x_{3-i}z_{3-i}$ $(z_{3-i} \in N(x_{3-i}))$, which are edges of G, can be constructed for each $i \in \{1, 2\}$.

Since the case $N(x_1) \nsubseteq V_2(G)$ and $N(x_2) \subseteq V_2(G)$ is symmetrical to *Case (C)* just considered, we are left with the consideration of one more large case for DT-graphs.

Case (D): $N(x_1) \nsubseteq V_2(G)$ and $N(x_2) \nsubseteq V_2(G)$.

Then $d_G(x_i) = 2$ for $i = 1, 2$. Let $N(x_i) = \{u_i, v_i\}$, $i \in \{1, 2\}$. Suppose C^+ is of the form $zx_1u_1 \dots u_3x_3v_3 \dots u_2x_2z$ as in *Case (C)* (2) above.

(1) Suppose $N(x_3) \subseteq V_2(G)$.

(a) Suppose $x_3 \notin (N(x_1) \cup N(x_2)) \cap V(C^+)$; That is, $x_3 \notin \{u_1, u_2\}$.

Let $S = E \cup P$ be an $[x_i; u_1, u_2, x_3]$-EPS-graph of G^+ with $C^+ \subseteq E$ which exists by Theorem B. Then a required hamiltonian cycle C_i in $(G^+)^2$ can be constructed for each $i \in \{1, 2\}$.

(b) Suppose $x_3 \in N(x_1) \cap N(x_2) \cap V(C^+)$; hence $x_3 = u_1 = u_2$.

Suppose first that $d_G(x_3) > 2$. Consider $G'_2 = G - x_2x_3$. Note that x_2 is a pendant vertex in G'_2. Let B_2 be the endblock of G'_2 with $x_1, x_3 \in V(B_2)$, and they are not cutvertices. Then $G'_2 - B_2 \neq \emptyset$ is a trivial or non-trivial block chain with $c_2 = V(B_2) \cap V(G'_2 - B_2)$ being the cutvertex of G'_2 in B_2. Using induction on B_2 we have a hamiltonian path $P(x_1, c_2)$ in $(B_2)^2$ starting with $x_1s_1 \in E(B_2)$ and containing another edge $x_3s_3 \in E(B_2)$. Moreover $(G'_2 - B_2)^2$ has a hamiltonian path $P(c_2, x_2)$ starting with $x_2s_2 \in E(G'_2 - B_2)$ by Corollary 1*(ii)* or if $G'_2 - B_2 = K_2$. Then $P(x_1, c_2)P(c_2, x_2)$ is a hamiltonian path as required.

If however, $d_G(x_3) = 2$, then we set $G'' = G - x_3$ which is a non-trivial block chain with pendant vertices x_1, x_2, otherwise $G = K_3$ and this case has been solved at the beginning of the proof. Thus $(G'')^2$ has a hamiltonian path $P(x_1, x_2)$ starting and ending with edges in G'', by Corollary 1*(ii)*. Now it is trivial to enlarge $P(x_1, x_2)$ to a hamiltonian path P of G^2 as required by appropriately using x_3x_i, $i \in \{1, 2\}$ as the last edge in P. This finishes case (b).

(c) Suppose $x_3 \in (N(x_i) - N(x_{3-i})) \cap V(C^+)$, $i \in \{1, 2\}$. Without loss of generality $i = 2$. Hence $x_3 \notin \{u_1, v_1\}$ and $x_3 = u_2$.

Consider $G' = G^+ - x_2x_3$.

If G' is 2-connected, then we consider a cycle C^* traversing x_1, z, x_2, v_2, x_3 in this order (observe that G' contains a cycle through x_2 and x_3, and $d_{G'}(z) = d_{G'}(x_2) = 2$). Because of case (b) before we may assume that $x_1x_3 \notin E(G')$. Therefore we denote $t_1 = N_{C^*}(x_1) - \{z\}$, hence $t_1 \notin \{v_2, x_3\}$.

Now we apply Theorem B to obtain an $[x_1; t_1, v_2, x_3]$-EPS-graph $S^* = E^* \cup P^*$ of G' with $C^* \subseteq E^*$. Now it is straightforward to see that $(S^*)^2$ has a hamiltonian cycle as required (containing an edge of G incident to x_i for both $i = 1$ and $i = 2$).

If G' is not 2-connected, we define B_z, G_3, and correspondingly S_3 as in *Case (C)*(1)(b)(ii).

Let C_z be a cycle in B_z containing z, x_1, x_2, v_2, t_1, c, where $t_1 = N_{C_z}(x_1) - \{z\}$.

If $c \notin \{x_1, v_2, t_1\}$, then by Theorem B let $S_z = E_z \cup P_z$ be an $[x_1; v_2, c, t_1]$-EPS-graph of B_z with $C_z \subseteq E_z$. Now we continue as in *Case (C)*(1)(b)(ii), additionally using that $d_{P_z} \leq 1$.

If $c = v_2$, then again v_2 is a cutvertex of G, a contradiction.

Now suppose $c = x_1$. Hence $d_{G_3}(x_1) = 1$ and G_3 is a non-trivial block chain (note that $G_3 = K_2$ is not possible because of $x_3 \notin N(x_1)$ in this case). Now we continue as in the corresponding subcase of *Case (C)*(1)(b)(ii) with both of x_1 and x_2 being pendant vertices of $G_2 = B_z - z$.

Finally suppose $c = t_1$. Because of C^+ we conclude that $t_1 = u_1$. We have $\{z, x_1, x_2, u_1, v_1, v_2\} \subset V(B_z)$. Thus there is a cycle $C_1 \subset B_z$ traversing x_1u_1, x_1v_1. Consequently, $z, x_2 \notin C_1$. In fact, $\widehat{C} = C^+ \triangle C_1$ is a cycle containing $u_1, v_1, x_1, z, x_2, x_3$ in this order; i.e., x_1u_1 is a chord of $\widehat{C}$. Hence $G'' = G^+ - x_1u_1$ is 2-connected. Therefore $G''' = G'' - x_2x_3$ is a non-trivial block chain with one endblock $B_z''' \subset B_z$ since $G_3 \subset G'''$ ($z \in V(B_z''')$). Thus we can write

$$G^+ - \{x_1u_1, x_2x_3\} = G''' = B_z''' \cup G_3''' \quad \text{with} \quad B_z''' \cap G_3''' = \{c_0\},$$

where c_0 is a cutvertex of G''' (possibly $c_0 = c$).

Then $(G_3''')^2$ has a hamiltonian cycle H_3 containing $c_0w_0, x_3w_3 \in E(G_3''')$ by Corollary 1 *(i)* if G_3''' is a non-trivial block chain, or by Theorem E if G_3''' is 2-connected. Note that $G_3''' = K_2 = u_1x_3$ is not possible because of $d_G(u_1) > 2$ and $N(x_3) \subseteq V_2(G)$ in this case.

Let C_z be a cycle in B_z''' containing z, x_1, x_2, c_0.

If $c_0 \notin \{v_1, v_2\}$, we operate with a $[v_i; v_{3-i}, c_0]$-EPS-graph $S''' = E''' \cup P'''$ of B_z''' with $C_z \subseteq E'''$, $i \in \{1, 2\}$, which exists by Theorem C. $(S''')^2$ contains a hamiltonian cycle H''' containing $zx_1, zx_2, x_1v_1, c_0z_0, x_2v_2 \in E(B_z)$. $(H_3 - c_0w_0) \cup (H''' - c_0z_0) \cup \{w_0z_0\}$ is a required hamiltonian cycle in $(G^+)^2$ with $x_1v_1, x_2v_2, x_3z_3 \in E(G)$.

If $c_0 = v_i$ and $c_0 \neq v_{3-i}$, $i \in \{1,2\}$, we operate with a $[v_i; v_{3-i}]$-*EPS*-graph $S''' = E''' \cup P'''$ of B_z''' with $C_z \subseteq E'''$, which exists by Theorem D. $(S''')^2$ contains a hamiltonian cycle H''' containing $zx_1, zx_2, x_1v_1, c_0z_0, x_2v_2 \in E(B_z)$. Again, $(H_3 - c_0w_0) \cup (H''' - c_0z_0) \cup \{w_0z_0\}$ is a required hamiltonian cycle in $(G^+)^2$ with $x_1v_1, x_2v_2, x_3z_3 \in E(G)$.

If $c_0 = v_1 = v_2$, then $B_z''' = zx_1c_0x_2z$. $(H_3 - c_0w_0) \cup (B_z''' - x_ic_0) \cup \{x_iw_0\}$ is a required hamiltonian cycle in $(G^+)^2$ with $x_{3-i}c_0, x_3z_3 \in E(G)$ for each $i \in \{1,2\}$.

(2) Suppose $N(x_3) \not\subseteq V_2(G)$.

Then $d_G(x_3) = 2$. Set $N(x_3) = \{u_3, v_3\}$. Now we set $x_3^* \in N(x_3) - V_2(G)$. As before, let C^+ be of the form $zx_1u_1 \cdots u_3x_3v_3 \cdots u_2x_2z$.

(a) Suppose $x_3 \notin N(x_1) \cup N(x_2)$.

(a1) $x_3^* \notin \{u_1, u_2\}$.

By Theorem B, let $S = E \cup P$ be an $[x_i; u_1, u_2, x_3^*]$-*EPS*-graph of G^+ with $C^+ \subseteq E$ for any $i \in \{1,2\}$. Since $d_P(x_i) = 0 = d_P(z)$, $d_P(u_1) \leq 1$ and $d_P(u_2) \leq 1$, a required hamiltonian cycle C_i in $(G^+)^2$ can be constructed in S^2 for each $i \in \{1,2\}$ due to the restriction on x_3^*.

(a2) $x_3^* = u_3 = u_1$ (the case $x_3^* = u_2$ is symmetrical and therefore does not need separate consideration).

(i) $v_3 \neq u_2$. In this case we operate with an $[x_i; u_1, v_3, u_2]$-*EPS*-graph $S_i = E_i \cup P_i$ with $C^+ \subseteq E_i$, $i = 1,2$ (see Theorem B). The restrictions on x_3^* and v_3 guarantee that $(S_1)^2$ and $(S_2)^2$ yield hamiltonian cycles as claimed by the theorem.

(ii) $v_3 = u_2$. That is, $C^+ = zx_1u_1x_3u_2x_2z$.

Assume first that one of u_1, u_2 is 2-valent, i.e., $d_G(u_2) = 2$ since $u_1 = x_3^* \notin V_2(G)$. Then we operate with an $[x_i; u_1, x_{3-i}]$-*EPS*-graph $S_i = E_i \cup P_i$ with $C^+ \subseteq E_i$ for each $i \in \{1,2\}$, which exists by Theorem C. A required hamiltonian cycle in $(G^+)^2$ containing $x_iu_i, x_3u_1 \in E(G)$ can be constructed for each $i \in \{1,2\}$.

Hence assume that $d_G(u_i) > 2$, $i = 1,2$. $G' = G - x_3$ is a trivial or non-trivial block chain.

Suppose G' is 2-connected. Using induction, $(G')^2$ has an x_1x_2-hamiltonian path $P_i'(x_1, x_2)$ containing $x_iw_i, u_iz_i \in E(G')$ for each $i \in \{1,2\}$. Then

$$(P_i(x_1, x_2) - u_iz_i) \cup \{u_ix_3z_i\}$$

is a required hamiltonian path in G^2 for each $i \in \{1,2\}$.

Finally assume that G' is a non-trivial block chain. The endblock B_i' in G' containing u_i, x_i is 2-connected (since $d_{G'}(u_i) \geq 2$), $i \in \{1,2\}$; it also contains v_i since $d_G(x_i) = 2$. Let $P(x_i, u_i)$ denote an x_iu_i-path in B_i' containing v_i for any $i \in \{1,2\}$. Define the cycle $\widetilde{C_i}$ by

$$E(\widetilde{C_i}) = (E(C^+) - x_iu_i) \cup E(P(x_i, u_i)).$$

Let $\widetilde{G_i} = G^+ - x_iu_i$, $i \in \{1,2\}$; $\widetilde{G_i}$ is 2-connected because x_iu_i is a chord of $\widetilde{C_i}$. $\widetilde{C_i}$ contains $z, x_1, x_2, x_3, u_1, u_2, v_i$. By Theorem C, there is a $[u_i; v_i, u_{3-i}]$-EPS-graph $\widetilde{S_i} = \widetilde{E_i} \cup \widetilde{P_i} \subset \widetilde{G_i}$ with $\widetilde{C_i} \subseteq \widetilde{E_i}$. A required hamiltonian cycle in $(\widetilde{S_i})^2$ containing $x_iv_i, x_3u_i \in E(G)$ can be constructed, for each $i \in \{1,2\}$.

(b) Suppose $x_3 \in N(x_1)$ but $x_3 \notin N(x_2)$; that is, $x_3 = u_1$. By definition of x_3^* we have $x_3^* = v_3$.

Suppose $v_3 \neq u_2$, i.e., $N(x_3) \cap N(x_2) \cap C^+ = \emptyset$. To get a required hamiltonian cycle C_i containing $x_iu_i, x_3x_3^*$, we operate with an $[x_i; u_2, x_3^*]$-EPS-graph $S_i = E_i \cup P_i$ of G^+ with $C^+ \subseteq E_i$, which exists by Theorem C.

Hence suppose $v_3 = u_2$, i.e., $N(x_3) \cap N(x_2) \cap C^+ \neq \emptyset$. Because of $d_G(x_3^*) > 2$ there exists $w_3 \in N(x_3^*) - \{x_3, x_2\}$. There is a w_3v_2-path $P(w_3, v_2)$ in G not containing x_3^* and therefore, $x_1, x_2 \notin P(w_3, v_2)$. Then $C^* = zx_1x_3u_2w_3P(w_3, v_2)v_2x_2z$ is a cycle in G^+ with $N(x_3) \cap N(x_2) \cap C^* = \emptyset$; thus we are back to the preceding case.

(c) Suppose $x_3 \notin N(x_1)$ but $x_3 \in N(x_2)$.

This case is symmetrical to case (b) above.

(d) Suppose $x_3 \in N(x_1) \cap N(x_2)$. This case is not possible because of $N(x_i) \nsubseteq V_2(G)$, $i = 1, 2, 3$.

(II) Assume that $D(G) \neq \emptyset$.

We apply Theorem G to G with respect to $\{x_1, x_2\}$ to conclude that $D(G)$ contains an edge e such that $G - e$ has a DT-endblock B_e such that $A = \{x_1, x_2\} \not\subset V(B_e)$, and if $x_i \in V(B_e)$, then it is a cutvertex of $G - e$. Let B_e' denote the other endblock of $G - e$. Also, let c and c' denote the cutvertices of $G - e$ belonging to B_e and B_e' respectively. If $c \neq c'$, set $G_0 = G - e - (B_e \cup B_e')$; it is a block chain containing c, c' which are not cutvertices of G_0. Also, for the above e, denote $e = xx'$ where $x \in V(B_e)$ and $x' \in V(B_e')$.

Let $X = \{x_1, x_2, x_3\}$. Suppose $X \cap V(B_e) = \emptyset$. Then we replace the subgraph B_e in G with a path of length 3 to obtain the 2-connected edge-critical graph H. By induction, H has the strong $\mathcal{F}_3$ property. Moreover any strong $\mathcal{F}_3$ x_1x_2-hamiltonian path in H^2 can be converted into a strong $\mathcal{F}_3$ x_1x_2-hamiltonian path in G^2 by the method used in [7]. Hence we can assume that $X \cap V(B_e) \neq \emptyset$. We also note that it is tacitly assumed that the hamiltonian paths/cycles in the square of the smaller graphs contain as many edges of the given graphs as possible. The purpose of this assumption (already formulated in [7] and subsequent papers) is to facilitate the induction step and to keep the various cases arising, under control.

With the same argument as above, we see that $X \cap V(B'_e) \neq \emptyset$ if B'_e is a DT-block. If B'_e is not a DT-block, then there is an edge $f \in E(B'_e) \cap D(G)$ such that one of the endblocks B_f of $G - f$ is a DT-block and $V(B_f) \subset V(B'_e)$. This means that $X \cap V(B_f) = \emptyset$ if $X \cap V(B'_e) = \emptyset$, and again the above argument can similarly be applied.

In the ensuing discussion we keep in mind that there are at least two DT-endblocks B^* and B^{**} defined by the same element e^* or by different elements $e^*, f^* \in D(G)$; and $B^* \cap B^{**} = \emptyset$, or $B^* \cap B^{**} = c^*$ where c^* is a cutvertex of $G - e^*$. Therefore, a case not considered in B^* implies a (sort of complementary) case in B^{**} which is being taken care of when it occurs in B^*.

Next, we consider two special cases.

Case (A): $X \cap (V(B_e) - c) = \emptyset$, or $x_3 \in V(B_e) - \{c, x\}$ and $A \cap V(B_e) = \emptyset$.

In the first case, it follows from the preceding argument that $c \in \{x_3, x_i\}$ for some $i \in \{1, 2\}$. As before, we replace the subgraph B_e in G with a path of length 3 to obtain the 2-connected edge-critical graph H. By induction, H has the strong $\mathcal{F}_3$ property. Moreover, by a careful study of the method used in [7] one sees that any strong $\mathcal{F}_3$ x_1x_2-hamiltonian path P_H in H^2 can be converted into a strong $\mathcal{F}_3$ x_1x_2-hamiltonian path in G^2. This applies, in particular, to the case where $c \in A$ and P_H contains an edge of H incident to c (here, some of the 13 cases listed in [7] need not be considered). Hence we are left with the case where $X \cap V(B_e) = x_3$ and $x_3 \neq c, x$.

We proceed as before, replacing B_e with a path P_3 of length 3; again, the resulting graph is denoted by H. By induction on $|D(G)|, H^2$ has a strong $\mathcal{F}_3$ x_1x_2-hamiltonian path containing $e_u \in E(G)$ incident to u for some $u \in V(G) - V(B_e)$. In fact, a careful study of the procedure employed before shows that P_H can be converted into a strong $\mathcal{F}_3$ x_1x_2-hamiltonian path $P_{1,2}$ of G^2 containing an edge of G incident to x_3. Namely, depending on the various cases of the traversal of $V(P_3)$ by P_H,

- one either applies Lemma 3 to use a hamiltonian cycle C_e of $(B_e)^2$ such that C_e traverses in c edges of B_e, and likewise, C_e traverses at least one edge in x and at least one edge in x_3, belonging to B_e (observe that $|V(B_e)| \geq 4$ since G is edge-critical and thus does not have a triangle);
- or one applies induction to use a hamiltonian path P_e of $(B_e)^2$ joining x and c and containing at least one edge of B_e incident to x_3 and an edge of B_e incident to any given $t \in \{x, c\}$.

Case (B): $X \cap V(B_e) = x_3 = x$.

It follows that $A \cap V(B'_e) \neq \emptyset$; without loss of generality $x_1 \in V(B'_e)$. Assume the notation chosen in such a way that $x_1 \neq c'$ if $A \subset V(B'_e)$. Moreover, if $A \cap V(B'_e) = x_1$ it follows from the preceding considerations that x_1 belongs to the DT-endblock $B_f \subseteq B'_e$ for some $f \in D(G)$ if $E(B'_e) \cap D(G) \neq \emptyset$, or else $B_{e'}$ is a DT-endblock; and $x_1 \neq c_f$ by Theorem G, where c_f is the cutvertex of $G - f$ in B_f. Also, $c_f = c'$ if $c' \in V(B_f)$. Hence $x_1 \neq c'$ can be assumed in any case.

Denote the blocks of $G - e$ by $B_0, \ldots, B_k$ according to their order in $bc(G - e)$ such that $B_0 = B'_e$, $B_k = B_e$, and let j be the smallest index such that $x_2 \in V(B_j)$; possibly $j = 0$. Set

$$G_{0,j} := \bigcup_{i=0}^{j} B_i \text{ and } G_{j+1,k} := \bigcup_{i=j+1}^{k} B_i.$$

(i) Suppose $j > 0$. By applying induction to the individual 2-connected blocks of $G_{0,j}$ it follows that $(G_{0,j})^2$ has a hamiltonian x_1x_2-path $P_{1,2}$ containing $x_iy_i \in E(G_{0,j}), i = 1, 2$, for some $y_1, y_2 \in V(G_{0,j})$, as well as $x'y' \in E(B'_e), c^*y^* \in E(B_j)$ where $c^* = B_j \cap B_{j+1}$, and $x'y' = x_1y_1$ if $x' = x_1$, $c^*y^* = x_2y_2$ if $c^* = x_2$. Likewise by Corollary 1*(i)*, $(G_{j+1,k})^2$ has a hamiltonian cycle $\tilde{C}$ containing x_3y_3, $x_3z_3 \in E(B_e)$ and $c^*z^* \in E(B_{j+1})$.

In any case,

$$(P_{1,2} \cup \tilde{C} - \{c^*y^*, c^*z^*\}) \cup \{y^*z^*\} \tag{1}$$

defines a hamiltonian x_1x_2-path of G^2 containing x_1y_1 and x_3y_3; it also contains x_2y_2 if $x_2 \neq c^*$. On the other hand, if $c^* = x_2$ we construct a hamiltonian x_1x_2-path of G^2 containing x_2y_2 and x_3y_3 as follows: if $j+1 < k$ and $\kappa(B_{j+1}) \geq 2$, then $\tilde{C}$ can be assumed to contain c^*z^*, $c^*x^* \in E(B_{j+1})$, and we set

$$\tilde{\tilde{C}} = (\tilde{C} - \{c^*z^*, c^*x^*\}) \cup \{x^*z^*\}$$

which defines a hamiltonian cycle of $(G_{j+1,k} - c^*)^2$ containing x_3y_3, x_3z_3. The same type of hamiltonian cycle is obtained if B_{j+1} is a bridge of $G - e$. Thus, in both cases

$$(P_{1,2} \cup \tilde{\tilde{C}} - \{x'y', x_3z_3\}) \cup \{x'z_3, y'x_3\} \tag{2}$$

defines a hamiltonian x_1x_2-path of G^2 containing x_2y_2 and x_3y_3.

Thus we are left with the case $j + 1 = k$ implying $x_2 \neq c$ and thus $\kappa(B_j) \geq 2$. We now proceed as in (1) above.

(ii) $j = 0$. That is, $G_{0,0} = B'_e$; $\{x_1, x_2\} \subseteq V(B'_e)$ follows.

If $\{x_1, x_2\} \neq \{x', c'\}$ we obtain by induction a hamiltonian x_1x_2-path $(P^{(i)}_{1,2})$ of $(B'_e)^2$ containing $x_iy_i \in E(B'_e)$ for any $i \in \{1, 2\}$, but also an edge $e_{t'}$ incident to $t' \in \{x', c'\} - \{x_1, x_2\}$. If $t' = c'$ we proceed as in (1), whereas we proceed as in (2) if $t' = x'$.

Thus we assume $\{x_1, x_2\} = \{x', c'\}$; by the initial choice of notation, $x_1 = x'$ and $x_2 = c'$ follows. Also, $c' \neq c$ by the hypothesis of this *Case (B)*.

Since $c' \neq c$, $G' := G - e - B'_e$ is a non-trivial block chain. By Corollary 1*(ii)*, $(G')^2$ has a hamiltonian x_2x_3-path $P_{2,3}$ containing edges $x_2y_2, x_3y_3 \in E(G')$. By Theorem E, $(B'_e)^2$ has an $[x_i; x_{3-i}]$-hamiltonian cycle C_i for every $i \in \{1, 2\}$. Denote the corresponding edges of $C_i \cap E(B'_e)$ by $x_1y_1^{(1)}, x_1z_1^{(1)}$ and $x_2z_2^{(1)}$ if $i = 1$, and by $x_1y_1^{(2)}$ and $x_2u_2^{(2)}, x_2z_2^{(2)}$ if $i = 2$.

Assume further the notation chosen in such a way that the x_1x_2-path in C_i containing $x_1y_1^{(i)}$ also contains $x_2z_2^{(i)}$; denote it by $P_1^{(i)}$ and set $P_2^{(i)} = C_i - P_1^{(i)}$.

$$(P_1^{(i)} - x_2) \cup \{z_2^{(1)}y_2\} \cup (P_{2,3} - x_2) \cup \{x_3z_1^{(1)}\} \cup (P_2^{(1)} - x_1)$$

defines a hamiltonian x_1x_2-path of G^2 starting with $x_1y_1^{(1)} \in E(G)$. Likewise

$$(P_2^{(2)} - x_2) \cup \{u_2^{(2)}z_2^{(2)}\} \cup (P_1^{(2)} - \{x_1, x_2\}) \cup \{y_1^{(2)}x_3\} \cup P_{2,3}$$

defines a hamiltonian x_1x_2-path of G^2 ending with $x_2y_2 \in E(G)$.

This finishes *Case (B)*.

For the remaining cases of the proof of Theorem 3 we consider $D_1(G)$ comprising those edges of $D(G)$ such that for every $g \in D_1(G)$ one of the endblocks of $G - g$, B_g say, is a DT-graph. Let c_g denote the cutvertex of $G - g$ in B_g. Having solved the *Cases (A)* and *(B)* we conclude that $X \cap (V(B_g) - c_g) \neq \emptyset$ in any case. Note that G has at least two DT-endblocks as just described: one is the aforementioned B_e, another one is either B'_e, or B'_e contains $f \in D_1(G)$ such that the corresponding DT-endblock B_f is a proper subgraph of B'_e. $B_e \cap B_f = \emptyset$ or $B_e \cap B_f = c$ where $c = c_f = c_e$.

We proceed analogous to *Case (B)* denoting the blocks of $G - e$ by $B_0, \ldots, B_m$ with $B_0 = B_e$, $B_m = B'_e$.

Suppose first that $|X \cap V(B_e)| = 1$. In view of *Cases (A)* and *(B)* we have

$$X \cap V(B_e) = A \cap V(B_e) = A \cap (V(B_e) - c) = x_i \ ;$$

without loss of generality $i = 1$. By the same token $x_2 \in V(B_f) \subseteq V(B'_e)$ and $x_2 \neq c'$.

Let $P(c, x_1)$ be a c, x_1-hamiltonian path of B_e^2 with $x_1z_1 \in (P(c, x_1)) \cap E(B_e)$, which exists by Theorem F *(ii)*.

If $x_3 \in V(B'_e) - \{c'\}$, we operate with an x_2c'-hamiltonian path $P(x_2, c')$ of $(B'_e)^2$ with $x_2z_2, x_3z_3 \in E(P(x_2, c')) \cap E(B'_e)$ using induction, and trivially with a $c'c$-hamiltonian path P_0 of G_0^2. Note that $P_0 = \emptyset$ if $c = c'$ in this case.

If $x_3 \notin V(B'_e) - \{c'\}$, we operate with an x_2c'-hamiltonian path $P(x_2, c')$ of $(B'_e)^2$ with $x_2 \in E(P(x_2, c')) \cap E(B'_e)$ which exists by Theorem F *(ii)*, and with a $c'c$-hamiltonian path P_0 of G_0^2 containing $x_3z_3 \in E(G_0)$ applying Theorem F to each 2-connected block of G_0. Note that $c = c' = x_3$ is not possible by the assumption $X \cap V(B_e) = x_1$ and it covers the case $x_3 = c' \neq c$.

Then

$$P(x_2, c'), P_0, P(c, x_1)$$

is a hamiltonian x_1x_2-path of G^2 containing $x_3z_3, x_iz_i \in E(G)$, for $i = 1, 2$. This settles the case $|X \cap V(B_e)| = 1$.

Now suppose $|X \cap V(B_e)| = 2$. Because of the case just settled we must also have $|X \cap V(B_f)| = 2$ implying $c_f = c' = c \in X$. Again, suppose without loss of generality that $x_1 \in B_e$.

If $c = x_3$, let P_0 be an x_1x_3-hamiltonian path of B_e^2 such that $x_1z_1 \in E(P_0) \cap E(B_e)$ and P_1 be an x_3x_2-hamiltonian path of $B_{e'}^2$ such that $x_3z_3 \in E(P_1) \cap E(B_{e'})$, which exist by Theorem F *(ii)*. Hence

$$P_0P_1$$

is an x_1x_2-hamiltonian path of G^2 with $x_1z_1, x_3z_3 \in E(G)$. We proceed analogously to obtain an x_2x_1-hamiltonian path of G^2 with $x_2z_2, x_3z_3 \in E(G)$ as required by the theorem.

Now suppose without loss of generality that $c = x_1$ and $x_3 \in V(B_e) - c$. Hence we have $x_1, x_2 \in V(B_f)$, i.e., $x_1, x_2 \in V(B'_e)$. We apply Theorem E to B_e and either Theorem F *(ii)* or Theorem 2 to B'_e.

By Theorem E, B_e^2 contains a hamiltonian cycle C_e with $x_1y_1, x_1z_1 \in E(C_e) \cap E(B_e)$ and $x_3u_3 \in E(C_e) \cap E(B_e)$.

As for $(B'_e)^2$, it has an x_1x_2-hamiltonian path $P_{1,2}$ with $x_1u_1 \in E(P_{1,2}) \cap E(B'_e)$, by Theorem F *(ii)*. Thus,

$$E(P_{1,2}) \cup E(C_e) \cup \{u_1y_1\} - \{x_1u_1, x_1y_1\}$$

defines on x_1x_2-hamiltonian path of G^2, containing $x_1z_1 \in E(G)$, but also $x_3u_3 \in E(G)$.

Likewise, Theorem 2 implies that $(B'_e)^2$ has either an x_1x_2-hamiltonian path $P_{1,2}$ with $x_1u_1, x_2z_2 \in E(P_{1,2}) \cap E(B'_e)$, or it has an x_1x_2-hamiltonian path $P_{1,2}$ with $x_2z_2 \in E(P_{1,2}) \cap E(B'_e)$ and $u_1v_1 \in E(P_{1,2})$ for some $u_1, v_1 \in N(x_1)$. In the first case, we define an x_1x_2-hamiltonian path of G^2 as above; it contains $x_2z_2, x_3u_3 \in E(G)$. In the second case we proceed similarly: here,

$$E(P_{1,2}) \cup E(C_e) \cup \{u_1y_1, v_1z_1\} - \{u_1v_1, x_1y_1, x_1z_1\}$$

defines an x_1x_2-hamiltonian path containing $x_2z_2, x_3u_3 \in E(G)$. Thus G has the strong $\mathcal{F}_3$ property.

The case $X \subset V(B_e)$ needs no separate consideration since it implies $|X \cap V(B_f)| \leq 1$, in which case we may consider B_f instead of B_e. This finishes the proof of Theorem 3.

3. Arbitrary 2-connected graphs

We now proceed to prove the main result of this paper.

Theorem 4. *Let G be a 2-connected graph. Then G has the $\mathcal{F}_4$ property.*

Proof: We may assume that G is an edge-critical block since otherwise we can delete edges of G until we reach an edge-critical block.

If G is a DT-block, then the result is true by Theorem 1. So assume that G is not a DT-block. The rest of the proof is by induction on $|D(G)|$, or on $|V(G)|$. That is, if H is an edge-critical block with $|D(H)| < |D(G)|$ or $|V(H)| < |V(G)|$, then H has the $\mathcal{F}_4$ property.

By [7, Theorem 1], $D(G)$ contains an edge e such that $G - e$ is a block chain with at least one of its endblocks, say B_e, being a DT-block. Let B'_e be the other endblock of $G - e$.

Throughout, we let $e = xx'$ where $x \in V(B_e)$ and $x' \in V(B'_e)$.

We claim that $D(G)$ contains an edge e^* such that $G-e^*$ has an endblock B_{e^*} which is a DT-block satisfying $|V(B_e)\cap V(B_{e^*})| \leq 1$. To see this, we note that if B'_e is also a DT-block, then $e^* = e$ and $B'_e = B_{e^*}$, and the inequality holds trivially. If B'_e is not a DT-block, then it is edge-critical and again [7, Theorem 1] applies and e^* is in $D(G) \cap B'_e$, and B_{e^*} is a subgraph of B'_e. Since $|V(B_e) \cap V(B'_e)| \leq 1$ the claimed inequality holds.

Let $X = \{x_1, x_2, x_3, x_4\}$ and let $k = \min \{|V(B_e) \cap X|,\ |V(B_{e^*}) \cap X|\}$. Then clearly $k \leq 2$. Without loss of generality, we assume that $|V(B_e)\cap X| = k$.

We first dispose of the case $k = 0$ proceeding as in the proof of Theorem 3: we replace B_e by a path of length 3. The resulting graph H is an edge-critical block and $|D(H)| < |D(G)|$. By induction H has the $\mathcal{F}_4$ property. Any $\mathcal{F}_4$ hamiltonian path in H^2 can then be converted into an $\mathcal{F}_4$ hamiltonian path in G^2 by the same method used in [7].

Hence we assume that $k \in \{1, 2\}$.

Let c and c' be the cutvertices of $G - e$ belonging to B_e and B'_e respectively. Note that if $c = c'$, then $G - e$ is a block chain with only 2 blocks B_e and B'_e.

If $k = 2$, then we may assume without loss of generality that either $V(B_e) \cap X = \{x_3, x_4\}$ or $V(B_e) \cap X = \{x_2, x_4\}$, or $V(B_e) \cap X = \{x_1, x_2\}$ (namely, if $c = c' = x_i \in \{x_1, x_2\}$, $\{x_3, x_4\} \subseteq V(B_{e^*})$, and $B'_e = B_{e^*}$).

If $k = 1$, then we may assume without loss of generality that either $V(B_e) \cap X = \{x_2\}$ or $V(B_e) \cap X = \{x_4\}$. In any case, we note that $c \notin \{x_2, x_4\}$: otherwise, we replace B_e by a path of length 3 to obtain H which has an $\mathcal{F}_4$ x_1x_2-hamiltonian path in H^2. Again, as before, we apply the method used in [7] to see that any corresponding $\mathcal{F}_4$ hamiltonian path in H^2 can be converted into an $\mathcal{F}_4$ hamiltonian path in G^2.

Case (A): $c = c'$

(1) Suppose $k = 2$.

(a) Suppose $x_3, x_4 \in V(B_e)$. Then $x_1, x_2 \in V(B'_e)$ and there is an x_1x_2-hamiltonian path $P'(x_1, x_2)$ in $(B'_e)^2$ containing an edge cw' of B'_e, by Theorem F (note that $c \notin \{x_1, x_2\}$ by assumption). Let $w \in N(c) \cap V(B_e)$. Then, by Theorem 1, there is an $\mathcal{F}_4$ cw-hamiltonian path $P(c, w)$ in $(B_e)^2$ containing x_3z_3, x_4z_4 which are edges of B_e if $\{c, w\} \cap \{x_3, x_4\} = \emptyset$.

Suppose $|\{c, w\} \cap \{x_3, x_4\}| = 1$. Without loss of generality, assume that $x_3 \in \{c, w\}$. By Theorem 3, B_e has the strong $\mathcal{F}_3$ property. Consequently,

B_e^2 has a cw-hamiltonian path $P(c,w)$ containing x_4z_4, $cz \in E(B_e)$, $cz \neq cw$ if $x_3 = c$; or it contains x_4z_4, $wv \in E(B_e)$, $wv \neq cw$, if $x_3 = w$.

Suppose $\{c,w\} = \{x_3,x_4\}$. If $d_{B_e}(c) > 2$, then consider $u \in N(c) \cap V(B_e) - w$ such that $u \notin \{x_3,x_4\}$ and argue with u in place of w as in the preceding case. Thus we may assume that $d_{B_e}(c) = 2$. By Theorem E, $(B_e)^2$ has a $[c;w]$-hamiltonian cycle C_w containing cw, vw, cz which are three different edges of B_e. Let $C_w - cw = P(c,w)$.

By deleting in all cases the edge cw' from $P'(x_1,x_2)$ and adding $ww' \in E(G^2)$, we have a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

(b) Suppose $x_2, x_4 \in V(B_e)$. Then $x_1, x_3 \in V(B'_e)$. Note that $c \notin \{x_1,x_3\}$ since $k=2$.

If $x_2 \neq c$, then by Theorem F, there is an x_2c-hamiltonian path $P(x_2,c)$ in $(B_e)^2$ containing an edge x_4z_4 of B_e (independent of $x_4 = c$ or $x_4 \neq c$) and there is an x_1c-hamiltonian path $P'(x_1,c)$ in $(B'_e)^2$ containing $x_3z_3 \in E(B'_e)$. $P'(x_1,c)$ and $P(x_2,c)$ form a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

If $x_2 = c$, we apply Theorem E to B_e to obtain a hamiltonian cycle C in $(B_e)^2$ containing x_2v_1, x_2v_2, x_4z_4 which are edges of B_e. By Theorem 3, $(B'_e)^2$ has a strong $\mathcal{F}_3$ x_1x_2-hamiltonian path $P'(x_1,x_2)$ containing x_2w', x_3z_3 which are edges of B'_e. A required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2 is given by $(C - x_2v_1) \cup (P'(x_1,x_2) - x_2w') \cup \{v_1w'\}$.

(c) Suppose $V(B_e) \cap X = \{x_1,x_2\}$ with $x_i = c$ for some $i \in \{1,2\}$; without loss of generality $x_2 = c = c'$. $x_3, x_4 \in V(B'_e)$ follows (note that the case $c \notin \{x_1,x_2\}$ can be treated symmetrically to case (a)). By Theorem F, $(B_e)^2$ has an x_1x_2-hamiltonian path $P(x_1,x_2)$ containing an edge $x_2v \in E(B_e)$.

Suppose $w' \in (N(c') - \{x_3,x_4\}) \cap V(B'_e)$ exists. By induction, $(B'_e)^2$ has a $w'x_2$-hamiltonian path P' containing edges $x_3z_3, x_4z_4 \in E(B'_e)$. Clearly,

$$(P(x_1,x_2) \cup P' - x_2v) \cup \{w'v\}$$

defines an x_1x_2-hamiltonian path of G^2 as required.

Finally suppose $N(c') \cap V(B'_e) = \{x_3,x_4\}$. That is, $c' = c = x_2$ is 2-valent in B'_e. In this case we apply Theorem E to obtain a $[c';x_3]$-hamiltonian cycle C' in $(B'_e)^2$ containing three different edges $c'x_3, c'x_4, x_3z_3 \in E(B'_e)$.

It follows that

$$(P(x_1,x_2) - x_2v) \cup (C' - x_3c') \cup \{vx_3\}$$

defines an $\mathcal{F}_4$ x_1y_2-hamiltonian path of G^2 as required. This settles case (1).

(2) Suppose $k = 1$.

(a) Suppose $x_2 \in V(B_e)$. Then $x_1, x_3, x_4 \in V(B'_e) - c$. Hence by induction there is an $\mathcal{F}_4$ x_1c-hamiltonian path $P'(x_1, c)$ in $(B'_e)^2$ containing $x_3z_3, x_4z_4 \in E(B'_e)$. In $(B_e)^2$, there is an x_2c-hamiltonian path $P(x_2, c)$ which together with $P'(x_1, c)$ form a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

(b) Suppose $x_4 \in V(B_e)$. Then $x_1, x_2, x_3 \in V(B'_e) - c$ and by induction there is an $\mathcal{F}_4$ x_1x_2-hamiltonian path $P'(x_1, x_2)$ in $(B'_e)^2$ containing $x_3z_3, cw' \in E(B'_e)$. In $(B_e)^2$, there is a hamiltonian cycle C_c containing three different edges $cw, cz, x_4z_4 \in E(B_e)$ by Theorem E. Delete cw' from $P'(x_1, x_2)$ and cw from C_c and join w' to w to obtain a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2. This settles case (2) and thus finishes the proof of *Case (A)*.

Case (B): $c \neq c'$

Let $G_0 = G - (e \cup B_e \cup B'_e)$.

(1) Suppose $k = 2$.

In this case $(V(G_0) - \{c, c'\}) \cap X = \emptyset$. By Corollary 1, $(G_0)^2$ has a hamiltonian cycle C_0 containing $c'w'_0, cw_0$ which are edges of G_0, provided G_0 is a non-trivial block chain. If, however, $G_0 \neq K_2$ is a block, then such hamiltonian cycle C_0 exists by Theorem E. Moreover, we only have to deal with the cases (1.1), (1.2) below; otherwise, we could consider $B_e* \subseteq B'_e$.

(1.1) Suppose $x_3, x_4 \in V(B_e)$.

Then $x_1, x_2 \in V(B'_e)$. If $c' \notin \{x_1, x_2\}$, then by Theorem F*(i)*, $(B'_e)^2$ has an $\mathcal{F}_3$ x_1x_2-hamiltonian path $P'(x_1, x_2)$ containing an edge $c'w'$ of B'_e. If $c' \in \{x_1, x_2\}$, say $c' = x_1$, then we let $P'(x_1, x_2)$ denote an x_1x_2-hamiltonian path in $(B'_e)^2$ containing an edge $x_1w' = c'w'$ of B'_e (see Theorem F(ii)).

(a) Suppose $c = x_i$ for some $i \in \{3, 4\}$. Let C_c denote an $[x_i; x_{7-i}]$-hamiltonian cycle in $(B_e)^2$ containing $x_iz_i, x_iw_i, x_{7-i}z_{7-i}$ which are edges of B_e. In this case,

$$(P'(x_1, x_2) - c'w') \cup (C_0 - \{c'w'_0, cw_0\}) \cup (C_c - x_iw_i) \cup \{w'w'_0, w_0w_i\}$$

defines a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2 provided $G_0 \neq K_2$. If, however, $G_0 = K_2$, then

$$(P'(x_1, x_2) - c'w') \cup (C_c - x_iw_i) \cup \{c'w_i, cw'\}$$

yields the required result.

(b) Suppose $c \neq x_i$ for any $i \in \{3, 4\}$.

(i) Suppose $N(c) \cap \{x_3, x_4\} = \emptyset$. Let $w \in N(c) \cap V(B_e)$. By induction, there is an $\mathcal{F}_4$ cw-hamiltonian path $P_e(c, w)$ in $(B_e)^2$ containing x_3z_3, x_4z_4 which are edges of B_e. In this case,

$$(P'(x_1, x_2) - c'w') \cup (C_0 - \{c'w'_0, cw_0\}) \cup P_e(c, w) \cup \{w'w'_0, ww_0\}$$

yields a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2; and if $G_0 = K_2$, then we obtain the required result analogously as in case (a).

(ii) Hence we assume that $N(c) \cap \{x_3, x_4\} \neq \emptyset$.

If there exists $w \in N(c) \cap V(B_e)$ such that $w \notin \{x_3, x_4\}$, then the argument used in (i) applies and we have a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2 as before.

So assume that $N(c) \cap V(B_e) = \{x_3, x_4\}$. Let C_c denote an $[c; x_3]$-hamiltonian cycle in $(B_e)^2$ containing x_3c, x_3z_3, x_4z_4 which are edges of B_e. Then

$$(P'(x_1, x_2) - c'w') \cup (C_0 - \{c'w'_0, cw_0\}) \cup (C_c - x_3c) \cup \{w'w'_0, w_0x_3\}$$

yields a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2 if $G_0 \neq K_2$; and the case $G_0 = K_2$ is treated analogously as before.

(1.2) Suppose $x_2, x_4 \in V(B_e)$.

Then $x_1, x_3 \in V(B'_e)$. If $x_1 \neq c'$, then by Theorem F, $(B'_e)^2$ has an $\mathcal{F}_3$ x_1c'-hamiltonian path $P'(x_1, c')$ containing an edge x_3z_3 of B'_e (even if $x_3 = c'$). If $c' = x_1$, then by Theorem E, $(B'_e)^2$ has an $[x_1; x_3]$-hamiltonian cycle C' containing three edges $x_1w_1, x_1z_1, x_3z_3 \in E(B'_e)$. In this case, let $P'(x_1, w_1) = C' - x_1w_1$.

Consider B_e. If $x_2 \neq c$, then by Theorem F, $(B_e)^2$ has an $\mathcal{F}_3$ x_2c-hamiltonian path $P(x_2, c)$ containing an edge x_4z_4 of B_e (even if $x_4 = c$). If $c = x_2$, then by Theorem E, $(B_e)^2$ has an $[x_2; x_4]$-hamiltonian cycle C containing x_2w_2, x_2z_2, x_4z_4 which are edges of B_e. In this case, let $P(x_2, w_2) = C - x_2w_2$.

(a) Suppose $G_0 \neq K_2$.

By Corollary 1 *(ii)*, Theorem F respectively, $(G_0)^2$ has a cc'-hamiltonian path $P_0(c, c')$ containing an edge cw_0 of G_0 incident to c, or an edge $c'w'_0$ of G_0 incident to c'. In the case that G_0 has 2 or more blocks, then $P_0(c, c')$ can be chosen to contain both cw_0 and $c'w'_0$.

(i) Suppose $c \neq x_2$ and $c' \neq x_1$. Then $P'(x_1, c') \cup P_0(c', c) \cup P(x_2, c)$ yields a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

(ii) Suppose $c \neq x_2$ and $c' = x_1$. Then $P'(x_1, w_1) \cup (P_0(c', c) - c'w_0') \cup P(x_2, c) \cup \{w_0'w_1\}$ yields a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

(iii) Suppose $c = x_2$ and $c' \neq x_1$. Then $P'(x_1, c') \cup (P_0(c', c) - cw_0) \cup P(x_2, w_2) \cup \{w_0w_2\}$ yields a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

(iv) Suppose $c = x_2$ and $c' = x_1$.

First assume that G_0 has 2 or more blocks. Then

$$P'(x_1, w_1) \cup (P_0(c', c) - \{c'w_0', cw_0\}) \cup P(x_2, w_2) \cup \{w_0'w_1, w_0w_2\}$$

yields a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

Next assume that G_0 is 2-connected. By Theorem 2, $(G_0)^2$ has a cc'-hamiltonian path $P_0(c, c')$ containing an edge cw_0 of G_0 and $P_0(c, c')$ either contains an edge $c'w_0'$ of G_0 or else contains an edge uv for some vertices $u, v \in N(c') \cap V(G_0)$. In the former case, we proceed as in the case where G_0 has 2 or more blocks, to obtain a required x_1x_2-hamiltonian path in G^2. In the latter case,

$$P(x_2, w_2) \cup (P_0(c, c') - \{cw_0, uv\}) \cup (P'(x_1, w_1) - \{z_1x_1\}) \cup \{w_2w_0, w_1v, z_1u\}$$

yields a required x_1x_2-hamiltonian path in G^2.

(b) Suppose $G_0 = K_2$.

If $c \neq x_2$ or $c' \neq x_1$, then the methods used in the above cases (a) (i), (ii), (iii) can be used to construct a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2. Hence we assume that $c = x_2, c' = x_1$. Then by Theorem F, $(B_e)^2$ (respectively $(B_e')^2$) has an $\mathcal{F}_3$ x_2x-hamiltonian path $P(x_2, x)$ containing x_4z_4 (respectively x_1x'-hamiltonian path $P'(x_1, x')$ containing x_3z_3) where $x_3z_3, x_4z_4 \in E(G)$ (even if $x_4 = x$ and $x_3 = x'$). Then $P(x_2, x) \cup \{xx'\} \cup P'(x_1, x')$ is a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

(2) Suppose $k = 1$.

Recall that, in this case, either $V(B_e) \cap X = \{x_2\}$ or else $V(B_e) \cap X = \{x_4\}$ and that $c \notin \{x_2, x_4\}$.

(2.1) Suppose $x_2 \in V(B_e)$ and $x_2 = x$.

Write the block chain $G - e$ as $B_1 \cup B_2 \cup \cdots \cup B_k$, $k > 2$ with $B_i \cap B_{i+1} = c_i$ for $i = 1, 2, \ldots, k-1$ where $B_1 = B_e'$ and $B_k = B_e$ so that $c_1 = c'$, $c_{k-1} = c$ and $G_0 = B_2 \cup \cdots \cup B_{k-1}$.

(a) Suppose $(V(B'_e) - c_1) \cap X = \emptyset$ and $c_1 = x_1$.

Then either (i) B'_e is a DT-graph or else (ii) B'_e contains an edge $f \in D(G)$ such that one of its endblocks B_f of $G - f$ is a DT-graph of B'_e (and thus of $G - e$). Moreover, if $x_1 \in V(B_f)$, then x_1 is a cutvertex of $G - f$ (see Theorem G).

In either case, we reduce G to the graph H by replacing either B'_e (in case (i)) or else B_f (in case (ii)) by a path of length 3. By induction, H^2 has an $\mathcal{F}_4$ x_1x_2-hamiltonian path. This hamiltonian path can be converted to an $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2 by the same method used in [7].

(b) Suppose $x_1 \in V(B'_e) - c_1$.

In $(B_1)^2$, we take an x_1c_1-hamiltonian path $P_1(x_1, c_1)$ containing edges of B_1 incident to x_r if x_r is in B_1 for every $r \in \{3, 4\}$. For each $i \in \{2, \ldots, k-1\}$, we take a $c_{i-1}c_i$-hamiltonian path $P_i(c_{i-1}, c_i)$ in $(B_i)^2$ containing edges of B_i incident to x_r if $x_r \in V(B_i)$, for every $r \in \{3, 4\}$. In $(B_k)^2$, we take a $c_{k-1}x_2$-hamiltonian path $P_k(c_{k-1}, x_2)$. Note that this is always possible either trivially or by induction (to get an $\mathcal{F}_4$ hamiltonian path in $(B_i)^2$) or by Theorem 3 (to get a strong $\mathcal{F}_3$ hamiltonian path in $(B_i)^2$).Then

$$P_1(x_1, c_1) \cup P_2(c_1, c_2) \cup \cdots \cup P_k(c_{k-1}, x_2)$$

yields a required x_1x_2-hamiltonian path in G^2.

Consequently, if $x_1 \notin V(B'_e) - c_1$, then $x_r \in V(B'_e) - c_1$ for at least one $r \in \{3, 4\}$.

(c) Suppose $x_1 = c_1$.

Because of case (a) settled already, we have $x_r \in V(B'_e) - c_1$ for at least one $r \in \{3, 4\}$.

(i) Suppose $x_3, x_4 \in V(B'_e) - c_1$.

Proceeding as in case (b), we can construct a c_1x_2-hamiltonian path $P_2(c_1, x_2)$ in $(G_0 \cup B_e)^2$ containing an edge c_1w_1 of B_2.

If there is a vertex $w \in N(c_1) \cap V(B_1)$ such that $w \notin \{x_3, x_4\}$, then by induction let $P_1(x_1, w)$ be an $\mathcal{F}_4$ x_1w-hamiltonian path in $(B_1)^2$ containing an edge of B_1 incident to x_r for each $r \in \{3, 4\}$. A required x_1x_2-hamiltonian path in G^2 is given by $P_1(x_1, w) \cup (P_2(c_1, x_2) - c_1w_1) \cup \{ww_1\}$.

So assume that $N(c_1) \cap V(B_1) = \{x_3, x_4\}$. Then by Theorem E let C_1 be an $[x_1; x_3]$-hamiltonian cycle in $(B_1)^2$ containing x_1x_3, x_1x_4, x_3w_3 which are edges of B_1. Let $P_1(x_1, x_3) = C_1 - x_1x_3$. Then a required x_1x_2-hamiltonian path in G^2 is given by $P_1(x_1, x_3) \cup (P_2(c_1, x_2) - c_1w_1) \cup \{x_3w_1\}$.

(ii) Suppose $x_3 \in V(B'_e) - c_1$ and $x_4 \in V(G_0)$.

Let $w \in N(x_1) \cap V(B_1)$. By Theorem F, there is an x_1w-hamiltonian path $P_1(x_1, w)$ in $(B_1)^2$ containing an edge x_3w_3 of B_1. As in case (i), we can construct an x_1x_2-hamiltonian path $P_2(x_1, x_2)$ in $(G_0 \cup B_e)^2$ containing an edge x_1w_1 of B_2 and an edge x_4w_4 of G_0 (apply Theorem 3 if $w_1, x_4 \in V(B_2)$ and apply Theorem F otherwise). Then a required x_1x_2-hamiltonian path in G^2 is given by $P_1(x_1, w) \cup (P_2(x_1, x_2) - x_1w_1) \cup \{ww_1\}$.

(d) Suppose $x_1 \in V(G_0) - c_1$.

Then $x_1 \in V(B_t)$ for some $t \in \{2, \ldots, k-1\}$. In the case that x_1 is a cutvertex of $G - e$, then $x_1 = c_t$ with $t < k-1$. Let $G_t = B_1 \cup \cdots \cup B_t$ and $H_t = B_{t+1} \cup \cdots \cup B_k$.

(i) Suppose $\{x_3, x_4\} \subseteq V(G_t)$.

Then by induction or by applying Theorem F or Theorem 3 to each 2-connected block of G_t, we can construct an x_1x'-hamiltonian path $P_1(x_1, x')$ in $(G_t)^2$ containing x_3z_3, x_4z_4 which are edges of G_t. Since $X \cap (V(H_t) - c_t) = \{x_2\}$, by applying Theorem E to each 2-connected block of H_t, we can construct a hamiltonian cycle C_e in $(H_t)^2 - c_t$ containing an edge x_2v of B_e. Then a required x_1x_2-hamiltonian path in G^2 is defined by $P_1(x_1, x') \cup (C_e - x_2v) \cup \{x'v\}$.

(ii) Suppose $\{x_3, x_4\} \cap V(G_t) \neq \emptyset$ and $\{x_3, x_4\} \cap V(H_t) \neq \emptyset$.

Assume without loss of generality that $x_3 \in V(G_t)$ and $x_4 \in (V(H_t) - V(B_k))$. Because of the preceding discussion, we have $x_3 \in V(B_1) - c_1$.

Suppose $x_4 \in V(B_q)$ where $t < q < k$. Split H_t into two block chains J_t and L_q where $J_t = B_{t+1} \cup \cdots \cup B_q$ and x_4 is not a cutvertex of J_t; and $L_q = B_{q+1} \cup \cdots \cup B_k$.

Let $P_1(x_1, x')$ denote an x_1x'-hamiltonian path in $(G_t)^2$ containing x_3w_3, c_tw_t which are edges of G_t. Note that this is possible because $x_3 \neq c_t$, $|V(B_t) \cap X| < 3$ and by applying Theorem 3 or Theorem F, respectively.

Let C_4 denote an hamiltonian cycle in $(J_t)^2$ containing c_tz_t, x_4z_4 which are edges of J_t. Note that this is possible by applying Theorem E to each block of J_t, provided J_t is not a bridge of H_t. In the case that J_t is a bridge c_tx_4, then C_4 denotes c_tx_4 in $(J_t)^2$.

Proceed analogously to case (i) to obtain a hamiltonian cycle C_e in $(L_q)^2 - c_q$ containing an edge x_2v of B_e. Then a required x_1x_2-hamiltonian path in G^2 is defined by

$$(P_1(x_1, x') - c_tw_t) \cup (C_4 - c_tz_t) \cup (C_e - x_2v) \cup \{w_tz_t, x'v\}$$

if J_t is not a bridge; otherwise, it is defined by

$$(P_1(x_1, x') - c_t w_t) \cup \{c_t x_4\} \cup (C_e - x_2 v) \cup \{w_t x_4, x' v\}.$$

This settles case (d) and thus finishes the proof of case (2.1).

For the remaining cases of the proof of the theorem, we adopt a different strategy of proof. For this purpose, let B^+ denote the graph obtained from $B'_e \cup G_0$ by adding a new edge cx'. Then B^+ is an edge-critical block. Since $|V(B^+)| < |V(G)|$, by induction, B^+ has the $\mathcal{F}_4$ property. Also, as before it is tacitly assumed that the hamiltonian paths constructed in the $(B^+)^2$ will traverse as many edges of B^+ as possible.

We note that, $E((B^+)^2) = E((B'_e \cup G_0)^2) \cup E^+$ where

$$E^+ = \{cx'\} \cup \{x' w_c, u' c \mid w_c \in N(c) \cap V(G_0), u' \in N(x') \cap V(B'_e)\}.$$

In what follows, any vertex in $N(c) \cap V(G_0)$ will be subscribed with c, and any vertex in $N(x') \cap V(B'_e)$ will be superscribed with $'$. Also, we use y to denote a neighbor of x in B_e.

(2.2) Suppose $x_2 \in V(B_e)$ and $x_2 \neq x$.

Then $x_1, x_3, x_4 \in V(B^+) - \{c\}$. Let $P^+(x_1, c)$ denote an $\mathcal{F}_4$ x_1c-hamiltonian path in $(B^+)^2$ containing x_3z_3, x_4z_4 which are different edges of B^+ using induction. Note that $x_i z_i = x'c$ is possible for $i \in \{3, 4\}$.

Set $E^* = E(P^+(x_1, c)) \cap E^+$ and set $|E^*| = r$. Clearly, $0 \leq r \leq 3$. Observe that $r = 4$ would imply that x' and c are internal vertices of the corresponding hamiltonian path, which is not possible. However, the case $r = 3$ could be reduced to the case (b) (i) below traversing more edges of B^+ than the original path. Thus $r = 3$ is also impossible.

(a) Suppose $r = 0$ in which case $x_i z_i \neq x'c$ for $i = 3, 4$.

Trivially $(B_e)^2$ has an x_2c-hamiltonian path $P_2(x_2, c)$ (since $x_2 \notin \{c, x\}$). Then $P^+(x_1, c) \cup P_2(x_2, c)$ defines a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

(b) Suppose $r = 1$.

By Theorem F *(i)*, $(B_e)^2$ has an x_2x-hamiltonian path $P_2(x_2, x)$ containing an edge cw of B_e.

(i) If $E^* = \{cx'\}$, then $(P^+(x_1, c) - cx') \cup P_2(x_2, x) \cup \{xx'\}$ defines a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2. Note that $x_i z_i = x'c$ for $i \in \{3, 4\}$ is not an obstacle.

From now on we can assume that $x_iz_i \neq x'c$ for $i = 3, 4$.

(ii) If $E^* = \{cu'\}$, then $(P^+(x_1, c) - cu') \cup P_2(x_2, x) \cup \{xu'\}$ defines a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

(iii) If $E^* = \{x'w_c\}$, then $(P^+(x_1, c) - x'w_c) \cup (P_2(x_2, x) - cw) \cup \{xx', ww_c\}$ defines a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2. This holds true even if $w_c \in \{x_3, x_4\}$ and $w_cc \in E(P^+(x_1, c))$.

(c) Suppose $r = 2$.

By Theorem F, $(B_e)^2$ has an x_2c-hamiltonian path $P_2(x_2, c)$ containing an edge xy of B_e.

(i) If $E^* = \{cx', x'w_c\}$, then

$$(P^+(x_1, c) - \{cx', x'w_c\}) \cup (P_2(x_2, c) - xy) \cup \{yx', xx', cw_c\}$$

defines a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2, even if $\{x_3, x_4\} \cap \{x', w_c\} \neq \emptyset$. Note that $x_iz_i = x'c$ for $i \in \{3, 4\}$ is not an obstacle.

From now on we ca assume that $x_iz_i \neq x'c$ for $i = 3, 4$.

(ii) If $E^* = \{x'u_c, x'w_c\}$, then

$$(P^+(x_1, c) - \{x'u_c, x'w_c\}) \cup (P_2(x_2, c) - xy) \cup \{yx', xx', w_cu_c\}$$

defines a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

(iii) If $E^* = \{x'u_c, w'c\}$, then

$$(P^+(x_1, c) - \{x'u_c, w'c\}) \cup (P_2(x_2, c)) \cup \{cu_c, w'x'\}$$

defines a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

(2.3) Suppose $x_4 \in V(B_e)$.

Then $x_1, x_2, x_3 \in V(B^+) - \{c\}$. Let $P^+(x_1, x_2)$ denote an $\mathcal{F}_4$ x_1x_2-hamiltonian path in $(B^+)^2$ containing x_3z_3, cc^*, where $c^* \in \{w_c, x'\}$, which are different edges of B^+ using induction. Note that $x_3z_3 = x'c$ is possible.

Now set $E^* = E(P^+(x_1, x_2)) \cap E^+$ and set $|E^*| = r$. Clearly, $0 \leq r \leq 4$. Note that the case $r = 4$ yields a contradiction just as did the case $r = 3$ in the subcase (2.2) above.

(a) Suppose $r = 0$, in which case $c^* = w_c$ and $x_3z_3 \neq x'c$.

By Theorem E, $(B_e)^2$ has a $[c; x_4]$-hamiltonian cycle C_e containing cw, cz, x_4z_4 which are edges of B_e. Then $(P^+(x_1, x_2) - cw_c) \cup (C_e - cw) \cup \{ww_c\}$ defines a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

(b) Suppose $r = 1$.

(i) Suppose $E^* = \{x'c\}$ or $E^* = \{w'c\}$. By Theorem F, $(B_e)^2$ has a cx-hamiltonian path $P_4(c, x)$ containing an edge x_4z_4 of B_e. Then $(P^+(x_1, x_2) - zc) \cup P_4(c, x) \cup \{zx\}$ is a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2 for any vertex $z \in \{x', w'\}$. Note that either $c^* = x'$ or $x_3z_3 = x'c$ is not an obstacle.

(ii) Suppose $E^* = \{x'u_c\}$. Hence $c^* = w_c$ and $x_3z_3 \neq x'c$. By Theorem 3, $(B_e)^2$ has a cx-hamiltonian path $P_4(c, x)$ containing cw, x_4z_4 which are edges of B_e. As for the x_1x_2-hamiltonian path $P^+(x_1, x_2)$ in $(B^+)^2$ we possibly have $u_c = w_c$. In any case, $(P^+(x_1, x_2) - x'u_c) \cup (P_4(c, x) - cw) \cup \{x'x, wu_c\}$ defines a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

(c) Suppose $r = 2$. Note that $x_4 \neq c$ in this case and $xc \notin E(B_e)$ because of G is edge-critical.

(c1) $E^* = \{x'c, w'c\}$, in which case $c^* = x'$ and hence $x_3z_3 \neq x'c$. By Theorem F, $(B_e)^2$ has an xy-hamiltonian path $P_4(x, y)$ containing an edge x_4z_4 of B_e. Then $(P^+(x_1, x_2) - \{x'c, w'c\}) \cup P_4(x, y) \cup \{x'y, w'x\}$ yields a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

(c2) $E^* = \{x'c, x'u_c\}$.

Let $y \in N(x) \cap V(B_e)$ where $y \neq x_4$, and let $P_4(x, y)$ be an $\mathcal{F}_4$ xy-hamiltonian path in $(B_e)^2$ containing x_4z_4, cw which are edges of B_e (by induction or by Theorem 3). Then $(P^+(x_1, x_2) - \{x'c, x'u_c\}) \cup (P_4(x, y) - cw) \cup \{x'x, x'y, wu_c\}$ results in a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2. Note that either $c^* = x'$ or $x_3z_3 = x'c$ is not an obstacle.

From now on we can assume that $c^* = w_c$ and $x_3z_3 \neq x'c$.

(c3) $E^* = \{x'y_c, x'u_c\}$.

By Lemma 3, there is a $[c; x, x_4]$-hamiltonian cycle C_e in $(B_e)^2$ containing cw, cu, xy, x_4z_4 which are edges of B_e provided $x_4 \neq x$; otherwise, let C_4 be an $[c; x_4]$-hamiltonian cycle of $(B_e)^2$ containing cw, cu, xy which are edges of B_e resulting from an application of Theorem E. Then

$$(P^+(x_1, x_2) - \{x'y_c, x'u_c\}) \cup (C_e - \{cw, cu, xy\}) \cup \{wy_c, uu_c, xx', yx'\}$$

defines a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2 independent of the position of x_4.

(c4) $E^* = \{x'y_c, w'c\}$. Then there are two subcases to consider.

(i) Suppose $y_c = w_c$. Let $P_4(x, y)$ be as defined in case (c2). Then

$$(P^+(x_1, x_2) - \{x'y_c, w'c\}) \cup (P_4(x, y) - cw) \cup \{w'x, x'y, wy_c\}$$

yields a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

(ii) Suppose $y_c \neq w_c$. There are four possibilities.

If $P^+(x_1, x_2)$ takes the form $x_1 \cdots x'y_c \cdots w_c c w' \cdots x_2$, then proceed as in (i) to obtain a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

If $P^+(x_1, x_2)$ takes the form

$$x_1 \cdots y_c x' \cdots w_c c w' \cdots x_2 \text{ or } x_1 \cdots x' y_c \cdots w' c w_c \cdots x_2,$$

then we can reduce this case to case (a) where $r = 0$ as follows. Delete $x'y_c, w'c$ from $P^+(x_1, x_2)$ and add to it the edges $x'w', cy_c$.

If $P^+(x_1, x_2)$ takes the form $x_1 \cdots y_c x' \cdots w' c w_c \cdots x_2$, then let $P_4(x, y)$ denote an xy-hamiltonian path in $(B_e)^2$ as defined in case (c2). Then

$$(P^+(x_1, x_2) - \{x'y_c, w'c\}) \cup (P_4(x, y) - cw) \cup \{w'x, x'y, y_c w\}$$

defines a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

The other cases are symmetrical.

(c5) $E^* = \{cv', cw'\}$. This case cannot happen since $E^* \cap E(B^+) = \emptyset$, but $cw_c \in E(P_2(x_1, x_2)) \cap E(B^+)$.

(d) Suppose $r = 3$. Thus E^* must be one of the following three sets: $E^* = \{x'u_c, x'c, w'c\}$, $E^* = \{x'u_c, x'y_c, w'c\}$, $E^* = \{w'c, v'c, x'u_c\}$. It is now straightforward to see that in each of these three cases the corresponding $P^+(x_1, x_2)$ can be modified so as to contain more edges of B^+ and satisfying $E^* = \{x'c\}$, i.e., $r = 1$. Namely, in the respective case
form $(\{x'u_c, x'c, w'c\} - \{x'u_c, w'c\}) \cup \{w'x', cu_c\}$;
replace $\{x'u_c, x'y_c, w'c\}$ with $\{w'x', x'c, u_c y_c\}$;
replace $\{w'c, v'c, x'u_c\}$ with $\{w'v', x'c, cu_c\}$.

Theorem 4 now follows. □

As a special case of Theorem 4 we obtain the following.

Corollary 2. *Let G be a 2-connected graph on $n \geq 4$ vertices, and let $e = xy \in E(G)$ and $u, v \in V(G)$ such that $\{x, y\} \cap \{u, v\} = \emptyset$. Then G^2 has a hamiltonian cycle C with $e \in E(C)$, and at least one of the edges of u in C at least one of the edges of v in C are edges of G.*

4. Final remarks

In subsequent papers we shall use some of the theorems of this paper to describe (among other results) the most general structure a graph may have such that its square is hamiltonian or hamiltonian connected, respectively. This will also solve a problem raised in [4] in the affirmative and proves a conjecture raised in [19]; we shall also present a partial solution of a conjecture stated in [5].

It is easy to see that the complete bipartite graph $K_{2,k-2}$ does not have the $\mathcal{F}_k$ property for every integer $k \geq 5$. For example, take x_1, x_2 to be the two vertices of degree $k-2$ and $x_3, \ldots, x_k$ to be the rest of the vertices. Hence, Theorem 4 is best possible.

A graph G is said to have the $\overline{\mathcal{F}}$ *property* if it has three 2-valent vertices x, y, z such that $N(x) = N(y) = N(z)$. From the above observation, we see that if G has the $\overline{\mathcal{F}}$ property, then G does not have the $\mathcal{F}_k$ property for any $k \geq 5$.

While it is now known that Theorem F(i) can be generalized to Theorem 4, it is also of interest to know whether or not Theorem E can be generalized to 3 given vertices. That is, given three arbitrary vertices v, w_1, w_2 of a 2-connected graph G, does G^2 contain a $[v; w_1, w_2]$-hamiltonian cycle C? The following example shows that this is not true in general.

Let $k \geq 5$ be an integer and let $v_1 v_2 \cdots v_n v_1$ be a cycle with n vertices where $n \geq k+3$. Take a new vertex v and join it to v_1 and v_k to get the graph H. Let $w_1 = v_1$ and $w_2 = v_k$. Then it is easy to see that H^2 admits no hamiltonian cycle C containing the edges vw_1, vw_2 and $w_i z_i$ where $z_i \in N(w_i)$, $i = 1, 2$.

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