Applying parabolic Peterson: affine algebras and the quantum cohomology of the Grassmannian

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The Peterson isomorphism relates the homology of the affine Grassmannian to the quantum cohomology of any flag variety. In the case of a partial flag, Peterson's map is only a surjection, and one needs to quotient by a suitable ideal on the affine side to map isomorphically onto the quantum cohomology. We provide a detailed exposition of this parabolic Peterson isomorphism in the case of the Grassmannian of m-planes in complex n-space, including an explicit recipe for doing quantum Schubert calculus in terms of the appropriate subset of non-commutative k-Schur functions. As an application, we recast Postnikov's affine approach to the quantum cohomology of the Grassmannian as a consequence of parabolic Peterson by showing that the affine nilTemperley-Lieb algebra arises naturally when forming the requisite quotient of the homology of the affine Grassmannian.

1. Introduction

The theory of quantum cohomology was developed in the early 1990s by physicists working in the field of superstring theory, who were able to provide a partial answer to the Clemens conjecture counting the number of rational curves of a given degree on a general quintic threefold. A mathematical formulation of quantum cohomology pioneered by Givental formally proved the answer proposed by the physicists [Giv96], and simultaneously introduced the Gromov-Witten invariants in joint work with Kim [GK95].

The (small) quantum cohomology ring is a deformation of the classical co-homology by a sequence of quantum parameters q, and the Schubert basis elements then form a $\mathbb{Z}[q]$-basis for the quantum cohomology. The theory of quantum Schubert calculus seeks non-recursive, positive combinatorial formulas for expressing the quantum product of two Schubert classes in terms of the Schubert basis.

The Grassmannian has emerged as a favorite test case among all homogeneous spaces for carrying out the quantum Schubert calculus program. In the type A Grassmannian $Gr(m, n)$, the Schubert classes are indexed by the elements of $S_n/(S_m \times S_{n-m})$, or equivalently by Young diagrams fitting inside an $m \times (n - m)$ rectangle. The product of two quantum Schubert classes in $QH^*(Gr(m, n))$ is a $\mathbb{Z}_{\geq 0}[q]$-linear combination of the Schubert basis elements, and the associated quantum Littlewood-Richardson coefficients count the number of puzzles corresponding to a certain 2-step flag variety, as proved by Buch, Kresch, Purbhoo, and Tamvakis in [BKPT16].

In a series of lectures delivered at MIT in the late 1990s, Peterson developed a deep connection between the equivariant quantum cohomology of any partial flag variety G/P and the equivariant homology of the affine Grassmannian $\mathcal{G}r_G$ [Pet96]. This result, later proved by Lam and Shimozono [LS10], says that there is a surjective homomorphism of Hopf algebras

$$H_*^T(\mathcal{G}r_G) \twoheadrightarrow QH_T^*(G/P),$$

up to localization. In the case of the complete flag variety G/B, the map above is in fact an isomorphism, and otherwise, one needs to mod out by a suitable ideal J_P on the affine side in order to map injectively onto the smaller ring $QH_T^*(G/P)$. Peterson's isomorphism also admits a non-equivariant shadow, and it is one goal of this paper to provide a detailed account of the parabolic Peterson isomorphism for the type A Grassmannian, which says that after a suitable localization,

$$H_*(\mathcal{G}r_{SL_n})/J_P \cong QH^*(Gr(m, n)).$$

In the type A setting, the Schubert classes in the homology of the affine Grassmannian were shown by Lam to be represented by the k-Schur functions of Lapointe and Morse [LM05]. The approach in [Lam08] proves that the non-commutative k-Schur functions defined in [Lam06] span the affine Fomin-Stanley subalgebra $\mathbb{B}$ of the nilHecke ring of Kostant and Kumar [KK86], and in turn that $H_*(\mathcal{G}r_{SL_n}) \cong \mathbb{B}$. The non-commutative k-Schur functions are indexed by minimal length elements in the quotient of the

affine symmetric group $\tilde{S}_n/S_n$. Any such $w \in \tilde{S}_n/S_n$ corresponds bijectively to a k-bounded partition $\lambda(w)$ having first part no larger than $k = n - 1$. The fundamental goal of affine Schubert calculus is to combinatorially describe the multiplication of two affine Schubert classes, as modeled by the non-commutative k-Schur functions $\mathbf{s}_{\lambda(w)}^{(k)}$ or some other variation; see the book [LLM$^+$14] for a comprehensive survey.

1.1. Statement of the main results

The first part of this paper is devoted to explaining how to perform calculations in $QH^*(Gr(m,n))$ using the parabolic Peterson isomorphism, which we suitably rephrase in Theorem 1.1 for this purpose. The following statement combines Propositions 3.5 and 3.9 with Theorem 3.11.

Theorem 1.1. *Let P be the maximal parabolic subgroup such that $SL_n(\mathbb{C})/P \cong Gr(m,n)$, and let v be any minimal length coset representative in $S_n/(S_m \times S_{n-m})$. The Peterson isomorphism gives a correspondence between (representatives of) non-commutative k-Schur functions and quantum Schubert classes:*

$$
(1) \qquad \Psi_P : (H_*(\mathcal{G}r_{SL_n})/J_P)[(\overline{\mathbf{s}}_{\lambda(u)}^{(k)})^{-1}] \longrightarrow QH^*(Gr(m,n))[q^{-1}]
$$
$$
\overline{\mathbf{s}}_{\lambda(vu^r)}^{(k)} \longmapsto q^{-r}\sigma_v,
$$

where

$$
(2) \qquad u = s_{m,m+1,\ldots,n-1,m-1,\ldots,1,0}
$$

is an element of the affine symmetric group $\tilde{S}_n$ whose corresponding k-bounded partition $\lambda(u)$ is a hook shape. Here, r is any positive integer such that vu^r is a minimal length coset representative in $\tilde{S}_n/S_n$. Moreover, if $w \in \tilde{S}_n/S_n$ supports a braid relation, then $\mathbf{s}_{\lambda(w)}^{(k)} \in J_P$ so that $\overline{\mathbf{s}}_{\lambda(w)}^{(k)} = 0$.

We illustrate the ideal J_P corresponding to $Gr(1,3)$ in Figure 1. See Section 2 for the definitions of all terms appearing in the statement of Theorem 1.1, the proof of which is found in Section 3.

Our primary motivation in proving Theorem 1.1 is not simply to reproduce formulas for quantum Schubert products in $QH^*(Gr(m,n))$. We further aim to demonstrate that certain existing results about the quantum cohomology of the Grassmannian which were discovered independently can be reinterpreted as corollaries of the parabolic Peterson isomorphism. To
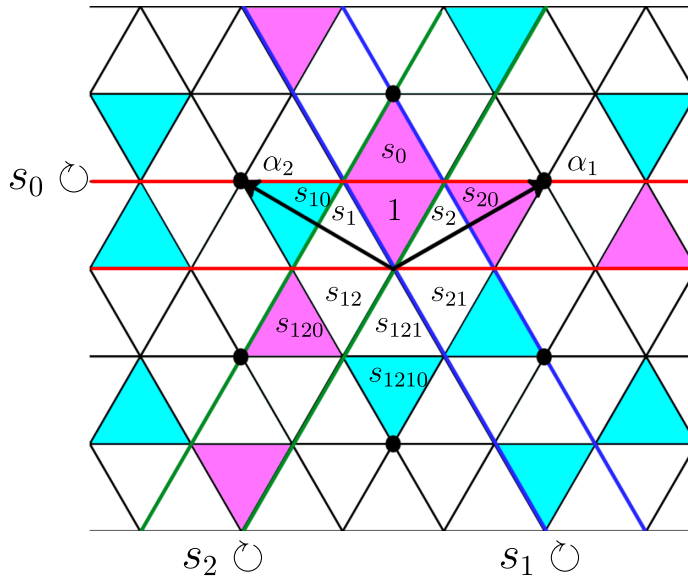
Figure 1: The elements of J_P for the parabolic P corresponding to $Gr(1,3)$ are colored blue. The minimal length coset representatives which do not lie in J_P are colored pink.

provide a concrete example, our primary application in Theorem 1.2 recasts Postnikov's affine approach to quantum Schubert calculus from [Pos05] as a consequence of Theorem 1.1.

In [Pos05], Postnikov proves that a localized subalgebra of the affine nilTemperley-Lieb algebra generated by the elements $\tilde{\mathbf{e}}_r^n$ and $\tilde{\mathbf{h}}_r^n$ is isomorphic to the localization $QH^*(Gr(m,n))[q^{-1}]$. Moreover, under this isomorphism, the generators $\tilde{\mathbf{e}}_r^n$ and $\tilde{\mathbf{h}}_r^n$ map to the Schubert classes $\sigma_{(1^r)}$ and $\sigma_{(r)}$, respectively. In Section 4, we prove that Postnikov's isomorphism is the composition of the parabolic Peterson isomorphism in Theorem 1.1, followed by two duality isomorphisms. The following result combines Theorems 4.7 and 4.9 with Corollary 4.10.

Theorem 1.2. *Let P be the maximal parabolic subgroup such that $SL_n(\mathbb{C})/P \cong Gr(m,n)$. The subalgebra $\mathbb{X}$ of the localized affine nil-Templerley-Lieb algebra $n\widehat{TL_{mn}}$ generated by $\tilde{\mathbf{e}}_r^n$ and/or $\tilde{\mathbf{h}}_r^n$ is isomorphic to the quotient of the affine Fomin-Stanley algebra $\mathbb{B}$ appearing in Theorem 1.1:*

$$(3) \qquad \chi : (H_*(\mathcal{G}r_{SL_n})/J_P)[(\bar{\mathbf{s}}_{\lambda(u)}^{(k)})^{-1}] \longrightarrow \mathbb{X},$$

and the isomorphism χ is induced by the natural projection $\mathbb{B} \longrightarrow \mathbb{X}$ killing any element which supports a braid relation.

Let T be the transpose map and ι the strange duality involution reviewed in Theorem 4.4. The following composition is also an isomorphism and maps the generators as follows:

$$
\begin{aligned}
T \circ \iota \circ \Psi_P : (H_*(\mathcal{G}r_{SL_n})/J_P)[(\bar{\mathbf{s}}_{\lambda(u)}^{(k)})^{-1}] &\longrightarrow QH^*(Gr(n-m,n))[q^{-1}] \\
\tilde{\mathbf{e}}_r &\longmapsto \sigma_{(1^r)} \\
\tilde{\mathbf{h}}_r &\longmapsto \sigma_{(r)}.
\end{aligned}
\tag{4}
$$

Therefore, if P' is the maximal parabolic subgroup such that $SL_n(\mathbb{C})/P' \cong Gr(n-m,n)$, then Postnikov's isomorphism is in fact the composition of these isomorphisms

$$
T \circ \iota \circ \Psi_{P'} \circ \chi^{-1} : \mathbb{X} \longrightarrow QH^*(Gr(m,n))[q^{-1}],
\tag{5}
$$

providing an independent proof of Theorem 4.3 from [Pos05].

1.2. Directions for future work

We provide several examples of results concerning $QH^*(Gr(m,n))$ which would be interesting to understand through the Peterson lens, and for which we expect that the machinery developed in this paper might be especially useful. In [Pos05], Postnikov argues that the affine nilTemperley-Lieb algebra is a Grassmannian analog of Fomin-Kirillov's quadratic algebra from [FK99], in which the Dunkl elements encode the multiplication in $QH^*(SL_n(\mathbb{C})/B)$. Reverse engineering the analog of the subalgebra of the affine nilTemperley-Lieb algebra considered in Theorem 1.2 is thus a likely first step toward extending Fomin and Kirillov's approach to other partial flags.

Lee has proved that cylindric skew Schur functions are cylindric Schur positive, and that these expansions contain all Gromov-Witten invariants, using a result of Lam from [Lam06] that the cylindric skew Schur functions are precisely those affine Stanley symmetric functions which are indexed by permutations which do not support braid relations [Lee17]. It would be interesting to further explore these connections between the cylindric Schur functions of Postnikov and the affine Stanley symmetric functions indexed by 321-avoiding affine permutations, perhaps using the crystal structure developed by Morse and Schilling [MS16].

2. Background on Schubert calculus

The majority of the material reviewed here is discussed in more detail in the book [LLM$^+$14], which presents a comprehensive exposition based on the original papers. For a more leisurely treatment of this background, we invite the reader to consult the version of this paper available on the arXiv.

2.1. Quantum cohomology of the Grassmannian

Throughout the paper, we fix an $n \in \mathbb{N}$ and define $k = n - 1$. The **Grassmannian** of m-planes in $\mathbb{C}^n$ is the set $Gr(m, n)$ of all m-dimensional subspaces of $\mathbb{C}^n$. The **quantum cohomology** of the Grassmannian is a ring $QH^*(Gr(m, n))$ which admits a $\mathbb{Z}[q]$-basis of Schubert classes indexed by the elements of $S_n/(S_m \times S_{n-m})$.

Every element $w \in S_n$ can be written as a product $w = s_{i_1} \cdots s_{i_r}$, where s_j is the simple transposition exchanging j and $j + 1$ in $[n] = \{1, 2, \ldots, n\}$. When the product $w = s_{i_1} \cdots s_{i_r}$ uses the minimum possible number of generators, we call this a **reduced expression** for w and define the **length** of w to be $\ell(w) = r$. We often abbreviate $w = s_{i_1} \cdots s_{i_r}$ as $w = s_{i_1 i_2 \cdots i_r}$.

As permutations in **one-line notation** recording only the image of $[n]$ under the bijection w, minimal length coset representatives of $S_n/(S_m \times S_{n-m})$ have the form $[a_1 \ a_2 \ \cdots \ a_m \ | \ a_{m+1} \cdots \ a_n]$ with a unique (possible) **descent** $a_{m+1} > a_m$. To each such permutation, we can associate a 01-**word**, by recording whether each element of $[n]$ is in the right or left batch in the one-line notation. Equivalently, this 01-word traces out a **Young diagram** which fits inside an $m \times (n - m)$ rectangle by starting in the bottom left corner and taking a right step for each 0 and an upward step for each 1. A Young diagram is often recorded as a **partition** which lists the number of left-justified boxes in each row of the diagram. We denote by $\mathcal{P}_{mn}$ the set of all such Young diagrams or partitions.

Example 2.1. We illustrate the reduced expression, the one-line notation, the 01-word, the Young diagram, and the partition for a minimal length coset representative in $S_9/(S_5 \times S_4)$:

$$s_{124354765} \quad \longleftrightarrow \quad [2\ 3\ 5\ 6\ 8 \mid 1\ 4\ 7\ 9] \quad \longleftrightarrow \quad 011011010$$

$$\longleftrightarrow \quad \begin{array}{c}\text{(Young diagram)}\end{array} \quad \longleftrightarrow \quad (3, 2, 2, 1, 1).$$

2.2. Homology of the affine Grassmannian

Let $F = \mathbb{C}((t))$ be the field of Laurent series with complex coefficients, and let $\mathcal{O} = \mathbb{C}[[t]]$ be the ring of integers in F. The **type A affine Grassmannian** is defined to be $\mathcal{G}r = SL_n(F)/SL_n(\mathcal{O})$, and this space admits a decomposition into affine Schubert cells indexed by minimal length coset representatives in the quotient of the affine symmetric group $\tilde{S}_n$ by the symmetric group S_n. As in the non-affine case, the (co)homology classes of the affine Schubert varieties form a $\mathbb{Z}$-basis for the (co)homology of the affine Grassmannian.

The **affine symmetric group** is obtained from S_n by adding a single additional generator s_0 which satisfies the following relations with the simple transpositions of S_n:

$$
\begin{aligned}
s_i^2 &= 1 \\
(6) \qquad s_i s_j &= s_j s_i & \Longleftrightarrow \; (i-j) \bmod n \notin \{-1, 1\} \\
s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} & \text{for all } i \text{ with indices modulo n.}
\end{aligned}
$$

Denote by $\tilde{S}_n^0$ the set of minimal length coset representatives in $\tilde{S}_n/S_n$. As in the non-affine context, there are also several combinatorial indexing sets for the affine Schubert cells. The k-**bounded partitions** are all Young diagrams with width less than k. Any box in a Young digram has an associated **hook**, consisting of all boxes to the right and below the given box, including the box itself. A hook containing n boxes corresponds to an n-**rim hook**, consisting of the n contiguous boxes running along the border of the diagram which connect the top and bottom most boxes of the n-hook. The **n-cores** are all Young diagrams with no removable n-rim hook.

There are bijections between the elements of $\tilde{S}_n^0$, the k-bounded partitions, and the n-cores. We review these bijections via example, referring the reader to Section 1.2 of [LLM$^+$14] for the details.

Example 2.2. Let $n = 4$ so that $k = 3$, and let $\lambda = (2, 1, 1)$ be a 3-bounded partition. Since the top left box has a hook length of four, slide the top row to the right until there are no hook lengths of four or more to obtain the corresponding 4-core $\mu = (3, 1, 1)$. Now, removing boxes:
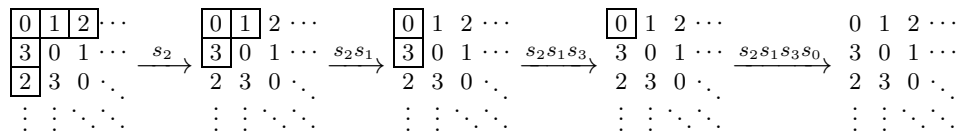
The 4-core $\mu = (3, 1, 1)$ thus corresponds to $s_2 s_1 s_3 s_0 \in \tilde{S}_n^0$.

Let $\mathbf{c}(\lambda)$ denote the n-core corresponding to a k-bounded partition λ, and define the k-**conjugate** of λ to be the k-bounded partition $\lambda^{\omega_k} = \mathbf{c}^{-1}[(\mathbf{c}(\lambda))^T]$. Note that k-conjugation maps the reduced expression $w = s_{i_1, i_2, \dots, i_r, 0} \in \tilde{S}_n^0$ to $w' = s_{i'_1, i'_2, \dots, i'_r, 0}$ where $i'_j = n - i_j$ for all $1 \leq j \leq r$.

The ring of symmetric functions $\Lambda \subset \mathbb{Z}[x_1, x_2, \dots]$ is generated by the **homogeneous symmetric functions**

$$(7) \qquad h_r = \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

Following [LM07], we consider the ring $\Lambda^{(k)} = \mathbb{Z}[h_1, \dots, h_k]$, which has a basis given by the k-**Schur functions**, denoted $s_\lambda^{(k)}$. We define the k-Schur functions recursively via the weak Pieri rule.

Theorem 2.3 (Weak Pieri rule, Section 2.2 [LLM$^+$14])**.** *Let λ be a k-bounded partition, and fix an integer $1 \leq r \leq k$. Then*

$$(8) \qquad h_r \cdot s_\lambda^{(k)} = \sum s_\nu^{(k)}$$

where the sum is over all k-bounded partitions ν satisfying two conditions: ν is obtained from λ by adding exactly r boxes, no two of which were added in the same column, and ν^{ω_k} is obtained from λ^{ω_k} by adding exactly r boxes, no two of which were added to the same row.

To begin the recursion, first note that $s_{(r)}^{(k)} = h_r$ for all $r \leq k$. Then use the following lemma, the inductive proof for which is a straightforward generalization of Example 2.11 in [LLM$^+$14].

Lemma 2.4 (Section 2.2 [LLM$^+$14])**.** *Let $\lambda = (\lambda_1, \dots, \lambda_p)$ be a k-bounded partition. Then,*

$$(9) \qquad s_\lambda^{(k)} = h_{\lambda_1} \cdot s_{(\lambda_2, \dots, \lambda_p)}^{(k)} + \sum_{i=1}^{n-1-\lambda_1} h_{i+\lambda_1} \cdot \left(\sum_j c_{ij} s_{\mu_{ij}}^{(k)} \right)$$

for some finite number of k-bounded partitions μ_{ij} and constants $c_{ij} \in \mathbb{Z}$.

The homology of the affine Grassmannian is isomorphic to a subalgebra of the affine nilCoxeter algebra, in which the affine Schubert classes correspond to the non-commutative k-Schur functions.

Definition 2.5. The **affine nilCoxeter algebra** $(\mathbb{A}_{\mathrm{af}})_0$ is generated by 1 and A_i for $i \in \{0, 1, ..., n-1\}$ satisfying the following relations:

$$A_i^2 = 0$$
(10)
$$A_i A_j = A_j A_i \qquad \Longleftrightarrow \quad (i-j) \bmod n \notin \{-1, 1\}$$
$$A_i A_{i+1} A_i = A_{i+1} A_i A_{i+1} \quad \text{for all } i \text{ with indices modulo n.}$$

More generally, we write $A_w = A_{w_1} A_{w_2} \cdots A_{w_p}$ where $w = s_{w_1} \cdots s_{w_p}$ is a reduced expression, or equivalently $w = (w_1, ..., w_p)$ is a **word**, for $w \in \tilde{S}_n$.

Definition 2.6. Let A be an algebra with generators $u_0, u_1, ..., u_{n-1}$ that satisfy

(11)
$$u_i u_j = u_j u_i \iff (i-j) \bmod n \notin \{-1, 1\},$$

with indices taken modulo n. Let $I = \{i_1, ..., i_r\} \subsetneq \mathbb{Z}/n\mathbb{Z}$ be any proper subset. Define the **cyclically increasing** (resp. **cyclically decreasing**) element of A corresponding to I, denoted u_I (resp. u_I'), to be the unique element satisfying $u_I = u_{i_1} u_{i_2} \cdots u_{i_r}$ (resp. $u_I' = u_{i_1} u_{i_2} \cdots u_{i_r}$) if and only if $i_a = i_b + 1$ implies $a > b$ (resp. $b > a$).

This notion of cyclically increasing and decreasing elements in $(\mathbb{A}_{\mathrm{af}})_0$ permits a generalization of the notion of the homogeneous and elementary symmetric functions in the affine nilCoxeter algebra.

Definition 2.7. For any $r \in \{1, ..., n-1\}$, define the **non-commutative homogeneous** and **elementary symmetric functions** in $(\mathbb{A}_{\mathrm{af}})_0$ by

$$\tilde{\mathbf{h}}_r = \sum_{\substack{w \in \tilde{S}_n, \, \ell(w) = r \\ A_w \text{ is cyclically decreasing}}} A_w \quad \text{and} \quad \tilde{\mathbf{e}}_r = \sum_{\substack{w \in \tilde{S}_n, \, \ell(w) = r \\ A_w \text{ is cyclically increasing}}} A_w.$$

By replacing each of the homogeneous symmetric functions h_i with their non-commutative counterpart $\tilde{\mathbf{h}}_i$, we obtain the non-commutative k-Schur functions.

Definition 2.8. Let $s_\lambda^{(k)} = f(h_1, ..., h_k)$ be a k-Schur function and $f(x_1, .., x_k) \in \mathbb{Z}[x_1, ..., x_k]$. The **non-commutative k-Schur function** $\mathbf{s}_\lambda^{(k)} \in (\mathbb{A}_{\mathrm{af}})_0$ is defined to be $\mathbf{s}_\lambda^{(k)} = f(\tilde{\mathbf{h}}_1, ..., \tilde{\mathbf{h}}_k)$. That is, simply express $s_\lambda^{(k)}$ in terms of h_i, and then replace h_i by $\tilde{\mathbf{h}}_i$.

Proposition 2.9 (Proposition 8.8, Theorem 8.9 [LLM$^+$14])**.** *The subalgebra* $\mathbb{B} \subset (\mathbb{A}_{af})_0$ *generated by* $\tilde{\mathbf{h}}_1, \ldots, \tilde{\mathbf{h}}_{n-1}$ *is isomorphic to the subalgebra* $\Lambda^{(k)}$ *of*

symmetric functions. Further, the non-commutative k-Schur functions $\mathbf{s}_\lambda^{(k)}$ form a basis of this subalgebra.

The subalgebra $\mathbb{B}$ defined in Proposition 2.9 is called the **affine Fomin-Stanley algebra**. The affine Fomin-Stanley algebra has another basis, which is often referred to as the j**-basis**.

Theorem 2.10 (Theorem 8.2 [LLM$^+$14])**.** *There is a basis $\{j_w^0 \mid w \in \tilde{S}_n^0\}$ for $\mathbb{B}$ which is uniquely defined by the following property*

$$(12) \qquad j_w^0 = A_w + \sum_{\substack{v \notin \tilde{S}_n^0 \\ \ell(w) = \ell(v)}} c_w^v A_v,$$

for some constants $c_w^v \in \mathbb{Z}$, and j_w^0 is the unique j-basis element containing the term A_w.

The j-basis is the one which Peterson identifies with the Schubert basis for $H_*(\mathcal{G}r)$, and Lam then proves in [Lam08] that the j-basis and the basis of non-commutative k-Schurs coincide.

Theorem 2.11 (Theorems 7.1 and 7.4 [Lam08])**.** *For $w \in \tilde{S}_n^0$, let $\lambda(w)$ denote the k-bounded partition corresponding to w. As elements of the affine Fomin-Stanley algebra, $j_w^0 = \mathbf{s}_{\lambda(w)}^{(k)}$. Moreover, both bases represent the Schubert classes under the isomorphism $H_*(\mathcal{G}r) \cong \mathbb{B}$.*

2.3. The parabolic Peterson isomorphism

This section aims to restate the Peterson isomorphism in the special case of a partial flag variety. In the case of $G = SL_n$, the quotient G/P is a partial flag variety, indexed by a strictly increasing sequence $\mathbf{i} = (i_1, \ldots, i_r) \in \mathbb{N}^r$. The **partial flag variety for i** is the set of all r**-step flags**

$$E = (\{0\} \subset E_{i_1} \subset E_{i_1} \subset \cdots \subset E_{i_r} \subset \mathbb{C}^n),$$

where $\dim E_{i_j} = i_j$. It is occasionally useful for the flag vector $\mathbf{i}$ to indicate the existence of the zero subspace and the whole space as part of the flag, and so we define $i_0 = 0$ and $i_{r+1} = n$ by convention. Each choice of r-step flag vector $\mathbf{i} = (i_1, ..., i_r)$ for some $1 \leq r \leq n$ corresponds to a choice of standard parabolic subgroup $P \subset SL_n(\mathbb{C})$, and vice versa. The extreme case of the Grassmannian with $\mathbf{i} = (m)$ is the primary example we study in the next section.

We first require some additional terminology associated to the underlying root system. Since we are working in type A, the roots and coroots coincide, and so we phrase everything in terms of roots. Let $\{\vec{e}_1, \ldots, \vec{e}_n\}$ be the standard orthonormal basis of $\mathbb{R}^n$. The vectors $\alpha_i = \vec{e}_i - \vec{e}_{i+1}$ are called the **simple roots** for type A_{n-1}. The **root lattice**, denoted by Q, is the $\mathbb{Z}$-span of the set Δ of simple roots. Let $V \subsetneq \mathbb{R}^n$ be the $(n-1)$-dimensional hyperplane of $\mathbb{R}^n$ in which the α_i all reside. For a vector $v = c_1 \vec{e}_1 + \ldots + c_n \vec{e}_n \in \mathbb{R}^n$ and a simple generator $s_i \in \tilde{S}_n$, define

$$s_i \cdot v = \begin{cases} c_1 \vec{e}_1 + \cdots + c_i \vec{e}_{i+1} + c_{i+1} \vec{e}_i + \cdots + c_n \vec{e}_n, & i \in \{1, 2, ..., n-1\} \\ (c_n + 1) \vec{e}_1 + c_2 \vec{e}_2 + \cdots + c_{n-1} \vec{e}_{n-1} + (c_1 - 1) \vec{e}_n, & i = 0. \end{cases}$$

For $w \in \tilde{S}_n$, choosing a reduced expression $w = s_{i_1} \cdots s_{i_r}$ and iterating this definition provides an action of any $w \in \tilde{S}_n$ on any vector $v \in \mathbb{R}^n$, which restricts to an action of $\tilde{S}_n$ on V. Further restricting this action to one of S_n on the simple roots $\Delta \subset V$ defines the set of all **(finite) roots**

$$(13) \qquad R = \{w \cdot \alpha_i \mid w \in S_n, \alpha_i \in \Delta\} = \{\vec{e}_i - \vec{e}_j \mid i \neq j\}.$$

The set of **positive roots** is defined as $R^+ = \{\vec{e}_i - \vec{e}_j \mid i < j\}$. The set of **negative roots** is then defined to be $R^- = R \backslash R^+$. There is a distinguished root

$$(14) \qquad \theta = \sum_{i=1}^{n-1} \alpha_i = \vec{e}_1 - \vec{e}_n,$$

which is called the **highest root**.

Denote by $\Delta_P = \Delta \backslash \{\alpha_{i_j} \mid 1 \leq j \leq r\}$ the simple roots corresponding to P. For ease of notation, we occasionally refer exclusively to the indexing set $I_P = \{i \in \mathbb{N} \mid \alpha_i \in \Delta_P\}$, which is a subset of $I = \{1, \ldots, n-1\}$. The parabolic subgroup P also corresponds to a subset of finite roots

$$(15) \qquad R_P = \{\vec{e}_i - \vec{e}_j \mid i_{a-1} < i, j \leq i_a \text{ for some } 1 \leq a \leq r+1\}.$$

By R_P^+ we denote the roots in R_P which are also positive, and by R_P^- those which are negative. The root lattice Q_P associated to the parabolic subgroup P is the $\mathbb{Z}$-span of the roots in R_P. Define $\eta_P : Q \longrightarrow Q/Q_P$ to be the natural projection.

The action of the affine symmetric group on $\mathbb{R}^n$ yields a realization of each element in $\tilde{S}_n$ as an affine transformation on V. Define a hyperplane
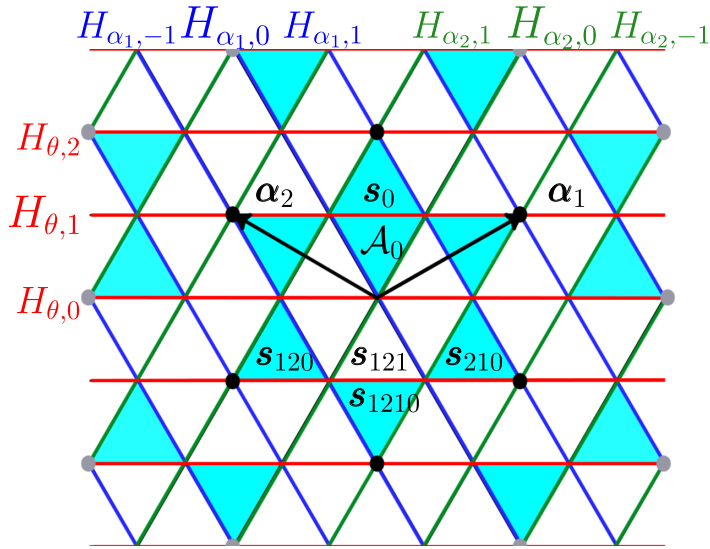
Figure 2: The type $\tilde{A}_2$ root lattice, affine hyperplanes, and alcoves for $\tilde{S}_3$.

in V indexed by a root α and an integer p by

$$(16) \qquad H_{\alpha,p} = \{v \in V \mid \langle v, \alpha \rangle = p\},$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product on $\mathbb{R}^n$. The action of s_i corresponds to a reflection across the hyperplane $H_{\alpha_i,0}$, and acting by s_0 reflects across the hyperplane $H_{\theta,1}$. Defining

$$\mathcal{H} = \bigcup_{\alpha \in R^+, p \in \mathbb{Z}} H_{\alpha,p},$$

the set $V \backslash \mathcal{H}$ consists of disjoint convex regions called **alcoves**. We will single out a particular alcove as the **fundamental alcove**

$$(17) \qquad \mathcal{A}_0 = \{v \in V \mid 0 < \langle v, \alpha \rangle < 1 \text{ for all } \alpha \in R^+\},$$

which we can identify with the identity of the group $\tilde{S}_n$.

Each alcove $\mathcal{A}$ then corresponds bijectively to the element $w \in \tilde{S}_n$ which maps $\mathcal{A}_0$ to $\mathcal{A}$ when we reflect $\mathcal{A}_0$ over the sequence of hyperplanes associated to any reduced expression for w. Certain elements $w \in \tilde{S}_n$ simply translate $\mathcal{A}_0$ by a vector $\lambda \in Q$. We denote these **translation elements** as $t_\lambda \in \tilde{S}_n$. This identification provides a natural embedding of the root lattice $Q \cong$

$\{t_\lambda \mid \lambda \in Q\}$ as a normal subgroup in $\tilde{S}_n$. In fact, $\tilde{S}_n \cong S_n \ltimes Q$, and we frequently use this decomposition to write $wt_\lambda \in \tilde{S}_n$, where $w \in S_n$ and $\lambda \in Q$. Note that the action on roots corresponds to conjugation in the group, in the sense that $t_{w \cdot \alpha} = wt_\alpha w^{-1}$ for all $\alpha \in R$ and $w \in S_n$.

An element $\lambda \in Q$ is called **dominant** if $\langle \lambda, \alpha_i \rangle > 0$ for all $\alpha_i \in \Delta$, and we denote the set of all dominant elements by Q^+. Similarly, an element $\lambda \in Q$ is called **antidominant** if $\langle \lambda, \alpha_i \rangle < 0$ for all $\alpha_i \in \Delta$. The antidominant elements of the root lattice will play a special role in the Peterson isomorphism, and so we denote this set by Q^-. We refer to the set of translations $t_\lambda \in \tilde{S}_n$ where $\lambda \in Q^-$ as the **antidominant translations**.

To define the ideal with which we form a quotient in the parabolic Peterson isomorphism, we require some additional terminology involving the affine roots in the type $\tilde{A}_{n-1}$ root system. Define the **null root** $\delta = \alpha_0 + \theta$ where α_0 is formally added to our collection of roots R. We then define the set of **affine roots** and **positive/negative affine roots** as:

$$
\begin{aligned}
(18) \qquad & R_{af} = \{\alpha + p\delta \mid \alpha \in R \text{ and } p \in \mathbb{Z}\} \\
& R_{af}^+ = \{\alpha + p\delta \mid \alpha \in R \text{ and } p \in \mathbb{Z}_{>0}\} \cup R^+ \\
& R_{af}^- = R_{af} \backslash R_{af}^+.
\end{aligned}
$$

We occasionally denote $\beta > 0$ and $\beta < 0$ for $\beta \in R_{af}^+$ and $\beta \in R_{af}^-$, respectively. Denote the projection $\delta \mapsto 0$ on an affine root $\beta = \alpha + p\delta$ by $\bar{\beta} = \alpha \in R$. We can also extend the definition of R_P from (15) to the affine root system:

$$
(19) \qquad (R_P)_{af}^+ = \{\beta \in R_{af}^+ \mid \bar{\beta} \in R_P\}.
$$

We now define the **level-zero action** of $\tilde{S}_n$ on the affine roots in R_{af} by declaring that $s_j \cdot \delta = \delta$. Let the action of $s_j \in \tilde{S}_n$ on any simple root $\alpha_i \in \Delta$ and $\alpha_0 = \delta - \theta$ be defined by

$$
(20) \qquad s_j \cdot \alpha_i = \begin{cases} -\alpha_i & \text{if } i = j \\ \alpha_i + \alpha_j & \text{if } (i-j) \equiv \pm 1 \bmod n \\ \alpha_i & \text{otherwise.} \end{cases}
$$

Since this action is linear, Equation (20) determines an action $s_j \cdot \beta$ for all affine roots $\beta \in R_{af}$ which extends to all $w \in \tilde{S}_n$. The **inversion set** of a word $w \in \tilde{S}_n$ is defined in terms of this action of w on the positive affine roots as follows:

$$
(21) \qquad \text{Inv}(w) = \{\alpha \in R_{af}^+ \mid w \cdot \alpha \in R_{af}^-\}.
$$

It is a standard fact that the length of the word is equal to the number of its inversions.

In general, the homology of the affine Grassmannian $H_*(\mathcal{G}r)$ admits only a surjection onto the quantum cohomology of a partial flag variety. In order to map isomorphically onto the smaller ring $QH^*(SL_n(\mathbb{C})/P)$, we will need to select certain affine Schubert classes to mod out by. The Weyl group $(S_n)_P$ for the Levi subgroup of P is the product of the symmetric groups generated by the simple reflections $\{s_{i_j} \mid 1 \leq j \leq r\}$ corresponding to Δ_P. The partial flag variety $SL_n(\mathbb{C})/P$ admits a decomposition into Schubert cells indexed by the elements of the quotient $S_n/(S_n)_P$, where we denote the set of minimal length coset representatives by S_n^P. The Schubert classes indexed by S_n^P form a $\mathbb{Z}[q_i : i \in I \backslash I_P]$-basis for $QH^*(SL_n(\mathbb{C})/P)$.

Definition 2.12. Define an ideal in $H_*(\mathcal{G}r)$ by

$$(22) \qquad J_P = \sum_{w \in \tilde{S}_n^0 \backslash \tilde{S}_n^P} \mathbb{Z}\, j_w^0,$$

where

$$(23) \qquad \tilde{S}_n^P = \{w \in \tilde{S}_n \mid w \cdot \beta \in R_{af}^+ \text{ for all } \beta \in (R_P)_{af}^+\}.$$

The following lemma permits a factorization of every element in $\tilde{S}_n$ into a product, the first component of which lies in this distinguished subgroup $\tilde{S}_n^P$, the second of which lies in the subgroup

$$(24) \qquad (\tilde{S}_n)_P = \{vt_\lambda \in \tilde{S}_n \mid \lambda \in Q_P, \ v \in (S_n)_P\}.$$

Lemma 2.13 (Lemma 10.6 [LS10])**.** *For all $w \in \tilde{S}_n$, there is a unique factorization $w = w_1 w_2$ where $w_1 \in \tilde{S}_n^P$ and $w_2 \in (\tilde{S}_n)_P$.*

Use Lemma 2.13 to define a projection onto the first factor

$$(25) \qquad \pi_P : \tilde{S}_n \to \tilde{S}_n^P \qquad \text{by} \qquad \pi_P(w) = w_1.$$

Theorem 2.14 (Parabolic Peterson Isomorphism [Pet96], [LS10])**.** *Let $v \in S_n^P$, and suppose that $v\pi_P(t_\lambda) \in \tilde{S}_n^0$. Then there is a $\mathbb{Z}$-algebra isomorphism*

$$\Psi_P : (H_*(\mathcal{G}r)/J_P)[(\overline{j}_{\pi_P(t_\mu)}^0)^{-1} \mid \mu \in Q^-] \longrightarrow QH^*(SL_n(\mathbb{C})/P)[q_i^{-1} \mid i \in I \backslash I_P]$$
$$\overline{j}_{v\pi_P(t_\lambda)}^0 \mapsto q^{\eta_P(\lambda)}\sigma_v,$$

where $\overline{j}_w^0$ denotes the image of j_w^0 in the quotient $H_*(\mathcal{G}r)/J_P$.

Here, writing $\nu = \sum_{i \in I \setminus I_P} c_i \alpha_i \in Q/Q_P$, the corresponding quantum parameter is $q^\nu = \prod_{i \in I \setminus I_P} q_i^{c_i}$.

3. The parabolic Peterson isomorphism for the Grassmannian

This section presents an effective means for using the parabolic Peterson isomorphism to do quantum Schubert calculus in the Grassmannian $Gr(m, n)$. The special case of the parabolic Peterson for the choice of $\mathbf{i} = (m)$ appears as Theorem 3.11.

3.1. The ideal J_P in the affine Fomin-Stanley algebra

We first aim to understand the set $\tilde{S}_n^0 \setminus \tilde{S}_n^P$ to find which classes get killed in the quotient $H_*(\mathcal{G}r)/J_P$. To visualize this set in a concrete example where $n = 3$, refer to Figure 3.
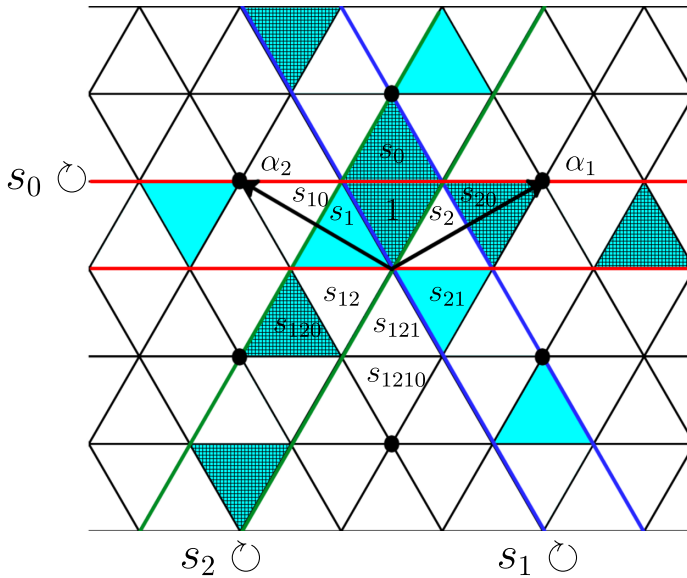

Figure 3: The elements of $\tilde{S}_3^P$ for the parabolic P corresponding to $Gr(1,3)$ are colored blue. The shaded blue alcoves are the elements of $\tilde{S}_3^P \cap \tilde{S}_3^0$, which are also minimal length coset representatives.

To understand the set $\tilde{S}_n^P$, we need to check a positivity condition on all of the affine roots in $(R_P)_{af}^+$. The following lemma provides a significant

reduction step to determine whether or not an element lies in $\tilde{S}_n^P$, enabling us to verify the positivity condition only on finitely many roots.

Lemma 3.1. *A word* $w \in \tilde{S}_n^P$ *if and only if for all* $\alpha \in R_P^+$, *we have* $w \cdot \alpha \in R^+ \cup (R^- + \delta)$.

Proof. We prove both statements by contrapositive.

($\Longleftarrow$) Since $w \notin \tilde{S}_n^P$, there exists $\beta' \in (R_P)_{af}^+$ such that $w \cdot \beta' = \eta \in R_{af}^-$. By definition, $\beta' = \beta + n\delta$ with $n \geq 0$ and $\beta \in R_P^+ \cup (R_P^- + \delta)$. Then $w \cdot \beta = w \cdot (\beta' - n\delta) = \eta - n\delta \in R_{af}^-$, so there exists $\beta \in R_P^+ \cup (R_P^- + \delta)$ such that $w \cdot \beta \in R_{af}^-$. If $\beta \in R_P^+$, then we have found $w \cdot \beta \notin R^+ \cup (R^- + \delta)$ as R_{af}^- contains no positive roots. Otherwise, $\beta = -\alpha + \delta$ for some $\alpha \in R_P^+$. By definition of R_{af}^-, either $w \cdot \beta = -\gamma - m\delta$ or $w \cdot \beta = \gamma - (m+1)\delta$ for some $\gamma \in R^+$ and $m \geq 0$, which implies that either $w \cdot \alpha = w \cdot (-\beta + \delta) = \gamma + (m+1)\delta \notin R^+ \cup (R^- + \delta)$ or $w \cdot \alpha = w \cdot (-\beta + \delta) = -\gamma + (m+2)\delta \notin R^+ \cup (R^- + \delta)$. In any case, we have found $\beta \in R_P^+$ or $\alpha = -\beta + \delta \in R_P^+$ such that either $w \cdot \beta$ or $w \cdot \alpha$ is not in $R^+ \cup (R^- + \delta)$.

($\Longrightarrow$) Let $\alpha \in R_P^+$ with $w \cdot \alpha \notin R^+ \cup (R^- + \delta)$. If $w \cdot \alpha = \beta - (n+1)\delta$ or $w \cdot \alpha = -\beta - n\delta$ for $\beta \in R^+$ and $n \geq 0$, then $w \notin \tilde{S}_n^P$. Otherwise, either $w \cdot \alpha = \beta + (n+1)\delta$ or $w \cdot \alpha = -\beta + (n+2)\delta$ for $\beta \in R^+$ and $n \geq 0$, whence $w \cdot (-\alpha + \delta) = -\beta - n\delta$ or $w \cdot (-\alpha + \delta) = \beta - (n+1)\delta$, so $w \notin \tilde{S}_n^P$. $\square$

Using Lemma 3.1, we can now identify a large family of words which lie in $\tilde{S}_n^P$.

Lemma 3.2. *Let* $I_P = I \backslash \{m\}$, *and let* $w = s_{i_1} \cdots s_{i_p} s_0 \in \tilde{S}_n^0$ *be a reduced expression with* $i_a \neq m$ *for all* a. *Then,* $w \in \tilde{S}_n^P$.

Proof. Because $i_a \neq m$, the element w corresponds to an n-core which fits in an $(n-m) \times m$ rectangle. The hook length of the upper-leftmost box is thus at most $n-1$, and so we know that the n-core coincides with the k-bounded partition. Suppose that the k-bounded partition has parts $\lambda = (\lambda_1, ..., \lambda_{n-m})$. Then for all $1 \leq i \leq n-m$, define a word w_i as follows

$$(26) \qquad w_i = s_{\lambda_i - i} s_{\lambda_i - 2 - i} \cdots s_{-i+2} s_{-i+1},$$

with indices taken modulo n. Define their product

$$(27) \qquad w = w_{\lambda_{n-m-1}} w_{\lambda_{n-m-1}} \cdots w_1,$$

which is the word constructed from the n-core λ using the algorithm discussed in Section 2.2. Here, since the shape fits inside the $(n-m) \times m$

rectangle, we have proceeded by removing only one box at a time, going row by row, removing rows from bottom to top.

Consider a root $\alpha = \vec{e}_i - \vec{e}_j \in R_P^+$ with $0 < i < j$ and $j \leq k$. Then,

$$(28) \qquad w_1 \cdot \alpha = \begin{cases} \vec{e}_i - \vec{e}_j & \text{if } \lambda_1 < i \\ \vec{e}_{i-1} - \vec{e}_j & \text{if } j \geq \lambda_1 \geq i \\ \vec{e}_{i-1} - \vec{e}_{\lambda_1 - 1} & \text{if } \lambda_1 > j \end{cases}$$

For each additional w_i applied to $w_1 \cdot \alpha$, we know that, because $\lambda_a \geq \lambda_{a+1}$ for all a, the resulting root will always be of the form $\vec{e}_x - \vec{e}_y$ for some $i \geq x > m-n$ and $y = \max(\{j, \lambda_1 - 1\})$. Indeed, the maximum possible value for λ_1 is m. Furthermore, applying w_i to $\vec{e}_x - \vec{e}_y$ will give either $\vec{e}_{x-1} - \vec{e}_y$ or $\vec{e}_x - \vec{e}_y$, so the minimum possible value for x in $w \cdot \alpha = \vec{e}_x - \vec{e}_y$ is $i - n + m > m - n$. In any case, $w \cdot \alpha = \beta$ or $w \cdot \alpha = -\beta + \delta$ for some $\beta \in R^+$. Indeed, if $x > 0$, we have $w \cdot \alpha = \alpha_x + \cdots + \alpha_{y-1}$, and if $x < 0$ then $m < x + n \leq n - 1$, and we have $w \cdot \alpha = -\alpha_y - \alpha_{y+1} - \cdots - \alpha_{x+n-1} + \delta$ since $x + n - 1 > m - 1$ and $y \leq m - 1$. Therefore, $w \cdot \alpha \in R^+ \cup (R^- + \delta)$.

Similarly, if $\alpha = \vec{e}_i - \vec{e}_j \in R_P^+$ with $i < j \leq n$ and $i \geq m + 1$, we instead consider the transpose λ^T having parts λ_i^T, and we write $w = w'_m w'_{m-1} \cdots w'_1$ where

$$(29) \qquad w'_i = s_{i - \lambda_i^T} s_{i - \lambda_i^T + 1} \cdots s_{i-2} s_{i-1}.$$

That is, we insert boxes column by column. The same argument then shows that $w \cdot \alpha \in R^+ \cup (R^- + \delta)$.

Having checked all of the elements of R_P^+, by Lemma 3.1 we know that $w \in \tilde{S}_n^P$. $\qquad \square$

Lemma 3.3. *Let $I_P = I \backslash \{m\}$. Then $w = s_{r-1,r-2,\ldots,1,0} \in \tilde{S}_n^P$ if and only if $r \leq m \leq n - 1$. Similarly, $w = s_{n-r+1,n-r+2,\ldots,n-1,0} \in \tilde{S}_n^P$ if and only if $r \leq n - m \leq n - 1$.*

Proof. ($\Longleftarrow$) In both cases this direction follows from Lemma 3.2 since w does not use s_m.

($\Longrightarrow$) Let $n > r > m$. Consider the root $\beta = \alpha_r + \ldots + \alpha_{n-1} = \vec{e}_r - \vec{e}_n \in (R_P)_{af}^+$. Then for $w = s_{r-1,r-2,\ldots,1,0}$, we can calculate that

$$(30) \quad w \cdot \beta = s_{r-1} \cdot (\alpha_r + \cdots + \alpha_{n-1} + \alpha_0 + \alpha_1 + \cdots + \alpha_{r-2}) = \alpha_{r-1} + \delta.$$

Therefore, $w \cdot (-\beta + \delta) \in R_{af}^-$, and so $w \notin \tilde{S}_n^P$ by definition.

For $n > r > n - m$, we consider the root $\beta = \alpha_1 + \ldots + \alpha_{n-r} = \vec{e}_1 - \vec{e}_{n-r} \in (R_P)^+_{af}$. Then for $w = s_{n-r+1, n-r+2, \ldots, n-1, 0}$, we can calculate that

$$(31) \quad w \cdot \beta = s_{n-r+1} \cdot (\alpha_1 + \cdots + \alpha_{n-r} + \alpha_{n-r+2} + \cdots + \alpha_{n-1} + \alpha_0) = \alpha_{n-r+1} + \delta.$$

Therefore, $w \cdot (-\beta + \delta) \in R^-_{af}$, and so $w \notin \tilde{S}^P_n$ by definition. $\qquad \square$

Remark 3.4. Note that, because $r \leq n - 1$, then $w = s_{r-1, r-2, \ldots, 1, 0}$ is a cyclically decreasing element in $\tilde{S}^0_n$, and so $j^0_w = \tilde{\mathbf{h}}_r$. Lemma 3.3 then says that $\tilde{\mathbf{h}}_r \in J_P$ if and only if $r > m$. For the second family of elements $w' = s_{n-r+1, n-r+2, \ldots, n-1, 0}$ considered in Lemma 3.3, we then have $\tilde{\mathbf{e}}_r = j^0_{w'}$ and $\tilde{\mathbf{e}}_r \in J_P$ if and only if $r > n - m$.

The next lemma identifies a different family of elements in J_P; namely, those which support braid relations.

Proposition 3.5. *Let $I_P = I \backslash \{m\}$. If $w \in \tilde{S}^0_n$ and $w = w_1 s_i s_{i+1} s_i w_2$ is a reduced expression for w so that $\ell(w) = \ell(w_1) + \ell(s_i s_{i+1} s_i) + \ell(w_2)$, then $w \notin \tilde{S}^P_n$.*

In order to prove Proposition 3.5, we need two technical lemmas.

Lemma 3.6. *Let $w = w_1 w_2 \in \tilde{S}_n$ where $\ell(w) = \ell(w_1) + \ell(w_2)$. If $w \in \tilde{S}^P_n$, then $w_2 \in \tilde{S}^P_n$.*

Proof. We argue by contradiction. Suppose there exists $\beta \in (R_P)^+_{af}$ such that $w_2 \cdot \beta \in R^-_{af}$. This means that $\beta \in \mathrm{Inv}(w_2)$. Since $\beta \notin \mathrm{Inv}(w)$ by hypothesis, then after applying some s_i which is a subexpression of w_1, the root β will no longer be in the inversion set. Therefore, since $\ell(w) = |\mathrm{Inv}(w)|$, the length of the expression has not increased with the addition of s_i, and so $w_1 w_2 = w$ cannot satisfy $\ell(w_1) + \ell(w_2) = \ell(w)$. $\qquad \square$

Elements of a certain form will arise throughout the next argument, and so we define

$$(32) \qquad s^p_{I,i,j} = s_{i,i+1,i+2,\ldots,j-1,j} \quad \text{and} \quad s^p_{D,i,j} = s_{i,i-1,i-2,\ldots,j+1,j}$$

to be a sequence of cyclically increasing, respectively decreasing, elements starting with s_i, ending with s_j, and containing p copies of s_0. For example, if $n = 6$, then

$$s_{I,1,3} = s_{1,2,3} \qquad s^1_{I,4,2} = s_{4,5,6,0,1,2} \qquad s^2_{D,3,5} = s_{3,2,1,0,6,5,4,3,2,1,0,6,5}.$$

Lemma 3.7. *Let $w \in \tilde{S}_n^0$. Suppose that $w = w_1 s_i s_{i+1} s_i w_2$ is a reduced expression with $\ell(w) = \ell(w_1) + \ell(s_i s_{i+1} s_i) + \ell(w_2)$ for some $i \in \{0, 1, ..., n - 1\}$. If i is chosen such that the braid $s_i s_{i+1} s_i$ is as far right as possible in the reduced expression, then $w_2 = s_{i+2, i+3, ..., n-1, i-1, i-2, ..., 1, 0}$ and $i \notin \{0, n-1\}$.*

Proof. Let $w_3 = s_i s_{i+1} s_i w_2$. Note that w_2 must begin with either s_{i+2} or s_{i-1}, since otherwise we could move the braid further to the right. Suppose w_2 begins with s_{i+2}. Because we cannot move the four terms $s_i s_{i+1} s_i s_{i+2}$ to the right, either s_{i-1} appears to the right or s_{i+3} does. Suppose that s_{i-1} does not appear, so s_{i+3} is to the right of $s_{i,i+1,i,i+2}$. Now assume that this same case occurs at each step where s_{c+1} appears to the right of s_c. Ultimately, we will get the word $w_3 = s_{i,i+1,i,i+2} s_{I,i+3,n-1}^a s_0$. If $a \neq 0$, then we can slide s_i to the right until we have $s_{i,i+1} s_{I,i+2,i-1}^1 s_{i,i-1,i} s_{I,i+1,n-1}^{a-1} s_0$, which is a contradiction since this subexpression of w contains a braid farther to the right than the original. Therefore, $w_3 = s_{i,i+1,i} s_{I,i+2,n-1}^0 s_0$, but this is also a contradiction as the second s_i commutes with everything to its right and $w \notin \tilde{S}_n^0$, unless $i = 1$, $i = n-1$, or $i = 0$. In the first case, w_2 has the claimed form. In the second case, $w_3 = s_{n-1,0,n-1,0} = s_{0,n-1}$ and so $w \notin \tilde{S}_n^0$. In the third case, we slide s_0 to the right to get a braid $s_{0,n-1,0}$, which is another contradiction. Therefore, $w_3 = s_{i,i+1,i} s_{I,i+2,c}^a s_{D,i-1,d}^b v s_0$ where $a \geq 0$, $b \geq 0$, and $v \in \tilde{S}_n$.

Now consider w_3 expressed in this form. We can always increase c or decrease d if there are still terms to the right of $s_{D,i-1,d}^b$ (besides s_0). Note that if we can slide $s_{D,i-1,d}^b$ to the right, then since we cannot slide $s_{I,i+2,c}^a$ to the right without shortening w_2, we have $w_3 = s_{i,i+1,i} s_{I,i+2,c}^a s_{c+1} s_{D,i-1,d}^b v' s_0$ for some new $v' \in \tilde{S}_n$ with $\ell(v') = \ell(v) - 1$ as any other term would cause a braid farther right (or would commute with everything). If we cannot move $s_{D,i-1,d}^b$ to the right, there must be an s_{d-1} to its right. Therefore, we increase the lengths of these chains at every step until we have $w_3 = s_{i,i+1,i} s_{I,i+2,c'}^{a'} s_{D,i-1,d'}^{b'} s_0$ for some new a', b', c', and d'.

If $d' \neq 1$ then $w \notin \tilde{S}_n^0$, and so $d' = 1$. If $b' > 0$, we know that by sliding $s_{c'}$ to the right, we will encounter the subexpression $s_{c',c'+1,c'}$, which gives us a braid further right than the original. Therefore, we also have that $b' = 0$. Similarly, $c' = n - 1$, since otherwise we can slide $s_{c'}$ to the right until we have $s_{c',c'+1,c'}$ if $0 \leq c' < i + 1$, or $w \notin \tilde{S}_n^0$ as $s_{c'}$ commutes with everything to its right. Finally, $a' = 0$ by the same argument as before. Therefore, $w_3 = s_{i,i+1,i} s_{I,i+2,n-1}^0 s_{D,i-1,1}^0 s_0 = s_{i,i+1,i,i+2,...,n-1,i-1,...,1,0}$, and w_2 has the desired reduced expression.

Finally, notice that, because $i + 2$ and $i - 1$ must appear to the right of the braid, i cannot be 0 or $n - 1$. If both $s_{I,i+2,n-1}^0$ and $s_{D,i-1,1}^0$ appear,

we will have cancellation between the increasing and decreasing portions (if $i = 0$) or a braid appearing further right (if $i = n - 1$). If only the increasing or decreasing portion appears instead, the expression will not be forced to end in s_0. Therefore, when the braid is farthest right, we know that $i \notin \{0, n - 1\}$. $\qquad\square$

With these two lemmas in hand, we are now prepared to prove Proposition 3.5.

Proof of Proposition 3.5. By Lemma 3.7, we can write $w = w_1 s_i s_{i+1} s_i w_2$, where we have moved the braid as far right as possible, and so therefore $w_2 = s_{i+2, i+3 \ldots, n-1, i-1, i-2 \ldots, 1, 0}$. Consider the following pair of affine roots:

$$(33) \qquad \beta_1 = \alpha_1 + \cdots + \alpha_i \quad \text{and} \quad \beta_2 = \alpha_{i+1} + \cdots + \alpha_{n-1}.$$

Since $I_P = I \backslash \{m\}$ we only remove one root to obtain Δ_P, namely α_m. Thus, one of the two roots β_1 or β_2 is guaranteed to be in $(R_P)^+_{af}$.

Now, calculate that $w_2 \cdot \beta_1 = -\alpha_{i+1} + \delta$ and $w_2 \cdot \beta_2 = -\alpha_i + \delta$. Therefore, when we act next by the braid $s_i s_{i+1} s_i$, we obtain $s_i s_{i+1} s_i w_2 \cdot \beta_1 = \alpha_i + \delta$ and $s_i s_{i+1} s_i w_2 \cdot \beta_2 = \alpha_{i+1} + \delta$. Therefore, $s_i s_{i+1} s_i w_2 \cdot (-\beta_j + \delta) \in R^-_{af}$ for both $j = 1, 2$. We have thus proved that there is a root $\beta \in (R_P)^+_{af}$ such that $s_i s_{i+1} s_i w_2 \cdot \beta \in R^-_{af}$. By Lemma 3.6 we have that $w \notin \tilde{S}^P_n$. $\qquad\square$

3.2. The projection π_P on the affine symmetric group

This section is be devoted to understanding the map π_P in the case of $Gr(m, n)$. We begin by recording several useful facts about how π_P behaves on translation elements.

Lemma 3.8 (Proposition 10.10 [LS10])**.** *Let* $\lambda, \mu \in Q$, *and let* $\nu \in Q_P$. *Then,*
 *1)* $\pi_P(t_{\lambda+\mu}) = \pi_P(t_\lambda)\pi_P(t_\mu)$
  2) $\pi_P(t_{\lambda+\nu}) = \pi_P(t_\lambda)$.

In practice, to use the parabolic Peterson isomorphism to do quantum Schubert calculus, it suffices to understand the image under π_P of a single translation element, namely $\pi_P(t_{-\theta})$, where θ is the highest root. The next proposition identifies this element explicitly.

Proposition 3.9. *For* $I_P = I \backslash \{m\}$, *we have*

$$(34) \qquad\qquad \pi_P(t_{-\theta}) = s_{m, m+1, \ldots, n-1, m-1, \ldots, 1, 0}.$$

Example 3.10. We illustrate the element $\pi_P(t_{-\theta})$ for P such that $SL_8(\mathbb{C})/P = Gr(3, 8)$ below.
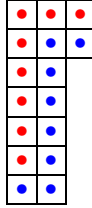
The red dots indicate the boxes which belong to the 7-bounded partition corresponding to $u = \pi_P(t_{-\theta}) = s_{34567210}$, and the 7-bounded partition outlined by taking both the red and blue dots together corresponds to the element $u^2 = \pi_P(t_{-2\theta})$. In general, the powers of the element $\pi_P(t_{-\theta})$ for $Gr(m, n)$ correspond to the k-bounded partitions obtained by adding successive n-rim hooks, each of which start in column 1 and end in column m; compare the construction of $\blacktriangle(\lambda)$ in [LM08].

Proof of Proposition 3.9. Using Lemma 3.8, since $\alpha_m - \theta \in Q_P$, we compute $\pi_P(t_{-\alpha_m}) = \pi_P(t_{-\theta})$.

It is easy to observe geometrically that $t_{-\theta} = s_\theta s_0$, where $s_\theta = \tau_{1n}$ is the reflection in S_n corresponding to θ, which is the transposition that interchanges positions 1 and n. Then, writing down a reduced expression for τ_{1n}, we obtain

$$t_{-\theta} = \tau_{1n}s_0 = s_{1,2\ldots n-2,n-1,n-2,\ldots 2,1,0},$$

Note that $t_{-\alpha_m}$ can be found using $t_{w\cdot\alpha} = wt_\alpha w^{-1}$. Letting

$$v = s_{m+1,m+2,\ldots,n-1,m-1,m-2,\ldots,1},$$

it is straightforward to verify that

$$(35) \qquad v \cdot (-\theta) = v \cdot (-\alpha_1 - \ldots - \alpha_{n-1}) = -\alpha_m,$$

and so $t_{-\alpha_m} = vt_{-\theta}v^{-1}$. We claim that $vt_{-\theta} = s_{m,m+1,\ldots,n-1,m-1,\ldots,1,0}$. Using

the notation of Equation (32), we can compute that

$$
\begin{aligned}
vt_{-\theta} &= vs^0_{I,1,n-2}s^0_{D,n-1,1}s_0 \\
&= s^0_{I,m+1,n-1}s^0_{I,m,n-2}s^0_{D,n-1,1}s_0 \\
&= s^0_{I,m+1,n-2}s^0_{I,m,n-3}s_{n-1,n-2,n-1,n-2}s^0_{D,n-3,1}s_0 \\
&= s^0_{I,m+1,n-2}s^0_{I,m,n-3}s_{n-2,n-1}s^0_{D,n-3,1}s_0 \\
&= s^0_{I,m+1,b+1}s^0_{I,m,n-1}s^0_{D,b,0} \\
&= s^0_{I,m+1,b}s^0_{I,m,b-1}s_{b+1,b,b+1,b}s^0_{I,b+2,n-1}s^0_{D,b-1,0} \\
&= s^0_{I,m+1,b}s^0_{I,m,n-1}s^0_{D,b-1,0} \qquad \text{(which we repeat until } b = m+1) \\
&= s_{m+1,m,m+1,m+2,\ldots,n-1,m,m-1,\ldots,1,0} \\
&= s_{m,m+1,\ldots,n-1,m-1,m-2,\ldots,1,0},
\end{aligned}
$$

as claimed. The idea is that the decreasing part of v cancels immediately, and, one-by-one, we can braid the terms in the increasing part of v to get a cancellation.

Clearly $v^{-1} \in (S_n)_P \subset (\tilde{S}_n)_P$, and so if we can show that $vt_{-\theta} \in \tilde{S}^P_n$, then we have factored $t_{-\alpha_m}$ as in Lemma 2.13. As in the proof of Lemma 3.2, we consider $\beta = e_i - e_j = \alpha_i + \cdots + \alpha_{j-1}$ with $0 < i < j$ and $j \le m$. Then,

$$
s_{m,m+1,\ldots,n-1,m-1,\ldots,1,0} \cdot \beta = \begin{cases} \alpha_{i-1} + \cdots + \alpha_{j-2} & \text{if } i > 1, \\ \alpha_m + \cdots + \alpha_{n-1} + \alpha_0 + \cdots + \alpha_{j-2} & \text{if } i = 1, \end{cases}
$$

since $j \le m$, we see that $vt_{-\theta} \cdot \beta \in R^+ \cup (R^- + \delta)$. With a similar calculation, if instead we have $\beta = e_i - e_j$ with $i < j \le n$ and $i \ge m+1$, we can show that $vt_{-\theta} \cdot \beta \in R^+ \cup (R^- + \delta)$. By Lemma 3.1, we then know that $vt_{-\theta} \in \tilde{S}^P_n$, and so $t_{-\alpha_m} = (vt_{-\theta})(v^{-1})$ is a factorization as in Lemma 2.13. Therefore,

$$
\pi_P(t_{-\theta}) = \pi_P(t_{-\alpha_m}) = vt_{-\theta} = s_{m,m+1,\ldots,n-1,m-1,\ldots,1,0},
$$

as desired. $\qquad \square$

3.3. The Peterson isomorphism for the Grassmannian

We are now prepared to present the statement of the parabolic Peterson isomorphism for the special case of the Grassmannian $Gr(m,n)$. Recall that the map $\eta_P : Q \mapsto Q/Q_P$ sends $\alpha_i \mapsto 0$ for any $\alpha_i \in \Delta_P$. In particular, for $I_P = I \backslash \{m\}$, this map preserves only α_m. We denote q_m simply by q and suppress the notation η_P, since there is only one quantum parameter in this case.

Theorem 3.11 (Peterson Isomorphism for the Grassmannian). *For $I_P = I\backslash\{m\}$, there is a $\mathbb{Z}$-algebra isomorphism*

$$(36) \qquad \Psi_P : (H_*(\mathcal{G}r)/J_P)[(\overline{\jmath}_u^0)^{-1}] \longrightarrow QH^*(Gr(m,n))[q^{-1}]$$
$$\overline{\jmath}_{vu^r}^0 \longmapsto q^{-r}\sigma_v$$

for $v \in S_n^P$ a minimal length coset representative and any $r \in \mathbb{N}$ such that $vu^r \in \tilde{S}_n^0$. Here, $u \in \tilde{S}_n^0$ is the hook shape corresponding to the element

$$(37) \qquad u = s_{m,m+1,\ldots,n-1,m-1,\ldots,1,0}.$$

Here we need only invert the single class $\overline{\jmath}_{\pi_P(t_{-\theta})}^0$ since by Lemma 3.8 this class generates the entire set around which we localize on the affine side.

Example 3.12. We illustrate how to use Theorem 3.11 to reproduce the multiplication table for $QH^*Gr(1,3)$:

$$(38) \quad \sigma_\square * \sigma_\square = \sigma_{\square\square} \qquad\qquad \sigma_\square * \sigma_{\square\square} = q\sigma_\emptyset \qquad\qquad \sigma_{\square\square} * \sigma_{\square\square} = q\sigma_\square.$$

First note that since $\mathbf{i} = (1)$, then here $I_P = \{1,2\}\backslash\{1\} = \{2\}$. The elements of S_3^P are those which are forced to end with an element of $\langle s_1 \rangle$ on the right, corresponding to each of the three Schubert classes in $QH^*(Gr(1,3))$:

$$(39) \qquad \sigma_1 = \sigma_\emptyset \qquad\qquad \sigma_{s_1} = \sigma_\square \qquad\qquad \sigma_{s_{21}} = \sigma_{\square\square}$$

From Proposition 3.9, $\pi_P(t_{-\theta}) = u = s_{120}$. Using Theorem 3.11 we have that

$$(40) \qquad
\begin{aligned}
\overline{\jmath}_u^0 = \overline{\jmath}_{s_{120}}^0 &\longmapsto q^{-1}\sigma_\emptyset \\
\overline{\jmath}_{s_1 u}^0 = \overline{\jmath}_{s_1 \cdot s_{120}}^0 = \overline{\jmath}_{s_{20}}^0 &\longmapsto q^{-1}\sigma_\square \\
\overline{\jmath}_{s_{21} u}^0 = \overline{\jmath}_{s_{21} \cdot s_{120}}^0 = \overline{\jmath}_{s_0}^0 &\longmapsto q^{-1}\sigma_{\square\square}.
\end{aligned}$$

Since $j_{s_0}^0 = \tilde{\mathbf{h}}_1$ and $j_{s_{20}}^0 = \tilde{\mathbf{e}}_2$, we can use Definition 2.7 to compute the

following affine products:

$$
\begin{aligned}
\bar{j}^0_{s_{20}} \cdot \bar{j}^0_{s_{20}} &= (A_{20} + A_{12} + A_{01})^2 = A_{2012} + A_{1201} + A_{0120} \\
&= \bar{j}^0_{s_{0120}} \\
\bar{j}^0_{s_{20}} \cdot \bar{j}^0_{s_0} &= (A_{20} + A_{12} + A_{01})(A_0 + A_1 + A_2) \\
&= A_{201} + A_{202} + A_{120} + A_{121} + A_{010} + A_{012} \\
&= \bar{j}^0_{s_{120}} \\
\bar{j}^0_{s_0} \cdot \bar{j}^0_{s_0} &= (A_0 + A_1 + A_2)^2 = A_{01} + A_{02} + A_{10} + A_{12} + A_{20} + A_{21} \\
&= \bar{j}^0_{s_{10}} + \bar{j}^0_{s_{20}}.
\end{aligned}
\tag{41}
$$

Next, recall from Lemma 3.1 that we can easily verify whether an element $w \in \tilde{S}_3$ lies in $\tilde{S}_3^P$ by checking whether or not the two positive affine roots α_2 and $-\alpha_2 + \delta = \alpha_0 + \alpha_1$ are inversions of w. As seen in Figure 3, we have that

$$
\{s_0, s_{20}, s_{120}, s_{0120}, s_{20120}, s_{120120}\} \subset \tilde{S}_3^P \cap \tilde{S}_3^0
\tag{42}
$$

are all of the elements in $\tilde{S}_3^P \cap \tilde{S}_3^0$ of length six or less. Note that $s_{10} \notin \tilde{S}_3^P$, which means that $\bar{j}^0_{s_{10}} \equiv 0 \mod J_P$. We can now map our products in (41) via Theorem 3.11 to $QH^*(Gr(1,3))$:

$$
\begin{aligned}
q^{-1}\sigma_\square * q^{-1}\sigma_\square \quad &\longleftrightarrow \quad \bar{j}^0_{s_{20}} \cdot \bar{j}^0_{s_{20}} = \bar{j}^0_{s_{0120}} = \bar{j}^0_{s_{21}\cdot s_{120120}} \\
&= \bar{j}^0_{s_{21}\pi_P(t_{-2\theta})} \longmapsto q^{-2}\sigma_{\square\square} \\
q^{-1}\sigma_\square * q^{-1}\sigma_{\square\square} \quad &\longleftrightarrow \quad \bar{j}^0_{s_{20}} \cdot \bar{j}^0_{s_0} = \bar{j}^0_{s_{120}} \longmapsto q^{-1}\sigma_\emptyset \\
q^{-1}\sigma_{\square\square} * q^{-1}\sigma_{\square\square} \quad &\longleftrightarrow \quad \bar{j}^0_{s_0} \cdot \bar{j}^0_{s_0} = \bar{j}^0_{s_{10}} + \bar{j}^0_{s_{20}} \longmapsto 0 + q^{-1}\sigma_\square
\end{aligned}
\tag{43}
$$

Multiplying everything through by q^2 to clear denominators, we exactly recover the three products in $QH^*(Gr(1,3))$ from (38).

4. Posnikov's approach and the affine nilTemperley-Lieb algebra

As an application of the parabolic Peterson isomorphism, we recast Postnikov's affine approach to the quantum cohomology of the Grassmannian from [Pos05] as a corollary of Theorem 3.11. In particular, we demonstrate that Postnikov's isomorphism is the composition of the parabolic Peterson

isomorphism, followed by two duality isomorphisms which correct for the fact that the parabolic Peterson isomorphism itself does not map $\tilde{\mathbf{h}}_r \mapsto \sigma_{(r)}$ or $\tilde{\mathbf{e}}_r \mapsto \sigma_{(1^r)}$.

4.1. The affine nilTemperley-Lieb algebra

Definition 4.1. The **affine nilTemperley-Lieb algebra** $n\widehat{\mathrm{TL}}_n$ is the quotient of the affine nilCoxeter algebra formed by sending $A_i A_{i+1} A_i \mapsto 0$ for all $i \bmod n$.

To distinguish the elements $\tilde{\mathbf{h}}_r, \tilde{\mathbf{e}}_r \in (\mathbb{A}_{af})_0$ from their images in this quotient, we denote by $\tilde{\mathbf{h}}_r^n$ and $\tilde{\mathbf{e}}_r^n$ the images of the non-commutative homogeneous and elementary symmetric functions in the affine nilTemperley-Lieb algebra. Following [Pos05], we further define certain products of the elementary and homogeneous functions as follows:

$$\text{(44)} \qquad \mathbf{z}_i = \tilde{\mathbf{e}}_i^n \cdot \tilde{\mathbf{h}}_{n-i}^n.$$

Lemma 8.4 in [Pos05] shows that the elements $\mathbf{z}_i$ are in the center of $n\widehat{\mathrm{TL}}_n$.

Definition 4.2. For any fixed $1 \leq m \leq n-1$, define the algebra

$$\text{(45)} \qquad n\widehat{\mathrm{TL}_{mn}} = n\widehat{\mathrm{TL}}_n[\mathbf{z}_m^{-1}]/\langle \mathbf{z}_1, ..., \mathbf{z}_{m-1}, \mathbf{z}_{m+1}, ...\mathbf{z}_{n-1}\rangle,$$

and then let $\mathbb{X} \subset n\widehat{\mathrm{TL}_{mn}}$ be the subalgebra generated by the $\tilde{\mathbf{e}}_r^n$ and/or $\tilde{\mathbf{h}}_r^n$ for all $1 \leq r \leq n-1$.

Postnikov proceeds to prove that the subalgebra $\mathbb{X}$ generated by the $\tilde{\mathbf{h}}_r^n$ and/or $\tilde{\mathbf{e}}_r^n$ is isomorphic to the quantum cohomology of the Grassmannian, localized around the quantum parameter.

Theorem 4.3 (Proposition 8.5 [Pos05]). *For any fixed $1 \leq m \leq n-1$, the following map is an isomorphism of $\mathbb{Z}$-algebras which maps the generators as follows for all $1 \leq r \leq n-1$:*

$$
\begin{aligned}
\phi_{Po} : \mathbb{X} \subset n\widehat{TL_{mn}} &\longrightarrow QH^*(Gr(m,n))[q^{-1}] \\
\tilde{\mathbf{h}}_r^n &\longmapsto \sigma_{(r)} \\
\tilde{\mathbf{e}}_r^n &\longmapsto \sigma_{(1^r)} \\
\mathbf{z}_m &\longmapsto q.
\end{aligned}
$$

(46)

4.2. Duality isomorphisms

The cohomology of the Grassmannian admits a natural duality isomorphism arising from the fact that $Gr(m,n) \cong Gr(n-m,n)$. Given a partition $\lambda \in \mathcal{P}_{mn}$, define the **transpose** of λ by interchanging the role of the rows and columns, and denote it by $\lambda^T \in \mathcal{P}_{n-m,n}$. The map

$$(47) \qquad \begin{aligned} T : QH^*(Gr(m,n)) &\longrightarrow QH^*(Gr(n-m,n)) \\ \sigma_\lambda &\longmapsto \sigma_{\lambda^T} \end{aligned}$$

is an isomorphism. Note in particular that the transpose interchanges the classes $\sigma_{(r)}$ and $\sigma_{(1^r)}$.

We now review the **strange duality** involution appearing in [Pos05], which was discovered independently in the special case of $q = 1$ by Hengelbrock [Hen02], and later generalized by Chaput, Manivel, and Perrin to other (co)miniscule homogeneous varieties [CMP07].

Theorem 4.4 (Theorem 6.5 [Pos05]). *There is an involution*

$$(48) \qquad \begin{aligned} \iota : QH^*(Gr(m,n)) &\longrightarrow QH^*(Gr(m,n)) \quad by \\ \sigma_\lambda &\longmapsto q^{-diag_m(\lambda)} \sigma_\mu \\ q &\longmapsto q^{-1} \end{aligned}$$

where if the 01-word corresponding to λ is $b = i_1 i_2 \cdots i_n$, then the 01-word corresponding to μ is $b' = i_m i_{m-1} \cdots i_1 i_n i_{n-1} \cdots i_{m+1}$, and $diag_m(\lambda)$ is the number of boxes on the main diagonal.

Example 4.5. Consider $Gr(5,9)$ and the partition $\lambda = (3,2,2,1,1)$. The 01-word for λ is given by $b = 011011010$. Therefore, the image μ has the 01-word 101100101, and the corresponding shape is $\mu = (4,3,1,1,0)$.

$$(49) \qquad \lambda = \quad \mu =$$

Similar to the transpose map on quantum cohomology, the k-Schur functions admit an automorphism under k-conjugation.

Theorem 4.6 (Theorem 38 [LM07]). *The k-conjugation map is an involution on the subalgebra of symmetric functions spanned by the homogeneous*

symmetric functions:

(50)
$$\omega : \Lambda^{(k)} \longrightarrow \Lambda^{(k)}$$
$$s_\lambda^{(k)} \longmapsto s_{\lambda^{\omega_k}}^{(k)}.$$

4.3. Relationship to the parabolic Peterson isomorphism

In the remainder of this section, we aim to show the following isomorphism naturally relating the subalgebra considered by Postnikov to the localized affine Fomin-Stanley algebra

(51)
$$(H_*(\mathcal{G}r)/J_P)[(\overline{j}_{\pi_P(t_{-\theta})}^0)^{-1}] \cong \mathbb{X} \subset n\widehat{T\overline{L}_{n-m,n}},$$

where $I_P = I \backslash \{m\}$. As a first step, we prove that the composition of the parabolic Peterson isomorphism Ψ_P, followed by the strange duality involution and the transpose isomorphism, map the generators in the same way as the map ϕ_{P_o} from Theorem 4.3.

Theorem 4.7. *Let $I_P = I \backslash \{m\}$. The following composition is an isomorphism which maps the generators as follows for all $1 \le r \le n-1$:*

(52)
$$T \circ \iota \circ \Psi_P : (H_*(\mathcal{G}r)/J_P)[(\overline{j}_{\pi_P(t_{-\theta})}^0)^{-1}] \longrightarrow QH^*(Gr(n-m,n))[q^{-1}]$$
$$\tilde{\mathbf{h}}_r \longmapsto \sigma_{(r)}$$
$$\tilde{\mathbf{e}}_r \longmapsto \sigma_{(1^r)}$$
$$\tilde{\mathbf{z}}_{n-m} = \tilde{\mathbf{e}}_{n-m} \cdot \tilde{\mathbf{h}}_m \longmapsto q.$$

Proof. This map is obviously an isomorphism since it combines the parabolic Peterson isomorphism with two duality isomorphisms. In addition, recall from Definition 2.8 and Theorem 2.11 that for every $w \in \tilde{S}_n^0$, we have $j_w^0 = f(\tilde{\mathbf{h}}_1, \tilde{\mathbf{h}}_2, ..., \tilde{\mathbf{h}}_{n-1})$, and so it is sufficient to check that the generators are mapped as claimed.

First note that $\tilde{\mathbf{h}}_r = j_{w(r)}^0$, where $w(r) = s_{r-1,r-2,...,1,0}$, which corresponds to a k-bounded shape given by a horizontal row of r boxes since $r \le n-1$. Now recall Lemma 3.3, which says that $\tilde{\mathbf{h}}_r \in \tilde{S}_n^P$ if and only if $r \le m$. Therefore, $\tilde{\mathbf{h}}_r \in J_P$ if and only if $m > r \ge n-1$, which means that $\Psi_P : \tilde{\mathbf{h}}_r \mapsto 0$ if the horizontal strip (r) does not fit in $\mathcal{P}_{mn}$.

Now for $\tilde{\mathbf{h}}_r$ with $1 \le r \le m$, recalling the notation of Eq. (32), we can clearly see that

$$w(r) = s_{r-1,...,0} = s_{I,r,m-1}^0 s_{D,n-1,m}^0 s_{I,m,n-1}^0 s_{D,m-1,1}^0 s_0$$

$$= s^0_{I,r,m-1} s^0_{D,n-1,m} \pi_P(t_{-\theta}),$$

so using the statement of the Peterson isomorphism from Theorem 3.11, we have

$$(53) \qquad\qquad \Psi_P : \overline{j}^{\,0}_{w(r)} \longmapsto q^{-1}\sigma_\mu,$$

where μ is a hook shape with width $n - m$ and height $m - r + 1$. The main diagonal of μ has 1 box, the first row has full width in $\mathcal{P}_{n-m,n}$, and the first column of is missing $r - 1$ boxes. Therefore, when we map next by ι, the shape will have one box in the first row and r boxes in the first column; i.e. $\iota(\mu) = (1^r)$. Summarizing the composition, we have proved that

$$(54) \qquad\qquad \tilde{\mathbf{h}}_r \xrightarrow{\Psi_P} q^{-1}\sigma_\mu \xrightarrow{\iota} \sigma_{(1^r)} \xrightarrow{T} \sigma_{(r)}.$$

Finally, since one can write $\sigma_{(1^r)}$ in terms of products and sums of $\sigma_{(r)}$, and since $s^{(k)}_\lambda = s_\lambda$ if $\lambda \in \mathcal{P}_{mn}$, we know that $\tilde{\mathbf{e}}_r$ will be mapped as claimed. In addition, one can easily check that, modulo the ideal J_P,

$$(55) \qquad \tilde{\mathbf{z}}_{n-m} = \tilde{\mathbf{e}}_{n-m} \cdot \tilde{\mathbf{h}}_m = \overline{j}_{\pi_P(t_{-\theta})} \qquad \text{and} \qquad \sigma_{(1^{n-m})} \cdot \sigma_{(m)} = q$$

in $QH^*(Gr(n - m, m))$, as desired. $\qquad\qquad\qquad\qquad\qquad\square$

Our final goal is to directly relate Theorems 4.3 and 4.7. We require one last technical lemma.

Lemma 4.8. *Let $I_P = I \backslash \{m\}$. Suppose that $j^0_w \in J_P$, where $w = w_1 \pi_P(t_{-\theta})$ with $w_1 \in \tilde{S}^0_n$ and $\ell(w) = \ell(w_1) + \ell(\pi_P(t_{-\theta}))$. Then,*

$$(56) \qquad\qquad \overline{j}^{\,0}_{w_1} \equiv 0 \in (H_*(\mathcal{G}r)/J_P)[(\overline{j}^{\,0}_{\pi_P(t_{-\theta})})^{-1}].$$

Proof. Recall by Theorem 2.10 that $j^0_v = A_v + \displaystyle\sum_{u \notin \tilde{S}^0_n, \, \ell(u)=\ell(v)} c^u_v A_u$ for some $c^u_v \in \mathbb{Z}$. Thus,

$$j^0_{w_1} \cdot j^0_{\pi_P(t_{-\theta})}$$

$$= \left(A_{w_1} + \sum_{\substack{w' \notin \tilde{S}^0_n \\ \ell(w') = \ell(w_1)}} c^{w'}_{w_1} A_{w'} \right) \left(A_{\pi_P(t_{-\theta})} + \sum_{\substack{v' \notin \tilde{S}^0_n \\ \ell(v') = \ell(\pi_P(t_{-\theta}))}} c^{v'}_{\pi_P(t_{-\theta})} A_{v'} \right)$$

$$= A_{w_1 \pi_P(t_{-\theta})} + \sum_{w'} c_{w_1}^{w'} A_{w' \pi_P(t_{-\theta})} + \sum_{u \notin \tilde{S}_n^0} d_{u,w} A_u$$

(57)
$$= j_{w_1 \pi_P(t_{-\theta})}^0 + \sum_{\substack{w' \pi_P(t_{-\theta}) \in \tilde{S}_n^0 \\ A_{w' \pi_P(t_{-\theta})} \neq 0}} c_{w_1}^{w'} j_{w' \pi_P(t_{-\theta})}^0.$$

Now, since $w' \notin \tilde{S}_n^0$, then each w' appearing in the sum must end in a letter s_i with $i \neq 0$. Note that, for $i < m-1$, we have by Proposition 3.9 that

(58)
$$\begin{aligned}
s_i \pi_P(t_{-\theta}) &= s_i s_{m,m+1,\ldots,n-1,m-1,\ldots,1,0} \\
&= s_{m,m+1,\ldots,n-1,m-1,\ldots,i,i+1,i,\ldots,1,0} \\
&= s_{m,m+1,\ldots,n-1,m-1,\ldots,i+1,i,i+1,\ldots,1,0} \\
&= s_{m,m+1,\ldots,n-1,m-1,\ldots,i+1,i,\ldots,1,0,i+1} \notin \tilde{S}_n^0,
\end{aligned}$$

which shows that w' cannot end in s_i with $0 < i < m-1$. A similar calculation shows w' cannot end in $i > m+1$. If $i = m$, then $A_{w' \pi_P(t_{-\theta})} = 0$ since we have two adjacent generators A_m. Therefore, the only remaining possibilities are that w' ends in either s_{m-1} or s_{m+1}. However, in either of these two cases, the element $w' \pi_P(t_{-\theta})$) supports a braid relation. Therefore, by Proposition 3.5 and our hypotheses, the sum in Eq. (57) lies in J_P. The claim follows by the invertibility of $\overline{j}_{\pi_P(t_{-\theta})}^0$. $\qquad\square$

Theorem 4.9. *Let $I_P = I \backslash \{m\}$. There is a $\mathbb{Z}$-algebra isomorphism*

(59)
$$\chi : (H_*(\mathcal{G}r)/J_P)[(\overline{j}_{\pi_P(t_{-\theta})}^0)^{-1}] \cong \mathbb{X} \subset n\widehat{TL_{n-m,n}},$$

where $\mathbb{X}$ is the subalgebra of the affine nilTemperley-Lieb algebra defined in Theorem 4.3.

Of course, the existence of this isomorphism follows immediately because it can be expressed using Theorem 4.3 and Theorem 4.7 as the following composition of isomorphisms

(60)
$$\chi = \phi_{Po}^{-1} \circ T \circ \iota \circ \Psi_P.$$

However, our goal is to prove that this isomorphism arises independently of Postnikov's work, so we provide a proof which does not use the map ϕ_{Po}, but rather only the parabolic Peterson isomorphism. Indeed, the natural projection $\mathbb{B} \longrightarrow \mathbb{B}/J_P$ induces an isomorphism between the localization of the affine Fomin-Stanley algebra and the subalgebra $\mathbb{X}$ of the affine nilTemperley-Lieb algebra considered by Postnikov.

Proof of Theorem 4.9. Consider the natural surjection

$$(61) \qquad p : H_*(\mathcal{G}r)[(\vec{j}^0_{\pi_P(t_{-\theta})})^{-1}] \longrightarrow nT\widehat{L_{n-m,n}} \qquad \text{by} \qquad \tilde{\mathbf{h}}_r \longmapsto \tilde{\mathbf{h}}_r^n.$$

By a standard localization theorem, we have

$$(62) \quad (H_*(\mathcal{G}r)/J_P)[(\vec{j}^0_{\pi_P(t_{-\theta})})^{-1}] \cong H_*(\mathcal{G}r)[(j^0_{\pi_P(t_{-\theta})})^{-1}]/J_P[(j^0_{\pi_P(t_{-\theta})})^{-1}].$$

To prove the isomorphism in Equation (59), it thus suffices to show that

$$(63) \qquad\qquad \ker(p) = J_P[(j^0_{\pi_P(t_{-\theta})})^{-1}].$$

Since p is the natural surjection, we know that $\ker(p)$ is the localization of the ideal that contains all elements a satisfying the condition that for every monomial A_w occurring in a, the element w necessarily supports a braid relation by the definition of $nT\widehat{L_{n-m,n}}$. Since the element of the j-basis indexed by $w \in \tilde{S}_n^0$ at least contains the term A_w, then all the j-basis terms that appear in the expansion of a support braids and are in J_P by Proposition 3.5. Therefore, $\ker(p) \subseteq J_P[(j^0_{\pi_P(t_{-\theta})})^{-1}]$ since localization respects inclusion.

Conversely, suppose that $j^0_w \in J_P[(j^0_{\pi_P(t_{-\theta})})^{-1}]$, and suppose by contradiction that $j^0_w \notin \ker(p)$. Further suppose that $\ell(w)$ is the minimum length of any word satisfying this condition. By Lemma 3.2, we know that s_m appears in every reduced expression for w. Pick the rightmost instance of s_m among all reduced expressions for w. There are clearly three cases: $w = w_1 s_{m,m-1,...,1,0}$ or $w = w_1 s_{m,m+1,...,n-1,0}$ or $w = w_1 s_{m,m+1,...,n-1,m-1,m-2,...,1,0}$ where in each case $\ell(w) = \ell(w_1) + \ell(s_{m,...,0})$.

Consider the first case $w = w_1 s_{m,m-1,...,1,0}$. We first argue using the bijections discussed in Section 2.2 that w corresponds to a k-bounded partition with at least $m + 1$ boxes in the first row. Let the n-core associated to w be denoted $\mu = (\mu_1, ..., \mu_p)$. Consider the $m + 1$ rightmost boxes in the first row, starting in column $i = \mu_1 - m$. In order for one of these boxes to be removed when we biject to the k-bounded partitions, the ith column must have $n - m + 1$ boxes. Thus, after inserting the first $m + 1$ boxes, we must have inserted i boxes in the $(n - m)^{\text{th}}$ row. The grid-numbers of this row begin with $m + 1$, so inserting i boxes into row $n - m$ also inserts i boxes in the first row since there are shared labels. This means the first row has $i + m + 1 = \mu_1 + 1$ boxes, a contradiction. Therefore, $\lambda(w)$, the corresponding k-bounded partition, has at least $m + 1$ boxes in the first row. Now, by

Lemma 2.4 and Definition 2.8, we know that for some $c_{ij} \in \mathbb{Z}$,

$$(64) \qquad j_w^0 = \mathbf{s}_{\lambda(w)}^{(k)} = \tilde{\mathbf{h}}_{\lambda_1} \mathbf{s}_{(\lambda_2,\ldots,\lambda_p)}^{(k)} + \sum_{i=1}^{n-1-\lambda_1} \tilde{\mathbf{h}}_{i+\lambda_1} \cdot \left(\sum_j c_{ij} \mathbf{s}_{\mu_{ij}}^{(k)} \right).$$

By Lemma 3.3, we know that $\tilde{\mathbf{h}}_r \in J_P$ for $r > m$. Since $p(\tilde{\mathbf{h}}_r) = \tilde{\mathbf{h}}_r^n \equiv 0$ if $r > m$, then we also have that $\tilde{\mathbf{h}}_r \in \ker(p)$. Since we have shown that $\lambda_1 > m$, we thus have $j_w^0 \in \ker(p)$.

Similarly, if $w = w_1 s_{m,m+1,\ldots,n-1,0}$, then the k-conjugate partition corresponds to the word $w^{\omega_k} = w_1^{\omega_k} s_{n-m,n-m-1,\ldots,1,0}$, since the k-conjugate partition is the transpose of the n-core, meaning that we reflect which diagonal we add the boxes to. Therefore, we can write

$$(65) \qquad j_{w^{\omega_k}}^0 = \mathbf{s}_{\lambda^{\omega_k}}^{(k)} = \tilde{\mathbf{h}}_{\lambda_1^{\omega_k}} \mathbf{s}_{(\lambda_2^{\omega_k},\ldots,\lambda_p^{\omega_k})}^{(k)} + \sum_{i=1}^{n-1-\lambda_1^{\omega_k}} \tilde{\mathbf{h}}_{i+\lambda_1^{\omega_k}} \cdot \left(\sum_j c_{ij} \mathbf{s}_{\mu_{ij}^{\omega_k}}^{(k)} \right)$$

using the same $c_{ij} \in \mathbb{Z}$ as above. Recalling from Theorem 4.6 that k-conjugation is an automorphism on $H_*(\mathcal{G}r)$, we now k-conjugate both sides of Equation (65), which expresses j_w^0 as a sum of terms $\tilde{\mathbf{e}}_i \cdot f_i$ for some elements $f_i \in \mathbb{B}$, where we know that $i > n - m$, since $\lambda_1^{\omega_k} > n - m$. By Lemma 3.3, we know that $\tilde{\mathbf{e}}_r \in J_P$ for $r > n - m$, and since $p(\tilde{\mathbf{e}}_r) = \tilde{\mathbf{e}}_r^n \equiv 0$ if $r > n - m$, then $\tilde{\mathbf{e}}_r \in \ker(p)$. Since we have shown that $\lambda_1^{\omega_k} > n - m$, we thus have that $j_w^0 \in \ker(p)$ in this case as well.

Finally, in the third case, we have $w = w_1 s_{m,m+1,\ldots,n-1,m-1,\ldots,1,0} = w_1 \pi_P(t_{-\theta})$ by Proposition 3.9. Therefore, by similar arguments to those made in the proof of Lemma 4.8, we see that w_1 must end in s_0, s_{m-1}, or s_{m+1}. In the latter case, we have that w ends with

$$s_{m+1} s_{m,m+1,\ldots,n-1,m-1,\ldots,1,0} = s_{m,m+1,m,m+2,\ldots,n-1,m-1,\ldots,1,0}$$
$$= s_{m,m+1,m+2,\ldots,n-1,m,m-1,\ldots,1,0},$$

which reduces to a previous case. This similarly holds if w_1 ends in s_{m-1}. When w_1 is forced to end in s_0, then by Lemma 4.8, we know that $j_{w_1}^0 \in J_P[(j_{\pi_P(t_{-\theta})}^0)^{-1}]$. Therefore, since $\ell(w_1) < \ell(w)$, we know that $j_{w_1}^0 \in \ker(p)$ by definition of w. Recall from Eq. (57) in the proof of Lemma 4.8 that

$$j_{w_1}^0 j_{\pi_P(t_{-\theta})}^0 = j_w^0 + \sum_v c_{w_1}^v j_{v\pi_P(t_{-\theta})}^0,$$

where v is forced to end in either s_{m-1} or s_{m+1}. This means that $j^0_{v\pi_P(t_{-\theta})} \in$ ker(p) because it reduces to a previous case argued above. Therefore, $j^0_w \in$ ker(p) in this third case as well.

We have reached a contradiction to the fact that $j^0_w \notin$ ker(p). Therefore $J_P[j^0_{\pi_P(t_{-\theta})})^{-1}] \subseteq$ ker(p), establishing that χ is an isomorphism. $\qquad\square$

Theorem 4.9 proves that the isomorphism established by Postnikov in [Pos05] between the localizations of the subalgebra $\mathbb{X}$ of the affine nilTemperley-Lieb algebra and the quantum cohomology of the Grassmannian is really a direct consequence of the parabolic Peterson isomorphism.

Corollary 4.10. *The map ϕ_{Po} from [Pos05] defined in Theorem 4.3 factors as*

$$(66) \qquad \phi_{Po} = T \circ \iota \circ \Psi_P \circ \chi^{-1} : \mathbb{X} \subset \widehat{nTL_{mn}} \longrightarrow QH^*(Gr(m,n))[q^{-1}],$$

where χ is the isomorphism established in Theorem 4.9 and $I_P = I \backslash \{n-m\}$.

Proof. Having already established the isomorphism, it suffices to check that the image of this composition has the same effect on the generators as ϕ_{Po}. By the proof of Theorem 4.9, if $r \le n-m$, then $\tilde{\mathbf{h}}_r = \chi(\tilde{\mathbf{h}}_r^n)$, which maps to $\sigma_{(r)}$ in the codomain by Theorem 4.7. Otherwise, $\tilde{\mathbf{h}}_r^n = \tilde{\mathbf{h}}_r = 0$. Since the $\tilde{\mathbf{h}}_r^n$ generate $\widehat{nTL_{mn}}$ and the $\sigma_{(r)}$ generate QH*(Gr(m,n)), this concludes the proof. $\qquad\square$

Acknowledgements

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