

Generating functions for shifted symmetric functions

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We construct the generating functions for shifted Schur functions and describe their vertex operator realization.

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1. Introduction

The algebra of shifted symmetric functions is a deformation of the classical algebra of symmetric functions. There are a lot of applications and connections of shifted symmetric functions in representation theory: the study of centers of universal enveloping algebras, Capelli-type identities, asymptotic characters for unitary groups and symmetric groups, infinite-dimensional quantum groups, and Yangians etc. In particular, the Harish-Chandra isomorphism identifies the center of the universal enveloping algebra of the general linear Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ with the algebra of shifted symmetric functions, sending a central element to its eigenvalue on a highest weight module. With a distinguished basis of the center the images of the elements of the basis under the Harish-Chandra isomorphism are exactly the shifted Schur functions [9], [10], [8], [11].

In this note we prove a new formula for the generating function of shifted Schur functions

$$(1.1) \quad \det \left[\frac{1}{(u_i | i - j)} \right] \prod_{i=1}^l H^*(u_i - i + 1) = \sum_{l(\lambda) \leq l} \frac{s_\lambda^*}{(u_1 | \lambda_1) \cdots (u_l | \lambda_l)},$$

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where $H^*(u)$ is the generating function of the homogeneous complete shifted symmetric functions $h_r^* = s_{(r)}^*$. This result can be compared with the formula for classical Schur functions s_λ

$$(1.2) \quad \prod_{i < j} \left(1 - \frac{u_j}{u_i}\right) \prod_{i=1}^l H(u_i) = \sum_{l(\lambda) \leq l} s_\lambda u_1^{\lambda_1} \cdots u_l^{\lambda_l},$$

where $H(u)$ is the generating function of the homogeneous complete symmetric functions $h_r = s_{(r)}$ (see e.g. [7, Ex. 29, Chapter 1]).

The origin of the formula (1.2) lies in the renowned vertex operator realization of Schur functions in the framework of the boson-fermion correspondence. Several important families of symmetric functions (such as Schur symmetric functions, Schur Q -functions and more generally, Hall-Littlewood symmetric functions) can be interpreted as coefficients of the generating functions defined by certain correlation factors. In such a case there is a simple and natural way to introduce the action of the Clifford algebra or a modified Clifford algebra on the vector space spanned by the coefficients of the generating function. This in turn gives rise to (in some cases modified) vertex operators. Generalization of this approach allowed us to construct the generating function (1.1) and describe the action of the Clifford algebra on the space of shifted symmetric functions. The generating function (1.1) is interpreted as a result of a successive applications of certain vertex operators to the vacuum vector.

The paper is organized as follows. In Sections 2 and 3, we review definitions related to shifted symmetric functions and the classical boson-fermion correspondence. In Section 4, we construct the generating function for shifted Schur functions. In Section 5, the Clifford algebra action and the resulting vertex operator presentation are described.

2. Shifted symmetric functions

We follow the notations and definitions of [10]. Combinatorially a shifted Schur polynomial $s_\lambda^*(x_1, \dots, x_n)$ can be defined as the ratio of determinants

$$s_\lambda^*(x_1, \dots, x_n) = \frac{\det(x_i + n - i | \lambda_j + n - j)}{\det(x_i + n - i | n - j)},$$

where the falling factorial power of x is defined by

$$(x|k) = \begin{cases} x(x-1)\cdots(x-k+1) & \text{for } k = 1, 2, \dots, \\ 1, & \text{for } k = 0, \\ \frac{1}{(x+1)\cdots(x+(-k))} & \text{for } k = -1, -2, \dots \end{cases}$$

The stability property of shifted Schur polynomials allows one to introduce the shifted Schur functions $s_\lambda^* = s_\lambda^*(x_1, x_2, \dots)$. In particular, the complete shifted Schur functions $h_r^* = s_{(r)}^*$ are

$$(2.1) \quad h_r^*(x_1, x_2, \dots) = \sum_{1 \leq i_1 \leq \dots \leq i_r < \infty} (x_{i_1} - r + 1)(x_{i_2} - r + 2) \cdots x_{i_r},$$

and elementary shifted Schur functions $e_r^* = s_{(1^r)}^*$ are

$$(2.2) \quad e_r^*(x_1, x_2, \dots) = \sum_{1 \leq i_1 < \dots < i_r < \infty} (x_{i_1} + r - 1)(x_{i_2} + r - 2) \cdots x_{i_r}.$$

By [10, Corollary 1.6], the shifted Schur functions s_λ^* form a linear basis in the ring \mathcal{B}^* of shifted symmetric functions, which is also a polynomial ring in the shifted complete or elementary symmetric functions:

$$\mathcal{B}^* = \mathbb{C}[h_1^*, h_2^*, \dots] = \mathbb{C}[e_1^*, e_2^*, \dots].$$

Theorem 13.1 in [10] states that for any partition λ of length $l(\lambda)$

$$(2.3) \quad s_\lambda^* = \det[\tau^{j-1} h_{\lambda_i - i + j}^*]_{1 \leq i, j \leq l}, \quad s_\lambda^* = \det[\tau^{1-j} e_{\lambda_i - i + j}^*]_{1 \leq i, j \leq m},$$

where l, m are arbitrary fixed integers such that $l \geq l(\lambda)$, $m \geq \lambda_1$, and τ is the automorphism of \mathcal{B}^* defined by the formula

$$\tau(h_k^*) = h_k^* + (k-1)h_{k-1}^*, \quad \tau^{-1}(e_k^*) = e_k^* + (k-1)e_{k-1}^*.$$

3. Boson-fermion correspondence

Consider the infinite-dimensional complex vector space $V = \bigoplus_{j \in \mathbb{Z}} \mathbb{C} v_j$ with a linear basis $\{v_j\}_{j \in \mathbb{Z}}$. Define $F^{(m)}$ ($m \in \mathbb{Z}$) as the linear span of semi-infinite monomials $v_{i_m} \wedge v_{i_{m-1}} \wedge \dots$ with the properties:

- (1) $i_m > i_{m-1} > \dots$,
- (2) $i_k = k$ for $k \ll 0$.

The monomial of the form $|m\rangle = v_m \wedge v_{m-1} \wedge \dots$ is called the m th vacuum vector. The elements of $F^{(m)}$ are linear combinations of monomials $v_I =$

$v_{i_1} \wedge v_{i_2} \wedge \dots$ that are different from $|m\rangle$ only at a finitely many places, and $I = \{i_1, i_2, \dots\}$. The fermionic Fock space is defined to be the graded space

$$\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} F^{(m)}.$$

Many important algebraic structures act on the Fock space, these include the (infinite dimensional) Clifford algebra, the Heisenberg algebra \mathcal{A} , the Virasoro algebra and the infinite-dimensional Lie algebra \mathfrak{gl}_∞ . Their actions are closely related to each other.

The Clifford algebra acts on the Fock space \mathcal{F} by wedge operators ψ_k^+ and contraction operators ψ_k^- ($k \in \mathbb{Z}$):

$$\begin{aligned} \psi_k^+(v_{i_1} \wedge v_{i_2} \wedge \dots) &= v_k \wedge v_{i_1} \wedge v_{i_2} \wedge \dots, \\ \psi_k^-(v_{i_1} \wedge v_{i_2} \wedge \dots) &= \delta_{k,i_1} v_{i_2} \wedge v_{i_3} \wedge \dots - \delta_{k,i_2} v_{i_1} \wedge v_{i_3} \wedge \dots + \delta_{k,i_3} v_{i_1} \wedge v_{i_2} \wedge \dots. \end{aligned}$$

The operators satisfy the relations

$$\psi_k^+ \psi_m^- + \psi_m^- \psi_k^+ = \delta_{k,m}, \quad \psi_k^+ \psi_m^+ + \psi_m^+ \psi_k^+ = 0, \quad \psi_k^- \psi_m^- + \psi_m^- \psi_k^- = 0.$$

Combine the operators ψ_k^\pm in the generating functions (formal distributions)

$$(3.1) \quad \Psi^+(u) = \sum_{k \in \mathbb{Z}} \psi_k^+ u^k \quad \text{and} \quad \Psi^-(u) = \sum_{k \in \mathbb{Z}} \psi_k^- u^{-k}.$$

Then the action of the Heisenberg algebra \mathcal{A} on the Fock fermionic space \mathcal{F} can be introduced with the help of the normal ordered product of these formal distributions. Set

$$\alpha(u) =: \Psi^+(u) \Psi^-(u) := \Psi^+(u)_+ \Psi^-(u) - \Psi^-(u) \Psi^+(u)_-,$$

where the cut-off parts are given by

$$\Psi^+(u)_+ = \sum_{k \geq 1} \psi_k^+ u^k, \quad \Psi^+(u)_- = \sum_{k \leq 0} \psi_k^+ u^k.$$

The coefficients α_k of the formal distribution $\alpha(u) = \sum \alpha_k u^{-k}$ and central element 1 then satisfy the relations of Heisenberg algebra \mathcal{A} (see e.g. [6, 16.3]):

$$[1, \alpha_k] = 0, \quad [\alpha_k, \alpha_m] = m \delta_{m,-k} \quad (k, m \in \mathbb{Z}).$$

On the other hand, there is also a natural action of the Heisenberg algebra \mathcal{A} on Fock boson space $\mathcal{B}^{(m)} = z^m \mathbb{C}[p_1, p_2, \dots]$:

$$\alpha_n = \frac{\partial}{\partial p_n}, \quad \alpha_{-n} = np_n, \quad \alpha_0 = m \quad (n \in \mathbb{N}, m \in \mathbb{Z}).$$

The boson–fermion correspondence identifies the spaces $\mathcal{B}^{(m)}$ and $\mathcal{F}^{(m)}$ as equivalent \mathcal{A} -modules (see e.g. [1], [2], [3], [4], [6]). The construction of the correspondence relies on the interpretation of $\mathcal{B} = \mathbb{C}[p_1, p_2, \dots]$ as a ring of symmetric functions, where p_k 's are interpreted as the k -th (normalized) power sums. Then each graded component $\mathcal{B}^{(m)}$ is viewed as a ring of symmetric functions, which is known to be the ring of polynomials in variables p_k 's. The linear basis of elements $v_\lambda = (v_{\lambda_1+m} \wedge v_{\lambda_2+m-1} \wedge v_{\lambda_3+m-2} \dots)$ of $\mathcal{F}^{(m)}$, labeled by partitions $\lambda = (\lambda_1, \geq \lambda_2, \geq \dots \geq \lambda_l \geq 0)$ corresponds to the linear basis $z^m s_\lambda$ of $\mathcal{B}^{(m)}$, where s_λ is the Schur functions associated with the partition λ (see e.g. [6] Theorem 6.1).

The correspondence carries the action of operators ψ_k^\pm on \mathcal{F} to the action on the graded space $\oplus_m \mathcal{B}^{(m)}$. It can be described by generating functions $\Psi^\pm(u, m)$, traditionally written in a vertex operator form

$$(3.2) \quad \Psi^+(u, m) = u^{m+1} z \exp\left(\sum_{j \geq 1} p_j u^j\right) \exp\left(-\sum_{j \geq 1} \frac{\partial p_j}{j} u^{-j}\right),$$

$$(3.3) \quad \Psi^-(u, m) = u^{-m} z^{-1} \exp\left(-\sum_{j \geq 1} p_j u^j\right) \exp\left(\sum_{j \geq 1} \frac{\partial p_j}{j} u^{-j}\right).$$

The formulae (3.2), (3.3) can be simplified if one changes the set of generators of the ring of symmetric functions. Namely, introduce generating functions $E(u)$, $H(u)$ for the operators of multiplication by elementary symmetric functions $e_r = s_{(1^r)}$ and complete symmetric functions $h_r = s_{(r)}$. Note that $H(u)E(-u) = 1$. The ring of symmetric functions possesses a natural scalar product, where the classical Schur functions s_λ constitute an orthonormal basis: $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu}$. Then for any symmetric function f one can define the adjoint operator D_f acting on the ring of symmetric functions by the standard rule: $\langle D_f g, w \rangle = \langle g, f w \rangle$, where $g, f, w \in \Lambda$. The properties of adjoint operators are described in [7, I.5]. Set

$$DE(u) = \sum_{k \geq 0} D_{e_k} u^k, \quad DH(u) = \sum_{k \geq 0} D_{h_k} u^k.$$

Then

$$(3.4) \quad \Psi^+(u, m) = u^{m+1} z H(u) DE(-u^{-1}),$$

$$(3.5) \quad \Psi^-(u, m) = u^{-m} z^{-1} E(-u) DH(u^{-1}).$$

4. Generating functions for s_λ^*

Let $f(u)$ be a formal series or a function in variable u in some general sense. We introduce the *shift operator*

$$e^{k\partial_u} (f(u)) = f(u + k).$$

This exponential notation is motivated by Taylor series expansion formula, where for an appropriate class of functions in the domain of convergence one can write

$$f(u + k) = \sum_{s=0}^{\infty} \frac{(k\partial_u)^s}{s!} (f(u)) = e^{k\partial_u} (f(u)).$$

We use the short notation $e^{k\partial_{u_i}} := e^{k\partial_{u_i}}$ for shifts along variable u_i acting on $f(u_1, \dots, u_l)$.

Note that a shifted k -th power sum is a result of application to the constant function 1 of the k -th power of the operator $(ue^{-\partial_u})^k$:

$$(ue^{-\partial_u})^k (1) = (u|k).$$

We will be interested in shifted generating functions, which will be infinite sums in monomials of shifted powers of formal variables u_i 's.

Consider¹

$$(4.1) \quad H^*(u) = \sum_{k=0}^{\infty} \frac{h_k^*}{(u|k)}, \quad E^*(u) = \sum_{k=0}^{\infty} (-1)^k e_k^*(u| - k).$$

It is proved in [10] (Corollary 12.3) that

$$H^*(u)E^*(u) = 1.$$

¹ $H^*(u)$ and $E^*(u)$ in this note correspond to $H^*(u)$ and $E^*(-u-1)$ respectively in [10].

Also consider the formal series of the following form:

$$YH^*(u) = \sum_{k \geq 0} h_k^* \left(\frac{1}{u} e^{-\partial_u} \right)^k, \quad YE^*(u) = \sum_{k \geq 0} (-1)^k e_k^* \left(e^{\partial_u} \frac{1}{u} \right)^k,$$

where h_k^* and e_k^* are viewed as the multiplication operators by these functions acting on the space \mathcal{B}^* . Then

$$H^*(u) = YH^*(u)(1), \quad E^*(u) = YE^*(u)(1).$$

For a matrix $A = (a_{ij})_{i,j=1,\dots,N}$ with non-commutative entries the determinant is defined by $\det(A) = \sum_{\sigma \in S_N} \text{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{N\sigma(N)}$. Set

$$(4.2) \quad YH^*(u_1, \dots, u_l) = \det \left[\left(\frac{1}{u_i} e^{-\partial_i} \right)^{i-j} e^{(1-j)\partial_i} \right] \circ \prod_{i=1}^l YH^*(u_i),$$

$$(4.3) \quad YE^*(u_1, \dots, u_l) = \det \left[\left(e^{\partial_i} \frac{1}{u_i} \right)^{i-j} e^{(j-1)\partial_i} \right] \circ \prod_{i=1}^l YE^*(u_i).$$

The result of application of (4.2) or (4.3) to the constant function 1 is a formal series in shifted powers of u with coefficients in \mathcal{B}^* :

$$(4.4) \quad H^*(u_1, \dots, u_l) = YH^*(u_1, \dots, u_l)(1),$$

$$(4.5) \quad E^*(u_1, \dots, u_l) = YE^*(u_1, \dots, u_l)(1).$$

Proposition 4.1.

$$(4.6) \quad H^*(u_1, \dots, u_l) = \det \left[\frac{1}{(u_i | i - j)} \right] \prod_{i=1}^l H^*(u_i - i + 1),$$

$$(4.7) \quad E^*(u_1, \dots, u_l) = \det [(u_i | j - i)] \prod_{i=1}^l E^*(u_i + i - 1).$$

Also

$$(4.8) \quad H^*(u_1, \dots, u_l) = \prod_{i=1}^l \left(\frac{1}{u_i} e^{-\partial_i} \right)^{i-1} \left(\prod_{i < j} (u_j - u_i) \prod_{i=1}^l H^*(u_i) \right),$$

$$(4.9) \quad E^*(u_1, \dots, u_l) = \prod_{i=1}^l \left(e^{\partial_i} \frac{1}{u_i} \right)^{i-1} \left(\prod_{i < j} (u_j - u_i) \prod_{i=1}^l E^*(u_i) \right).$$

Proof. Since $(\frac{1}{u} e^{-\partial_u})^k = \frac{1}{(u|k)} e^{-k\partial_u}$ for any $k \in \mathbb{Z}$, one has

$$YH^*(u_1, \dots, u_l) = \det \left[\frac{1}{(u_i|i-j)} \right] \prod_{i=1}^l e^{(1-i)\partial_i} \circ YH^*(u_i),$$

and similarly,

$$YE^*(u_1, \dots, u_l) = \det [(u_i|j-i)] \prod_{i=1}^l e^{(i-1)\partial_i} \circ YE^*(u_i).$$

Then (4.6) and (4.7) follow by application of $YH^*(u_1, \dots, u_l)$ and $YE^*(u_1, \dots, u_l)$ to the vacuum vector 1.

For the proof of (4.8) and (4.9), we see that

$$\begin{aligned} & \prod_{i=1}^l \left(\frac{1}{u_i} e^{-\partial_i} \right)^{i-1} \left(\prod_{i=1}^l H^*(u_i) \prod_{i < j} (u_j - u_i) \right) \\ &= \prod_{i=1}^l H^*(u_i - i + 1) \prod_{i=1}^l \left(\frac{1}{u_i} e^{-\partial_i} \right)^{i-1} \left(\prod_{i < j} (u_j - u_i) \right). \end{aligned}$$

Lemma 4.1.

$$\begin{aligned} \prod_{i=1}^l \left(\frac{1}{u_i} e^{-\partial_i} \right)^{i-1} \left(\prod_{i < j} (u_j - u_i) \right) &= \det \left[\frac{1}{(u_i|i-j)} \right], \\ \prod_{i=1}^l \left(e^{\partial_i} \frac{1}{u_i} \right)^{i-1} \left(\prod_{i < j} (u_j - u_i) \right) &= \det [(u_i|j-i)]. \end{aligned}$$

Proof. We can rewrite the first product using the Vandermonde determinant:

$$\begin{aligned} & \prod_{i=1}^l \left(\frac{1}{u_i} e^{-\partial_i} \right)^{i-1} \left(\prod_{i < j} (u_j - u_i) \right) = \prod_{i=1}^l \frac{1}{(u_i | i - 1)} \prod_{i < j} (u_j - j - u_i + i) \\ & = \prod_{i=1}^l \frac{1}{(u_i | i - 1)} \prod_{i < j} ((u_j - j + 2) - (u_i - i + 2)) \\ & = \prod_{i=1}^l \frac{1}{(u_i | i - 1)} \det [(u_i - i + 2)^{j-1}]. \end{aligned}$$

Recall that shifted and ordinary powers of a variable x are related by

$$x^m = \sum_{k=0}^m (-1)^{m-k} S(m, k) x(x+1) \cdots (x+k-1),$$

where $S(m, k)$ are the Stirling numbers of the second kind, and $S(m, m) = 1$. Therefore, with $x = u_i - i + 2$ we can expand by linearity the columns of the determinant

$$\prod_{i=1}^l \frac{1}{(u_i | i - 1)} \det [(u_i - i + 2)(u_i - i + 3) \cdots (u_i - i + j)] = \det \left[\frac{1}{(u_i | i - j)} \right].$$

Similarly,

$$\begin{aligned} & \prod_{i=1}^l \left(e^{\partial_i} \frac{1}{u_i} \right)^{i-1} \left(\prod_{i < j} (u_j - u_i) \right) = \prod_{i=1}^l (u_i | 1 - i) \det [(u_i + i - 1)^{j-1}] \\ & = \prod_{i=1}^l (u_i | 1 - i) \det [(u_i + i - 1 | j - 1)] = \det [(u_i | j - i)]. \end{aligned}$$

□

Then (4.8) and (4.9) follow from Lemma 4.1. This completes the proof of the proposition. □

Remark 4.1. Note from (4.8), (4.9) that the change of order of variables leads to the following equalities:

$$\begin{aligned} \frac{1}{u} e^{-\partial_v} H^*(u, v, u_2, u_3, \dots) &= -\frac{1}{v} e^{-\partial_v} H^*(v, u, u_2, u_3, \dots), \\ e^{\partial_u} \frac{1}{u} E^*(u, v, u_2, u_3, \dots) &= -e^{\partial_v} \frac{1}{v} E^*(v, u, u_2, u_3, \dots). \end{aligned}$$

Our next goal is to identify coefficients of the shifted expansion of the generating functions $H^*(u_1, \dots, u_l)$ and $E^*(u_1, \dots, u_l)$ with shifted Schur functions. Following [10], Section 13, observe that the formal action of the shift operator $e^{-\partial_u}$ on the generating function $H^*(u)$ corresponds to an automorphism $\tau : \mathcal{B}^* \rightarrow \mathcal{B}^*$ of shifted symmetric functions. Namely, write for $a \in \mathbb{Z}_{\geq 0}$,

$$(4.10) \quad H^*(u-a) = e^{-a\partial_u}(H^*(u)) = \sum_{k=0}^{\infty} \frac{h_k^*}{(u-a|k)} = \sum_{k=0}^{\infty} \frac{\tau^a(h_k^*)}{(u|k)}.$$

Note that for $k = 1, 2, \dots$,

$$(4.11) \quad \frac{1}{(u-1|k)} = \frac{1}{(u|k)} + \frac{k}{(u|k+1)},$$

$$(u+1|-k) = (u|-k) - k(u|-k-1).$$

Hence, the explicit action of τ on the generators h_k^* ($k = 1, 2, \dots$) is given by

$$(4.12) \quad \tau(h_k^*) = h_k^* + (k-1)h_{k-1}^*,$$

$$(4.13) \quad \tau^a(h_k^*) = \sum_{i=0}^a \binom{a}{i} (k-1|i) h_{k-i}^* \quad (a = 1, 2, \dots).$$

Similarly, for $k = 1, 2, \dots$,

$$(4.14) \quad \tau^{-1}(e_k^*) = e_k^* + (k-1)e_{k-1}^*,$$

$$(4.15) \quad \tau^{-a}(e_k^*) = \sum_{i=0}^a \binom{a}{i} (k-1|i) e_{k-i}^*, \quad (a = 1, 2, \dots),$$

which by (4.11) corresponds to a shift of a variable of the generating function $E^*(u)$ for $a \in \mathbb{Z}_{\geq 0}$:

$$e^{a\partial_u} E^*(u) = E^*(u+a) = \sum_k (-1)^k e_k^*(u+a|-k) = \sum_k (-1)^k \tau^{-a}(e_k^*)(u|-k).$$

We need the following statement.

Lemma 4.2. *Let $\tau^{\pm 1}$ be the automorphisms of \mathcal{B}^* defined by (4.12), (4.14), and let $k_i \in \mathbb{Z}_{\geq 0}$, $m_i \in \mathbb{Z}$ ($i = 1, \dots, l$). Then in the shifted expansions*

$$\prod_{i=1}^l \frac{1}{(u_i|m_i)} H^*(u_i - k_i - m_i) = \sum_{\lambda} C_{\lambda} \frac{1}{(u_1|\lambda_1) \cdots (u_l|\lambda_l)},$$

$$\prod_{i=1}^l (u_i - m_i) E^*(u_i + k_i + m_i) = \sum_{\lambda} D_{\lambda} (u_1 - \lambda_1) \cdots (u_l - \lambda_l),$$

where the coefficient C_{λ} in the first expansion is the monomial given by

$$C_{\lambda} = \tau^{k_1}(h_{\lambda_1 - m_1}^*) \cdots \tau^{k_l}(h_{\lambda_l - m_l}^*),$$

and the coefficient D_{λ} in the second expansion is the monomial given by

$$D_{\lambda} = (-1)^{(\sum \lambda_i - \sum m_i)} \tau^{-k_1}(e_{\lambda_1 - m_1}^*) \cdots \tau^{-k_l}(e_{\lambda_l - m_l}^*).$$

Proof. The statement is implied by the following argument for $k \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}$:

$$\begin{aligned} \frac{1}{(u|m)} H^*(u - k - m) &= \sum_p \frac{\tau^k(h_p^*)}{(u|m)(u - m|p)} = \sum_p \frac{\tau^k(h_p^*)}{(u|m + p)} \\ &= \sum_a \frac{\tau^k(h_{a-m}^*)}{(u|a)}, \end{aligned}$$

$$\begin{aligned} (u - m) E^*(u + k + m) &= \sum_p (-1)^p \tau^{-k}(e_p^*) (u - m)(u + m - p) \\ &= \sum_p (-1)^p \tau^{-k}(e_p^*) (u - m - p) = \sum_a (-1)^{a-m} \tau^{-k}(e_{a-m}^*) (u - a). \end{aligned}$$

□

The following theorem states that $H^*(u_1, \dots, u_l)$ and $E^*(u_1, \dots, u_l)$ are generating functions for the shifted Schur functions.

Theorem 4.1. *Let λ be an integer vector with at most l non-zero parts, let $N = \sum_i \lambda_i$, and let λ' be a conjugate vector. The coefficient of $\frac{1}{(u_1|\lambda_1) \cdots (u_l|\lambda_l)}$ in a shifted expansion of $H^*(u_1, \dots, u_l)$ is a shifted Schur function s_{λ}^* , and the coefficient of $(u_1 - \lambda_1) \cdots (u_l - \lambda_l)$ in $E^*(u_1, \dots, u_l)$ is a shifted Schur function $(-1)^N s_{\lambda'}^*$:*

$$H^*(u_1, \dots, u_l) = \sum_{l(\lambda) \leq l} \frac{s_{\lambda}^*}{(u_1|\lambda_1) \cdots (u_l|\lambda_l)},$$

$$E^*(u_1, \dots, u_l) = \sum_{l(\lambda) \leq l} (-1)^N s_{\lambda}^*(u_1 | -\lambda_1) \cdots (u_l | -\lambda_l).$$

Proof. The expansion of determinant (4.6) gives

$$\begin{aligned} H^*(u_1, \dots, u_l) &= \sum_{\sigma \in S_l} \operatorname{sgn}(\sigma) \frac{1}{(u_1 | 1 - \sigma(1))} \cdots \frac{1}{(u_l | l - \sigma(l))} \prod_{i=1}^l H^*(u_i - i + 1) \\ &= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \frac{1}{(u_1 | 1 - \sigma(1))} \cdots \frac{1}{(u_l | l - \sigma(l))} \\ &\quad \times \prod_{i=1}^l H^*(u_i - (i - \sigma(i)) - (\sigma(i) - 1)). \end{aligned}$$

Set $k_i = \sigma(i) - 1$ and $m_i = i - \sigma(i)$ for $i = 1, \dots, l$. Observe that $k_i \geq 0$ for any $i = 1, \dots, l$, so by Lemma 4.2, we obtain the coefficient of $\frac{1}{(u_1 | \lambda_1) \cdots (u_l | \lambda_l)}$

$$\sum_{\sigma \in S_l} \operatorname{sgn}(\sigma) \tau^{\sigma(1)-1} (h_{\lambda_1-1+\sigma(1)}^*) \cdots \tau^{\sigma(l)-1} (h_{\lambda_l-l+\sigma(l)}^*) = \det[\tau^{j-1} h_{\lambda_i-i+j}^*].$$

Then Jacobi–Trudi identity (2.3) provides identification of this coefficient with s_{λ}^* . The second statement is proved along the same lines. \square

Remark 4.2. Note that (2.3) allows us to extend the definition of s_{λ} to any integer vector. Namely, define

$$(4.16) \quad s_{\alpha}^* = \det[\tau^{j-1} h_{\alpha_i-i+j}^*]_{1 \leq i, j \leq l}$$

for any vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_l)$ with entries $\alpha_i \in \mathbb{Z}$. Then it is clear that

$$(4.17) \quad s_{(\dots, \alpha_i, \alpha_{i+1}, \dots)}^* = -s_{(\dots, \alpha_{i+1}-1, \alpha_i+1, \dots)}^*.$$

This follows from permutation of the rows of the determinant (4.16). Let $\rho_l = (l-1, l-2, \dots, 0)$, where l is such a number that at most l entries of the integer vector α are non-zero. It is easy to see that $s_{\alpha}^* \neq 0$ if and only if $\alpha - \rho_l = \sigma(\lambda - \rho_l)$ for some permutation $\sigma \in S_l$ and some partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l \geq 0)$, $(\lambda_i \in \mathbb{Z}_{\geq 0})$. In such a case,

$$(4.18) \quad s_{\alpha}^* = \operatorname{sgn}(\sigma) s_{\lambda}^*.$$

For example, $s_{(1,3)}^* = s_{(1,3,0)}^* = -s_{(2,2)}^* = s_{(2,-1,3)}^* = -s_{(1,-1,4)}^*$.

5. Creation and annihilation operators for shifted Schur functions

We introduce creation and annihilation operators Ψ_k^\pm , $k \in \mathbb{Z}$, acting on the set of s_λ^* 's labeled by partitions λ , as

$$(5.1) \quad \Psi_k^+(s_\lambda^*) = s_{(k,\lambda)}^*, \quad \Psi_{-k}^-(s_\lambda^*) = (-1)^k s_{(k,\lambda)'}^*,$$

where $s_{\alpha'}^* = \text{sgn}(\sigma)s_{\lambda'}^*$, if $\alpha - \rho_l = \sigma(\lambda - \rho_l)$ for some σ and partition λ , and $s_{\alpha'}^* = 0$ otherwise. Shifted Schur functions s_λ^* span the space \mathcal{B}^* , hence (5.1) defines the action of linear operators Ψ_k^\pm on \mathcal{B}^* . The lowering-raising property (4.17) implies exactly the same commutation relations of operators Ψ_k^\pm as in the classical case:

$$\begin{aligned} \Psi_k^+ \Psi_l^+ + \Psi_{l-1}^+ \Psi_{k+1}^+ &= 0, \\ \Psi_k^- \Psi_l^- + \Psi_{l+1}^- \Psi_{k-1}^- &= 0, \\ \Psi_k^- \Psi_l^+ + \Psi_l^+ \Psi_k^- &= \delta_{-k,l}. \end{aligned}$$

Let us rewrite these relations in terms of shifted generating functions. Define

$$\Psi^+(v) = \sum_{k \in \mathbb{Z}} \frac{\Psi_k^+}{(v|k)}, \quad \Psi^-(v) = \sum_{k \in \mathbb{Z}} \Psi_k^-(v| - k).$$

Then

$$\begin{aligned} \Psi^+(v) (H^*(u_1, \dots, u_l)) &= H^*(v, u_1, \dots, u_l), \\ \Psi^-(v) (E^*(u_1, \dots, u_l)) &= E^*(v, u_1, \dots, u_l), \end{aligned}$$

and generating functions of shifted Schur functions can be viewed as a result of application of $\Psi^\pm(v)$ to vacuum vector:

$$\begin{aligned} H^*(u_1, \dots, u_l) &= \Psi^+(u_1) \circ \dots \circ \Psi^+(u_l) (1), \\ E^*(u_1, \dots, u_l) &= \Psi^-(u_1) \circ \dots \circ \Psi^-(u_l) (1). \end{aligned}$$

The commutation relations are

$$(5.2) \quad \frac{1}{u} e^{-\partial_u} \circ \Psi^+(u) \circ \Psi^+(v) + \frac{1}{v} e^{-\partial_v} \circ \Psi^+(v) \circ \Psi^+(u) = 0,$$

$$(5.3) \quad e^{\partial_u} \circ \frac{1}{u} \Psi^-(u) \circ \Psi^-(v) + e^{\partial_v} \circ \frac{1}{v} \Psi^-(v) \circ \Psi^-(u) = 0,$$

$$(5.4) \quad \Psi^+(u) \circ \Psi^-(v) + \Psi^-(v) \circ \Psi^+(u) = \sum_{k \in \mathbb{Z}} \frac{(u|k)}{(v|k)} \cdot Id.$$

Our next goal is to find a “normally ordered form” of $\Psi^\pm(u)$, similar to (3.4), (3.5). Let

$$DE^*(u) = \sum_{m=0}^{\infty} DE_m^*(u|m), \quad DH^*(u) = \sum_{m=0}^{\infty} DH_m^* \frac{1}{(u|-m)},$$

be the formal shifted series with DE_m^*, DH_m^* being linear operators acting on \mathcal{B}^* , such that series $DE^*(u)$ and $DH^*(u)$ have the property

$$(5.5) \quad DE^*(u)(H^*(v)) = \frac{1}{v} e^{-\partial_v} ((v-u)H^*(v)) = \left(1 - \frac{u+1}{v}\right) H^*(v-1),$$

$$(5.6) \quad DH^*(u)(E^*(v)) = e^{\partial_v} \frac{1}{v} ((v-u)E^*(v)) = \left(1 - \frac{u}{v+1}\right) E^*(v+1).$$

Formulae (5.5), (5.6) describe the action of DE_m^*, DH_m^* on generators h_k^* and e_k^* respectively. For example, $DE_m^*(h_k^*)$ is the coefficient of $\frac{(u|m)}{(v|k)}$ in the expansion of the first equation of (5.6):

$$H^*(v-1) - \frac{u+1}{v} H^*(v-1) = \sum_{k=0}^{\infty} \frac{\tau(h_k^*)}{(v|k)} - (u+1) \sum_{k=0}^{\infty} \frac{h_{k-1}^*}{(v|k)}.$$

Therefore, for $k = 0, 1, 2, \dots$,

$$\begin{aligned} DE_0^*(h_k^*) &= \tau(h_k^*) - h_{k-1}^* = h_k^* + (k-2)h_{k-1}^*, \\ DE_1^*(h_k^*) &= -h_{k-1}^*, \quad DE_m^*(h_k^*) = 0 \quad (m = 2, 3, \dots). \end{aligned}$$

Similarly, $(-1)^k DH_m^*(e_k^*)$ is the coefficient of $\frac{(v|-k)}{(u|-m)}$ in the expansion of the second equation of (5.6), which gives for $k = 0, 1, 2, \dots$

$$\begin{aligned} DH_0^*(e_k^*) &= \tau^{-1}(e_k^*) - e_{k-1}^* = e_k^* + (k-2)e_{k-1}^*, \\ DH_1^*(e_k^*) &= e_{k-1}^*, \quad DH_m^*(e_k^*) = 0 \quad (m = 2, 3, \dots). \end{aligned}$$

Next, we extend the action of DE_m^*, DH_m^* to all of \mathcal{B}^* by linearity and by the rule

$$(5.7) \quad DE^*(u) \left(\prod_{i=1}^l H^*(u_i) \right) = \prod_{i=1}^l DE^*(u)(H^*(u_i)),$$

$$(5.8) \quad DH^*(u) \left(\prod_{i=1}^l E^*(u_i) \right) = \prod_{i=1}^l DH^*(u)(E^*(u_i)).$$

In particular, write

$$\begin{aligned} \prod_{i=1}^l H^*(u_i) &= \sum_{0 \leq \lambda_1, \dots, \lambda_l < \infty} \frac{h_{\lambda_1}^* \cdots h_{\lambda_l}^*}{(u_1|\lambda_1) \cdots (u_l|\lambda_l)}, \\ \prod_{i=1}^l E^*(u_i) &= \sum_{0 \leq \lambda_1, \dots, \lambda_l < \infty} (-1)^{|\lambda|} e_{\lambda_1}^* \cdots e_{\lambda_l}^* (u_1 - \lambda_1) \cdots (u_l - \lambda_l), \end{aligned}$$

(here $|\lambda| = \sum \lambda_i$) to get the action of $DE^*(u)$ on the monomials that span \mathcal{B}^* :

$$\begin{aligned} DE^*(u)(h_{\lambda_1}^* \cdots h_{\lambda_l}^*) &= DE^*(u)(h_{\lambda_1}^*) \cdots DE^*(u)(h_{\lambda_l}^*) \\ &= \prod_{i=1}^l (h_{\lambda_i} + (\lambda_i - 2)h_{\lambda_i-1} - h_{\lambda_i}u), \\ DH^*(u)(e_{\lambda_1}^* \cdots e_{\lambda_l}^*) &= DE^*(u)(e_{\lambda_1}^*) \cdots DE^*(u)(e_{\lambda_l}^*) \\ &= \prod_{i=1}^l (e_{\lambda_i} + (\lambda_i - 2)e_{\lambda_i-1} + e_{\lambda_i}u), \end{aligned}$$

and expand these products in shifted powers of u to get the explicit values of $DE_m^*(h_{\lambda_1}^* \cdots h_{\lambda_l}^*)$ and $DH_m^*(e_{\lambda_1}^* \cdots e_{\lambda_l}^*)$. For example,

$$\begin{aligned} DE^*(u)(h_a^* h_b^*) &= (h_a + (a-2)h_{a-1})(h_b + (b-2)h_{b-1}) \\ &\quad - u(h_a h_b + (a-2)h_b h_{a-1} + (b-2)h_a h_{b-1}) + (u|2)h_a h_b. \end{aligned}$$

Hence,

$$\begin{aligned} DE_0^*(h_a^* h_b^*) &= (h_a + (a-2)h_{a-1})(h_b + (b-2)h_{b-1}), \\ DE_1^*(h_a^* h_b^*) &= -(h_a h_b + (a-2)h_b h_{a-1} + (b-2)h_a h_{b-1}), \\ DE_2^*(h_a^* h_b^*) &= h_a h_b, \quad DE_m^*(h_a^* h_b^*) = 0, \quad (m = 3, 4, \dots). \end{aligned}$$

Finally, from (5.7), (5.8) we write the action of $DE^*(v)$ on $H^*(u_1, \dots, u_l)$ and of $DH^*(v)$ on $E^*(u_1, \dots, u_l)$. Formulae (5.5), (5.6), (4.8), (4.9) immediately imply

$$(5.9) \quad H^*(v) \circ DE^*(v)(H^*(u_1, \dots, u_l)) = H^*(v, u_1, \dots, u_l),$$

$$(5.10) \quad E^*(v) \circ DH^*(v)(E^*(u_1, \dots, u_l)) = E^*(v, u_1, \dots, u_l),$$

and from (5.9), (5.10) follows the “normally ordered” presentation of $\Psi^\pm(u)$ analogous to (3.4), (3.5) (see also [5]).

Proposition 5.1.

$$\Psi^+(v) = H^*(v) \circ DE^*(v), \quad \Psi^-(v) = E^*(v) \circ DH^*(v).$$

References

- [1] Frenkel E., Ben-Zvi D. (2004). *Vertex algebras and algebraic curves*. Mathematical Surveys and Monographs, **88**. American Mathematical Society, Providence, RI. [MR2082709](#)
- [2] Frenkel, I. (1981). *Two constructions of affine Lie algebra representations and boson-fermion correspondence in quantum field theory*, J. Funct. Anal. **44**, 259–327. [MR0643037](#)
- [3] Frenkel I., Lepowsky J., Meurman A. (1988). *Vertex operator algebras and the Monster*. Pure and Applied Mathematics, **134**. Academic Press, Inc., Boston, MA.
- [4] Jimbo M., Miwa T. (1983). *Solitons and infinite dimensional Lie algebras*, Publ. RIMS, Kyoto Univ. **19**, 943–1001. [MR0723457](#)
- [5] Jing N., Rozhkovskaya N. (2016). *Vertex operators arising from Jacobi-Trudi identities*, Comm. Math. Phys. **346**, 679–701.
- [6] Kac V., Raina A., Rozhkovskaya N. (2013). *Bombay lectures on highest weight representations of infinite dimensional Lie algebras*, 2nd ed., World Scientific, Hackensack, NJ.
- [7] Macdonald I. (1995) *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Univ. Press, New York.
- [8] Nazarov M. (1998). *Yangians and Capelli identities*, Kirillov’s seminar on representation theory, Amer. Math. Soc. Transl. Ser. 2, 181, Adv. Math. Sci., **35**, Amer. Math. Soc., Providence, RI, 139–163. [MR1618751](#)
- [9] Okounkov A. (1996). *Quantum immanants and higher Capelli identities*, Transform. Groups **1**, no. 1–2, 99–126.
- [10] Okounkov A., Olshanski G. (1998). *Shifted Schur functions*, St. Petersburg Math. J. **9**, 239–300.

- [11] Sahi S. (1994). *The spectrum of certain invariant differential operators associated to a Hermitian symmetric space*, Lie theory and geometry, Progr. Math., **123**, Birkhäuser Boston, Boston, MA, 569–576. [MR1327549](#)

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