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On sequences covering all rainbow k-progressions

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Let $\mathrm{ac}(n,k)$ denote the smallest positive integer with the property that there exists an n-colouring f of $\{1,\dots,\mathrm{ac}(n,k)\}$ such that for every k-subset $R \subseteq \{1,\dots,n\}$ there exists an (arithmetic) k-progression A in $\{1,\dots,\mathrm{ac}(n,k)\}$ with $\{f(a) : a \in A\} = R$.

Determining the behaviour of the function $\mathrm{ac}(n,k)$ is a previously unstudied problem. We use the first moment method to give an asymptotic upper bound for $\mathrm{ac}(n,k)$ for the case $k = o(n^{1/5})$.

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1. Introduction

Let $a,k,d \in \mathbb{N}$. The set $A = \{a, a+d, a+2d, \dots, a+(k-1)d\}$ is called an (arithmetic) k-progression. We say A has *common difference* d.

Let $n, N \in \mathbb{N}$ $(n \leq N)$ and let $f : [N] \to [n]$ be an n-colouring of $[N]$. Let $R \in \binom{[n]}{k}$ be a k-subset of $[n]$. We say a k-progression A in $[N]$ is *R-coloured* if $\{f(a) : a \in A\} = R$. We call such a k-progression a *rainbow k-progression*. We say f *covers* R if there is a k-progression in $[N]$ that is R-coloured.

Example. The 6-colouring $f = (4,6,5,1,3,4,2,5,6,3,1,4)$ of the interval $\{1,2,\dots,14\}$ covers every 3-subset of $\{1,\dots,6\}$; we give examples for some subsets:

$$\begin{aligned}
\{1,2,3\}&: (4,6,5,\mathbf{1},3,4,\mathbf{2},5,6,\mathbf{3},1,4)\\
\{3,4,5\}&: (\mathbf{4},6,\mathbf{5},1,\mathbf{3},4,2,5,6,3,1,4)\\
\{3,4,6\}&: (\mathbf{4},6,5,1,\mathbf{3},4,2,5,\mathbf{6},3,1,4)\\
\{2,5,6\}&: (4,6,5,1,3,4,\mathbf{2},\mathbf{5},\mathbf{6},3,1,4)
\end{aligned}$$

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For $n, k \in \mathbb{N}$ (where $k \leq n$), let $\mathrm{ac}(n, k)$ denote the smallest positive integer such that there exists an n-colouring f of $[\mathrm{ac}(n, k)] = \{1, 2, \ldots, \mathrm{ac}(n, k)\}$ that covers every k-subset of $[n]$.

Among related problems, the anti-van der Waerden numbers $\mathrm{aw}([N], k)$ are well-studied in Ramsey theory. The number $\mathrm{aw}([N], k)$ is defined to be the smallest positive integer r such that every surjective r-colouring of $[N]$ contains at least one rainbow k-progression.

Butler et al. [1] calculate exact values of $\mathrm{aw}([N], k)$ for small values of N and k and give asymptotic results. Berikkyzy et al. [2] give an exact formula for $\mathrm{aw}([N], 3)$, proving a conjecture of Butler et al. [1]. Young [3] and Schulte et al. [4] study generalizations of this problem to finite abelian groups and graphs, respectively.

The problem of studying anti-van der Waerden numbers is about finding colourings avoiding all rainbow k-progressions. Conversely, the problem we study in this work is about finding colourings that *do not* avoid *any* rainbow k-progressions.

A wide range of problems about covering all k-subsets of $[n]$, on various structures, are studied [5, 6, 7].

We prove the following asymptotic result.

Theorem. *As n tends to infinity, we have*

$$ac(n, k) = \Omega\left(\sqrt{k\binom{n}{k}}\right).$$

If $k = k(n) = o(n^{1/5})$, we have

$$ac(n, k) = \mathcal{O}\left(\log n \cdot e^{k/2} \cdot k^{-k/2+5/4} \cdot n^{k/2}\right).$$

Comparing the asymptotic upper and the asymptotic lower bound for the case $k = o(n^{1/5})$, we see that the bounds differ by the factor $k \log n$.

The proof of the theorem is given in Section 2. The main tool of the proof (Lemma 1) is shown in Section 3. We achieve this by finding a lower bound on the expected number of k-subsets of $[n]$ covered by a random colouring.

2. Proof of theorem

All asymptotics are to be understood with respect to n, where n tends to infinity.

The lower bound in the theorem is a consequence of the fact that an n-colouring of $[N]$ can only cover all k-subsets of $[n]$ if $[N]$ contains at least $\binom{n}{k}$ k-progressions.

The remainder of this section is dedicated to proving the upper bound given in the theorem. To this end, as claimed let $k = k(n) = o(n^{1/5})$ and $N = N(n) = \left\lceil \sqrt{2}\sqrt{\frac{k-1}{k!}} \cdot n^{k/2} \right\rceil$.

The proof of the following lemma is given in Section 3.

Lemma 1. *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a family of k-subsets of $[n]$. There exists an n-colouring f^* of $[N]$ such that the number of sets of $\mathcal{F}$ that are covered by f^* is at least $|\mathcal{F}|\left(\frac{1}{2} + o(1)\right)$.*

It follows that there exists an n-colouring g_0 of $[N]$ that covers at least $\binom{n}{k}\left(\frac{1}{2} + o(1)\right)$ of the sets of $\mathcal{F}_0 := \binom{[n]}{k}$.

Let $\mathcal{F}_1$ be the family of sets of $\mathcal{F}_0$ that have not been covered by g_0. Applying Lemma 1 again, we obtain an n-colouring g_1 of $[N]$ that covers at least $|\mathcal{F}_1|\left(\frac{1}{2} + o(1)\right)$ of the sets of $\mathcal{F}_1$. We repeat this process r times, by defining $\mathcal{F}_i$ to be the family of k-subsets of $[n]$ not yet covered by any of the colourings $g_0, \ldots, g_{i-1}$.

After r iterations, the number of k-subsets of $[n]$ that are not covered by any of the constructed colourings is at most $|\mathcal{F}_0|\left(\frac{1}{2} + o(1)\right)^r$. Setting $r = r(n,k) = \lceil \alpha \cdot k \log n \rceil$, where $\alpha > \frac{1}{\log(2)}$, we get

$$|\mathcal{F}_0|\left(\frac{1}{2} + o(1)\right)^{r(n)} = \binom{n}{k}\left(\frac{1}{2} + o(1)\right)^{r(n)} = o(1).$$

Thus, for sufficiently large n, after $r(n)$ iterations, every k-subset of $[n]$ is covered by at least one of the colourings

$$g_0, g_1, \ldots, g_{r(n)-1}.$$

From the colourings $g_0, g_1, \ldots, g_{r(n)-1}$ we construct an n-colouring g of $S := [r(n) \cdot N]$. We split S into $r(n)$ intervals of length N and colour each of these intervals with the corresponding colouring g_i. Formally, we set

$$g\,(i \cdot N + s) = g_i(s) \quad i \in \{0, \ldots, r(n) - 1\},\ s \in [N].$$

The colouring g is an n-colouring of $S = \left[\lceil \alpha \cdot k \log n \rceil \cdot \left\lceil \sqrt{2}\sqrt{\frac{k-1}{k!}} \cdot n^{k/2} \right\rceil\right]$

that covers all k-subsets of $[n]$. It follows that

$$\mathrm{ac}(n,k) = \mathcal{O}\left(k \cdot \log n \cdot \sqrt{\frac{k-1}{k!}} \cdot n^{k/2}\right).$$

If $k = o(n^{1/5})$ tends to infinity as $n \to \infty$,

$$\mathrm{ac}(n,k) = \mathcal{O}\left(\log n \cdot e^{k/2} \cdot k^{-k/2+5/4} \cdot n^{k/2}\right)$$

holds.

3. Proof of Lemma 1 using the probabilistic method

For $n, N, k \in \mathbb{N}$ (where $k \le n \le N$) let f be a random n-colouring of $[N]$ (chosen uniformly at random from all such colourings). For each $R \in \binom{[n]}{k}$ let X_R be the indicator variable of the event *“f covers R”*. Given a k-progression A in $[N]$, let $Y_{A,R}$ be the event *“The progression A is R-coloured”*.

We are interested in the random variable $\sum_{R \in \binom{[n]}{k}} X_R$, which counts the number of k-subsets of $[n]$ that are covered by f.

For the sake of brevity, let $\mathrm{AP}_k(N)$ denote the set of all k-progressions in $[N]$ and $\mathcal{H}_k(N) = \binom{\mathrm{AP}_k(N)}{2}$ denote the set of all unordered pairs of k-progressions in $[N]$. Note that X_R is the indicator variable of the event $\bigcup_{A \in \mathrm{AP}_k(N)} Y_{A,R}$.

Using a Bonferroni inequality we obtain the following lower bound for $\mathbb{E}X_R$.

Lemma 2. *For every k-subset R of $[n]$, the following holds:*

$$\begin{aligned}
\mathbb{E}X_R &= \mathbb{P}(X_R = 1) = \mathbb{P}\Bigg(\bigcup_{A \in AP_k(N)} Y_{A,R}\Bigg) \\
&\ge \sum_{A \in AP_k(N)} \mathbb{P}(Y_{A,R}) - \sum_{\{A,B\} \in \mathcal{H}_k(N)} \mathbb{P}\left(Y_{A,R} \cap Y_{B,R}\right) \\
&= \sum_{A \in AP_k(N)} \frac{k!}{n^k} - \sum_{i=0}^{k-1} \sum_{\substack{\{A,B\} \in \mathcal{H}_k(N) \\ |A \cap B| = i}} \frac{k!(k-i)!}{n^{2k-i}}.
\end{aligned}$$

□

To evaluate the lower bound from Lemma 2, we need to count the number $h(N,k) = |\mathrm{AP}_k(N)|$ of k-progressions in $[N]$ and the numbers $h_i(N,k)$, defined as the number of unordered pairs of k-progressions in $[N]$ that intersect in exactly i positions.

Lemma 3. *As N tends to infinity, the following asymptotic bounds hold:*

- $h(N,k) = \frac{N^2}{2k-2} + \mathcal{O}(N)$,
- $h_0(N,k) \le \binom{h(N,k)}{2} = \frac{N^4}{8(k-1)^2} + \mathcal{O}(N^3/k)$,
- $h_1(N,k) \le h(N,k)k^2N = \mathcal{O}\left(N^3 k\right)$,
- $h_j(N,k) \le \binom{N}{2}\binom{\binom{k}{2}}{2} = \mathcal{O}\left(N^2k^4\right)$ *for* $j \ge 2$.

Proof. The formula for $h(N,k)$ is obtained by counting the number of ways to choose the initial term and common difference of a k-progression. We bound $h_0(N,k)$ by the number of unordered pairs of k-progressions. The bound for $h_1(N,k)$ is obtained by fixing a k-progression and an element of that progression; there are at most kN k-progressions containing this element. For each $j \ge 2$, $h_j(N,k)$ is bounded by the total number of pairs of k-progressions intersecting in at least two positions. For each pair of distinct elements there are at most $\binom{k}{2}$ k-progressions containing both of them. □

We are ready to evaluate the lower bound from Lemma 2.

Lemma 4. *Let $k = k(n) = o(n^{1/5})$, $N = N(n) = \left\lceil \sqrt{2}\sqrt{\frac{k-1}{k!}} \cdot n^{k/2} \right\rceil$ and f be a random n-colouring of $[N]$. Then, for every $R \in \binom{[n]}{k}$ the inequality*

$$\mathbb{E}X_R \ge \frac{1}{2} + o(1)$$

holds.

Proof. Using Lemma 2 and the asymptotic bounds for h and the h_i's we get

$$\begin{aligned}\mathbb{E}X_R &\ge h(N)\frac{k!}{n^k} - h_0(N)\frac{k!k!}{n^{2k}} - h_1(N)\frac{k!(k-1)!}{n^{2k-1}} - \sum_{i=2}^{k-1} h_i(N)\frac{k!(k-i)!}{n^{2k-i}} \\ &\ge \left(\frac{N^2}{2k-2} + \mathcal{O}(N)\right)\frac{k!}{n^k} - \left(\frac{N^4}{8(k-1)^2} + \mathcal{O}(N^3/k)\right)\frac{k!k!}{n^{2k}} \\ &\quad + \mathcal{O}(N^3k)\frac{k!(k-1)!}{n^{2k-1}} + \mathcal{O}(N^2k^4)\sum_{i=2}^{k-1}\frac{k!(k-i)!}{n^{2k-i}} =: L(n).\end{aligned}$$

Only the terms $\frac{N^2}{2k-2}\frac{k!}{n^k}$ and $\frac{N^4}{8(k-1)^2}\frac{k!k!}{n^{2k}}$ are asymptotically relevant. It follows from Stirling's formula that $\mathcal{O}(N)\frac{k!}{n^k} = o(1)$, $\mathcal{O}(N^2/k)\frac{k!k!}{n^{2k}} = o(1)$, and

$\mathcal{O}(N^3k)\frac{k!(k-1)!}{n^{2k-1}} = o(1)$. To see that $\mathcal{O}(N^2k^4)\sum_{i=2}^{k-1}\frac{k!(k-i)!}{n^{2k-i}} = o(1)$, we use the fact that the last term of the sum asymptotically dominates the sum of all other terms and the assumption $k = o(n^{1/5})$.

We are thus left with the following representation of $L(n)$:

$$L(n) = \frac{N^2}{2k-2}\frac{k!}{n^k} - \frac{N^4}{8(k-1)^2}\frac{k!k!}{n^{2k}} + o(1),$$

which, by our choice of N, gives $L(n) = \frac{1}{2} + o(1)$. □

Lemma 1 follows from Lemma 4 by linearity of expectation.

4. Conclusion

Various generalizations of the problem we studied are possible, by replacing $[N]$ by another structure endowed with a sensible definition of k-progression. Structures of interest include cycles $\mathbb{Z}_N$, abelian groups and graphs, which are already studied for anti-van der Waerden numbers.

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