Journal of Combinatorics
Volume 9, Number 4, 721–738, 2018

Forbidden subgraphs for k vertex-disjoint stars

Michitaka Furuya* and Naoki Matsumoto

For a connected graph H, a graph G is said to be *H-free* if G does not contain H as an induced subgraph. In this context, H is called a *forbidden subgraph*. In this paper, we study a transition of forbidden subgraphs for the existence of vertex-disjoint stars. For $t \geq 1$, $k \geq 1$ and $d \geq t$, let $\mathcal{H}(t, k, d)$ be the family of connected graphs H such that every H-free graph G of sufficiently large order with $\delta(G) \geq d$ has k vertex-disjoint $K_{1,t}$. We characterize the family $\mathcal{H}(t, k, d)$ for almost all triples (t, k, d). In particular, we give a complete characterization of $\mathcal{H}(t, k, d)$ for $t \leq 4$.

AMS 2000 subject classifications: Primary 05C70.
Keywords and phrases: Vertex-disjoint star, forbidden subgraph, star-free graph.

1. Introduction

For a connected graph H, a graph G is said to be *H-free* if G contains no induced subgraph isomorphic to H. In this context, H is called a *forbidden subgraph*. Let $K_{1,r}$ denote the star of order $r + 1$.

A star has been widely studied as one of the most important forbidden subgraphs. For example, Sumner [13] proved that every m-connected $K_{1,m+1}$-free graph of even order has a perfect matching, and Matthews and Sumner [11] gave a well-known conjecture that every 4-connected $K_{1,3}$-free graph is Hamiltonian. Moreover, the star-free condition itself has been studied (for example, see [3, 7]).

Here one may estimate that if a graph H has similar properties like the star from the point of view of forbidden subgraphs, then a result concerning star-free graphs will provide useful information to H-free graphs. To find a graph H satisfying such an assumption, we study a transition of forbidden subgraphs. For example, it has been known that a transition of the star-free condition for the existence of a perfect matching depends on the connectivity as mentioned above. Our main aim is to find a larger transition of forbidden subgraphs.

*This work was supported by JSPS KAKENHI Grant number 26800086.

Now we focus on the problem concerning the existence of vertex-disjoint stars in a graph. The problem comes from a famous result which gives a relationship between the size of a matching and the minimum degree condition. We let $\delta(G)$ denote the minimum degree of a graph G.

Theorem A (Erdős and Pósa [6]). *Let k be a positive integer, and let G be a graph with $|V(G)| \geq 2k$ and $\delta(G) \geq k$. Then G has a matching of size k.*

We can regard a matching in a graph as special vertex-disjoint stars. Egawa and Ota [5] and Ota [12] studied the minimum degree condition for the existence of k vertex-disjoint $K_{1,t}$. (After that Fujita [8] and Chiba [1] improved the order condition in Theorem D.)

Theorem B (Ota [12]). *Let k be a positive integer, and let G be a graph with $|V(G)| \geq 3k+2$ and $\delta(G) \geq k+1$. Then G has k vertex-disjoint $K_{1,2}$.*

Theorem C (Egawa and Ota [5]). *Let k be a positive integer, and let G be a graph with $|V(G)| \geq 4k+6$ and $\delta(G) \geq k+2$. Then G has k vertex-disjoint $K_{1,3}$.*

Theorem D (Ota [12]). *Let t and k be positive integers with $t \geq 4$, and let G be a graph with $|V(G)| \geq (t+1)k + 2t^2 - 3t - 1$ and $\delta(G) \geq t+k-1$. Then G has k vertex-disjoint $K_{1,t}$.*

On the other hand, Fujita [9, 10] gave the forbidden subgraph condition for the existence of k vertex-disjoint $K_{1,t}$ as follows.

Theorem E (Fujita [9, 10]). *Let t and k be positive integers with $t \geq 3$ and $k \geq 3$, and let H be a connected graph. Then there exists an integer $n = n(H)$ such that every H-free graph G with $|V(G)| \geq n$ and $\delta(G) \geq t$ has k vertex-disjoint $K_{1,t}$ if and only if H is a star.*

However, for positive integers t, k and d with $t+1 \leq d \leq t+k-2$, it has not been known what kind of forbidden subgraphs H assure the existence of k vertex-disjoint $K_{1,t}$ in an H-free graph with minimum degree at least d. We formally consider the following families $\mathcal{H}(t,k,d)$: Let $\mathcal{G}$ be the set of connected graphs of order at least three. For positive integers t, k and d with $d \geq t$, let $\mathcal{H}(t,k,d)$ be the family of graphs $H \in \mathcal{G}$ satisfying that there exists an integer $n = n(H)$ such that every H-free graph G with $|V(G)| \geq n$ and $\delta(G) \geq d$ has k vertex-disjoint $K_{1,t}$.

We let K_n denote the complete graph of order n, and let K_{n_1,n_2} denote the complete bipartite graph with partite sets having cardinalities n_1 and n_2. For two disjoint graphs H_1 and H_2, we let $H_1 \cup H_2$ and $H_1 + H_2$ denote the union and the join of H_1 and H_2, respectively. For a graph H and an integer s, we let sH denote the union of s disjoint copies of H. Let $\mathcal{K} = \{K_{1,r} : r \geq 2\}$, and for a positive integer j, let $\mathcal{K}(j) = \{K_1 + (r_1K_1 \cup r_2K_2) : r_1 \geq 0, r_2 \geq 0, r_1 + 2r_2 \geq 2\} \cup \{K_2 + rK_1 : 1 \leq r \leq j\}$. Note that $\mathcal{K}(1) = \{K_1 + (r_1K_1 \cup r_2K_2) : r_1 \geq 0, r_2 \geq 0, r_1 + 2r_2 \geq 2\}$. Our main result is the following. (Note that Theorem 1.1(i) includes Theorem E.)

Theorem 1.1. *Let t, k and d be positive integers with $d \geq t$. Then the following hold:*

(i) *If $d \leq \max\{k-1, t + \lfloor \frac{k-1}{2} \rfloor - 1\}$, then $\mathcal{H}(t,k,d) = \mathcal{K}$.*
(ii) *If $\max\{k, t + \lfloor \frac{k-1}{2} \rfloor\} \leq d \leq t + k - 2$, then*

$$\mathcal{K}(2d - 2t - k + 3) \subseteq \mathcal{H}(t,k,d) \subseteq \mathcal{K}(\max\{2d - 2t - k + 3, t - 1\}).$$

Furthermore, if $d \geq \min\{\frac{3t+k-4}{2}, \frac{t^2+(k-2)t-k+1}{t}\}$, then $\mathcal{H}(t,k,d) = \mathcal{K}(2d - 2t - k + 3)$.
(iii) *If $t \geq 4$, then $\mathcal{H}(t,4,t+1) = \mathcal{K}(2)$.*
(iv) *If $t \geq 4$, then $\mathcal{H}(t, 2t-2, 2t-2) = \mathcal{K}(1)$.*
(v) *If $d \geq t + k - 1$, then $\mathcal{H}(t,k,d) = \mathcal{G}$.*

By Theorem 1.1, we get a transition of forbidden subgraphs (and so we suspect that $\mathcal{K}(j)$ is one of natural generalizations of the family $\mathcal{K}$). Hence our main purpose is attained.

We continue to investigate $\mathcal{H}(t,k,d)$. The family $\mathcal{H}(t,k,d)$ has not characterized in Theorem 1.1 if and only if the triple (t,k,d) satisfies

(H1) $\max\{k, t + \lfloor \frac{k-1}{2} \rfloor\} \leq d < \min\{\frac{3t+k-4}{2}, \frac{t^2+(k-2)t-k+1}{t}\}$, and
(H2) $(t,k,d) \notin \{(t,4,t+1), (t,2t-2,2t-2)\}$.

By simple calculations, for a fixed integer $t \geq 1$, we check that the number of triples (t,k,d) satisfying (H1) and (H2) is finite (and we omit its detail). Hence for an integer $t \geq 1$, Theorem 1.1 determines $\mathcal{H}(t,k,d)$ with finite exceptions. On the other hand, it seems difficult to completely characterize $\mathcal{H}(t,k,d)$ for every triple (t,k,d). So one may pose a natural problem: For a fixed t, find some properties of $\mathcal{H}(t,k,d)$. In this paper, by a few additional proofs together with Theorem 1.1, we could completely characterize $\mathcal{H}(t,k,d)$ for $1 \leq t \leq 4$.

Theorem 1.2. *Let t, k and d be positive integers with $1 \leq t \leq 4$ and $d \geq t$. Then*

$$\mathcal{H}(t,k,d) = \begin{cases} \mathcal{K} & (d \le \max\{k-1, t+\lfloor \frac{k-1}{2} \rfloor - 1\}) \\ \mathcal{K}(2d-2t-k+3) & (\max\{k, t+\lfloor \frac{k-1}{2} \rfloor\} \le d \le t+k-2 \\ & \ \textit{and } (t,k,d) \ne (4,4,5)) \\ \mathcal{K}(2) & ((t,k,d) = (4,4,5)) \\ \mathcal{G} & (d \ge t+k-1). \end{cases}$$

We will use the following notation and terminology. Let G be a graph, and let $x \in V(G)$. For an integer $i \ge 1$, we let $N_G^{(i)}(x) = \{y \in V(G)$: the distance between x and y is $i\}$. We write $N_G(x)$ for $N_G^{(1)}(x)$. We let $d_G(x)$ denote the *degree* of x in G. For $X \subseteq V(G)$, we let $G[X]$ be the subgraph of G which is induced by X. For $F \subseteq E(G)$, we let $V(F)$ denote the set of vertices incident with an edge in F. For terms and symbols not defined here, we refer the reader to [4].

2. Triples (t, k, d) with either $\mathcal{H}(t, k, d) = \mathcal{G}$ or $\mathcal{H}(t, k, d) = \mathcal{K}$

In this section, we study triples (t,k,d) with either $\mathcal{H}(t,k,d) = \mathcal{G}$ or $\mathcal{H}(t,k,d) = \mathcal{K}$. By the definition of $\mathcal{H}(t,k,d)$, we have $\mathcal{H}(t,k,d) \subseteq \mathcal{G}$. Let $H \in \mathcal{G}$ be a graph, and let t and k be positive integers. Then by Theorems A–D, every H-free graph G with $|V(G)| \ge (t+1)k + 2t^2 - 3t + 1$ and $\delta(G) \ge t+k-1$ has k vertex-disjoint $K_{1,t}$. Hence we get the following proposition.

Proposition 2.1. *Let t, k and d be positive integers with $d \ge t+k-1$. Then $\mathcal{H}(t,k,d) = \mathcal{G}$.*

Now we consider triples (t,k,d) with $\mathcal{H}(t,k,d) = \mathcal{K}$. Let G be a graph with $\delta(G) \ge t$. A family $\mathcal{X} \subseteq \binom{V(G)}{t+1}$ is *t-proper* if $X \cap X' = \emptyset$ and $G[X]$ contains a spanning $K_{1,t}$ for any $X, X' \in \mathcal{X}$ with $X \ne X'$. Note that G has a non-empty t-proper family. We start with the following lemma which will be used in the proof of Propositions 2.4 and 3.2.

Lemma 2.2. *Let t, k and d be positive integers with $d \ge t$. Let G be a graph with $\delta(G) \ge d$, and let $\mathcal{X}$ be a maximum t-proper family of G. If $|\mathcal{X}| \le k-1$, then there exists a set $S \subseteq \bigcup_{X \in \mathcal{X}} X$ such that $|S \cap X| = 1$ for each $X \in \mathcal{X}$ and the number of vertices y in $V(G) - (\bigcup_{X \in \mathcal{X}} X)$ with $N_G(y) \cap (\bigcup_{X \in \mathcal{X}} X) \subseteq S$ is at least $|V(G)| - (k-1)(2t^2+1)$.*

Proof. Set $X_0 = \bigcup_{X \in \mathcal{X}} X$. For each $X \in \mathcal{X}$, choose a vertex $x_X \in X$ so that $|N_G(x_X) \cap (V(G) - X_0)|$ is as large as possible. Let $S = \{x_X : X \in \mathcal{X}\}$. We show that S is a desired set.

Suppose that $|N_G(x) \cap (V(G) - X_0)| \geq 2t$ for some $x \in X_0 - S$, and let $U \subseteq N_G(x) \cap (V(G) - X_0)$ be a set with $|U| = t$. Let $X \in \mathcal{X}$ be the set containing x. By the choice of x_X, $|N_G(x_X) \cap (V(G) - X_0)| \geq 2t$. Let $U' \subseteq N_G(x_X) \cap (V(G) - (X_0 \cup U))$ be a set with $|U'| = t$. Then $(\mathcal{X} - \{X\}) \cup \{U \cup \{x\}, U' \cup \{x_X\}\}$ is a t-proper family of G, which contradicts the maximality of $\mathcal{X}$. Thus $|N_G(x) \cap (V(G) - X_0)| \leq 2t-1$ for every $x \in X_0 - S$. In particular, the number of vertices $y \in V(G) - X_0$ satisfying $N_G(y) \cap (X_0 - S) \neq \emptyset$ is at most $(k-1)t(2t-1)$, and hence the number of vertices $y \in V(G) - X_0$ satisfying $N_G(y) \cap X_0 \subseteq S$ is at least $|V(G)| - (k-1)(t+1) - (k-1)t(2t-1) (= |V(G)| - (k-1)(2t^2+1))$. □

We also use the following lemma.

Lemma 2.3 (Chvátal [2]). *Let t and r be positive integers. Then $R(K_{1,t}, K_r) \leq t(r-1)+1$ where $R(K_{1,t}, K_r)$ is the Ramsey number for $K_{1,t}$ and K_r.*

Our main result in this section is the following.

Proposition 2.4. *Let t, k and d be positive integers with $t \leq d \leq \max\{k-1, t + \lfloor \frac{k-1}{2} \rfloor - 1\}$. Then $\mathcal{H}(t,k,d) = \mathcal{K}$.*

Proof. We first show that $\mathcal{H}(t,k,d) \supseteq \mathcal{K}$. Let $H \in \mathcal{K}$; that is $H = K_{1,r}$ for some $r \geq 2$. Let G be a graph with $|V(G)| \geq (k-1)(2t^2 + tr - t + 2)$ and $\delta(G) \geq t$, and assume that G has no k vertex-disjoint $K_{1,t}$. We show that G contains $K_{1,r}$ as an induced subgraph. Let $\mathcal{X}$ be a maximum t-proper family of G, and set $X_0 = \bigcup_{X \in \mathcal{X}} X$. Then $|\mathcal{X}| \leq k-1$ and $|X_0| \leq (k-1)(t+1)$. By Lemma 2.2, there exists a set $S \subseteq X_0$ such that $|S \cap X| = 1$ for each $X \in \mathcal{X}$ and the number of vertices y in $V(G) - X_0$ with $N_G(y) \cap X_0 \subseteq S$ is at least $|V(G)| - (k-1)(2t^2+1)$. Let $Y = \{y \in V(G) - X_0 : N_G(y) \cap X_0 \subseteq S\}$, and take $x_0 \in S$ so that $|N_G(x_0) \cap Y|$ is as large as possible. Since $\delta(G) \geq t$ and $\delta(G - X_0) \leq t-1$ by the maximality of $\mathcal{X}$, $N_G(y) \cap S \neq \emptyset$ for every $y \in Y$. It follows from Lemma 2.3 that $|N_G(x_0) \cap Y| \geq \frac{|Y|}{|S|} \geq \frac{|V(G)|-(k-1)(2t^2+1)}{k-1} \geq t(r-1)+1 \geq R(K_{1,t}, K_r)$. Since $\delta(G - X_0) \leq t-1$, $G[N_G(x_0) \cap Y]$ contains an independent set Z with $|Z| = r$. Since x_0 is adjacent to all vertices in Z, $G[\{x_0\} \cup Z]$ contains $K_{1,r}$ as an induced subgraph. Consequently $H = K_{1,r} \in \mathcal{H}(t,k,d)$. Since H is arbitrary, we have $\mathcal{H}(t,k,d) \supseteq \mathcal{K}$.

We next show that $\mathcal{H}(t,k,d) \subseteq \mathcal{K}$. Let $H \in \mathcal{H}(t,k,d)$. By the definition of H, there exists an integer $n = n(H)$ such that every H-free graph G with $|V(G)| \geq n$ and $\delta(G) \geq d$ has k vertex-disjoint $K_{1,t}$.

Now we show that there exists a K_3-free connected graph G_1 with $|V(G_1)| \geq n$ and $\delta(G_1) \geq d$ having no k vertex-disjoint $K_{1,t}$. If $d \leq k-1$, then the graph $G_1 = K_{k-1,\max\{n,d\}}$ satisfies the above conditions. Thus we may assume that $d \geq k$. Then $d \leq t + \lfloor \frac{k-1}{2} \rfloor - 1$. Let X_1 and X_2 be disjoint sets with $|X_1| = \lceil \frac{k-1}{2} \rceil$ and $|X_2| = \lfloor \frac{k-1}{2} \rfloor$. For $i \in \{1, 2\}$ and $1 \leq j \leq \max\{n, d\}$, let $A_i^{(j)}$ be a set with $|A_i^{(j)}| = t - 1$. Let G_1 be the graph defined by

$$V(G_1) = \bigcup_{i \in \{1,2\}} \left(X_i \cup \left(\bigcup_{1 \leq j \leq \max\{n,d\}} A_i^{(j)} \right) \right)$$

and

$$E(G_1) = \bigcup_{1 \leq j \leq \max\{n,d\}} \Big\{ x_1a_1, x_2a_2, a_1a_2 : x_1 \in X_1, x_2 \in X_2, \\ a_1 \in A_1^{(j)}, a_2 \in A_2^{(j)} \Big\}.$$

Then G_1 is a K_3-free graph with $|V(G_1)| \geq n$ and $\delta(G_1) \geq d$. By considering the range of d, we have $k \geq 2$, and so $X_1 \neq \emptyset$. In particular, G_1 is connected. Furthermore, since any subgraphs $K_{1,t}$ of G_1 contain a vertex in $X_1 \cup X_2$, G_1 has no k vertex-disjoint $K_{1,t}$. Consequently G_1 is a desired graph. Hence G_1 is not H-free (i.e., G_1 contains H as an induced subgraph). Since G_1 is K_3-free, H is also K_3-free.

Let $G_2 = K_{k-1} + nK_t$. Then G_2 is a connected graph with $|V(G_2)| \geq n$ and $\delta(G_2) \geq d$ having no k vertex-disjoint $K_{1,t}$. Hence G_2 is not H-free. Since H is connected and K_3-free, this implies that H is a star. Since H is arbitrary, we have $\mathcal{H}(t, k, d) \subseteq \mathcal{K}$.

This completes the proof of Proposition 2.4. □

3. A subfamily of $\mathcal{H}(t, k, d)$

In this section, we focus on subfamilies of $\mathcal{H}(t, k, d)$ for the triples (t, k, d) considered in Theorem 1.1(ii).

A matching M of a graph G is *induced* if $E(G[V(M)]) = M$. We give a lemma concerning induced matchings.

Lemma 3.1. *Let j be a positive integer, and let $H \in \mathcal{K}(j)$. Let G be a graph, and let $T_0 \subseteq V(G)$ be a set with $|T_0| \geq j$. Let M be an induced matching of G with $V(M) \cap T_0 = \emptyset$ and $|V(M)| \geq 2|V(H)|$. If every vertex in $V(M)$ is adjacent to all vertices in T_0, then G contains H as an induced subgraph.*

Proof. Note that $G[V(M) \cup \{x\}]$ contains $K_1 + |V(H)|K_2$ as an induced subgraph, where $x \in T_0$. If $H = K_1 + (r_1K_1 \cup r_2K_2)$ for some integers r_1 and r_2, then $K_1 + |V(H)|K_2$ contains H as an induced subgraph, and hence G also contains H as an induced subgraph, as desired. Thus we may assume that $H = K_2 + mK_1$ for some integer m $(1 \leq m \leq j)$. If T_0 is an independent set of G, then $G[\{u, v\} \cup T_0]$ contains H as an induced subgraph, where $uv \in M$, as desired. Thus we may assume that $G[T_0]$ has an edge xy. Since $|V(M)| \geq 2|V(H)|$, there exists an independent set $A \subseteq V(M)$ of G with $|A| = |V(H)| - 2$. Then $\{x, y\} \cup A$ induces H in G. □

Our main result in this section is the following.

Proposition 3.2. *Let t, k and d be positive integers with* $\max\{k, t + \lfloor \frac{k-1}{2} \rfloor\} \leq d \leq t + k - 2$. *Then the following hold.*

(a) $\mathcal{H}(t, k, d) \supseteq \mathcal{K}(2d - 2t - k + 3)$.
(b) *If* $t \geq 4$ *and* $(k, d) = (4, t + 1)$, *then* $\mathcal{H}(t, k, d) \supseteq \mathcal{K}(2)$.

Proof. Let G be a graph with $\delta(G) \geq d$, and assume that G has no k vertex-disjoint $K_{1,t}$. Suppose for the moment that $|V(G)| \geq (k-1)(2t^2+1)+1$. Let $\mathcal{X}$ be a maximum t-proper set of G, and let $X_0 = \bigcup_{X \in \mathcal{X}} X$. Then $|\mathcal{X}| \leq k-1$ and $|X_0| \leq (k-1)(t+1)$. This together with Lemma 2.2 implies that there exists a set $S \subseteq X_0$ such that $|S \cap X| = 1$ for each $X \in \mathcal{X}$ and the number of vertices y in $V(G) - X_0$ with $N_G(y) \cap X_0 \subseteq S$ is at least $|V(G)| - (k-1)(2t^2+1)$. Let $Y = \{y \in V(G) - X_0 : N_G(y) \cap X_0 \subseteq S\}$. Note that $|Y| \geq |V(G)| - (k-1)(2t^2+1)$ (≥ 1).

Since $\delta(G) \geq d$ and $\delta(G - X_0) \leq t - 1$ by the maximality of $\mathcal{X}$, $|N_G(y) \cap S| \geq d - t + 1$ for every $y \in Y$. In particular, $|S| \geq d - t + 1$. For each $y \in Y$, let $T_y \in \binom{S}{d-t+1}$ be a set so that $T_y \subseteq N_G(y)$ (without regard to the intersection of $S - T_y$ and $N_G(y)$). For each $T \in \binom{S}{d-t+1}$, set $Y_T = \{y \in Y : T_y = T\}$. Then $\bigcup_{T \in \binom{S}{d-t+1}} Y_T = Y$ and $Y_T \cap Y_{T'} = \emptyset$ for all $T, T' \in \binom{S}{d-t+1}$ with $T \neq T'$. For each $T \in \binom{S}{d-t+1}$, let $Z_T \subseteq Y_T$ be the set of vertices which are adjacent to no vertex in $V(G) - (S \cup Y)$.

For two sets $U_1, U_2 \subseteq V(G)$ (which might not be disjoint), an edge $e \in E(G)$ is a U_1-U_2 *edge* if one endvertex of e belongs to U_1 and the other belongs to U_2.

Claim 3.1. *Let m be a positive integer, and suppose that* $|V(G)| \geq (k-1)(2m\binom{k-1}{d-t+1}^2(t^2-3t+3)+2t^3+t)$. *Then for some sets* $T_1, T_2 \in \binom{S}{d-t+1}$, *there exists an induced matching M of $G[Y]$ with $|V(M)| \geq 2m$ which consists of Z_{T_1}-Y_{T_2} edges.*

Proof of Claim 3.1. By the maximality of $\mathcal{X}$, every vertex in $V(G)-(X_0\cup Y)$ is adjacent to at most $t-1$ vertices in $V(G)-X_0$. Hence

$$\begin{aligned}
\left|\bigcup_{T\in\binom{S}{d-t+1}} Z_T\right| &= |Y| - \left|\bigcup_{T\in\binom{S}{d-t+1}} (Y_T - Z_T)\right| \\
&\geq |Y| - (t-1)(|V(G)| - |X_0| - |Y|) \\
&\geq t|Y| - (t-1)|V(G)| \\
&\geq t(|V(G)| - (k-1)(2t^2+1)) - (t-1)|V(G)| \\
&= |V(G)| - (k-1)(2t^3+t) \\
&\geq 2m(k-1)\binom{k-1}{d-t+1}^2(t^2-3t+3). \qquad (3.1)
\end{aligned}$$

Choose $T_1 \in \binom{S}{d-t+1}$ so that $|Z_{T_1}|$ is as large as possible. Then by (3.1),

$$|Z_{T_1}| \geq \frac{\left|\bigcup_{T\in\binom{S}{d-t+1}} Z_T\right|}{\left|\binom{S}{d-t+1}\right|} \geq 2m(k-1)\binom{k-1}{d-t+1}(t^2-3t+3). \qquad (3.2)$$

Since $\delta(G) \geq d \geq k > |S|$, every vertex in Z_{T_1} is adjacent to a vertex in Y. In particular, $G[Y]$ has an edge which is incident with a vertex in Z_{T_1}. Let M be an induced matching of $G[Y]$ such that every edge in M is incident with a vertex in Z_{T_1}. Choose M so that $|V(M)|$ is as large as possible.

Suppose that $|V(M)| < 2m\binom{k-1}{d-t+1}$. Let $W = \bigcup_{z\in V(M)}(N_{G[Y]}(z) \cup N^{(2)}_{G[Y]}(z))$. Note that $V(M) \subseteq W$. Since every vertex in Y is adjacent to at most $t-1$ vertices in Y, $|W| \leq |V(M)| + (t-2)|V(M)| + (t-2)^2|V(M)| = |V(M)|(t^2-3t+3) < 2m\binom{k-1}{d-t+1}(t^2-3t+3)$. On the other hand, $|Z_{T_1}| \geq 2m\binom{k-1}{d-t+1}(t^2-3t+3)$ by (3.2). Hence $Z_{T_1} - W \neq \emptyset$. Let $z_1 \in Z_{T_1} - W$. Since $d_G(z_1) \geq d \geq k$, $N_{G[Y]}(z_1) \neq \emptyset$ by the definition of Z_{T_1}. Let $z_1' \in N_{G[Y]}(z_1)$. Then $M' = M \cup \{z_1z_1'\}$ is an induced matching of $G[Y]$ such that every edge in M' is incident with a vertex in Z_{T_1}, which contradicts the maximality of M. Consequently $|V(M)| \geq 2m\binom{k-1}{d-t+1}$.

For $T \in \binom{S}{d-t+1}$, let $M_T = \{uv \in M : u \in Z_{T_1}, v \in Y_T\}$. Note that $\bigcup_{T\in\binom{S}{d-t+1}} M_T = M$ and $M_{T_1} = \{uv \in M : u, v \in Y_{T_1}\}$. Let $T_2 \in \binom{S}{d-t+1}$ be a set so that $|V(M_{T_2})|$ is as large as possible. Then

$$|V(M_{T_2})| \geq \frac{|V(M)|}{|\binom{S}{d-t+1}|} \geq 2m.$$

Since every edge in M_{T_2} is Z_{T_1}-Y_{T_2} edge, T_1 and T_2 are desired sets. □

We first show (a). Let $H \in \mathcal{K}(2d-2t-k+3)$, and set $m = |V(H)|$. Assume that $|V(G)| \geq (k-1)(2m\binom{k-1}{d-t+1}^2(t^2-3t+3)+2t^3+t)$. We show that G contains H as an induced subgraph. By Claim 3.1, for some sets $T_1, T_2 \in \binom{S}{d-t+1}$, there exists an induced matching M of $G[Y]$ with $|V(M)| \geq 2m$ which consists of Z_{T_1}-Y_{T_2} edges. Since $|T_1 \cup T_2| \leq |S| \leq k-1$, $|T_1 \cap T_2| = |T_1| + |T_2| - |T_1 \cup T_2| \geq 2(d-t+1)-(k-1) = 2d-2t-k+3$. Furthermore, every vertex in $V(M)$ is adjacent to all vertices in $T_1 \cap T_2$. Hence, applying Lemma 3.1 with T_0 replaced by $T_1 \cap T_2$, G contains H as an induced subgraph. Since H is arbitrary, (a) holds.

We next consider (b). Assume that $t \geq 4$ and $(k, d) = (4, t+1)$. We show that $\mathcal{H}(t, k, d) = \mathcal{H}(t, 4, t+1) \supseteq \mathcal{K}(2)$. By (a), $\mathcal{H}(t, 4, t+1) \supseteq \mathcal{K}(1)$. Since $\mathcal{H}(t, 3, t+1) \supseteq \mathcal{K}(2)$ by (a), if G has no 3 vertex-disjoint $K_{1,t}$ and the order of G is sufficiently large, then G contains $K_2 + 2K_1$ as an induced subgraph. Thus it suffices to show that if G has 3 vertex-disjoint $K_{1,t}$ and $|V(G)| \geq 6t^3 + 108t^2 - 321t + 324$, then G contains $K_2 + 2K_1$ as an induced subgraph. Note that $|\mathcal{X}| = |S| = 3$, $d-t+1 = 2$ and $|V(G)| \geq 6t^3 + 108t^2 - 321t + 324 = (k-1)(2 \cdot 2\binom{k-1}{d-t+1}^2(t^2-3t+3)+2t^3+t)$. Then by Claim 3.1, for some sets $T_1, T_2 \in \binom{S}{2}$, there exists an induced matching M of $G[Y]$ with $|V(M)| \geq 4$ consisting of Z_{T_1}-Y_{T_2} edges. For each edge $e \in M$, fix an endvertex u_e of M belonging to Z_{T_1}. Note that $\{u_e : e \in M\}$ is independent.

Claim 3.2. *If Y_{T_1} is not independent, then G contains $K_2 + 2K_1$ as an induced subgraph.*

Proof of Claim 3.2. Assume that Y_{T_1} is not independent, and let $uv \in G[Y_{T_1}]$. If T_1 is independent, then $\{u, v\} \cup T_1$ induces $K_2 + 2K_1$ in G, as desired. Thus we may assume that $G[T_1]$ has an edge (i.e., $G[T_1] \simeq K_2$). Then $T_1 \cup \{u_e, u_{e'}\}$ induces $K_2 + 2K_1$ in G, where $e, e' \in M$ with $e \neq e'$. $\square$

By Claim 3.2, we may assume that Y_{T_1} is independent.

Claim 3.3. *For an edge $e \in M$, if $S \not\subseteq N_G(u_e)$, then G contains $K_2 + 2K_1$ as an induced subgraph.*

Proof of Claim 3.3. Let $e \in M$, and suppose that $S \not\subseteq N_G(u_e)$. Since $T_1 \subseteq N_G(u_e)$, $S - N_G(u_e)$ consists of exactly one vertex, say s_0. Since $d_G(u_e) \geq t+1 \geq 5$, $|N_G(u_e) \cap Y| \geq 3$. This together with the assumption that Y_{T_1} is independent leads to $|N_G(u_e) \cap Y_T| \geq 2$ for some $T \in \binom{S}{2} - \{T_1\}$. Let $y_1, y_2 \in N_G(u_e) \cap Y_T$ with $y_1 \neq y_2$. Note that $T = (T_1 \cap T) \cup \{s_0\}$ (i.e., $(T_1 \cap T) \cup \{s_0\} \subseteq N_G(y_i)$ for $i \in \{1, 2\}$). If $y_1y_2 \in E(G)$, $\{u_e, s_0, y_1, y_2\}$ induces $K_2 + 2K_1$ in G; if $y_1y_2 \notin E(G)$, then $(T_1 \cap T) \cup \{u_e, y_1, y_2\}$ induces $K_2 + 2K_1$ in G. In either case, G contains $K_2 + 2K_1$ as an induced subgraph. $\square$

By Claim 3.3, we may assume that $S \subseteq N_G(u_e)$ for every $e \in M$. If $G[S]$ contains an edge xx', then $\{u_e, u_{e'}, x, x'\}$ induces K_2+2K_1 in G, where $e, e' \in M$ with $e \neq e'$, as desired. Thus we may assume that S is an independent set of G. Let $uv \in M$. Then both u and v are adjacent to all vertices in T_2. Hence $T_2 \cup \{u, v\}$ induces $K_2 + 2K_1$ in G. Consequently (b) holds.

This completes the proof of Proposition 3.2. □

4. Proof of Theorems 1.1 and 1.2

In this section, we complete the proof of Theorems 1.1 and 1.2. By Propositions 2.1 and 2.4, we obtain Theorem 1.1(v) and (i), respectively. Furthermore, the following two propositions which will be proved in this section imply Theorem 1.1(ii)(iii).

Proposition 4.1. *Let t, k and d be positive integers with* $\max\{k, t+\lfloor\frac{k-1}{2}\rfloor\} \leq d \leq t+k-2$. *Then* $\mathcal{H}(t, k, d) \subseteq \mathcal{K}(\max\{2d-2t-k+3, t-1\})$. *Furthermore, if $k = 2$, then $\mathcal{H}(t, k, d) \subseteq \mathcal{K}(1)$.*

Proposition 4.2. *Let t, k and d be positive integers with* $\max\{k, t+\lfloor\frac{k-1}{2}\rfloor\} \leq d \leq t+k-2$. *Then the following hold:*

(a) *If* $d \geq \min\{\frac{3t+k-4}{2}, \frac{t^2+(k-2)t-k+1}{t}\}$, *then* $\mathcal{H}(t, k, d) \subseteq \mathcal{K}(2d-2t-k+3)$.
(b) *If $t \geq 4$, then $\mathcal{H}(t, 4, t+1) = \mathcal{K}(2)$.*

Proof of Proposition 4.1. We let $H \in \mathcal{H}(t, k, d)$ and show that $H \in \mathcal{K}(\max\{2d-2t-k+3, t-1\})$. By the definition of H, there exists an integer $n = n(H)$ such that every H-free graph G with $|V(G)| \geq n$ and $\delta(G) \geq d$ has k vertex-disjoint $K_{1,t}$.

Now we construct two graphs G_1 and G_2 similar to graphs in the proof of Proposition 2.4. Let X be a set with $|X| = k-1$, and for each i $(1 \leq i \leq n)$, let Y_i be a complete graph of order t. Let G_1 be the graph defined by

$$V(G_1) = X \cup \left(\bigcup_{1\leq i\leq n} V(Y_i)\right)$$

and

$$E(G_1) = \{xx' : x, x' \in X, x \neq x'\} \cup \left(\bigcup_{1\leq i\leq n} (E(Y_i) \cup \{xy : x \in X, y \in V(Y_i)\})\right);$$

that is to say $G_1 \simeq K_{k-1} + nK_t$. Since $|X| = k-1$, we have $d-t+1 \leq |X| \leq 2(d-t+1)$. Hence there exist two sets $X_1, X_2 \subseteq X$ with $|X_i| = d-t+1$ ($i \in \{1,2\}$) and $X_1 \cup X_2 = X$. Note that $|X_1 \cap X_2| = 2d-2t-k+3 \geq 2(t+\lfloor\frac{k-1}{2}\rfloor) - 2t - k + 3 > 0$. For $i \in \{1,2\}$ and $1 \leq j \leq \max\{n,d\}$, let $A_i^{(j)}$ be a set with $|A_i^{(j)}| = t-1$. Let G_2 be the graph defined by

$$V(G_2) = X \cup \left(\bigcup_{1 \leq j \leq \max\{n,d\}} (A_1^{(j)} \cup A_2^{(j)}) \right)$$

and

$$E(G_2) = \bigcup_{1 \leq j \leq \max\{n,d\}} \Big\{ x_1a_1, x_2a_2, a_1a_2 : x_1 \in X_1, x_2 \in X_2, a_1 \in A_1^{(j)}, a_2 \in A_2^{(j)} \Big\}.$$

Then G_h ($h \in \{1,2\}$) is a connected graph with $|V(G_h)| \geq n$ and $\delta(G_h) \geq d$. Furthermore, since any subgraphs $K_{1,t}$ of G_h contain a vertex in X, G_h has no k vertex-disjoint $K_{1,t}$. Hence G_1 and G_2 are not H-free (i.e., H is a common induced subgraph of G_1 and G_2).

Let $U_1 \subseteq V(G_1)$ be a set with $G_1[U_1] \simeq H$. Since G_2 contains no K_4, H also contains no K_4. This implies that if $|U_1 \cap Z| \geq 3$ for some $Z \in \{X\} \cup \{V(Y_i) : 1 \leq i \leq n\}$, then H is a triangle (i.e., $H \in \mathcal{K}(\max\{2d-2t-k+3, t-1\})$), as desired. Thus we may assume that $|U_1 \cap X| \leq 2$ and $|U_1 \cap V(Y_i)| \leq 2$ for every $1 \leq i \leq n$. Since $|V(H)| \geq 3$ and H is connected, $U_1 \cap X \neq \emptyset$. If $|U_1 \cap X| = 1$, then H is an induced subgraph of $K_1 + nK_2$ (i.e., $H \in \mathcal{K}(\max\{2d-2t-k+3, t-1\})$), as desired. In particular, if $k=2$, then $\mathcal{H}(t,k,d) \subseteq \mathcal{K}(1)$. Thus we may assume that $|U_1 \cap X| = 2$. Since H contains no K_4, we see that $|U_1 \cap V(Y_i)| \leq 1$ for every $1 \leq i \leq n$, and hence $H = K_2 + mK_1$ for some $m \geq 1$.

Now we fix an edge uv of G_2. Since $G_2[X]$ contains no edge, we may assume that $u \in V(G_2) - X$. If $v \notin X$, then $N_{G_2}(u) \cap N_{G_2}(v) = X_1 \cap X_2$, and hence $|N_{G_2}(u) \cap N_{G_2}(v)| = 2d-2t-k+3$; if $v \in X$, then $N_{G_2}(u) \cap N_{G_2}(v) \subseteq N_{G_2}(u) - X$, and hence $|N_{G_2}(u) \cap N_{G_2}(v)| \leq t-1$. In either case, we have $|N_{G_2}(u) \cap N_{G_2}(v)| \leq \max\{2d-2t-k+3, t-1\}$. Since uv is arbitrary, if $K_2 + mK_1$ is an induced subgraph of G_2, then $m \leq \max\{2d-2t-k+3, t-1\}$. Therefore $H \in \mathcal{K}(\max\{2d-2t-k+3, t-1\})$. Since H is arbitrary, we have $\mathcal{H}(t,k,d) \subseteq \mathcal{K}(\max\{2d-2t-k+3, t-1\})$.

This completes the proof of Proposition 4.1. □

Now we give a lemma which is useful when we construct some examples. Let t, k and d be positive integers with $\max\{k, t+\lfloor\frac{k-1}{2}\rfloor\} \leq d \leq t+k-2$, and let $[k-1] = \{1, 2, \cdots, k-1\}$. For a K_3-free $(t-1)$-regular graph G, a labeling $f : V(G) \to \binom{[k-1]}{d-t+1}$ of G is (t, k, d)-*good* if

(F1) for every $i \in [k-1]$, there exists a vertex $u \in V(G)$ with $i \in f(u)$,
(F2) for every $uv \in E(G)$, $|f(u) \cap f(v)| \leq 2d-2t-k+3$, and
(F3) for every $i \in [k-1]$, if $i \in f(u)$, then $|\{v \in N_G(u) : i \in f(v)\}| \leq 2d-2t-k+3$.

Lemma 4.3. *Let t, k and d be positive integers with $\max\{k, t+\lfloor\frac{k-1}{2}\rfloor\} \leq d \leq t+k-2$. If there exists a K_3-free $(t-1)$-regular graph having a (t, k, d)-good labeling, then $\mathcal{H}(t, k, d) = \mathcal{K}(2d-2t-k+3)$.*

Proof. By Proposition 3.2, it suffices to show that $\mathcal{H}(t, k, d) \subseteq \mathcal{K}(2d-2t-k+3)$. Let $H \in \mathcal{H}(t, k, d)$. We show that $H \in \mathcal{K}(2d-2t-k+3)$. If H is an induced subgraph of K_1+nK_2 for some $n \geq 1$, then $H \in \mathcal{K}(2d-2t-k+3)$, as desired. Thus by Proposition 4.1, we may assume that $H = K_2+mK_1$ for some integer m $(1 \leq m \leq \max\{2d-2t-k+3, t-1\})$. By the definition of H, there exists an integer $n = n(H)$ such that every H-free graph G with $|V(G)| \geq n$ and $\delta(G) \geq d$ has k vertex-disjoint $K_{1,t}$.

Let A be a K_3-free $(t-1)$-regular graph having a (t, k, d)-good labeling. Let $s = \max\{n, d\}$. Let $A_1, \cdots, A_s$ be s disjoint copies of A, and for each i $(1 \leq i \leq s)$, let f_i be a (t, k, d)-good labeling of A_i. Let G be the graph defined by

$$V(G) = [k-1] \cup \left(\bigcup_{1\leq i\leq s} V(A_i) \right)$$

and

$$E(G) = \bigcup_{1\leq i\leq s} (E(A_i) \cup \{uj : u \in V(A_i), j \in [k-1], j \in f(u)\}).$$

Then $|V(G)| \geq n$ and $\delta(G) = d$. Furthermore, since any subgraphs $K_{1,t}$ of G contain a vertex in $[k-1]$, G has no k vertex-disjoint $K_{1,t}$. Hence G is not H-free. Let $U \subseteq V(G)$ be a set such that $G[U] \simeq H$, and let $uv \in E(G[U])$ be an edge which is contained in all triangles of $G[U]$. We may assume that $u \in \bigcup_{1\leq i\leq s} V(A_i)$. If $v \in [k-1]$, then $|N_G(u) \cap N_G(v)| \leq 2d-2t-k+3$ by the condition (F3); if $v \notin [k-1]$, then $|N_G(u) \cap N_G(v)| \leq 2d-2t-k+3$ by the condition (F2) since A is K_3-free. In either case, we have $|N_G(u) \cap N_G(v)| \leq 2d-2t-k+3$, and hence $H = K_2+mK_1$ for some $1 \leq m \leq 2d-2t-k+3$. Consequently, $H \in \mathcal{K}(2d-2t-k+3)$. Since H is arbitrary, we have $\mathcal{H}(t, k, d) \subseteq \mathcal{K}(2d-2t-k+3)$. □

Proof of Proposition 4.2. We first prove (a). If $d \geq \frac{3t+k-4}{2}$, then $t-1 \leq 2d-2t-k+3$, and hence $\mathcal{H}(t,k,d) \subseteq \mathcal{K}(2d-2t-k+3)$ by Proposition 4.1. Thus we may assume that $d < \frac{3t+k-4}{2}$ (and so $d \geq \frac{t^2+(k-2)t-k+1}{t}$). Then $t(t+k-2-d) \leq k-1$. We let $H \in \mathcal{H}(t,k,d)$, and show that $H \in \mathcal{K}(2d-2t-k+3)$. By Proposition 4.1, $H \in \mathcal{K}(\max\{2d-2t-k+3, t-1\})$. If H is an induced subgraph of K_1+nK_2 for some $n \geq 1$, then $H \in \mathcal{K}(2d-2t-k+3)$, as desired. Thus we may assume that $H = K_2+mK_1$ for some m ($1 \leq m \leq \max\{2d-2t-k+3, t-1\}$). By the definition of H, there exists an integer $n = n(H)$ such that every H-free graph G with $|V(G)| \geq n$ and $\delta(G) \geq d$ has k vertex-disjoint $K_{1,t}$.

Case 1: $t \leq 3$.

By simple calculations, $(t,k,d) = (3,2,3)$ and $(t,k,d) = (3,4,4)$ are the only triples satisfying all conditions. If $(t,k,d) = (3,2,3)$, then $\mathcal{H}(t,k,d) \subseteq \mathcal{K}(1)$ by Proposition 4.1, as desired. Thus we may assume that $(t,k,d) = (3,4,4)$. Let $C = x_1x_2\cdots x_6$ be the cycle of order 6, and let $f : V(G) \to \binom{[3]}{2}$ be a labeling with

$$f(x) = \begin{cases} \{1,2\} & (x \in \{x_1,x_4\}) \\ \{2,3\} & (x \in \{x_2,x_5\}) \\ \{1,3\} & (x \in \{x_3,x_6\}). \end{cases}$$

Then C is a K_3-free 2-regular graph and f is a $(3,4,4)$-good labeling of C. Hence by Lemma 4.3, we have $\mathcal{H}(3,4,4) = \mathcal{K}(1)$.

Case 2: $t \geq 4$.

Let X be a set with $|X| = k-1$. Since $t(t+k-2-d) \leq k-1$, there exist disjoint t sets $X_1, \cdots, X_t \in \binom{X}{t+k-2-d}$. Note that if $d = t+k-2$, then $X_i = \emptyset$ for each $1 \leq i \leq t$. Let $s = \max\{n,d\}$. For each $1 \leq j \leq s$, let Y_j be a complete graph of order t, and write $V(Y_j) = \{y_1^{(j)}, \cdots, y_t^{(j)}\}$. Let G_1 be the graph defined by

$$V(G_1) = X \cup \left(\bigcup_{1 \leq j \leq s} V(Y_j) \right)$$

and

$$E(G_1) = \bigcup_{1 \leq j \leq s} \left(E(Y_j) \cup \left(\bigcup_{1 \leq i \leq t} \{xy_i^{(j)} : x \in X - X_i\} \right) \right).$$

Then G_1 is a connected graph with $|V(G_1)| \geq n$ and $\delta(G_1) = d_{G_1}(y_1^{(1)}) = (t-1)+(k-1-(t+k-2-d)) = d$. Furthermore, since any subgraphs $K_{1,t}$ of G_1 contain a vertex in X, G_1 has no k vertex-disjoint $K_{1,t}$. Hence G_1 is not H-free.

Now we fix an edge uv of G_1. Since $G_1[X]$ contains no edge, we may assume that $u \in V(G_1)-X$. If $v \in X$, then $N_{G_1}(u) \cap N_{G_1}(v)$ induces a complete graph in G_1; if $v \notin X$, then the independence number of $G_1[N_{G_1}(u) \cap N_{G_1}(v)]$ is exactly $(k-1)-2(t+k-2-d) = 2d-2t-k+3$ because $t \geq 4$. In either case, the independence number of $G_1[N_{G_1}(u) \cap N_{G_1}(v)]$ is at most $2d-2t-k+3$. Since uv is arbitrary, if K_2+mK_1 is an induced subgraph of G_1, then $m \leq 2d-2t-k+3$. Therefore $H \in \mathcal{K}(2d-2t-k+3)$. Since H is arbitrary, (a) holds.

We next show (b). By (a), $\mathcal{H}(t,3,t+1) \subseteq \mathcal{K}(2)$. Furthermore, we see that $\mathcal{H}(t,4,t+1) \subseteq \mathcal{H}(t,3,t+1)$, and hence $\mathcal{H}(t,4,t+1) \subseteq \mathcal{K}(2)$. This together with Proposition 3.2(b) implies that $\mathcal{H}(t,4,t+1) = \mathcal{K}(2)$.

This completes the proof of Proposition 4.1. □

Now we complete the proof of Theorem 1.1. It suffices to show Theorem 1.1(iv). Let p and q be positive integers with $p \geq 2q+1$. Let $f_{p,q} : \binom{[p]}{q} \to \binom{[p]}{p-q}$ be a mapping with $f_{p,q}(A) = [p]-A$ for all $A \in \binom{[p]}{q}$. Then we can easily verify the following observation.

Observation 4.4. *Let p and q be positive integers with $p \geq 2q+1$. Then $f_{p,q}$ satisfies the following:*

(1) *for every $i \in [p]$, there exists $A \in \binom{[p]}{q}$ with $i \in f_{p,q}(A)$,*
(2) *for every $A_1, A_2 \in \binom{[p]}{q}$ with $A_1 \cap A_2 = \emptyset$, $|f_{p,q}(A_1) \cap f_{p,q}(A_2)| = p-2q$, and*
(3) *for every $i \in [p]$, if $i \in f_{p,q}(A)$, then $|\{A' : A \cap A' = \emptyset, i \in f_{p,q}(A')\}| = \binom{p-q-1}{q}$.*

The *Kneser graph*, denoted by $\mathrm{KN}(p,q)$, is the graph on $\binom{[p]}{q}$ such that for $A_1, A_2 \in \binom{[p]}{q}$, A_1 and A_2 are adjacent in $\mathrm{KN}(p,q)$ if and only if $A_1 \cap A_2 = \emptyset$. By the definition, $\mathrm{KN}(3,1)$ is isomorphic to K_3 and $\mathrm{KN}(5,2)$ is isomorphic to the Petersen graph. Furthermore, we have the following observation.

Observation 4.5. *Let p and q be positive integers with $p \geq 2q+1$. Then $KN(p,q)$ is $\binom{p-q}{q}$-regular. Furthermore, if $p \leq 3q-1$, then $KN(p,q)$ is K_3-free.*

We let $t \geq 2$, and focus on $f_{2t-3,t-2}$ and $\mathrm{KN}(2t-3,t-2)$. By Observation 4.4, we have the following:

(1) for every $i \in [2t-3]$, there exists $A \in \binom{[2t-3]}{t-2}$ with $i \in f_{2t-3,t-2}(A)$,
(2) for every $A_1, A_2 \in \binom{[2t-3]}{t-2}$ with $A_1 \cap A_2 = \emptyset$, $|f_{2t-3,t-2}(A_1) \cap f_{2t-3,t-2}(A_2)| = 1$, and
(3) for every $i \in [2t-3]$, if $i \in f_{2t-3,t-2}(A)$, then $|\{A' : A \cap A' = \emptyset, i \in f_{2t-3,t-2}(A')\}| = 1$.

By Observation 4.5, $\mathrm{KN}(2t-3, t-2)$ is K_3-free and $(t-1)$-regular. In particular, $f_{2t-3,t-2}$ is a $(t, 2t-2, 2t-2)$-good labeling of $\mathrm{KN}(2t-3, t-2)$. This together with Lemma 4.3 implies $\mathcal{H}(t, 2t-2, 2t-2) = \mathcal{K}(1)$. Consequently, we obtain Theorem 1.1(iv).

Finally, we show Theorem 1.2. By Theorem 1.1, it suffices to show that $\mathcal{H}(4, k, k) = \mathcal{K}(k-5)$ for $k \in \{7, 8\}$. For each $k \in \{7, 8\}$, let Y_k be the graph, vertices of which are labeled by $k-3$ elements of $[k-1]$, as in Figure 1 (to simplify the labeling, we use sequences instead of sets). Then Y_k is a K_3-free 3-regular graph having a $(4, k, k)$-good labeling. Hence it follows from Lemma 4.3 that $\mathcal{H}(4, k, k) = \mathcal{K}(k-5)$ for $k \in \{7, 8\}$, as desired.

5. Concluding remarks

In this paper, we characterize $\mathcal{H}(t, k, d)$ for almost all triples (t, k, d). By Theorems 1.1 and 1.2, $\mathcal{H}(t, k, d)$ have not been determined yet for triples (t, k, d) with $t \geq 5$ satisfying (H1) and (H2).

As we checked above, it is an important problem to find K_3-free $(t-1)$-regular graphs having (t, k, d)-good labelings, and the Kneser graphs have nice properties for good labeling. On the other hand, there exist non-Kneser graphs having a good labeling (for example, Y_7 and Y_8 are such graphs). However, Y_8 is a subgraph of $\mathrm{KN}(7, 2)$ and its good labeling can be obtained from $f_{7,2}$. Hence Kneser graphs might be strong tools.

By observing Proposition 4.1, such families $\mathcal{H}(t, k, d)$ may equal to $\mathcal{K}(2d-2t-k+3)$. On the other hand, for example, we can easily check that every K_3-free 4-regular graph has no $(5, 6, 7)$-good labeling. So we cannot judge whether $\mathcal{H}(5, 6, 7)$ is equal to $\mathcal{K}(1)$ or not from Lemma 4.3. (Indeed, we suspect that $\mathcal{H}(5, 6, 7) \neq \mathcal{K}(1)$.) We conclude this paper by presenting a problem related to the determination of $\mathcal{H}(t, k, d)$.

Problem 1. *Let t, k and d be positive integers with $t \geq 5$ satisfying (H1) and (H2). Is it true that $\mathcal{H}(t, k, d) = \mathcal{K}(2d-2t-k+3)$ if and only if there exists a K_3-free $(t-1)$-regular graph having a (t, k, d)-good labeling?*

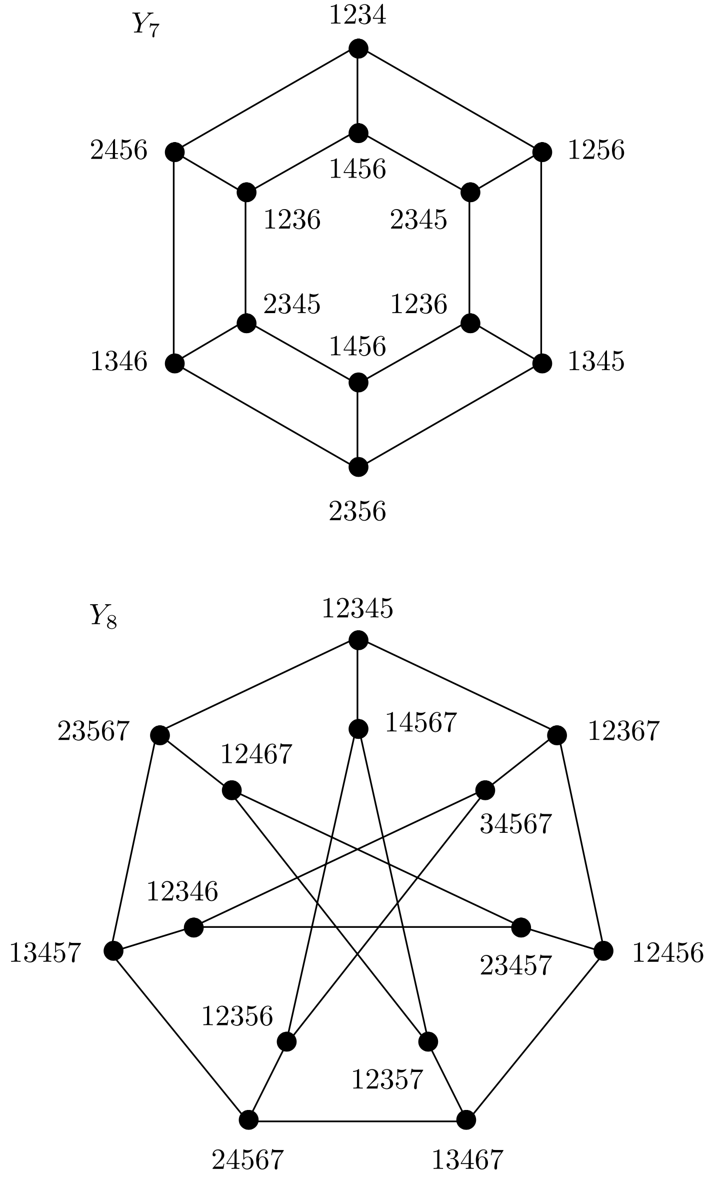

Figure 1: Graphs Y_7 and Y_8.

Acknowledgments

The authors would like to thank the referees for valuable suggestions. In particular, one of the referees found a (t, k, d)-good labeling of Kneser graphs.

References

[1] S. Chiba, Vertex-disjoint t-claws in graphs, SUT J. Math. **43** (2007) 149–172. MR2417538

[2] V. Chvátal, Tree-complete graph Ramsey numbers, J. Graph Theory **1** (1977) 93. MR0465920

[3] M. Chudnovsky and P. Seymour, The structure of claw-free graphs, *Surveys in combinatorics 2005*, London Math. Soc. Lecture Note Ser., **327**, Cambridge Univ. Press, (2005) pp. 153–171. MR2187738

[4] R. Diestel, "Graph Theory" (4th edition), Graduate Texts in Mathematics **173**, Springer (2010). MR2744811

[5] Y. Egawa and K. Ota, Vertex-disjoint claws in graphs, Discrete Math. **197/198** (1999) 225–246. MR1674865

[6] P. Erdős and L. Pósa, On the maximal number of disjoint circuits of a graph, Publ. Math. Debrecen **9** (1962) 3–12. MR0150756

[7] J. Fujisawa, K. Ota, K. Ozeki and G. Sueiro, Forbidden induced subgraphs for star-free graphs, Discrete Math. **311** (2011) 2475–2484. MR2832146

[8] S. Fujita, Vertex-disjoint $K_{1,t}$'s in graphs, Ars Combin. **64** (2002) 211–223. MR1914209

[9] S. Fujita, Forbidden pairs for vertex-disjoint claws, Far East J. Appl. Math. **18** (2005) 209–213. MR2145762

[10] S. Fujita, Disjoint stars and forbidden subgraphs, Hiroshima Math. J. **36** (2006) 397–403. MR2290665

[11] M. M. Matthews and D. P. Sumner, Hamiltonian results in $K_{1,3}$-free graphs, J. Graph Theory **8** (1984) 139–146. MR0732027

[12] K. Ota, Vertex-disjoint stars in graphs, Discuss. Math. Graph Theory **21** (2001) 179–185. MR1892809

[13] D. P. Sumner, 1-factors and antifactor sets, J. London Math. Soc. **13** (1976) 351–359. MR0409287

Michitaka Furuya
College of Liberal Arts and Science
Kitasato University
1-15-1 Kitasato, Minami-ku
Sagamihara, Kanagawa 252-0373
Japan
E-mail address: michitaka.furuya@gmail.com

Naoki Matsumoto
Department of Computer and Information Science
Seikei University
3-3-1 Kichijoji-Kitamachi
Musashino-shi, Tokyo 180-8633
Japan
E-mail address: naoki.matsumo10@gmail.com

Received 8 June 2014