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Monochromatic homeomorphically irreducible trees in 2-edge-colored complete graphs*

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It has been known that every 2-edge-colored complete graph has a monochromatic connected spanning subgraph. In this paper, we study a condition which can be imposed on such a monochromatic subgraph, and show that almost all 2-edge-colored complete graphs have a monochromatic spanning tree with no vertices of degree 2. As a corollary of our main theorem, we obtain a Ramsey type result: Every 2-edge-colored complete graph of order $n \geq 8$ has a monochromatic tree T with no vertices of degree 2 and $|V(T)| \geq n-1$.

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1. Introduction

All graphs considered here are finite simple graphs.

It has been known that every 2-edge-colored complete graph has a monochromatic connected spanning subgraph. One may impose an additional condition on monochromatic spanning trees in a 2-edge-colored complete graph. For example, the following theorems are known (where a *broom* is a tree obtained from a star and a path by identifying the center of the star and one endpoint of the path).

Theorem A ([3]). *Every 2-edge-colored complete graph has a monochromatic spanning broom.*

Theorem B ([1, 8, 9]). *Every 2-edge-colored complete graph has a monochromatic spanning subgraph of diameter at most three.*

However, for a property P of graphs, it is not always true that every 2-edge-colored complete graph has a monochromatic connected spanning

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subgraph satisfying P. Thus a natural Ramsey type problem arises: For a property P of graphs and a natural number m_0, determine the minimum number n_0 such that every 2-edge-colored complete graph of order $n \geq n_0$ has a monochromatic subgraph of order $m \geq m_0$ satisfying P. To put it simply, we want to find a large monochromatic subgraph satisfying a given property in 2-edge-colored complete graphs. In 2-edge-colored complete graphs, Gyárfás [6] found a large monochromatic path, Erdös and Fowler [5] found a large monochromatic subgraph of diameter at most two, and Bollobás and Gyárfás [2] found a large monochromatic 2-connected subgraph. Furthermore, Gyárfás and Sárközy [7] proved the following theorem.

Theorem C ([7])**.** *Every* 2*-edge-colored complete graph of order* n *has a monochromatic tree of diameter at most three and order at least* $(3n+1)/4$*.*

Considering Theorem C, it is natural to find a large subtree having small diameter in a 2-edge-colored complete graph. Now we focus on a special class of trees having small diameter. A tree T is a *homeomorphically irreducible tree* (or *HIT*) if T has no vertices of degree 2. The diameter of HITs tends to be small. Indeed, many trees of diameter at most three are HITs, and so it seems that the class of HITs is wider than the class of trees of diameter at most three. In particular, one may expect that a 2-edge-colored complete graph of order n has a HIT of order larger than $(3n+1)/4$. In this paper, we give an affirmative result for this expectation.

Theorem 1. *Every* 2*-edge-colored complete graph of order* $n \geq 8$ *has a monochromatic HIT of order at least* $n-1$*.*

In fact, we prove a stronger theorem which gives a necessary and sufficient condition for a 2-edge-colored complete graph to have a monochromatic spanning HIT (or *HIST*). Let $K_{m,n}$ denote the complete bipartite graph with partite sets having cardinality m and n. Let $K^-_{m,n}$ denote the graph obtained from $K_{m,n}$ by deleting one edge, and Z_n denote the complement of $K^-_{2,n-2}$ (see Figure 1). Our main result is the following.

Theorem 2. *Let* G *be a* 2*-edge-colored complete graph of order* $n \geq 8$ *colored with* 1 *and* 2*, and for each* $i \in \{1,2\}$*, let* G_i *be the spanning subgraph of* G *induced by all edges of color* i*. Then* G *has a monochromatic HIST if and only if* G_i *is isomorphic to neither* $K_{2,n-2}$ *nor* $K^-_{2,n-2}$ *for each* $i \in \{1,2\}$*.*

Since we can easily check that $K_{2,n-2}$ and $K^-_{2,n-2}$ have a HIT of order $n-1$, Theorem 1 follows from Theorem 2.

Remark 1*. If* $n = 7$*, Theorem 2 does not hold. For example, the graph* G *of order* 7 *depicted in Figure 2 is a counterexample. In fact, both* G *and*

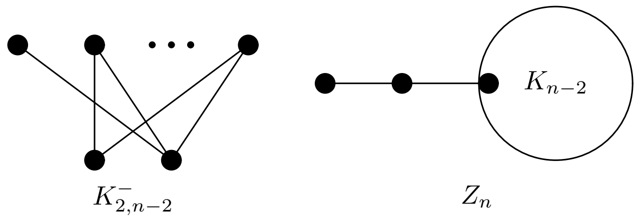

Figure 1: Graphs $K^-_{2,n-2}$ and Z_n.

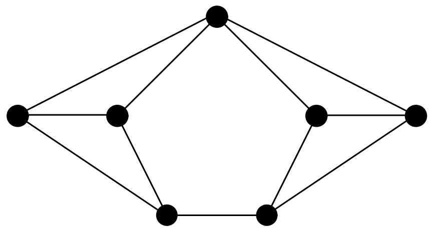

Figure 2: A graph G of order 7.

the complement of G are not isomorphic to $K_{2,n-2}$ or $K^-_{2,n-2}$ and have no HIST.

Our notation and terminology are standard, and mostly taken from [4]. Possible exceptions are as follows. Let G be a graph. For $x, y \in V(G)$, we let $d_G(x, y)$ denote the distance between x and y. When G is connected, we define the *diameter* of G by $\mathrm{diam}(G) := \max\{d_G(x, y) \mid x, y \in V(G)\}$. For $x \in V(G)$, we let $N_G(x)$ denote the *neighborhood* of x, and $d_G(x)$ denote the *degree* of x in G. We let $\Delta(G)$ and $\delta(G)$ denote the *maximum degree* and the *minimum degree* of G, respectively. For two disjoint sets $X, Y \subseteq V(G)$, we let $E_G(X, Y) = \{xy \in E(G) \mid x \in X, y \in Y\}$. A maximal 2-connected subgraph of G containing at most one cutvertex of G is called an *endblock* of G.

2. Fundamental properties

In this section, we give two useful lemmas.

Let G be a graph. A pair (x_1, x_2) of vertices of G is *branchable* if

(P1) $x_1x_2 \in E(G)$,
(P2) $N_G(x_1) \cup N_G(x_2) = V(G)$, and
(P3) for $i \in \{1, 2\}$, if $d_G(x_i) = 2$, then $d_G(x_{3-i}) = |V(G)| - 1$.

A quadruplet $(x_1, x_2, x_3; y)$ of vertices of G is *branchable* if

(Q1) $x_1x_3, x_2x_3, x_3y \in E(G)$,
(Q2) $N_G(x_1) \cup N_G(x_2) \supseteq V(G) - \{x_1, x_2, y\}$, and
(Q3) $|N_G(x_i) - \{x_{3-i}, x_3, y\}| \geq 2$ for each $i \in \{1, 2\}$.

Lemma 1. *Let G be a graph of order $n \geq 8$. If G has a branchable pair or a branchable quadruplet, then G has a HIST.*

Proof. Assume that G has a branchable pair or a branchable quadruplet.

Case 1: G has a branchable pair (x_1, x_2).

We may assume that $|N_G(x_1) - (N_G(x_2) \cup \{x_2\})| \geq |N_G(x_2) - (N_G(x_1) \cup \{x_1\})|$. If $d_G(x_1) = n - 1$, then G has a spanning star with the center x_1, as desired. Thus we may assume that $d_G(x_1) \leq n - 2$. By (P2) and (P3), we have $N_G(x_2) - (N_G(x_1) \cup \{x_1\}) \neq \emptyset$ and $d_G(x_2) \geq 3$. Then there exists a subset X_2 of $N_G(x_2) - \{x_1\}$ satisfying $X_2 \supseteq N_G(x_2) - (N_G(x_1) \cup \{x_1\})$ and $|X_2| \geq 2$. Choose X_2 so that $|X_2|$ is as small as possible. Let $X_1 = N_G(x_1) - (X_2 \cup \{x_2\})$. Then X_1 and X_2 are disjoint and by (P2), $X_1 \cup X_2 = V(G) - \{x_1, x_2\}$. If $|X_2| = 2$, then $|X_1| \geq 2$ because $|V(G)| \geq 8$; if $|X_2| \geq 3$, then $|N_G(x_2) - (N_G(x_1) \cup \{x_1\})| \geq 3$, and hence $|X_1| \geq |N_G(x_1) - (N_G(x_2) \cup \{x_2\})| \geq |N_G(x_2) - (N_G(x_1) \cup \{x_1\})| \geq 3$. Hence the spanning subgraph T of G with $E(T) = \{x_1x_2\} \cup (\bigcup_{i \in \{1,2\}} \{x_iu \mid u \in X_i\})$ is a HIST of G.

Case 2: G has a branchable quadruplet $(x_1, x_2, x_3; y)$.

We may assume that $|N_G(x_1) - (N_G(x_2) \cup \{x_2, x_3, y\})| \geq |N_G(x_2) - (N_G(x_1) \cup \{x_1, x_3, y\})|$. By (Q3), there exists a subset X_2 of $N_G(x_2) - \{x_1, x_3, y\}$ satisfying $X_2 \supseteq N_G(x_2) - (N_G(x_1) \cup \{x_1, x_3, y\})$ and $|X_2| \geq 2$. Choose X_2 so that $|X_2|$ is as small as possible. Let $X_1 = N_G(x_1) - (X_2 \cup \{x_2, x_3, y\})$. Then X_1 and X_2 are disjoint and by (Q2), $X_1 \cup X_2 \supseteq V(G) - \{x_1, x_2, x_3, y\}$. If $|X_2| = 2$, then $|X_1| \geq 2$ because $|V(G)| \geq 8$; if $|X_2| \geq 3$, then $|N_G(x_2) - (N_G(x_1) \cup \{x_1, x_3, y\})| \geq 3$, and hence $|X_1| \geq |N_G(x_1) - (N_G(x_2) \cup \{x_2, x_3, y\})| \geq |N_G(x_2) - (N_G(x_1) \cup \{x_1, x_3, y\})| \geq 3$. Hence the spanning subgraph T of G with $E(T) = \{x_1x_3, x_2x_3, x_3y\} \cup (\bigcup_{i \in \{1,2\}} \{x_iu \mid u \in X_i\})$ is a HIST of G. □

Lemma 2. *Let G be a graph, and let $x, y \in V(G)$ be two distinct vertices with $N_G(x) \cap N_G(y) \neq \emptyset$. If $G - \{x, y\}$ has a HIST, then G also has a HIST.*

Proof. Let $z \in N_G(x) \cap N_G(y)$, and let T be a HIST of $G - \{x, y\}$. Then the spanning graph T' of G with $E(T') = E(T) \cup \{xz, yz\}$ is a HIST of G. □

3. Proof of Theorem 2

In this section, we prove Theorem 2. We start with a key lemma.

Lemma 3. *Let G, n, G_1 and G_2 be as in Theorem 2. If G has no monochromatic HIST and for each $i \in \{1,2\}$, G_i is isomorphic to neither $K_{2,n-2}$ nor $K^-_{2,n-2}$, then for each $i \in \{1,2\}$, G_i is connected, $\delta(G_i) \geq 3$ and $diam(G_i) = 2$.*

Proof. Assume that G has no monochromatic HIST and for each $i \in \{1,2\}$, G_i is isomorphic to neither $K_{2,n-2}$ nor $K^-_{2,n-2}$. By Lemma 1, for each $i \in \{1,2\}$, G_i has no branchable pair and no branchable quadruplet. We first show some claims.

Claim 1. *For each $i \in \{1,2\}$, G_i is connected.*

Proof. Suppose that G_i is disconnected for some $i \in \{1,2\}$. Then there exist disjoint non-empty subsets X and Y of $V(G)$ with $X \cup Y = V(G)$ and $E_{G_i}(X,Y) = \emptyset$. We may assume that $|X| \geq |Y|$. Let $x \in X$ and $y \in Y$. Then $N_{G_{3-i}}(x) \supseteq Y$ and $N_{G_{3-i}}(y) \supseteq X$, and in particular, $N_{G_{3-i}}(x) \cup N_{G_{3-i}}(y) = V(G_{3-i})$. Since (x,y) is not branchable and $|X| \geq |Y|$, this implies that $d_{G_{3-i}}(x) = 2$ and $d_{G_{3-i}}(y) \leq n-2$. Hence $|Y| \leq d_{G_{3-i}}(x) = 2$ and $|X| \leq d_{G_{3-i}}(y) \leq n-2$. Since $|X| + |Y| = n$, this forces $|X| = n-2$, $|Y| = 2$, $N_{G_{3-i}}(x) = Y$ and $N_{G_{3-i}}(y) = X$. Since x and y are arbitrary, G_{3-i} is isomorphic to $K_{2,n-2}$, which is a contradiction. □

Claim 2. *For each $i \in \{1,2\}$, G_i has no endblock C which is a clique of order at most three. In particular, $\delta(G_i) \geq 2$ for each $i \in \{1,2\}$.*

Proof. Suppose that G_i has an endblock C which is a clique of order at most three for some $i \in \{1,2\}$. Since $|V(C)| \leq 3$ and G_i is connected by Claim 1, C contains a cutvertex z of G_i. If $d_{G_i}(z) = n-1$, then G_i has a HIST, which is a contradiction. Thus $V(G) - (N_{G_i}(z) \cup \{z\}) \neq \emptyset$. Let $x \in V(C) - \{z\}$ and $y \in V(G) - (N_{G_i}(z) \cup \{z\})$. Since $d_{G_i}(x,y) \geq 3$, $xy \in E(G_{3-i})$ and $N_{G_{3-i}}(x) \cup N_{G_{3-i}}(y) = V(G)$. Since $d_{G_{3-i}}(x) \geq n-3$ and (x,y) is not branchable in G_{3-i}, $d_{G_{3-i}}(y) = 2$, and hence $N_{G_{3-i}}(y) = \{x,z\}$. This implies that $V(C) = \{x,z\}$ and y is adjacent to all vertices in $V(G) - \{x,y,z\}$ in G_i. Since y is arbitrary, every vertex in $N_{G_i}(z) - \{x\}$ is adjacent to all vertices in $V(G) - (N_{G_i}(z) \cup \{z\})$ in G_i and $V(G) - (N_{G_i}(z) \cup \{z\})$ induces a clique in G_i. Let $w \in N_{G_i}(z) - \{x\}$. Since $N_{G_i}(z) \cup N_{G_i}(w) = V(G)$ and (z,w) is not branchable in G_i, either $d_{G_i}(z) = 2$ or $d_{G_i}(w) = 2$. If $d_{G_i}(z) = 2$, then $N_{G_i}(z) = \{x,w\}$, and hence G_i is isomorphic to Z_n, which is a contradiction. Thus $d_{G_i}(w) = 2$. Since w is arbitrary, $|V(G) - (N_{G_i}(z) \cup \{z\})| = 1$ and $N_{G_i}(z) - \{x\}$ is an independent set of G_i. This implies that G_i is isomorphic to $K^-_{2,n-2}$, which is a contradiction. □

Claim 3. *For each $i \in \{1,2\}$, $\delta(G_i) \geq 3$.*

Proof. We first show that for each $i \in \{1,2\}$, G_i has no edge x_1x_2 with $d_{G_i}(x_1) = d_{G_i}(x_2) = 2$. Suppose that for some $i \in \{1,2\}$, G_i has an edge x_1x_2 with $d_{G_i}(x_1) = d_{G_i}(x_2) = 2$. For each $j \in \{1,2\}$, write $N_{G_i}(x_j) = \{x_{3-j}, y_j\}$. If $y_1 = y_2$, then $\{x_1, x_2, y_1\}$ induces an endblock of G_i, which contradicts Claim 2. Thus $y_1 \neq y_2$. For $j \in \{1,2\}$, if there exists a vertex $w \in V(G) - (N_{G_i}(y_j) \cup \{x_{3-j}, y_1, y_2\})$, then $N_{G_{3-i}}(x_j) = V(G) - \{x_{3-j}, y_j\}$ and $\{x_1, x_2, y_j\} \subseteq N_{G_{3-i}}(w)$, and hence (x_j, w) is a branchable pair of G_{3-i}, which is a contradiction. Thus for each $j \in \{1,2\}$, $V(G) - \{x_{3-j}, y_1, y_2\} \subseteq N_{G_i}(y_j)$ (i.e., $N_{G_{3-i}}(y_j) \subseteq \{x_{3-j}, y_{3-j}\}$). If $y_1y_2 \in E(G_i)$, then (y_1, y_2) is a branchable pair of G_i, which is a contradiction. Thus $y_1y_2 \in E(G_{3-i})$. Let $z, z' \in V(G) - \{x_1, x_2, y_1, y_2\}$ with $z \neq z'$. Then either $zz' \in E(G_1)$ or $zz' \in E(G_2)$. If $zz' \in E(G_i)$, then $(y_1, y_2, z; z')$ is a branchable quadruplet of G_i; if $zz' \in E(G_{3-i})$, then $(x_1, x_2, z; z')$ is a branchable quadruplet of G_{3-i}. In either case, we get a contradiction. Thus

(1) G_i has no edge x_1x_2 with $d_{G_i}(x_1) = d_{G_i}(x_2) = 2$ for each $i \in \{1,2\}$.

Suppose that for some $i \in \{1,2\}$, G_i has a vertex x of degree 2, and write $N_{G_i}(x) = \{y_1, y_2\}$. Since $N_{G_{3-i}}(x) = V(G) - \{x, y_1, y_2\}$, if there exists a vertex $z \in N_{G_{3-i}}(y_1) \cap N_{G_{3-i}}(y_2)$, then (x, z) is a branchable pair of G_{3-i}, which is a contradiction. Thus $N_{G_{3-i}}(y_1) \cap N_{G_{3-i}}(y_2) = \emptyset$, and hence $N_{G_i}(y_1) \cup N_{G_i}(y_2) \supseteq V(G) - \{y_1, y_2\}$. Since $d_{G_i}(y_j) \geq 3$ for each $j \in \{1,2\}$ by (1), if $y_1y_2 \in E(G_i)$, then (y_1, y_2) is a branchable pair of G_i, which is a contradiction. Thus $y_1y_2 \notin E(G_i)$, and so $y_1y_2 \in E(G_{3-i})$. For each $j \in \{1,2\}$, write $N_{G_{3-j}}(y_j) = \{y_{3-j}, z_1^{(j)}, \cdots z_{s_j}^{(j)}\}$, where $s_j = d_{G_{3-i}}(y_j) - 1$.

Subclaim 3.1. *For some $j \in \{1,2\}$, if $d_{G_{3-i}}(y_j) \geq 3$, then for each $s \in \{1,2\}$, $N_{G_{3-i}}(z_s^{(j)}) \subseteq \{x, y_j, z_{3-s}^{(j)}\}$.*

Proof. Suppose that $N_{G_{3-i}}(z_s^{(j)}) - \{x, y_j, z_{3-s}^{(j)}\} \neq \emptyset$, and let $w \in N_{G_{3-i}}(z_s^{(j)}) - \{x, y_j, z_{3-s}^{(j)}\}$. Since $N_{G_{3-i}}(y_1) \cap N_{G_{3-i}}(y_2) = \emptyset$, $w \neq y_{3-j}$. Since $N_{G_{3-i}}(x) = V(G) - \{x, y_1, y_2\}$ and $\{y_{3-j}, z_{3-s}^{(j)}\} \subseteq N_{G_{3-i}}(y_j)$, $N_{G_{3-i}}(x) \cup N_{G_{3-i}}(y_j) \supseteq V(G) - \{x, y_j, w\}$ and $|N_{G_{3-i}}(y_j) - \{x, z_s^{(j)}, w\}| \geq 2$. Hence $(x, y_j, z_s^{(j)}; w)$ is a branchable quadruplet of G_{3-i}, which is a contradiction. □

Since $y_1y_2 \in E(G_{3-i})$, $d_{G_{3-i}}(y_j) \geq 3$ for some $j \in \{1,2\}$ by (1). We may assume that $d_{G_{3-i}}(y_1) \geq 3$. Suppose that $d_{G_{3-i}}(y_2) = 2$. Then $N_{G_i}(y_2) = V(G) - \{y_1, y_2, z_1^{(2)}\}$. Since $|V(G)| \geq 8$, there exists a vertex $u \in V(G) - \{x, y_1, y_2, z_1^{(1)}, z_2^{(1)}, z_1^{(2)}\}$. Then by Subclaim 3.1, $y_2z_1^{(1)}, z_1^{(1)}u, z_1^{(1)}z_1^{(2)}, z_1^{(2)}y_1$,

$z_1^{(2)}z_2^{(1)} \in E(G_i)$. This implies that $(y_2, z_1^{(2)}, z_1^{(1)}; u)$ is a branchable quadruplet of G_i, which is a contradiction. Thus $d_{G_{3-i}}(y_2) \geq 3$. Then by Subclaim 3.1, $N_{G_i}(z_1^{(2)}) \supseteq V(G)-\{x, y_2, z_2^{(2)}\}$. Furthermore, $y_2x, y_2z_2^{(1)} \in E(G_i)$. This implies that $(y_2, z_1^{(2)}, z_1^{(1)}; z_2^{(2)})$ is a branchable quadruplet of G_i, which is a contradiction. □

By Claims 1 and 3, it suffices to show that $\mathrm{diam}(G_i) = 2$ for each $i \in \{1, 2\}$. Suppose that $\mathrm{diam}(G_i) \neq 2$ for some $i \in \{1, 2\}$. Since G_i has no vertex of degree $n-1$, $\mathrm{diam}(G_i) \neq 1$, and so $\mathrm{diam}(G_i) \geq 3$. Then there exist vertices $x, y \in V(G)$ with $d_{G_i}(x, y) = 3$. Note that $xy \in E(G_{3-i})$ and $N_{G_{3-i}}(x) \cup N_{G_{3-i}}(y) = V(G)$ because $N_{G_i}(x) \cap N_{G_i}(y) = \emptyset$. Since $\delta(G_{3-i}) \geq 3$ by Claim 3, (x, y) is a branchable pair of G_{3-i}, which is a contradiction. Consequently $\mathrm{diam}(G_i) = 2$ for each $i \in \{1, 2\}$.

This completes the proof of Lemma 3. □

Now we prove Theorem 2.

Proof of Theorem 2. By the definition of HISTs, if a graph is disconnected, then the graph has no HIST. Also, if a graph has a cutset each of whose vertices has degree 2, then the graph has no HIST. So, we obtain the following fact which guarantees the "only if" part of Theorem 2.

Fact 4. *Let G, n, G_1 and G_2 be as in Theorem 2. For some $i \in \{1, 2\}$, if either $G_i \simeq K_{2,n-2}$ or $G_i \simeq K_{2,n-2}^-$, then G has no monochromatic HIST.*

Thus it suffices to show the "if" part of Theorem 2. Let G, n, G_1 and G_2 as in Theorem 2, and assume that for each $i \in \{1, 2\}$, G_i is isomorphic to neither $K_{2,n-2}$ nor $K_{2,n-2}^-$. Suppose that G has no monochromatic HIST. Choose n (≥ 8) so that n is as small as possible. By Lemma 3, G_i is connected, $\delta(G_i) \geq 3$ and $\mathrm{diam}(G_i) = 2$. Furthermore, by Lemma 1, for each $i \in \{1, 2\}$, G_i has no branchable pair and no branchable quadruplet.

Claim 5. $n \geq 10$.

Proof. Suppose that $n \in \{8, 9\}$. We may assume that $\Delta(G_1) \geq \Delta(G_2)$. Let $x \in V(G)$ with $d_{G_1}(x) = \Delta(G_1)$. Since $d_{G_1}(x) + d_{G_2}(x) = n-1$ and $d_{G_2}(x) \geq 3$, one of the following holds;

- $d_{G_1}(x) = n-4$; or
- $n = 9$ and G_1 is 4-regular.

Write $N_{G_1}(x) = \{y_1, \cdots, y_l\}$, where $l = d_{G_1}(x)$, and write $V(G)-(N_{G_1}(x) \cup \{x\}) = \{z_1, \cdots, z_{n-l-1}\}$. Note that if $d_{G_1}(x) = n-4$, then $n-l-1 = 3$; if $n = 9$ and G_1 is 4-regular, then $n-l-1 = 4$. If a vertex $y \in N_{G_1}(x)$ is adjacent to all of $z_1, \cdots, z_{n-l-1}$ in G_1, then (x, y) is a branchable pair, which is a contradiction. Thus no vertex in $N_{G_1}(x)$ is adjacent to all of $z_1, \cdots, z_{n-l-1}$ in G_1.

We first suppose that a vertex in $N_{G_1}(x)$ is adjacent to $n-l-2$ of $z_1, \cdots, z_{n-l-1}$ in G_1. We may assume that $\{z_1, \cdots, z_{n-l-2}\} \subseteq N_{G_1}(y_1)$. For each i $(2 \leq i \leq l)$, the graph T_i on $V(G) - \{y_i, z_{n-l-1}\}$ with $E(T_i) = \{xy_j \mid j \neq i\} \cup \{y_1 z_j \mid 1 \leq j \leq n-l-2\}$ is a HIST of $G - \{y_i, z_{n-l-1}\}$. Hence by Lemma 2,

$$(2) \qquad N_{G_1}(z_{n-l-1}) \cap N_{G_1}(y_i) = \emptyset \text{ for each } i \ (2 \leq i \leq l).$$

Since $\mathrm{diam}(G_1) = 2$, this leads to $\{y_i \mid 2 \leq i \leq l\} \subseteq N_{G_1}(z_{n-l-1})$. Again by (2), $\{y_i \mid 2 \leq i \leq l\}$ is an independent set of G_1. If $y_1 y_i \in E(G)$ for some i $(2 \leq i \leq l)$, then $(y_1, z_{n-l-1}, y_i; x)$ is a branchable quadruplet of G_1, which is a contradiction. Hence $\{y_i \mid 1 \leq i \leq l\}$ is an independent set of G_1. Since $d_{G_1}(y_1, z_{n-l-1}) \leq 2$ and $y_1 z_{n-l-1} \notin E(G_1)$, z_{n-l-1} is adjacent to one of $z_1, \cdots, z_{n-l-2}$ in G_1. We may assume that $z_1 z_{n-l-1} \in E(G_1)$. By (2), $y_i z_1 \notin E(G_1)$ for every i $(2 \leq i \leq l)$. Since $\{y_i \mid 1 \leq i \leq l\}$ is an independent set of G_1, $N_{G_1}(y_i) \subseteq \{x\} \cup \{z_j \mid 2 \leq j \leq n-l-1\}$ for each i $(2 \leq i \leq l)$. Suppose that $d_{G_1}(x) = n-4$ (i.e., $n-l-1 = 3$). Since $d_{G_1}(y_2) \geq 3$, this forces $N_{G_1}(y_2) = \{x, z_2, z_3 (= z_{n-l-1})\}$. Then $(x, z_3, y_2; z_2)$ is a branchable quadruplet of G_1, which is a contradiction. Thus $n = 9$ and G_1 is 4-regular (i.e., $n-l-2 = 4$). Then $N_{G_1}(y_i) = \{x, z_2, z_3, z_4 (= z_{n-l-1})\}$ for each i $(2 \leq i \leq l)$. This forces $N_{G_1}(z_2) = N_{G_1}(z_3) = N_{G_1}(x)$, and hence $N_{G_1}(z_1) = \{y_1, z_4\}$, which contradicts the 4-regularity of G. Therefore

(3)

every vertex in $N_{G_1}(x)$ is adjacent to at most $n-l-3$ of $z_1, \cdots, z_{n-l-1}$ in G_1.

Case 1: $d_{G_1}(x) = n-4$ (i.e., $n-l-1 = 3$).

By (3), the number of edges of G_1 between $N_{G_1}(x)$ and $\{z_1, z_2, z_3\}$ is at most l (≤ 5). This together with the fact that $d_{G_1}(z_1) + d_{G_1}(z_2) + d_{G_1}(z_3) \geq 9$ implies that the subgraph of G_1 induced by $\{z_1, z_2, z_3\}$ has at least two edges. We may assume that $z_1 z_2, z_1 z_3 \in E(G_1)$. Since $\mathrm{diam}(G_1) = 2$, $N_{G_1}(z_1) \cap N_{G_1}(x) \neq \emptyset$. We may assume that $z_1 y_1 \in E(G_1)$. Since $d_{G_1}(y_1) \geq 3$, $N_{G_1}(y_1) \cap N_{G_1}(x) \neq \emptyset$ by (3). We may assume that $y_1 y_2 \in E(G_1)$. Then $(x, z_1, y_1; y_2)$ is a branchable quadruplet of G_1, which is a contradiction.

Case 2: $n = 9$ and G_1 is 4-regular (i.e., $n-l-1 = 4$).

Suppose that a vertex in $N_{G_1}(x)$ is adjacent to two of $z_1, \cdots, z_4$ in G_1. We may assume that $y_1 z_1, y_1 z_2 \in E(G_1)$. For each $i \in \{3, 4\}$, since G_1 is 4-regular and $z_i y_1 \notin E(G_1)$ by (3), $|N_{G_1}(z_i) \cap \{y_2, y_3, y_4, z_1, z_2\}| \geq 3$.

In particular, $N_{G_1}(z_3) \cap N_{G_1}(z_4) \neq \emptyset$. On the other hand, the graph T on $V(G) - \{z_3, z_4\}$ with $E(T) = \{xy_i \mid 1 \leq i \leq 4\} \cup \{y_1z_1, y_1z_2\}$ is a HIST of $G_1 - \{z_3, z_4\}$. Hence by Lemma 2, G also has a HIST, which is a contradiction. Thus every vertex in $N_{G_1}(x)$ is adjacent to at most one of $z_1, \cdots, z_4$ in G_1. Since $\mathrm{diam}(G_1) = 2$, $N_{G_1}(z_i) \cap N_{G_1}(x) \neq \emptyset$ for each i $(1 \leq i \leq 4)$. We may assume that $y_iz_i \in E(G_1)$ for each i $(1 \leq i \leq 4)$. Since G_1 is 4-regular, this implies that $N_{G_1}(z_1) = \{y_1, z_2, z_3, z_4\}$ and $N_{G_1}(y_1) \cap N_{G_1}(x) \neq \emptyset$. We may assume that $y_1y_2 \in E(G_1)$. Then $(x, z_1, y_1; y_2)$ is a branchable quadruplet of G_1, which contradicts Lemma 1.

This completes the proof of Claim 5. □

Claim 6. *There exist two distinct vertices $x, y \in V(G)$ with $N_{G_1}(x) \cap N_{G_1}(y) \neq \emptyset$ and $N_{G_2}(x) \cap N_{G_2}(y) \neq \emptyset$.*

Proof. Let p and q be vertices of G with $p \neq q$. Without loss of generality, we may assume that $pq \in E(G_1)$. Since $\mathrm{diam}(G_2) = 2$ and $pq \notin E(G_2)$, $N_{G_2}(p) \cap N_{G_2}(q) \neq \emptyset$. If $N_{G_1}(p) \cap N_{G_1}(q) \neq \emptyset$, then the desired conclusion holds. Thus we may assume that $N_{G_1}(p) \cap N_{G_1}(q) = \emptyset$. Since $\delta(G_1) \geq 3$, there exist two distinct vertices $u, v \in N_{G_1}(p) - \{q\}$. Note that $p \in N_{G_1}(u) \cap N_{G_1}(v)$, and so $N_{G_1}(u) \cap N_{G_1}(v) \neq \emptyset$. Since $N_{G_1}(p) \cap N_{G_1}(q) = \emptyset$, $u, v \notin N_{G_1}(q)$, and hence $u, v \in N_{G_2}(q)$. Therefore $N_{G_2}(u) \cap N_{G_2}(v) \neq \emptyset$. □

Let $x, y \in V(G)$ be vertices assured in Claim 6. Let G' be the 2-edge-colored complete graph obtained from G by deleting x and y, and for each $i \in \{1, 2\}$, let G'_i be the spanning subgraph of G' induced by all edges of color i. Note that $|V(G')| = n-2 \geq 8$ by Claim 5. If G' has a monochromatic HIST, then G also has a monochromatic HIST by Lemma 2. Thus we may assume that G' has no monochromatic HIST. Then by the minimality of n, for some $i \in \{1, 2\}$, either $G'_i \simeq K_{2,n-4}$ or $G'_i \simeq K^-_{2,n-4}$.

Case 1: $G'_i \simeq K_{2,n-4}$.

Let A and B be the bipartition of G'_i with $|A| = 2$ and $|B| = n - 4$. Suppose that $N_{G_i}(x) \cap N_{G_i}(y) \cap B \neq \emptyset$. Let $a \in A$ and $b \in N_{G_i}(x) \cap N_{G_i}(y) \cap B$. Then (a, b) is a branchable pair of G_i, which is a contradiction. Thus $N_{G_i}(x) \cap N_{G_i}(y) \cap B = \emptyset$. Since $N_{G_i}(x) \cap N_{G_i}(y) \neq \emptyset$, there exists a vertex $a' \in A$ with $a'x, a'y \in E(G_i)$. Let $b' \in B$. Since $\delta(G_i) \geq 3$, b' is adjacent to at least one of x and y. Then (a', b') is a branchable pair of G_i, which is a contradiction.

Case 2: $G'_i \simeq K^-_{2,n-4}$.

Note that $G'_{3-i} \simeq Z_{n-2}$. Let a be the unique vertex of G'_{3-i} with $d_{G'_{3-i}}(a) = 1$. Write $N_{G'_{3-i}}(a) = \{b\}$, and write $N_{G'_{3-i}}(b) = \{a, c\}$. Note that $d_{G'_{3-i}}(c) = n-4$. Since $\delta(G_{3-i}) \geq 3$, $xa, ya \in E(G_{3-i})$ and b is adjacent to at least one of x and y in G_{3-i}. We may assume that $xb \in E(G_{3-i})$. If either $yb \in E(G_{3-i})$ or $yc \in E(G_{3-i})$, then (b, c) is a branchable pair of G_{3-i}, which is a contradiction. Thus $yb, yc \notin E(G_{3-i})$. Since $\delta(G_{3-i}) \geq 3$, y is adjacent to a vertex $u \in V(G) - \{a, b, c, x, y\}$. Let $v \in V(G) - \{a, b, c, x, y, u\}$. Note that $N_{G_{3-i}}(u) \supseteq V(G) - \{a, b, x\}$ and $cv \in E(G_{3-i})$. Then we can check that $(b, u, c; v)$ is a branchable quadruplet of G_{3-i}, which is a contradiction.

This completes the proof of Theorem 2. □

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