Journal of Combinatorics
Volume 9, Number 4, 619–629, 2018

Rainbow arithmetic progressions in finite abelian groups

Michael Young

For positive integers n and k, the *anti-van der Waerden number* of $\mathbb{Z}_n$, denoted by $\mathrm{aw}(\mathbb{Z}_n, k)$, is the minimum number of colors needed to color the elements of the cyclic group of order n and guarantee there is a rainbow arithmetic progression of length k. Butler et al. showed a reduction formula for $\mathrm{aw}(\mathbb{Z}_n, 3)$ in terms of the prime divisors of n. In this paper, we analagously define the anti-van der Waerden number of a finite abelian group G and show $\mathrm{aw}(G, 3)$ is determined by the order of G and the number of groups with even order in a direct sum isomorphic to G. The *unitary anti-van der Waerden number* of a group is also defined and determined.

1. Introduction

Let G be a finite additive abelian group. A *k-term arithmetic progression* (k-AP) of G is a sequence of the form

$$a, a+d, a+2d, \dots, a+(k-1)d,$$

where $a, d \in G$. For the purposes of this paper, an arithmetic progression is referred to as a set of the form $\{a, a+d, a+2d, \dots, a+(k-1)d\}$. A k-AP is *non-degenerate* if the arithmetic progression contains k distinct elements; otherwise, the arithmetic progression is *degenerate*.

An *r-coloring* of G is a function $c : G \to [r]$, where $[r] := \{1, \dots, r\}$. An r-coloring is *exact* if c is surjective. Given $c : G \to [r]$, an arithmetic progression is called *rainbow* (under c) if $c(a+id) \neq c(a+jd)$ for all $0 \leq i < j \leq k-1$. Given $P \subseteq G$, $c(P)$ denotes the set of colors assigned to the elements of P, i.e. $c(P) = \{c(i) : i \in P\}$.

The *anti-van der Waerden number* $\mathrm{aw}(G, k)$ is the smallest r such that every exact r-coloring of G contains a rainbow k-term arithmetic progression. If G contains no k-AP, then $\mathrm{aw}(G, k) = |G| + 1$ to be consistent with

arXiv: 1603.08153

the property that there is a coloring with $\mathrm{aw}(G,k)-1$ colors that has no rainbow k-AP.

Throughout the paper, $\mathbb{Z}_n$ will denote the cyclic group of order n consisting of the set $\{0,1,\ldots,n-1\}$ under the operation of addition modulo n. Define the direct product $[\mathbb{Z}_n]^s := \underbrace{\mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n}_{s \text{ times}}$.

Jungić, Licht, Mahdian, Nešetril, and Radoičić established several results on the existence of rainbow 3-APs in [4]. Jungić et al. proved that every 3-coloring of $\mathbb{N}$, where each color class has density at least $1/6$, contains a rainbow 3-AP. They also prove results about rainbow 3-APs in $\mathbb{Z}_n$. Other results on colorings of the integers with no rainbow 3-APs have been obtained in [1] and [2].

Anti-van der Waerden numbers were first defined by Uherka in a preliminary study (see [5]). Butler et. al., in [3], proved upper and lower bounds for anti-van der Waerden numbers of $[n]$ and $\mathbb{Z}_n$ for k-APs, for $3 \le k$.

Many of the extremal colorings that are constructed to prove lower bounds of $\mathrm{aw}(G,3)$ require colorings that use some color exactly once, which leads to the need of the following definitions.

An r-coloring of G is *unitary* if there is an element of G that is uniquely colored, which will be referred to as a *unitary color*. (A unitary coloring is referred to as a singleton coloring in [3].) The smallest r such that every exact r-coloring of G that is unitary contains a rainbow k-term arithmetic progression is denoted by $\mathrm{aw}_u(G,k)$. Similar to the anti-van der Waerden number, $\mathrm{aw}_u(G,k) = |G|+1$ if G has no k-AP.

Butler et al. use Proposition 1 to determine the exact value of $\mathrm{aw}(\mathbb{Z}_n,3)$.

Proposition 1. *[3, Proposition 3.5], For every prime number p,*

$$3 \le \mathrm{aw}_u(\mathbb{Z}_p,3) = \mathrm{aw}(\mathbb{Z}_p,3) \le 4.$$

Let $n = 2^{e_0}p_1^{e_1}p_2^{e_2}\cdots p_s^{e_s}$ such that p_j is prime and $0 \le e_j$ for $0 \le j \le s$, $\mathrm{aw}(\mathbb{Z}_{p_j},3) = 3$ for $1 \le j \le \ell$, and $\mathrm{aw}(\mathbb{Z}_{p_j},3) = 4$ for $\ell+1 \le j \le s$. Then Corollary 3.15 in [3] can be stated as follows:

Theorem 1. *[3, Corollary 3.15] For any integer $n \ge 2$,*

$$\mathrm{aw}(\mathbb{Z}_n,3) = \begin{cases} 2 + \sum_{j=1}^{\ell} e_j + \sum_{j=\ell+1}^{s} 2e_j & \textit{if } e_0 = 0, \\ 3 + \sum_{j=1}^{\ell} e_j + \sum_{j=\ell+1}^{s} 2e_j & \textit{if } 1 \le e_0. \end{cases}$$

In this paper, Theorem 1 is extended to all finite abelian groups while the following special case is generalized to finite abelian groups with order that is a power of 2.

Theorem 2. *[4, Theorem 3.5] For all positive integers m,*

$$aw(\mathbb{Z}_{2^m}, 3) = 3.$$

In Section 2, a closed formula for $\mathrm{aw}(G, 3)$ is given. This closed formula is determined by the order of G and the number of groups with even order in a direct sum isomorphic to G. In Section 3, a similar closed formula for $\mathrm{aw}_u(G, 3)$ is given.

2. Anti-van der Waerden numbers

In this section, a reduction formula for the anti-van der Waerden number of groups that have odd order is created. Determining the anti-van der Waerden number of an abelian group with odd order is equivalent to determining the anti-van der Waerden number of $\mathbb{Z}_m \times \mathbb{Z}_n$ for some positive odd integers m and n. First we provide a proof of a useful remark from [3].

Proposition 2. *[4, Remark 3.16] For all positive integers n,*

$$\mathrm{aw}_u(\mathbb{Z}_n, 3) = \mathrm{aw}(\mathbb{Z}_n, 3).$$

Proof. It is obvious that $\mathrm{aw}_u(\mathbb{Z}_n, 3) \leq \mathrm{aw}(\mathbb{Z}_n, 3)$. The inequality $\mathrm{aw}(\mathbb{Z}_n, 3) \leq \mathrm{aw}_u(\mathbb{Z}_n, 3)$ will be shown by induction on the number of odd prime divisors of n. It is obviously true if n is a power of 2. Assume n is not a power of 2.

Let $\mathbb{Z}_n = G \times \mathbb{Z}_p$, where G is a finite cyclic group and p be an odd prime. Let c_G be a unitary coloring of G with exactly $\mathrm{aw}_u(G, 3) - 1$ colors and no rainbow 3-AP, and c_p be a unitary coloring of $\mathbb{Z}_p$ with exactly $\mathrm{aw}_u(\mathbb{Z}_p, 3) - 1$ different colors. Without loss of generality, let 0 be uniquely colored by c_G and c_p. For each $(g, h) \in G \times \mathbb{Z}_p$, define c as follows:

$$c(g, h) = \begin{cases} c_G(g) & \text{if } h = 0, \\ c_p(h) & \text{if } h \neq 0. \end{cases}$$

Let $\{(a_1, a_2), (a_1 + d_1, a_2 + d_2), (a_1 + 2d_1, a_2 + 2d_2)\}$ be a 3-AP of $G \times \mathbb{Z}_p$. Since p is odd, $\{a_2, a_2 + d_2, a_2 + 2d_2\}$ is a non-degenerate 3-AP in $\mathbb{Z}_p$. Therefore, 0, 1, or 3 elements of $\{(a_1, a_2), (a_1+d_1, a_2+d_2), (a_1+2d_1, a_2+2d_2)\}$ will be assigned a color by c_G.

If $a_2 = 0$ and $d_2 = 0$, then the 3-AP is colored by c_G and is not rainbow. If $a_2 \neq 0$ and $d_2 = 0$, then all the elements of the 3-AP are colored with the same color. If $d_2 \neq 0$, then the 3-AP is colored by c_p (since $c_p(0)$ is a unitary color) and is not rainbow. Therefore no 3-AP in $G \times \mathbb{Z}_p$ is rainbow under c.

The color $c(0,0)$ is unique; therefore, c is a unitary coloring of $G \times \mathbb{Z}_p$. So,

$$\begin{aligned} \mathrm{aw}_u(G \times \mathbb{Z}_p, 3) - 1 &\geq |c(G \times \mathbb{Z}_p)| \\ &= \mathrm{aw}_u(G,3) + \mathrm{aw}_u(\mathbb{Z}_p, 3) - 3 \\ \text{(by induction hypothesis)} &= \mathrm{aw}(G,3) + \mathrm{aw}(\mathbb{Z}_p, 3) - 3 \\ \text{(by Theorem 1)} &= \mathrm{aw}(G \times \mathbb{Z}_p, 3) - 1. \end{aligned}$$

Therefore, $\mathrm{aw}_u(\mathbb{Z}_n, 3) \geq \mathrm{aw}(\mathbb{Z}_n, 3)$. □

Now a coloring with no rainbow 3-APs is constructed to determine a lower bound.

Proposition 3. *For all positive integers n,*

$$\mathrm{aw}(G,3) + \mathrm{aw}(\mathbb{Z}_n, 3) - 2 \leq \mathrm{aw}(G \times \mathbb{Z}_n, 3).$$

Proof. It suffices to show that $\mathrm{aw}(G,3) + \mathrm{aw}_u(\mathbb{Z}_n,3) - 2 \leq \mathrm{aw}(G \times \mathbb{Z}_n, 3)$. For each $g \in G$, let $P_g = \{(g,h) : h \in \mathbb{Z}_n\}$. Let c_G be a coloring of G with $\mathrm{aw}(G,3) - 1$ colors with no rainbow 3-AP and c_n be a unitary coloring of $\mathbb{Z}_n$ with $\mathrm{aw}_u(\mathbb{Z}_n, 3) - 1$ colors with no rainbow 3-AP. Without loss of generality, assume that 0 is an element of $\mathbb{Z}_n$ that is uniquely colored by c_n.

Now define a coloring of $G \times \mathbb{Z}_n$ with $aw(G,3) + \mathrm{aw}_u(\mathbb{Z}_n, 3) - 2$ colors as follows:

$$c(g,h) = \begin{cases} c_G(g) & \text{if } h \neq 0, \\ c_n(h) & \text{if } h = 0. \end{cases}$$

Under the coloring c, there can be no rainbow 3-AP in any P_g. Since n is odd, every other 3-AP must contain an element from P_a, P_{a+d}, and P_{a+2d} for some $a, d \in \mathbb{Z}_n$. However, such a 3-AP is not rainbow because $\{a, a+d, a+2d\}$ is not a rainbow 3-AP under c_G. □

The main tool used for determining the anti-van der Waerden number of abelian groups with odd order is applying Lemma 1 to create a well-defined auxiliary coloring of a specific subgroup.

Let G be a group and n be an odd positive integer. Partition $G \times \mathbb{Z}_n$ by letting $P_g = \{(g,x) | x \in \mathbb{Z}_n\}$ for each $g \in G$. Without loss of generality, let $|c(P_g)| \leq |c(P_0)|$ for all $g \in G$.

Since n is odd, 2 has a unique multiplicative inverse in $\mathbb{Z}_n$. Therefore, for every $x \in \mathbb{Z}_n$ there exists a $d \in \mathbb{Z}_n$ such that $x = 2d$. So given an AP in G, say $\{x_1, y_1, z_1\}$, and $x_2, z_2 \in \mathbb{Z}_n$, there exists a unique $y_2 \in \mathbb{Z}_n$ such that $x_2 + z_2 = 2y_2$, which yields $\{(x_1, x_2), (y_1, y_2), (z_1, z_2)\}$, a 3-AP in $G \times \mathbb{Z}_n$.

Lemma 1. *If c is a coloring of $G \times \mathbb{Z}_n$ with no rainbow 3-AP, then $|c(P_g) \setminus c(P_0)| \leq 1$ for all $g \in G$.*

Proof. Assume there is a $g \in G$ such that $2 \leq |c(P_g) \setminus c(P_0)|$. Let $\alpha, \beta \in c(P_g) \setminus c(P_0)$ and $\gamma, \rho \in c(P_0) \setminus c(P_g)$. By maximality of $c(P_0)$, γ and ρ exists, and neither are equal to α or β.

If there exists a $z \in P_{2g}$ such that $c(z)$ is not a color in $c(P_0)$, then there is a $y \in P_g$ such that $c(y) \in \{\alpha, \beta\}$ and $c(y) \neq c(z)$. Therefore, there is an $x \in P_0$ such that $\{x, y, z\}$ is a 3-AP in $G \times \mathbb{Z}_n$ and x does not have the same color as y or z. This is a contradiction since this arithmetic progression is rainbow.

If there exists a $z \in P_{2g}$ such that $c(z)$ is not a color in $c(P_g)$ and $g \neq |G|/2$, then there is an $x \in P_0$ such that $c(x) \in \{\gamma, \rho\}$ and $c(x) \neq c(z)$. Therefore, there is a $y \in P_0$ such that $\{x, y, z\}$ is a 3-AP in $G \times \mathbb{Z}_n$ and y does not have the same color as x or z. This is a contradiction since this arithmetic progression is rainbow.

Therefore, P_{2g} must contain every color in $c(P_0)$ and $c(P_g)$, which is a contradiction to the maximality of $c(P_0)$. □

If $|c(P_g) \setminus c(P_0)| \leq 1$ for all $g \in G$, then the following auxiliary coloring of G is well defined:

$$\overline{c}(g) = \begin{cases} \alpha & \text{if } c(P_g) \subset c(P_0), \\ c(P_g) \setminus c(P_0) & \text{otherwise.} \end{cases}$$

The next lemma goes on to show that if $G \times \mathbb{Z}_n$ does not contain a rainbow 3-AP, then $\overline{c}$ can not create a rainbow 3-AP in G.

Lemma 2. *If $\overline{c}$ contains a rainbow 3-AP in G, then there exists a rainbow 3-AP in $G \times \mathbb{Z}_n$.*

Proof. Let $\{a, a+d, a+2d\}$ be a rainbow arithmetic progression colored by $\overline{c}$ in G. Without loss of generality, there are two cases to consider: $\overline{c}(a+d) \neq \alpha$ and $\overline{c}(a+d) = \alpha$.

If $\overline{c}(a+d) = \beta$ and $\overline{c}(a+2d) = \gamma$, then there exists an $x \in P_a$, $y \in P_{a+d}$, and $z \in P_{a+2d}$ such that $\{x, y, z\}$ is a 3-AP in $G \times \mathbb{Z}_n$, $c(y) = \beta$, and $c(z) = \gamma$. However, $\overline{c}(a) \neq \beta, \gamma$, which implies $\beta, \gamma \notin P_a$, so $c(x) \neq \beta, \gamma$. This implies that $\{x, y, z\}$ is a rainbow arithmetic progression in $G \times \mathbb{Z}_n$, which is a contradiction.

If $\overline{c}(a) = \beta$, $\overline{c}(a+d) = \alpha$, and $\overline{c}(a+2d) = \gamma$, then there exists an $x \in P_a$, $y \in P_{a+d}$, and $z \in P_{a+2d}$ such that $\{x, y, z\}$ is a 3-AP in $G \times \mathbb{Z}_n$, $c(x) = \beta$ and $c(z) = \gamma$. However, $c(y) \neq \beta, \gamma$ because $\overline{c}(a+d) = \alpha$. This implies that $\{x, y, z\}$ is a rainbow arithmetic progression in $G \times \mathbb{Z}_n$, which is a contradiction. □

Theorem 3. *If G is a finite abelian group and n is an odd positive integer, then*

$$\mathrm{aw}(G \times \mathbb{Z}_n, 3) = \mathrm{aw}(G, 3) + \mathrm{aw}(\mathbb{Z}_n, 3) - 2.$$

Proof. The lower bound for $\mathrm{aw}(G \times \mathbb{Z}_n, 3)$ is by Proposition 3. For the upper bound, it suffices to show $\mathrm{aw}(G \times \mathbb{Z}_n, 3) \leq \mathrm{aw}(G, 3) + \mathrm{aw}_u(\mathbb{Z}_n, 3) - 2$. Let c be a coloring of $G \times \mathbb{Z}_n$ with $\mathrm{aw}(G, 3) + \mathrm{aw}_u(\mathbb{Z}_n, 3) - 2$ colors and no rainbow 3-APs. The statement will be proved by contradiction, showing that no such coloring exists.

For each $g \in G$, let $P_g = \{(g, h) : h \in \mathbb{Z}_n\}$. Without loss of generality, let $|c(P_g)| \leq |c(P_0)|$ for all $g \in G$. Since there are no rainbow 3-APs and P_0 is isomorphic to $\mathbb{Z}_n$, $|c(P_0)| \leq \mathrm{aw}(\mathbb{Z}_n) - 1$. Also, by Lemma 1, $|c(P_g) \backslash c(P_0)| \leq 1$, for all $g \in G$. Define a coloring of G as follows:

$$\overline{c}(g) = \begin{cases} \alpha & \text{if } c(P_g) \subset c(P_0), \\ c(P_g) \backslash c(P_0) & \text{otherwise.} \end{cases}$$

The total number of colors used by c is $|c(P_0)| + |\overline{c}(G)| - 1 \leq (\mathrm{aw}(\mathbb{Z}_n, 3) - 1) + (\mathrm{aw}(G, 3) - 1) - 1$. Therefore $|c(P_0)| \leq \mathrm{aw}(\mathbb{Z}_n, 3)$ or $|\overline{c}(G)| \leq \mathrm{aw}(G, 3)$ □

This leads to the following corollary which implies that for positive odd integers m and n, $\mathrm{aw}(\mathbb{Z}_m \times \mathbb{Z}_n, 3) = \mathrm{aw}(\mathbb{Z}_{mn}, 3)$.

Corollary 1. *Let n be the largest odd divisor of the order of G. There exists a finite abelian group G' such that the order of G' is a power of 2 and*

$$\mathrm{aw}(G, 3) = \mathrm{aw}(G', 3) + \mathrm{aw}(\mathbb{Z}_n, 3) - 2.$$

Proof. For each odd prime p and positive integer e, $\mathrm{aw}(\mathbb{Z}_{p^e}, 3) = (\mathrm{aw}(\mathbb{Z}_p, 3) - 2)e + 2$, by Theorem 1. Theorem 3 implies $\mathrm{aw}(\mathbb{Z}_{p^{e_1}} \times \mathbb{Z}_{p^{e_2}} \cdots \mathbb{Z}_{p^{e_\ell}}, 3) = 2 + (\mathrm{aw}(\mathbb{Z}_p, 3) - 2) \sum_{i=1}^{\ell} e_i$. So the anti-van der Waerden number is the same for any two finite abelian groups having the same odd order.

Now let $G = G' \times \mathbb{Z}_n$ and $n = \prod_{i=1}^{\ell} p_i^{e_i}$, where p_i is an odd prime for all i, where $1 \leq i \leq \ell$. Then

$$\begin{aligned}\mathrm{aw}(G,3) &= \mathrm{aw}(G' \times \mathbb{Z}_n, 3)\\ &= \mathrm{aw}(G',3) + \sum_{i=1}^{\ell} e_i(\mathrm{aw}(\mathbb{Z}_{p_i},3) - 2)\\ &= \mathrm{aw}(G',3) + \mathrm{aw}(\mathbb{Z}_n,3) - 2.\end{aligned}$$ □

2.1. Groups with power of 2 order

In order to completely use Corollary 1 the anti-van der Waerden number of groups with order that is a power of 2 must be determined.

Proposition 4. *For any finite abelian group G,*

$$\mathrm{aw}(G \times \mathbb{Z}_2, 3) \leq 2\,\mathrm{aw}(G,3) - 1.$$

Proof. Let $A = \{(g,0) : g \in G\}$ and $B = \{(g,1) : g \in G\}$. In any exact $(2\,\mathrm{aw}(G,3)-1)$-coloring of $G \times \mathbb{Z}_2$, either A or B will have at least $\mathrm{aw}(G,3)$ colors. Therefore, a rainbow 3-AP will exist since A and B are both isomorphic to G. □

An inductive argument, using Proposition 4 as the base case, gives the following corollary.

Corollary 2. *For all positive integers s,*

$$\mathrm{aw}([\mathbb{Z}_2]^s, 3) \leq 2^s + 1.$$

Theorem 4. *For $1 \leq i \leq s$, let m_i be a positive integer. Then*

$$aw(\mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_s}}, 3) = 2^s + 1.$$

Proof. Let $m_1 \leq m_2 \leq \ldots \leq m_s$ and $x = (x_1, x_2, \ldots, x_s) \in \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_s}}$. Define $c(x) = (x_1, x_2, \ldots, x_s) \mod 2$. The function c is an exact 2^s-coloring of $\mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_s}}$. Since $c(x) = c(x+2d)$, for any $d \in \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_s}}$, this coloring does not contain any rainbow arithmetic progressions. Therefore, $2^s + 1 \leq \mathrm{aw}(\mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_s}}, 3)$.

The proof of the upper bound is inductive on (s, m_s). The base case of $(1, m)$ is true for all positive integers m by Theorem 2 and the base case of $(s, 1)$ is true for all positive integers s by Corollary 2. Assume the statement is true for parameters (s', m) for all $1 \leq s' < s$ and $1 \leq m < m_s$.

It will be shown that the statement is true for parameters (s, m_s) by assuming there exists a coloring of $\mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_s}}$ with exactly $2^s + 1$ colors and no rainbow 3-AP, then arriving at a contradiction.

For each $i \in \mathbb{Z}_{2^{m_s}}$, let $P_i = \{(x, i) : x \in \mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_{(s-1)}}}\}$. So P_i is isomorphic to $\mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_{(s-1)}}}$ for all i. Let A be the set of P_i with i even, and B be the set of P_i with i odd. So A and B are both isomorphic to $\mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{(m_s-1)}}$. By the induction hypothesis, A and B both have at most 2^s colors. So there exists $\alpha \in c(A)\backslash c(B)$ and $\beta \in c(B)\backslash c(A)$.

Assume without loss of generality, $x_\alpha := c^{-1}(\alpha) \in P_0$ and $x_\beta := c^{-1}(\beta) \in P_j$, where j is odd. Then $\{x_\alpha, x_\beta, 2x_\beta - x_\alpha\}$ is a 3-AP in $\mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_s}}$. So $2x_\beta - x_\alpha \in P_{2j}$. Since there are no rainbow 3-APs $c(2x_\beta - x_\alpha)$ must be α.

Similarly, $\{(i-1)x_\beta - (i-2)x_\alpha, ix_\beta - (i-1)x_\alpha, (i+1)x_\beta - ix_\alpha\}$ is a 3-AP for all i and $c((i-1)x_\beta - (i-2)x_\alpha)$ must be equal to $c((i+1)x_\beta - ix_\alpha)$ if it is not rainbow. This implies that $\alpha \in c(P_i)$ for all even i, and $\beta \in c(P_i)$, for all odd i.

By the induction hypothesis, P_i has at most 2^{s-1} colors for all i. Therefore, $|c(P_0) \cup c(P_j)| \leq 2^s$. So there exists a color γ that is not in $c(P_0)$ or $c(P_j)$. Now define an exact 3-coloring of $\mathbb{Z}_{2^{m_s}}$ as follows:

$$\overline{c}(i) = \begin{cases} \alpha & \text{if } \alpha \in c(P_i) \text{ and } \gamma \notin c(P_i), \\ \beta & \text{if } \beta \in c(P_i) \text{ and } \gamma \notin c(P_i), \\ \gamma & \text{if } \gamma \in c(P_i). \end{cases}$$

The coloring $\overline{c}$ is an exact 3-coloring and creates a rainbow 3-AP in $\mathbb{Z}_{2^{m_s}}$ by Theorem 2. Let $\{a, a+d, a+2d\}$ be such a rainbow arithmetic progression. Without loss of generality, there are two cases to consider: $\overline{c}(a+d) \neq \gamma$ and $\overline{c}(a+d) = \gamma$.

If $\overline{c}(a) = \alpha$, $\overline{c}(a+d) = \gamma$, and $\overline{c}(a+2d) = \beta$, then a must be even and $a + 2d$ must be odd, which is a contradiction.

If $\overline{c}(a) = \alpha$, $\overline{c}(a+d) = \beta$, and $\overline{c}(a+2d) = \gamma$, then there exists an $x \in P_a$, $y \in P_{a+d}$, and $z \in P_{a+2d}$ such that $\{x, y, z\}$ is a 3-AP in $\mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_s}}$, $c(y) = \beta$ and $c(z) = \gamma$. However, $c(x) \neq \beta$ or γ because $\overline{c}(a) = \alpha$. This implies that $\{x, y, z\}$ is a rainbow arithmetic progression in $\mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_s}}$, which is a contradiction.

Therefore, $aw(\mathbb{Z}_{2^{m_1}} \times \mathbb{Z}_{2^{m_2}} \times \cdots \times \mathbb{Z}_{2^{m_s}}, 3) \leq 2^s + 1$. ☐

3. Unitary anti-van der Waerden numbers

Proposition 5. *For all positive integers p and q,*

$$\mathrm{aw}_u(\mathbb{Z}_p, 3) + \mathrm{aw}_u(\mathbb{Z}_q, 3) - 2 \leq \mathrm{aw}_u(\mathbb{Z}_p \times \mathbb{Z}_q, 3).$$

Proof. This is the same as the proof of Proposition 3 with $\overline{c}_{\mathbb{Z}_p}$ changed to a unitary coloring of $\mathbb{Z}_p$ with $\mathrm{aw}_u(\mathbb{Z}_p, 3) - 1$ colors and no rainbow 3-AP. □

Theorem 5. *For any positive odd integers n,*

$$\mathrm{aw}_u(G \times \mathbb{Z}_n, 3) = \mathrm{aw}_u(G, 3) + \mathrm{aw}_u(\mathbb{Z}_n, 3) - 2.$$

Proof. The lower bound is a direct result of Proposition 5. So it suffices to show the upper bound. Assume c is a coloring of $G \times \mathbb{Z}_n$ that is unitary with exactly $\mathrm{aw}_u(G, 3) + \mathrm{aw}_u(\mathbb{Z}_n, 3) - 2$ colors and no rainbow 3-AP. For all $h \in \mathbb{Z}_n$, let $P_h = \{(g, h) \mid g \in G\}$. Without loss of generality, let $|c(P_h)| \leq |c(P_0)|$ for all $h \in \mathbb{Z}_n$.

By Lemma 1, $|c(P_h) \backslash c(P_0)| \leq 1$, for all $h \in \mathbb{Z}_n$. Define a coloring of $\mathbb{Z}_n$ as follows:

$$\overline{c}(g) = \begin{cases} \alpha & \text{if } c(P_h) \subset c(P_0), \\ c(P_h) \backslash c(P_0) & \text{otherwise.} \end{cases}$$

Let ρ be a color used exactly once by c to color $G \times \mathbb{Z}_n$. Now consider the two cases in which $\rho \in P_0$ and $\rho \notin P_0$.

Case 1: If $\rho \in c(P_0)$, then $|c(P_0)| \leq \mathrm{aw}_u(G) - 1$. Therefore $\mathrm{aw}_u(\mathbb{Z}_n, 3) = (\mathrm{aw}_u(G, 3) + \mathrm{aw}_u(\mathbb{Z}_n, 3) - 2) - (\mathrm{aw}_u(G) - 1) + 1 \leq |\overline{c}(\mathbb{Z}_n)|$. Since $\mathrm{aw}(\mathbb{Z}_n, 3) = \mathrm{aw}_u(\mathbb{Z}_n, 3)$, Lemma 2 implies that c creates a rainbow 3-AP.

Case 2: If $\rho \in c(P_d)$, where $0 \neq d$, then $\overline{c}$ must be a unitary coloring of $\mathbb{Z}_n$ and not have any 3-APs by Lemma 2. So $|\overline{c}(\mathbb{Z}_n)| \leq \mathrm{aw}_u(\mathbb{Z}_n, 3) - 1$, which implies $\mathrm{aw}_u(G, 3) = (\mathrm{aw}_u(G, 3) + \mathrm{aw}_u(\mathbb{Z}_n, 3) - 2) - (\mathrm{aw}_u(\mathbb{Z}_n) - 2) \leq |c(P_0)|$.

If there exists $\gamma \in c(P_0) \setminus c(P_{-d})$, then there is an $x \in G$ such that $c(x, 0) = \gamma$. Now choose (y, d) such that $c(y, d) = \rho$. Then $\{(2x - y, -d), (x, 0), (y, d)\}$ is a rainbow 3-AP. Therefore, $|c(P_0)| = |c(P_{-d})|$. If $|c(P_0)| > |c(P_d)|$, then there exist $\beta, \gamma \in c(P_0) \setminus c(P_d)$ because $\rho \notin c(P_0)$. Now a rainbow 3-AP can be attained by choosing elements of P_0 and P_{-d} that are assigned β and γ, respectively, and the corresponding element of P_d. Hence, $\mathrm{aw}_u(G, 3) \leq |c(P_0)| = |c(P_d)|$. However, since there is only one element in P_d with the color ρ, this implies that P_d contains a rainbow 3-AP, which is a contradiction. □

Theorem 5 yields the following Corollary that is analogous to Corollary 1.

Corollary 3. *Let n be the largest odd divisor of the order of G. There exists a finite abelian group G' such that the order of G' is a power of 2 and*

$$\mathrm{aw}_u(G, 3) = \mathrm{aw}_u(G', 3) + \mathrm{aw}_u(\mathbb{Z}_n, 3) - 2.$$

3.1. Groups with power of 2 order

Proposition 6. *For all positive integers s,*

$$\mathrm{aw}_u([\mathbb{Z}_2]^s, 3) = s + 2.$$

Proof. This proof is by induction on s. The base case of $s = 1$ is trivial since $2 < \mathrm{aw}_u([\mathbb{Z}_2]^s, 3) \leq \mathrm{aw}([\mathbb{Z}_2]^s, 3) = 3$. Assume $1 < s$.

Let c' be a coloring of $[\mathbb{Z}_2]^{s-1}$ with $s+1$ colors, no rainbow 3-AP and a be an element of $[\mathbb{Z}_2]^{s-1}$ that does not share a color with any other element. For all $g \in [\mathbb{Z}_2]^{s-1}$ and $h \in \mathbb{Z}_2$, except $(a, 0)$, let $c(g, h) = c'(g)$ and assign $c(a, 0)$ a new color. Then c is a unitary $(s+1)$-coloring of $[\mathbb{Z}_2]^s$ with no rainbow 3-AP. So, $s + 2 \leq \mathrm{aw}_u([\mathbb{Z}_2]^s, 3)$.

Now assume $[\mathbb{Z}_2]^s$ is colored with a unitary coloring that has exactly $s + 2$ colors and no rainbow 3-AP. Let $A = \{(g, 0) : g \in [\mathbb{Z}_2]^{s-1}\}$ and $B = \{(g, 1) : g \in [\mathbb{Z}_2]^{s-1}\}$. Without loss of generality, let $(a, 0)$ be an element of $[\mathbb{Z}_2]^s$ that does not share a color with any other element. Therefore, by induction, $|c(A)| \leq s$. So there exists 2 colors, $\alpha, \beta \in c(B) \setminus c(A)$. Then there exists an element $(b, 1)$ such that $c(b, 1) \in \{\alpha, \beta\}$ and $2a \neq 2b$. Therefore, the 3-AP $\{(a, 0), (b, 1), (2b - a, 0)\}$ must be rainbow, which is a contradiction. So, $\mathrm{aw}_u([\mathbb{Z}_2]^s, 3) \leq s + 2$. □

Theorem 6. *For $1 \leq i \leq s$, let m_i be a positive integer. Then*

$$\mathrm{aw}_u(\mathbb{Z}_{2^{m_1}} \times \cdots \times \mathbb{Z}_{2^{m_s}}, 3) = s + 2.$$

Proof. This proof is inductive on $\sum_{i=1}^{s} m_i$. The base case of $\sum_{i=1}^{s} m_i = s$ is true by Proposition 6. So assume $s < \sum_{i=1}^{s} m_i$ and $2 \leq m_s$.

Let c' be a coloring of $\mathbb{Z}_{2^{m_1}} \times \cdots \times \mathbb{Z}_{2^{m_s}}$ with s+1 colors, no rainbow 3-AP and a be an element of $\mathbb{Z}_{2^{m_1}} \times \cdots \times \mathbb{Z}_{2^{m_{(s-1)}}}$ that does not share a color with any other element. For all $g \in \mathbb{Z}_{2^{m_1}} \times \cdots \times \mathbb{Z}_{2^{m_{(s-1)}}}$ and $h \in \mathbb{Z}_{2^{m_s}}$, except $(a, 0)$, let $c(g, h) = c'(g)$ and assign $c(a, 0)$ a new color. Then c is a unitary $(s + 1)$-coloring of $\mathbb{Z}_{2^{m_1}} \times \cdots \times \mathbb{Z}_{2^{m_s}}$ with no rainbow 3-AP. So, $s + 2 \leq \mathrm{aw}_u(\mathbb{Z}_{2^{m_1}} \times \cdots \times \mathbb{Z}_{2^{m_s}}, 3)$.

Now assume $\mathbb{Z}_{2^{m_1}} \times \cdots \times \mathbb{Z}_{2^{m_s}}$ is colored with a unitary coloring that has exactly $s + 2$ colors and no rainbow 3-AP. Let $A = \{(g, h) : g \in \mathbb{Z}_{2^{m_1}} \times \cdots \times \mathbb{Z}_{2^{m_{s-1}}}, h \in \mathbb{Z}_{2^{m_s}},$ and h is even$\}$ and $B = \{(g, h) : g \in \mathbb{Z}_{2^{m_1}} \times \cdots \times \mathbb{Z}_{2^{m_{s-1}}}, h \in \mathbb{Z}_{2^{m_s}},$ and h is odd$\}$. Without loss of generality, let $(a, 0)$ be an element of $\mathbb{Z}_{2^{m_1}} \times \cdots \times \mathbb{Z}_{2^{m_s}}$ that does not share a color with any other element. Therefore, by induction, $|c(A)| \leq s$. So there exists 2 colors, $\alpha, \beta \in c(B) \setminus c(A)$. Then there exists an element $(b, 2j + 1)$ such that $c(b, 2j + 1) \in \{\alpha, \beta\}$ and $2a \neq 2b$. Therefore, the 3-AP $\{(a, 0), (b, 2j + 1), (2b - a, 4j + 2)\}$ must be rainbow, which is a contradiction. So, $\mathrm{aw}_u(\mathbb{Z}_{2^{m_1}} \times \cdots \times \mathbb{Z}_{2^{m_s}}, 3) \leq s + 2$. □

Corollary 4. *Let G be a finite abelian group. Then $\mathrm{aw}(G,3) = \mathrm{aw}_u(G,3)$ if and only if the order of G is odd or G is cyclic.*

Proof. By Corollary 1 and Theorem 6,

$$\mathrm{aw}(G,3) = 2^s + \mathrm{aw}(\mathbb{Z}_n,3) - 1,$$

for some nonnegative integer s and odd integer n. By Corollary 3 and Theorem 4,

$$\mathrm{aw}_u(G,3) = s + \mathrm{aw}_u(\mathbb{Z}_n,3),$$

for the same s and n. Therefore, $\mathrm{aw}(G,3) = \mathrm{aw}_u(G,3)$ if and only if $2^s - 1 = s$; hence, $\mathrm{aw}(G,3) = \mathrm{aw}_u(G,3)$ if and only if s is 0 or 1. □

References

[1] M. Axenovich, D. Fon-Der-Flaass, On rainbow arithmetic progressions. *Electronic Journal of Combinatorics* **11**(1) (2004), Research Paper 1, 7pp. MR2034415

[2] M. Axenovich and R. R. Martin, Sub-Ramsey numbers for arithmetic progressions. *Graphs Comb.* **22** (2006), no. 1, 297–309. MR2264853

[3] S. Butler, C. Erickson, L. Hogben, K. Hogenson, L. Kramer, R. Kramer, J. Lin, R. Martin, D. Stolee, N. Warnberg, and M. Young, Rainbow arithmetic progressions. *Journal of Combinatorics* **7**(4) (2016), 595–626. MR3538156

[4] V. Jungić, J. Licht (Fox), M. Mahdian, J. Nešetril, and R. Radoičić, Rainbow arithmetic progressions and anti-Ramsey results. *Comb. Probab. Comput.* **12**(5–6) (2003), 599–620. MR2037073

[5] K. Uherka, An introduction to Ramsey theory and anti-Ramsey theory on the integers. Master's Creative Component (2013), Iowa State University.

Michael Young
Iowa State University
Mathematics Department
396 Carver Hall
Ames, IA, 50021
USA
E-mail address: myoung@iastate.edu

Received 26 March 2016