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PBW bases and marginally large tableaux in type D

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We give an explicit description of the unique crystal isomorphism between two realizations of $B(\infty)$ in type D: that using marginally large tableaux and that using PBW monomials with respect to one particularly nice reduced expression of the longest word.

1. Introduction

For any symmetrizable Kac–Moody algebra, the crystal $B(\infty)$ is a combinatorial object that contains information about the corresponding universal enveloping algebra and its integrable highest weight representations. Kashiwara's definition of $B(\infty)$ uses some intricate algebraic constructions, but it can often be realized in quite simple ways. We consider two such realizations in type D_n.

1. The construction using marginally large tableaux from [8] (and the closely related earlier work [4]), which is a limiting case of constructions in [11].
2. The recent construction using bracketing rules on Kostant partitions from [15], which is naturally identified with the algebraic crystal structure on PBW monomials for one particularly nice reduced expression of w_0.

We explicitly describe the unique crystal isomorphism between these two realizations (see Theorem 3.1). This is a type D_n analogue of a type A_n result that can be found in [3], although the type D_n situation is a little different.[1] Most notably, the isomorphism is not as "local": in type A_n, the map from tableaux to Kostant partitions simply maps each box in the tableau to a root, but in type D_n one must consider multiple boxes at once.

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[1]The types B_n and C_n analogues of these results can be found in [5].

Table 2.1: Positive roots of type D_n, expressed both as a linear combination of simple roots and in the canonical realization following [2]

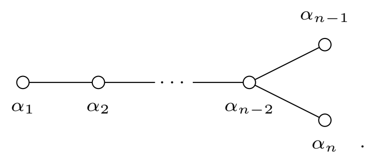
$\beta_{i,k} = \alpha_i + \cdots + \alpha_k,$	$1 \le i \le k \le n-1$
$\gamma_{i,k} = \alpha_i + \cdots + \alpha_{n-2} + \alpha_n + \alpha_{n-1} + \cdots + \alpha_k,$	$1 \le i < k \le n$
$\beta_{i,k} = \epsilon_i - \epsilon_{k+1},$	$1 \le i \le k \le n-1$
$\gamma_{i,k} = \epsilon_i + \epsilon_k,$	$1 \le i < k \le n$

In the final section we give a diagrammatic description of Kostant partitions and the crystal operators on them motivated by the multisegment realization of $B(\infty)$ in type A_n [3, 9, 12, 16].

2. Background

Let $\mathfrak{g}$ be the Lie algebra of type D_n with Cartan matrix and Dynkin diagram

$$A = (a_{ij}) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 2 & -1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 2 \end{pmatrix},$$



Let $\{\alpha_1, \dots, \alpha_n\}$ be the simple roots and $\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ the simple coroots, related by the inner product $\langle \alpha_j^\vee, \alpha_i \rangle = a_{ij}$. Define the fundamental weights $\{\omega_1, \dots, \omega_n\}$ by $\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}$. Then the weight lattice is $P = \mathbf{Z}\omega_1 \oplus \cdots \oplus \mathbf{Z}\omega_n$ and the coroot lattice is $P^\vee = \mathbf{Z}\alpha_1^\vee \oplus \cdots \oplus \mathbf{Z}\alpha_n^\vee$. The Cartan subalgebra $\mathfrak{h}$ is given by $\mathbf{C} \otimes_{\mathbf{Z}} P^\vee$. Let Φ denote the roots associated to $\mathfrak{g}$, with the set of positive roots denoted Φ^+. The list of positive roots is given in Table 2.1. The Weyl group associated to $\mathfrak{g}$ is the group generated by $s_1, \dots, s_n$, where $s_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i$ for all $\lambda \in P$. There exists a unique longest element of W, which we denote as w_0. For notational brevity, set $I = \{1, 2, \dots, n\}$.

Let $B(\infty)$ be the infinity crystal associated to $\mathfrak{g}$ as defined in [10]. This is a countable set along with operators e_i and f_i which roughly correspond to the Chevalley generators of $\mathfrak{g}$. We don't need the details of the definition of $B(\infty)$, as we just consider two explicitly defined ways to realize it.

2.1. Type D marginally large tableaux

Definition 2.1. A marginally large tableau of type D_n is an $n-1$ row tableau on the alphabet

$$J(D_n) := \left\{ 1 \prec \cdots \prec n-1 \prec \begin{matrix} n \\ \overline{n} \end{matrix} \prec \overline{n-1} \prec \cdots \prec \overline{1} \right\}$$

which satisfies the following conditions.

1. The first column has entries $1, 2, \dots, n-1$ in that order.
2. Entries weakly increase along rows.
3. The number of i-boxes in the ith row is exactly one more than the total number of boxes in the $(i+1)$st row. We call this condition "marginal largeness."
4. Every entry in the ith row is $\preceq \bar{\imath}$.
5. The entries n and $\overline{n}$ do not appear in the same row.

Denote by $\mathcal{T}(\infty)$ the set of marginally large tableaux.

For $1 \leq i \leq n-1$, the boxes in the ith row with content i will be called *shaded boxes*; all other boxes will be called *unshaded*. Given a tableau $T \in \mathcal{T}(\infty)$, define its weight as follows. Let $\boxed{k}_j$ be an unshaded box containing k in the jth row of T. Set

$$\mathrm{wt}\left(\boxed{k}_j\right) = \begin{cases} -\beta_{j,k-1} & \text{if } k \text{ is an unbarred letter,} \\ -\gamma_{j,k} & \text{if } k \text{ is a barred letter and } k \neq \bar{\jmath}, \\ -\gamma_{j,j+1} - \beta_{j,j} & \text{if } k = \bar{\jmath}. \end{cases}$$

Then the weight $\mathrm{wt}(T)$ of T is the sum over all unshaded boxes $\boxed{k}_j$ in T of $\mathrm{wt}(\boxed{k}_j)$.

Example 2.2. In type D_4, the elements of $\mathcal{T}(\infty)$ all have the form

$T =$

1	$1 \cdots 1$	$1 \cdots 1$	1	$1 \cdots 1$	$1 \cdots 1$	$1 \cdots 1$	$1 \cdots 1$	1	$2 \cdots 2$	$3 \cdots 3$	$x_1 \cdots x_1$	$\overline{3} \cdots \overline{3}$	$\overline{2} \cdots \overline{2}$	$\overline{1} \cdots \overline{1}$
2	$2 \cdots 2$	$2 \cdots 2$	2	$3 \cdots 3$	$x_2 \cdots x_2$	$\overline{3} \cdots \overline{3}$	$\overline{2} \cdots \overline{2}$							
3	$x_3 \cdots x_3$	$\overline{3} \cdots \overline{3}$												

,

where $x_i \in \{4, \overline{4}\}$ for each $i = 1, 2, 3$. In particular, the unique element of weight zero is

$$T_\infty = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} .$$

Definition 2.3. Fix a type D_n marginally large tableau. The reading word $\mathrm{read}(T)$ is obtained by reading right to left along rows, starting at the top and working down.

Definition 2.4. For each $1 \leq i \leq n$, the bracketing sequence $\mathrm{br}_i(T)$ is the sequence obtained by placing a ')' under each letter for which there

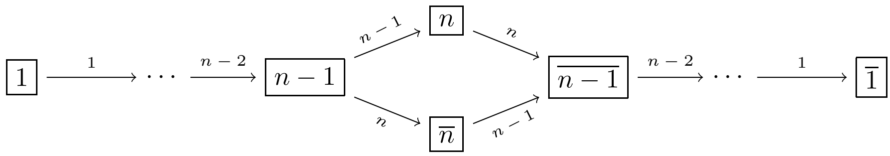


Figure 2.1: The fundamental crystal of type D_n.

is an i-colored arrow entering the corresponding box in Figure 2.1, and a '(' under each letter for which there is an i-colored arrow leaving the corresponding box. Sequentially cancel all ()-pairs to obtain a sequence of the form $)\cdots)(\cdots($. The remaining brackets are called *uncanceled*.

Remark 2.5. The sequence $\mathrm{br}_i(T)$ factors as $\mathrm{br}_i(R_1)\mathrm{br}_i(R_2)\cdots\mathrm{br}_i(R_{n-1})$, where $\mathrm{br}_i(R_j)$ is the sequence of brackets coming from the j^{th} row of T, counting from the top.

Definition 2.6. Let $T \in \mathcal{T}(\infty)$ and $i \in I$.

1. Let x be the letter in T corresponding to the rightmost uncanceled ')' in $\mathrm{br}_i(T)$. Then e_iT is the tableau obtained from T by replacing the box containing x by the box containing the letter at the other end of the i-arrow from x in Figure 2.1. If the result is not marginally large, then delete exactly one column containing the elements $1, \dots, i$ so that the result is marginally large. If no such ')' exists, then define $e_iT = 0$.
2. Let y be the letter in T corresponding to the leftmost uncanceled '(' in $\mathrm{br}_i(T)$. Then f_iT is the tableau obtained from T by replacing the box containing y by the box containing the letter at the other end of the i-arrow from y in Figure 2.1. If the result is not marginally large, then insert exactly one column containing the elements $1, \dots, i$ so that the result is marginally large.

Remark 2.7. For $i \leq n-1$, e_i changes the content of exactly one box in T either from $\bar{\imath}$ to $\overline{i+1}$ or from $i+1$ to i. The marginal largeness condition is preserved unless e_iT changes an $i+1$ to an i on the ith row. Then e_iT contains two adjacent columns with entries $1, 2, \dots, i$, one of which must be removed to satisfy the marginally large condition. The situation is similar for $i = n$.

Example 2.8. Consider D_4 and

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&1&1&2&2&\overline{3}&\overline{1}&\overline{1}&\overline{1}\\ \hline 2&2&2&2&3&\overline{4}&\overline{3}&\overline{3}\\ \cline{1-8} 3&\overline{4}&\overline{3}\\ \cline{1-3}\end{array}\ .$$

To calculate e_4 and f_4, the relevant arrows from Figure 2.1 are

$$\boxed{3} \xrightarrow{4} \boxed{\overline{4}}\ , \qquad\qquad \boxed{4} \xrightarrow{4} \boxed{\overline{3}}\ .$$

Thus each 3 and 4 will contribute '(', each $\overline{4}$ and $\overline{3}$ will contribute ')', and all other letters will contribute nothing. The reading word and bracketing sequence are

$$\begin{aligned}\mathrm{read}(T) &= \overline{1}\ \overline{1}\ \overline{1}\ \overline{3}\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \overline{3}\ \overline{3}\ \overline{4}\ 3\ 2\ 2\ 2\ 2\ \overline{3}\ \overline{4}\ 3\\ \mathrm{br}_4(T) &= \qquad\ \ )\qquad\qquad\qquad\qquad\qquad)\ )\ )\ \not(\ \ \ \not)\ )\ (.\end{aligned}$$

The rightmost uncanceled ')' is the one shown in blue, so e_4 changes the corresponding $\overline{4}$-box in the third row to a 3-box. To maintain marginal largeness, we must delete one of the columns containing 123:

$$e_4T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&1&2&2&\overline{3}&\overline{1}&\overline{1}&\overline{1}\\ \hline 2&2&2&3&\overline{4}&\overline{3}&\overline{3}\\ \cline{1-7} 3&\overline{3}\\ \cline{1-2}\end{array}\ .$$

Similarly,

$$f_4T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&1&1&1&2&2&\overline{3}&\overline{1}&\overline{1}&\overline{1}\\ \hline 2&2&2&2&2&3&\overline{4}&\overline{3}&\overline{3}\\ \cline{1-9} 3&\overline{4}&4&\overline{3}\\ \cline{1-4}\end{array}\ .$$

In Definition 2.3, we are using the so-called middle-Eastern reading, as defined in [7]. This differs from the definition of the signature rule for element of $\mathcal{T}(\infty)$ given in [8] which uses the far-Eastern reading in which the tableau is read from top to bottom in columns starting with the rightmost column. (The subsequent crystal structure is constructed using the same bracketing sequence as above and the same operators, just with $\mathrm{read}(T)$ replaced by this new reading.) However, the resulting operators are identical:

Proposition 2.9. *The operators e_i and f_i on $\mathcal{T}(\infty)$ defined using the far-Eastern reading and the middle-Eastern reading, respectively, are identical.*

Proof. Fix $T \in \mathcal{T}(\infty)$ and let $c_{i,j}$ be the number of j-boxes in row i of T. First assume $1 \le i \le n-2$. Then all brackets used in calculating f_i come from rows $1, \dots, i+1$. The brackets corresponding to unshaded boxes come in exactly the same order for the two readings. Thus the only difference between the two bracket orders is the suffix of the sequence, where one has:

$$\begin{aligned} &\text{far-Eastern: } \cdots (^{c_{i,i} - c_{i+1,i+1} + c_{\overline{i+1},i+1}} \underbrace{() \cdots ()}_{c_{i+1,i+1}}, \\ &\text{middle-Eastern: } \cdots (^{c_{i,i}} (^{c_{\overline{i+1},i+1}})^{c_{i+1,i+1}}. \end{aligned} \tag{2.10}$$

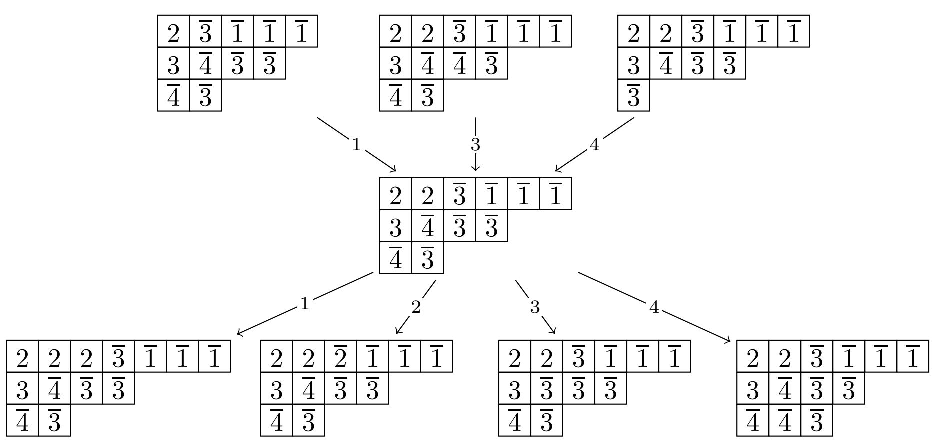
Since $c_{i,i} > c_{i+1,i+1}$, the portions shown each have no uncanceled, '),' and they have the same number of uncanceled '(,' with the first uncanceled '(' corresponding to a shaded i. It follows that the first uncanceled bracket of each type in the two sequences corresponds to a box of the same type (i.e., same content and on same row). Clearly both rules always apply f_i to the rightmost box of a given type, and e_i to the leftmost, so the two rules agree.

The argument for $i = n-1, n$ is similar, and in fact simpler, since the only shaded boxes that are relevant are the shaded $n-1$. □

Remark 2.11. Unlike in type A_n, the operators on finite type D_n tableaux using these two readings are different. They only agree for marginally large tableaux.

For a marginally large tableau T, we sometimes consider its *reduced form*, which is obtained by removing all shaded boxes and sliding the rows so that the result is left-justified. Note that we can recover T from it's reduced form.

Example 2.12. Continuing Example 2.8, the crystal graph around T using tableaux in reduced form is



2.2. Crystal structure on Kostant partitions

Here we review the crystal structure on Kostant partitions from [15]. As explained there, this is naturally identified with the crystal structure on PBW monomials from, for example, [1, 13] for the reduced expression

$$w_0 = (s_1 s_2 \cdots s_{n-2} s_{n-1} s_n s_{n-2} \cdots s_1) \cdots (s_{n-2} s_{n-1} s_n s_{n-2}) s_{n-1} s_n.$$

Let $\mathcal{R}$ be the set of symbols $\{(\beta) : \beta \in \Phi^+\}$. Let $\mathrm{Kp}(\infty)$ be the free $\mathbf{Z}_{\geq 0}$-span of $\mathcal{R}$. This is the set of *Kostant partitions*. We denote elements of $\mathrm{Kp}(\infty)$ by $\boldsymbol{\alpha} = \sum_{(\beta)\in\mathcal{R}} c_\beta(\beta)$.

Definition 2.13. Consider the following subsets of positive roots depending on $i \in I$.

1. For $1 \leq i \leq n-1$, define

$$\Phi_i = \{\beta_{k,i-1}, \beta_{k,i} : 1 \leq k \leq i\} \cup \{\gamma_{k,i}, \gamma_{k,i+1} : 1 \leq k \leq i-1\}$$

and order the roots in Φ_i by

$$\beta_{1,i} < \beta_{1,i-1} < \gamma_{1,i} < \gamma_{1,i+1} < \cdots \\ < \beta_{i-1,i} < \beta_{i-1,i-1} < \gamma_{i-1,i} < \gamma_{i-1,i+1} < \beta_{i,i}.$$

2. For $i = n$, define

$$\Phi_n = \{\beta_{k,n-2}, \beta_{k,n-1} : 1 \leq k \leq n-2\} \\ \cup \{\gamma_{k,n-1}, \gamma_{k,n} : 1 \leq k \leq n-2\} \cup \{\gamma_{n-1,n}\}$$

and order the roots in Φ_n by

$$\gamma_{1,n} < \beta_{1,n-2} < \gamma_{1,n-1} < \beta_{1,n-1} < \cdots \\ < \gamma_{n-2,n} < \beta_{n-2,n-2} < \gamma_{n-2,n-1} < \beta_{n-2,n-1} < \gamma_{n-1,n}.$$

The *bracketing sequence* $S_i(\boldsymbol{\alpha})$ consists of, for each $\beta \in \Phi_i$, c_β-many ')' if $\beta - \alpha_i$ is a positive root or if $\beta = \alpha_i$ and c_β-many '(' if $\beta + \alpha_i$ is a positive root, ordered as above. Successively cancel ()-pairs to obtain sequence of the form $)\cdots)(\cdots($. We call the remaining brackets *uncanceled*.

Definition 2.14. Let $i \in I$ and $\boldsymbol{\alpha} = \sum_{(\beta)\in\mathcal{R}} c_\beta(\beta) \in \mathrm{Kp}(\infty)$.

- Let β be the root corresponding to the rightmost uncanceled ')' in $S_i(\boldsymbol{\alpha})$. Define
$$e_i\boldsymbol{\alpha} = \boldsymbol{\alpha} - (\beta) + (\beta - \alpha_i).$$
If $\beta = \alpha_i$, we interpret (0) as the additive identity in $\mathrm{Kp}(\infty)$. If no such ')' exists, then $e_i\boldsymbol{\alpha}$ is undefined.
- Let γ be the root corresponding to the leftmost uncanceled '(' in $S_i(\boldsymbol{\alpha})$. Define
$$f_i\boldsymbol{\alpha} = \boldsymbol{\alpha} - (\gamma) + (\gamma + \alpha_i).$$
If no such '(' exists, set $f_i\boldsymbol{\alpha} = \boldsymbol{\alpha} + (\alpha_i)$.
- $\mathrm{wt}(\boldsymbol{\alpha}) = -\displaystyle\sum_{\beta\in\Phi^+} c_\beta\beta$.
- $\varepsilon_i(\boldsymbol{\alpha})$ = number of ')' in the bracketing sequence of $\boldsymbol{\alpha}$.
- $\varphi_i(\boldsymbol{\alpha}) = \varepsilon_i(\boldsymbol{\alpha}) + \langle \alpha_i^\vee, \mathrm{wt}(\boldsymbol{\alpha})\rangle$.

Proposition 2.15 ([15]). *With the operations defined above,* $\mathrm{Kp}(\infty)$ *realizes* $B(\infty)$. □

Example 2.16. Let $i = n = 4$ and consider

$$\boldsymbol{\alpha} = 5(\alpha_1) + (\alpha_1+\alpha_2+\alpha_3+\alpha_4) + 3(\alpha_1+2\alpha_2+\alpha_3+\alpha_4) + 2(\alpha_2+\alpha_4) + (\alpha_2+\alpha_3) + (\alpha_2+\alpha_3+\alpha_4) + (\alpha_3) + 2(\alpha_4).$$

Look at the coefficients c_β of $\boldsymbol{\alpha}$ corresponding to $\beta \in \Phi_4$.

$$\begin{array}{ccccccccc} 0\gamma_{1,4} & 0\beta_{1,2} & \gamma_{1,3} & 0\beta_{1,3} & 2\gamma_{2,4} & 0\beta_{2,2} & \gamma_{2,3} & \beta_{2,3} & 2\gamma_{3,4} \\ & &) & &)) & &) & \multicolumn{2}{c}{\cancel{(\quad)}\ )} \end{array}$$

Hence, $e_4\boldsymbol{\alpha} = \boldsymbol{\alpha} - (\gamma_{3,4}) + (0) = \boldsymbol{\alpha} - (\alpha_4)$ and $f_4\boldsymbol{\alpha} = \boldsymbol{\alpha} + (\alpha_4)$.

3. The isomorphism

Theorem 3.1. *The unique crystal isomorphism* $\Psi\colon \mathcal{T}(\infty) \longrightarrow \mathrm{Kp}(\infty)$ *can be described as follows. For a tableaux* $T \in \mathcal{T}(\infty)$*, let* $R_1, \dots, R_{n-1}$ *denote the rows of the reduced form of* T *starting at the top. Set* $\Psi(T) = \sum_{j=1}^{n-1}\Psi(R_j)$*, where* $\Psi(R_j)$ *is defined from the unshaded boxes in* R_j *as follows:*

1. *Each* $\overline{\jmath}$ *is sent to* $(\beta_{j,j}) + (\gamma_{j,j+1})$*;*
2. *each pair* $k, \overline{k}$*, where* $j < k \le n-1$ *is sent to* $(\beta_{j,k}) + (\gamma_{j,k+1})$*;*

3. each remaining $k \in \{j, j+1, \ldots, n\}$ *is sent to* $(\beta_{j,k-1})$*;*
4. each remaining $\overline{k} \in \{\overline{n}, \overline{n-1}, \ldots, \overline{j+1}\}$ *is sent to* $(\gamma_{j,k})$*.*

Example 3.2. Let $n = 4$ and

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & \overline{3} & \overline{1} & \overline{1} & \overline{1} \\ \hline 2 & 2 & 2 & 2 & 3 & \overline{4} & \overline{3} & \overline{3} \\ \cline{1-8} 3 & \overline{4} & \overline{3} \\ \cline{1-3} \end{array} .$$

Then

$$\begin{aligned} \Psi(R_1) &= 3\big((\beta_{1,1}) + (\gamma_{1,2})\big) + (\gamma_{1,3}) + 2(\beta_{1,1}), \\ \Psi(R_2) &= \big((\beta_{2,3}) + (\gamma_{2,4})\big) + (\gamma_{2,3}) + (\gamma_{2,4}), \\ \Psi(R_3) &= \big((\beta_{3,3}) + (\gamma_{3,4})\big) + (\gamma_{3,4}), \end{aligned}$$

so

$$\Psi(T) = 5(\beta_{1,1}) + (\gamma_{1,3}) + 3(\gamma_{1,2}) + 2(\gamma_{2,4}) + (\beta_{2,3}) + (\gamma_{2,3}) + (\beta_{3,3}) + 2(\gamma_{3,4}).$$

Compare with Example 2.16.

The proof of Theorem 3.1 will occupy the rest of this section. Denote by $e_i^{\mathcal{T}}$ and $f_i^{\mathcal{T}}$ the Kashiwara operators on $\mathcal{T}(\infty)$ from Definition 2.6, and by e_i^{Kp} and f_i^{Kp} those on $\mathrm{Kp}(\infty)$ from Definition 2.14.

Lemma 3.3. *Fix* $T \in \mathcal{T}(\infty)$ *and* $1 \le j < i < n$*. The strings* $\mathrm{br}_i(R_j)$ *and* $S_i\big(\Psi(R_j)\big)$ *have the same number of uncanceled brackets, both left and right. Here, as before,* R_j *means the* j^{th} *row of* T*. Furthermore, if the leftmost uncanceled left bracket in* $\mathrm{br}_i(T)$ *and* $S_i\big(\Psi(T)\big)$ *are in* $\mathrm{br}(R_j)$ *and* $S_i\big(\Psi(R_j)\big)$ *respectively, then* $f_i^{\mathrm{Kp}}\Psi(T) = \Psi(f_i^{\mathcal{T}}T)$*.*

Proof. It suffices to consider the case when the only unshaded boxes of T are in R_j, so the condition on left brackets holds exactly if there is an uncanceled left bracket in $\mathrm{br}_i(R_j)$. First assume $i \le n-1$. We are only interested in entries $i, i+1, \overline{i+1}$, and $\overline{\imath}$ in R_j, and pairs $i-1, \overline{i-1}$, since only these give rise to brackets in $\mathrm{br}_i(R_j)$ or $S_i\big(\Psi(R_j)\big)$.

A pair $i-1, \overline{i-1}$ corresponds to no brackets in $\mathrm{br}_i(R_j)$, and to $(\beta_{j,i-1})$, $(\gamma_{j,i})$ in $\Psi(R_j)$, which gives a canceling pair of brackets in $S_i\big(\Psi(R_j)\big)$. So the statement is true if and only if it is true with these removed. Thus we can assume R_j has no such pairs.

Assume R_j has p boxes of $\overline{\imath+1}$, q of $i+1$, r of i, s of $\bar\imath$:

$$R_j = \underbrace{\boxed{i}\boxed{\cdots}\boxed{i}}_{r}\underbrace{\boxed{i+1}\boxed{\cdots}\boxed{i+1}}_{q}\boxed{\cdots}\underbrace{\boxed{\overline{\imath+1}}\boxed{\cdots}\boxed{\overline{\imath+1}}}_{p}\underbrace{\boxed{\bar\imath}\boxed{\cdots}\boxed{\bar\imath}}_{s}\,.$$

We consider four cases.

Case 1: Assume $p > q$ and $r > s$. Then

$$\Psi(R_j) = (r-s)(\beta_{j,i-1}) + s(\beta_{j,i}) + q(\beta_{j,i+1}) + q(\gamma_{j,i+2}) + (s+p-q)(\gamma_{j,i+1}),$$

so

$$\mathrm{br}_i(R_j) = \,)^s\ (^p\ )^q\ (^r \qquad \text{and} \qquad S_i\big(\Psi(R_j)\big) = \,)^s\ (^{r-s}\ (^{s+p-q}.$$

Both $\mathrm{br}_i(R_j)$ and $S_i\big(\Psi(T)\big)$ have s uncanceled ')' and $r+p-q > 0$ uncanceled '(,' and

$$f_i^{\mathcal{T}} R_j = \underbrace{\boxed{i}\boxed{\cdots}\boxed{i}}_{r}\underbrace{\boxed{i+1}\boxed{\cdots}\boxed{i+1}}_{q}\boxed{\cdots}\underbrace{\boxed{\overline{\imath+1}}\boxed{\cdots}\boxed{\overline{\imath+1}}}_{p-1}\underbrace{\boxed{\bar\imath}\boxed{\cdots}\boxed{\bar\imath}}_{s+1}\,.$$

Then

$$\begin{aligned} f_i^{\mathrm{Kp}}\Psi(R_j) &= (r-s-1)(\beta_{j,i-1}) + (s+1)(\beta_{j,i}) + q(\beta_{j,i+1}) \\ &\qquad + q(\gamma_{j,i+2}) + (s+p-q)(\gamma_{j,i+1}) \\ &= \Psi(f_i^{\mathcal{T}} R_j). \end{aligned}$$

Case 2: Assume $p > q$ and $r \le s$. Then

$$\Psi(R_j) = r(\beta_{j,i}) + q(\beta_{j,i+1}) + q(\gamma_{j,i+2}) + (r+p-q)(\gamma_{j,i+1}) + (s-r)(\gamma_{j,i}).$$

Both $\mathrm{br}_i(R_j)$ and $S_i\big(\Psi(T)\big)$ have s uncanceled ')' and $r+p-q > 0$ uncanceled '(,' and

$$f_i^{\mathcal{T}} R_j = \underbrace{\boxed{i}\boxed{\cdots}\boxed{i}}_{r}\underbrace{\boxed{i+1}\boxed{\cdots}\boxed{i+1}}_{q}\boxed{\cdots}\underbrace{\boxed{\overline{\imath+1}}\boxed{\cdots}\boxed{\overline{\imath+1}}}_{p-1}\underbrace{\boxed{\bar\imath}\boxed{\cdots}\boxed{\bar\imath}}_{s+1}$$

giving

$$\begin{aligned} f_i^{\mathrm{Kp}}\Psi(R_j) &= r(\beta_{j,i}) + q(\beta_{j,i+1}) + q(\gamma_{j,i+2}) \\ &\qquad + (r+p-q-1)(\gamma_{j,i+1}) + (s-r+1)(\gamma_{j,i}) \\ &= \Psi(f_i^{\mathcal{T}} R_j). \end{aligned}$$

Case 3: Assume $p \le q$ and $r > s$. Then

$$\Psi(R_j) = (r-s)(\beta_{j,i-1}) + (q-p+s)(\beta_{j,i}) + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) + s(\gamma_{j,i+1}).$$

Both $\mathrm{br}_i(R_j)$ and $S_i\big(\Psi(T)\big)$ have $s+q-p$ uncanceled ‘)’ and $r>0$ uncanceled ‘(,’ and

$$f_i^{\mathcal{T}} R_j = \underbrace{\boxed{i}\boxed{\cdots}\boxed{i}}_{r-1}\underbrace{\boxed{i+1}\boxed{\cdots}\boxed{i+1}}_{q+1}\boxed{\cdots}\underbrace{\boxed{\overline{\imath+1}}\boxed{\cdots}\boxed{\overline{\imath+1}}}_{p}\underbrace{\boxed{\bar\imath}\boxed{\cdots}\boxed{\bar\imath}}_{s}$$

giving

$$\begin{aligned} f_i^{\mathrm{Kp}}\Psi(R_j) &= (r-s-1)(\beta_{j,i-1}) + (q-p+s+1)(\beta_{j,i}) \\ &\qquad + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) + s(\gamma_{j,i+1}) \\ &= \Psi(f_i^{\mathcal{T}} R_j). \end{aligned}$$

Case 4: Assume $p \le q$ and $r \le s$. Then

$$\Psi(R_j) = (q-p+r)(\beta_{j,i}) + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) + r(\gamma_{j,i+1}) + (s-r)(\gamma_{j,i}).$$

Again both $\mathrm{br}_i(R_j)$ and $S_i\big(\Psi(T)\big)$ have $s+q-p$ uncanceled ‘)’ and r uncanceled ‘(.’ If $r=0$ then the leftmost uncanceled left bracket in $\mathrm{br}_i(T)$ is not in $\mathrm{br}_i(R_j)$. If $r \neq 0$,

$$f_i^{\mathcal{T}} R_j = \underbrace{\boxed{i}\boxed{\cdots}\boxed{i}}_{r-1}\underbrace{\boxed{i+1}\boxed{\cdots}\boxed{i+1}}_{q+1}\boxed{\cdots}\underbrace{\boxed{\overline{\imath+1}}\boxed{\cdots}\boxed{\overline{\imath+1}}}_{p}\underbrace{\boxed{\bar\imath}\boxed{\cdots}\boxed{\bar\imath}}_{s}$$

giving

$$\begin{aligned} f_i^{\mathrm{Kp}}\Psi(R_j) &= (q-p+r)(\beta_{j,i}) + p(\beta_{j,i+1}) + p(\gamma_{j,i+2}) \\ &\qquad + (r-1)(\gamma_{j,i+1}) + (s-r+1)(\gamma_{j,i}) \\ &= \Psi(f_i^{\mathcal{T}} R_j). \end{aligned}$$

□

Example 3.4. Consider type D_4 and $i=2$, and the tableau

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 3 & 4 & \bar{3} & \bar{1} & \bar{1} \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}\ .$$

Then the reading word and bracketing sequence are

$$\operatorname{br}_2(T) = \begin{array}{ccccccccccccc} \overline{1} & \overline{1} & \overline{3} & 4 & 3 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 3 \\ & & (& &) & (& (& & & & (& (&) \end{array}$$

so

$$f_2^{\mathcal{T}} T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 & 3 & 4 & \overline{3} & \overline{1} & \overline{1} \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} .$$

Direct calculation gives

$$\begin{aligned} \Psi(T) &= 2\big((\beta_{1,1}) + (\gamma_{1,2})\big) + \big((\beta_{1,3}) + (\gamma_{1,4})\big) + 2(\beta_{1,1}) + (\beta_{1,3}) \\ &= 4(\beta_{1,1}) + 2(\beta_{1,3}) + (\gamma_{1,4}) + 2(\gamma_{1,2}) \end{aligned}$$

and

$$\begin{aligned} \Psi(f_2^{\mathcal{T}} T) &= 2\big((\beta_{1,1}) + (\gamma_{1,2})\big) + \big((\beta_{1,3}) + (\gamma_{1,4})\big) + (\beta_{1,1}) + (\beta_{1,2}) + (\beta_{1,3}) \\ &= 3(\beta_{1,1}) + (\beta_{1,2}) + 2(\beta_{1,3}) + (\gamma_{1,4}) + 2(\gamma_{1,2}). \end{aligned}$$

The bracketing sequence on Kostant partitions is

$$S_2\big(\Psi(T)\big) = \begin{array}{ccccc} 0\beta_{1,2} & 4\beta_{1,1} & 2\gamma_{1,2} & 0\gamma_{1,3} & 0\beta_{2,2}, \\ & ((((&)) & & \end{array}$$

so $f_2^{\mathrm{Kp}}\Psi(T) = \Psi(T) - (\beta_{1,1}) + (\beta_{1,1} + \alpha_2)$. Since $(\beta_{1,1} + \alpha_2) = (\beta_{1,2})$ this agrees with $\Psi(f_2^{\mathcal{T}} T)$.

Here $\operatorname{br}_2(T)$ and $S_2\big(\Psi(T)\big)$ have a different number of uncanceled left brackets. This is the $i = j$ case excluded from Lemma 3.3.

Proof of Theorem 3.1. We first consider the case in which $i < n$. If suffices to show that $f_i^{\mathrm{Kp}}\Psi(T) = \Psi(f_i^{\mathcal{T}} T)$. By the definition of the bracketing sequences and of Ψ,

$$\begin{gathered} \operatorname{br}_i(T) \text{ factors as } \operatorname{br}_i(R_1)\operatorname{br}_i(R_2)\cdots\operatorname{br}_i(R_{n-1}), \text{ and} \\ S_i(\Psi(T)) \text{ factors as } S_i\big(\Psi(R_1)\big)S_i\big(\Psi(R_2)\big)\cdots S_i\big(\Psi(R_{n-1})\big). \end{gathered}$$

By Lemma 3.3, for $j < i$, each $\operatorname{br}_i(R_j)$ has the same number of uncanceled left and right brackets as $S_i\big(\Psi(R_j)\big)$. For $j > i + 1$, it is clear that both $\operatorname{br}_i(R_j)$ or $S_i\big(\Psi(T)\big)$ are empty. As in Lemma 3.3, let p be the number of

$\overline{i+1}$ in R_i, q the number of $i+1$, r the number of i, and s the number of $\bar{\imath}$. Also, let r' be the number of $i+1$ on row $i+1$, and s' the number of $\overline{i+1}$ on row $i+1$. By direct calculation:

- $S_i\big(\Psi(R_i)\big) = {)}^{s+q-\min\{p,q\}}$, and $\mathrm{br}_i(R_i) = {)}^{s}{(}^{p}{)}^{q}{(}^{r}$, which in particular have the same number of uncanceled right brackets.
- $S_i\big(\Psi(R_{i+1})\big) = \varnothing$ and $\mathrm{br}_i(R_{i+1}) = {(}^{s'}\ {)}^{r'}$.

Since T is marginally large $r > r'$, so both subsequences $\mathrm{br}_i(R_i)\mathrm{br}_i(R_{i+1})$ and $S_i\big(\Psi(R_i)\big)S_i\big(\Psi(R_{i+1})\big)$ have the same number of uncanceled right brackets. Using this, if the leftmost uncanceled left bracket in $\mathrm{br}_i(T)$ comes from row j for $j < i$, then, by Lemma 3.3, this also holds for $S_i\big(\Psi(T)\big)$, and $\Psi\big(f_i^{\mathcal{T}}T\big) = f_i^{\mathrm{Kp}}\big(\Psi(T)\big)$.

Since $r > r'$ the leftmost uncanceled left bracket in $\mathrm{br}_i(T)$ cannot come from $\mathrm{br}_i(R_{i+1})$, so it remains to consider the case where it comes from $\mathrm{br}_i(R_i)$. Then $S_i\big(\Psi(T)\big)$ has no uncanceled left brackets, so f_i^{Kp} just adds a new $\alpha_i = \beta_{i,i}$. There are two cases for what can happen in $\mathrm{br}_i(T)$:

If $p > q$, then

$$\Psi(R_i) = s(\beta_{i,i}) + q(\beta_{i,i+1}) + q(\gamma_{i,i+2}) + (s+p-q)(\gamma_{i,i+1}),$$

$$f_i^{\mathcal{T}}R_i = \underbrace{\boxed{i}\boxed{\cdots}\boxed{i}}_{r}\underbrace{\boxed{i+1}\boxed{\cdots}\boxed{i+1}}_{q}\boxed{\cdots}\underbrace{\boxed{\overline{i+1}}\boxed{\cdots}\boxed{\overline{i+1}}}_{p-1}\underbrace{\boxed{\bar{\imath}}\boxed{\cdots}\boxed{\bar{\imath}}}_{s+1}$$

and

$$\Psi(f_i^{\mathcal{T}}R_i) = (s+1)(\beta_{i,i}) + q(\beta_{i,i+1}) + q(\gamma_{i,i+2}) + (s+p-q)(\gamma_{i,i+1}).$$

If $p \le q$, then

$$\Psi(R_i) = (s+q-p)(\beta_{i,i}) + p(\beta_{i,i+1}) + p(\gamma_{i,i+2}) + s(\gamma_{i,i+1}),$$

$$f_i^{\mathcal{T}}R_i = \underbrace{\boxed{i}\boxed{\cdots}\boxed{i}}_{r}\underbrace{\boxed{i+1}\boxed{\cdots}\boxed{i+1}}_{q+1}\boxed{\cdots}\underbrace{\boxed{\overline{i+1}}\boxed{\cdots}\boxed{\overline{i+1}}}_{p}\underbrace{\boxed{\bar{\imath}}\boxed{\cdots}\boxed{\bar{\imath}}}_{s}$$

and

$$\Psi(f_i^{\mathcal{T}}R_i) = (s+q-p+1)(\beta_{i,i}) + p(\beta_{i,i+1}) + p(\gamma_{i,i+2}) + s(\gamma_{i,i+1}).$$

In both cases, $\Psi(f_i^{\mathcal{T}}R_i) = \Psi(R_i) + (\beta_{i,i})$, as expected.

The $i = n$ case follows from the $i = n-1$ case using the Dynkin automorphism exchanging nodes $n-1$ and n, which acts on tableaux by interchanging the symbols $\bar{n}$ and n and on Kostant partitions by interchanging the roots $(\beta_{j,n-1})$ and $(\gamma_{j,n})$. $\square$

4. Stack notation

As mentioned in the introduction, this work is a type D_n analogue of a type A_n result in [3]. That result is described using the multisegments from [9, 12, 16], which are a convenient diagrammatic notation that makes the crystal structure apparent. By analogy, we now introduce *stack* notation for Kostant partitions in type D_n.

For $1 \le j \le k \le n-1$ and $1 \le \ell < m \le n$, make the association,

$$\beta_{j,k} = \begin{array}{c} k \\ \vdots \\ j \end{array}, \qquad \gamma_{\ell,m} = \begin{array}{c} m{+}1 \\ \vdots \\ n{-}2 \\ n{-}1 \;\; n \\ n{-}2 \\ \vdots \\ \ell \end{array},$$

Given $i \in I$, the set Φ_i from Definition 2.13 is the set of roots for which i may be either added or removed from the top of the stack to obtain a stack for another root. If $i \neq n$, the order imposed on Φ_i in Definition 2.13 is

$$\begin{array}{c} i \\ \vdots \\ 1 \end{array} < \begin{array}{c} i{-}1 \\ \vdots \\ 1 \end{array} < \begin{array}{c} i \\ \vdots \\ n{-}2 \\ n{-}1 \;\; n \\ n{-}2 \\ \vdots \\ 1 \end{array} < \begin{array}{c} i{+}1 \\ \vdots \\ n{-}2 \\ n{-}1 \;\; n \\ n{-}2 \\ \vdots \\ 1 \end{array} < \cdots < \begin{array}{c} i \\ i{-}1 \end{array} < i-1 < \begin{array}{c} i \\ \vdots \\ n{-}2 \\ n{-}1 \;\; n \\ n{-}2 \\ \vdots \\ i{-}1 \end{array} < \begin{array}{c} i{+}1 \\ \vdots \\ n{-}2 \\ n{-}1 \;\; n \\ n{-}2 \\ \vdots \\ i{-}1 \end{array} < i.$$

If $i = n$, then the order on Φ_n is

$$\begin{array}{c} n \\ n{-}2 \\ \vdots \\ 1 \end{array} < \begin{array}{c} n{-}2 \\ \vdots \\ 1 \end{array} < \begin{array}{c} n{-}1 \;\; n \\ n{-}2 \\ \vdots \\ 1 \end{array} < \begin{array}{c} n{-}1 \\ n{-}2 \\ \vdots \\ 1 \end{array} < \begin{array}{c} n \\ n{-}2 \\ \vdots \\ 2 \end{array} \cdots < \begin{array}{c} n \\ n{-}2 \end{array} < n-2 < \begin{array}{c} n{-}1 \;\; n \\ n{-}2 \end{array} < \begin{array}{c} n{-}1 \\ n{-}2 \end{array} < n.$$

The brackets in $S_i(\boldsymbol{\alpha})$ correspond to the stacks, and the crystal operators from Definition 2.14 act by adding or removing i from the top of an appropriate stack: f_i adds i to the top of the stack corresponding to the leftmost uncanceled '('.

Example 4.1. The Kostant partition from Example 2.16 written in stack notation is

$$\boldsymbol{\alpha} = 1\ 1\ 1\ 1\ 1\ \substack{34\\2\\1}\ \substack{2\\34\\2\\1}\ \substack{2\\34\\2\\1}\ \substack{2\\34\\2\\1}\ \substack{4\\2}\ \substack{4\\2}\ \substack{3\\2}\ \substack{34\\2}\ 3\ 4\ 4\ .$$

To find the bracket string $S_4(\boldsymbol{\alpha})$, we consider only those roots in Φ_4, and associate to each a bracket, in the following order:

$$\begin{array}{rccccccc} & \substack{34\\2\\1} & \substack{4\\2} & \substack{4\\2} & \substack{34\\2} & \substack{3\\2} & 4 & 4 \\ S_4(\boldsymbol{\alpha}) = &) &) &) &) & \cancel{(} & \cancel{)} &) \end{array}\ .$$

There is no uncanceled left bracket, so

$$f_4\boldsymbol{\alpha} = 1\ 1\ 1\ 1\ 1\ \substack{34\\2\\1}\ \substack{2\\34\\2\\1}\ \substack{2\\34\\2\\1}\ \substack{2\\34\\2\\1}\ \substack{4\\2}\ \substack{4\\2}\ \substack{3\\2}\ \substack{34\\2}\ 3\ 4\ 4\ 4\ .$$

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