Journal of Combinatorics
Volume 9, Number 3, 489–497, 2018

On the maximum number of colorings of a graph

Aysel Erey

Let $\mathcal{C}_k(n)$ be the family of all connected k-chromatic graphs of order n. Given a natural number $x \geq k$, we consider the problem of finding the maximum number of x-colorings among graphs in $\mathcal{C}_k(n)$. When $k \leq 3$ the answer to this problem is known, and when $k \geq 4$ the problem is wide open. For $k \geq 4$ it was conjectured that the maximum number of x-colorings is $x(x-1)\cdots(x-k+1)\,x^{n-k}$. In this article, we prove this conjecture under the additional condition that the independence number of the graphs is at most 2.

AMS 2000 subject classifications: Primary 05C15, 05C30, 05C31, 05C35.
Keywords and phrases: x-coloring, chromatic number, k-chromatic, chromatic polynomial.

1. Introduction

All graphs in this article are *simple*, that is, they do not have loops or multiple edges. Let $V(G)$ and $E(G)$ be the vertex set and edge set of a graph G, respectively. The *order* of G is $|V(G)|$ which is denoted by n_G, and the *size* of G is $|E(G)|$. For a nonnegative integer x, an *x-coloring* of G is a function $f : V(G) \to \{1, \dots, x\}$ such that $f(u) \neq f(v)$ for every $uv \in E(G)$. The *chromatic number* $\chi(G)$ is smallest x for which G has an x-coloring and G is called *k-chromatic* if $\chi(G) = k$. Let $\pi(G, x)$ denote the *chromatic polynomial of* G. For nonnegative integers x, $\pi(G, x)$ counts the number of x-colorings of G.

There has been a great interest in maximizing or minimizing the number of x-colorings over various families of graphs. Here we shall focus on the family of all connected graphs with fixed chromatic number and fixed order. Let $\mathcal{C}_k(n)$ be the family of all connected k-chromatic graphs of order n. Given a natural number $x \geq k$, we consider the problem of finding the maximum number of x-colorings among graphs in $\mathcal{C}_k(n)$. When $k \leq 3$ the answer to this problem is known. It is well known that (see, for example, [2]) for $k = 2$ and $x \geq 2$, the maximum number of x-colorings of a graph in $\mathcal{C}_2(n)$ is equal to

arXiv: 1610.07208

$x(x-1)^{n-1}$, and extremal graphs are trees when $x \geq 3$. Also, for $x \geq k = 3$, the maximum number of x-colorings of a graph in $\mathcal{C}_3(n)$ is

$$(x-1)^n - (x-1) \quad \text{for odd } n$$

and

$$(x-1)^n - (x-1)^2 \quad \text{for even } n$$

and furthermore the extremal graph is the odd cycle C_n when n is odd and odd cycle with a vertex of degree 1 attached to the cycle (denoted C^1_{n-1}) when n is even [4]. For $k \geq 4$, the problem is wide open. For $k \geq 4$, Tomescu [4] (see also [2, 3]) conjectured that the maximum number of x-colorings of a graph in $\mathcal{C}_k(n)$ is $(x)_{\downarrow k}(x-1)^{n-k} = x(x-1)\cdots(x-k+1)(x-1)^{n-k}$, and the extremal graphs are those which belong to the family of all connected k-chromatic graphs of order n with clique number k and size $\binom{k}{2} + n - k$, denoted by $\mathcal{C}^*_k(n)$.

Conjecture 1.1. *[2, pg. 315] Let G be a graph in $\mathcal{C}_k(n)$ where $k \geq 4$. Then for every $x \in \mathbb{N}$ with $x \geq k$*

$$\pi(G, x) \leq (x)_{\downarrow k}(x-1)^{n-k}.$$

Moreover, the equality holds if and only if G belongs to $\mathcal{C}^*_k(n)$.

Several authors studied this conjecture. Tomescu [4] proved this conjecture for $k = 4$ under the additional condition that graphs are planar. In [1], the authors proved this conjecture for every $k \geq 4$, provided that $x \geq n - 2 + \left(\binom{n}{2} - \binom{k}{2} - n + k\right)^2$. Our main result in this article is Theorem 2.1 which proves this conjecture for graphs whose independence numbers are at most 2 (i.e. complements of triangle-free graphs).

Let G/e be the graph formed from G by *contracting* edge e, that is, by identifying the ends of e (and taking the underlying simple graph). For $e \notin E(G)$, observe that

$$\chi(G) = \min\{\chi(G+e)\,,\, \chi(G/e)\}$$

and the well known *Edge Addition-Contraction Formula* says that

$$\pi(G, x) = \pi(G+e, x) + \pi(G/e, x).$$

Also, the chromatic polynomial of a graph can be computed by using the *Complete Cut-set Theorem*: If G_1 and G_2 are two graphs such that $G_1 \cap G_2 \cong$

K_r, then

$$\pi(G_1 \cup G_2, x) = \frac{\pi(G_1, x)\,\pi(G_2, x)}{(x)_{\downarrow r}}.$$

Let $G \cupdot H$ be the disjoint union of G and H, and $G \vee H$ be their join. It is easy to see that

$$\pi(G \vee K_1, x) = x\,\pi(G, x-1).$$

The maximum degree of a graph G is $\Delta(G)$, and a vertex v of G is *universal* if it is joined to all other vertices. In [1], Conjecture 1.1 was proven for graphs which contain a universal vertex.

Lemma 1.1. *[1] Let $G \in \mathcal{C}_k(n)$ and $\Delta(G) = n-1$. Then, for all $x \in \mathbb{N}$ with $x \geq k$, the inequality $\pi(G, x) \leq (x)_{\downarrow k}\,(x-1)^{n-k}$ holds. Furthermore, the equality is achieved if and only if $G = K_1 \vee (K_{k-1} \cupdot (n-k)K_1)$.*

Lastly, let $\omega(G)$ and $\alpha(G)$ be the clique number and independence number of G respectively.

2. Main results

Lemma 2.1. *Let $G \in \mathcal{C}_k(n)$ and $\omega(G) = k$. Then for all $x \in \mathbb{N}$ with $x \geq k$,*

$$\pi(G, x) \leq (x)_{\downarrow k}\,(x-1)^{n-k}$$

with equality if and only if $G \in \mathcal{C}_k^*(n)$.

Proof. Let H be a k-clique of G. If G has no cycle C such that $E(C)\backslash E(H) \neq \emptyset$ then $G \in \mathcal{C}_k^*(n)$ and the result is clear. So we assume that there exists a cycle C of G such that $E(C) \setminus E(H) \neq \emptyset$ (i.e. $G \notin \mathcal{C}_k^*(n)$). We may choose a cycle C such that $|E(C) \cap E(H)| \leq 1$, as H is a clique. Let G' be a minimal spanning connected subgraph of G which contains H and C. First we shall show that $\pi(G', x) = (x-1)_{\downarrow k-1}\,\pi(C, x)\,(x-1)^{n-k-n_C+1}$. If $|E(C) \cap E(H)| = 0$ (resp. $|E(C) \cap E(H)| = 1$), let G_1 and G_2 be two subgraphs of G' such that G_1 contains H, G_2 contains C, $G_1 \cup G_2 = G'$, and G_1 and G_2 intersect in a single vertex (resp. edge) of H. By the Complete Cut-set Theorem, if $|E(C) \cap E(H)| = 0$ then $\pi(G', x) = \frac{\pi(G_1,x)\pi(G_2,x)}{x}$, and if $|E(C) \cap E(H)| = 1$ then $\pi(G', x) = \frac{\pi(G_1,x)\pi(G_2,x)}{x(x-1)}$. In each case, $G_1 \in \mathcal{C}_k^*(n_{G_1})$ and G_2 is a connected unicyclic graph. Therefore,

$$\pi(G_1, x) = (x)_{\downarrow k}(x-1)^{n_{G_1}-k}$$

and

$$\pi(G_2, x) = \pi(C, x)(x-1)^{n_{G_2} - n_C}.$$

If $|E(C) \cap E(H)| = 0$ then $n_{G_1} + n_{G_2} = n + 1$, and if $|E(C) \cap E(H)| = 1$ then $n_{G_1} + n_{G_2} = n + 2$. Thus, we obtain $\pi(G', x) = (x-1)_{\downarrow k-1}\, \pi(C, x)\, (x-1)^{n-k-n_C+1}$. Also,

$$\begin{aligned}
\pi(G', x) &= (x-1)_{\downarrow k-1}\, \pi(C, x)\, (x-1)^{n-k-n_C+1} \\
&= (x-1)_{\downarrow k-1}\, \left((x-1)^{n_C} + (-1)^{n_C}(x-1)\right)\, (x-1)^{n-k-n_C+1} \\
&= (x-1)_{\downarrow k-1}\, \left((x-1)^{n-k+1} + (-1)^{n_C}(x-1)^{n-k-n_C+2}\right) \\
&< (x-1)_{\downarrow k-1}\, \left((x-1)^{n-k+1} + (x-1)^{n-k}\right) \\
&= (x)_{\downarrow k}\, (x-1)^{n-k}
\end{aligned}$$

where the inequality holds as $n_C \geq 3$. Now the result follows since $\pi(G', x) \geq \pi(G, x)$. □

A *cut-set* of a connected graph is a subset of the vertex set whose removal disconnects the graph. To prove our main result, we first deal with graphs which have a cut-set of size at most 2.

Proposition 2.1. *Let G be a connected k-chromatic graph with $\alpha(G) = 2$. If G has a stable cut-set S of size at most 2 then*

- *(i)* *$G \setminus S$ has exactly two connected components, say, G_1 and G_2,*
- *(ii)* *G_1 and G_2 are complete graphs,*
- *(iii)* $\max\{\chi(G_1), \chi(G_2)\} \geq k-1$,
- *(iv)* *For every u in S, either $V(G_1) \subseteq N_G(u)$ or $V(G_2) \subseteq N_G(u)$.*

Proof. (i) If $G \setminus S$ had more than two components then we could pick a vertex from each component and get a stable set of size at least 3. And this would contradict with the assumption that $\alpha(G) = 2$.

(ii) Suppose on the contrary that G_1 or G_2 is not a complete graph. Without loss, we may assume G_1 has two nonadjacent vertices u and v. Let w be a vertex of G_2. Then $\{u, v, w\}$ is a stable set of size 3 and again this contradicts with $\alpha(G) = 2$.

(iii) Suppose that $\chi(G_1)$ and $\chi(G_2)$ are at most $k-2$. Then we can properly color G_1 and G_2 with colors $1, \ldots, k-2$ and we can assign a new color $k-1$ to all vertices in S. This yields a proper $(k-1)$-coloring of G and this contradicts with the assumption that G is k-chromatic.

(iv) If there exists a vertex u in S such that u has a non-neighbor v in G_1 and a non-neighbor w in G_2 then we get a stable set $\{u, v, w\}$ of size 3 and this contradicts with $\alpha(G) = 2$. □

Note that if $G \in \mathcal{C}_k^*(n)$ and $\alpha(G) = 2$ then either G is a k-clique with a path of size one hanging off a vertex of the clique (denoted by $F_{1,k}$) or G is a k-clique with a path of size two hanging off a vertex of the clique (denoted by $F_{2,k}$).

Lemma 2.2. *Let $G \in \mathcal{C}_k(n)$ with $\alpha(G) = 2$. Let $x \in \mathbb{N}$ with $x \geq k$ and u be a cut-vertex of G. Then, $\pi(G, x) \leq (x)_{\downarrow k} (x-1)^{n-k}$. Furthermore, the equality holds if and only if $G \cong F_{1,k}$ or $G \cong F_{2,k}$.*

Proof. By Proposition 2.1, $G - u$ has exactly two connected components and they are complete graphs. Now it is easy to see that G is chordal and hence $\omega(G) = k$. Thus, the result follows by Lemma 2.1. □

Lemma 2.3. *Let G be a graph in $\mathcal{C}_k(n)$ with $\alpha(G) = 2$ and $k \geq 4$. If G has a stable cut-set of size 2 then*

$$\pi(G, x) \leq (x)_{\downarrow k} (x-1)^{n-k}$$

for all $x \in \mathbb{N}$ with $x \geq k$. Furthermore, the equality is achieved if and only if $G \cong F_{2,k}$.

Proof. Let $S = \{u, v\}$ be a stable cut-set of G. If $\omega(G) = k$ then the result follows from Lemma 2.1, so we may assume that $\omega(G) < k$. By Proposition 2.1, the graph $G \setminus S$ has exactly two connected components, say G_1 and G_2, and we may assume $G_1 \cong K_p$, $G_2 \cong K_q$ where $p \geq q$. Now, $p \geq k-1$ by Proposition 2.1 and $\omega(G) < k$ by the assumption. Therefore, $p = k-1$. Since $\omega(G) < k$, every vertex in S has at least one non-neighbor in G_1. Let u' and v' be two vertices of G_1 which are non-neighbors of u and v respectively.

Since $V(G_1) \nsubseteq N_G(u)$ and $V(G_1) \nsubseteq N_G(v)$, all vertices in S are adjacent to all vertices in G_2 by Proposition 2.1. The graph G_2 has at most $k-2$ vertices, as $\omega(G) < k$. If G_2 has less than $k-2$ vertices, then we can find a proper $k-1$ coloring c of G (we can first properly color the vertices of G_1 with colors $1, 2, \ldots k-1$ and assign $c(u')$ (resp. $c(v')$) to u (resp. v) and then we can properly color the vertices of G_2 with colors $\{1, 2, \ldots k-1\} \setminus \{c(u), c(v)\}$ which yields a proper $k-1$ coloring of G). Therefore G_2 has exactly $k-2$ vertices and $q = k-2$.

Since $\alpha(G) = 2$, the vertices u and v have no common non-neighbor. Therefore,

$$G/uv \cong K_1 \vee (K_{k-1} \cup K_{k-2}).$$

Now it is easy to see that

$$\pi(G/uv, x) = (x-1)_{\downarrow k-1}\,(x)_{\downarrow k-1}. \tag{1}$$

Let H_1 (resp. H_2) be the subgraph of $G + uv$ induced by the vertex set $V(G_1) \cup S$ (resp. $V(G_2) \cup S$). Now, the graphs H_1 and H_2 intersect at the edge uv in $G + uv$. Therefore,

$$\pi(G + uv, x) = \frac{\pi(H_1, x)\,\pi(H_2, x)}{x(x-1)}.$$

Since $H_2 \cong K_k$, we get $\pi(H_2, x) = (x)_{\downarrow k}$. Also, one of the vertices of S has a neighbor in G_1, as G is connected. So, H_1 contains a spanning subgraph which is isomorphic to a graph in $\mathcal{C}^*_{k-1}(k+1)$. Thus, $\pi(H_1, x) \le (x)_{\downarrow k-1}(x-1)^2$. Now,

$$\pi(G + uv, x) \le \frac{(x)_{\downarrow k}\,(x)_{\downarrow k-1}\,(x-1)^2}{x(x-1)} = (x-1)\,(x)_{\downarrow k-1}\,(x-1)_{\downarrow k-1}. \tag{2}$$

Using the edge addition-contraction formula and (1) and (2) we get

$$\begin{aligned}
\pi(G, x) &= \pi(G + uv, x) + \pi(G/uv, x) \\
&\le (x-1)\,(x)_{\downarrow k-1}\,(x-1)_{\downarrow k-1} \,+\, (x-1)_{\downarrow k-1}\,(x)_{\downarrow k-1} \\
&= (x)_{\downarrow k}\,(x)_{\downarrow k-1}.
\end{aligned}$$

The graph G has $2k-1$ vertices, so $(x)_{\downarrow k}\,(x-1)^{n-k} = (x)_{\downarrow k}\,(x-1)^{k-1}$. Now it is clear that

$$(x)_{\downarrow k}\,(x)_{\downarrow k-1} < (x)_{\downarrow k}\,(x-1)^{k-1}$$

holds for $k \ge 4$, as $(x)_{\downarrow k-1} = x(x-1)(x-2)\cdots$ and $x(x-2) < (x-1)^2$. □

Theorem 2.1. *Let G be a graph in $\mathcal{C}_k(n)$ with $\alpha(G) \le 2$ and $k \ge 4$. Then, for every $x \in \mathbb{N}$ with $x \ge k$,*

$$\pi(G, x) \le (x)_{\downarrow k}\,(x-1)^{n-k}.$$

Furthermore, the equality is achieved if and only if $G \cong F_{1,k}$, $G \cong F_{2,k}$ or $k = n$.

Proof. Since $\alpha(G)\chi(G) \ge n$, the equality $k = 4$ implies $n \le 8$. Computations show that the result holds to be true when $n \le 8$. So we may assume that $k \ge 5$. We proceed by induction on the number of vertices. For the basis step, $n = k$ and G is a complete graph. Hence, $\pi(G, x) = (x)_{\downarrow k}$ and now the result is clear.

Now we may assume that G is a k-chromatic graph of order at least $k+1$. By Lemma 2.2 and Lemma 2.3, we may assume that G has no stable cut-set of size at most 2. Also, if $\Delta(G) = n-1$ then the result follows by Lemma 1.1. Hence, we shall assume that $\Delta(G) < n-1$. Let u be a vertex of maximum degree. Set $t = n - 1 - \Delta(G)$ and let $\{v_1, \ldots, v_t\}$ be the set of non-neighbors of u in G, (that is, $\{v_1, \ldots, v_t\} = V(G) \setminus N_G[u]$). We set $G_0 = G$ and

$$G_i = G_{i-1} + uv_i$$
$$H_i = G_i / uv_i$$

for $i = 1, \ldots, t$. By applying the Edge Addition-Contraction Formula successively,

$$\pi(G, x) = \pi(G_t, x) + \sum_{i=1}^{t} \pi(H_i, x). \tag{3}$$

Note that $k \leq \chi(G_t) \leq k+1$ and $k \leq \chi(H_i) \leq k+1$ for $i = 1, 2, \ldots, t$. Since u is a universal vertex of G_t, we have

$$\pi(G_t, x) = x\,\pi(G - u, x - 1). \tag{4}$$

Clearly, $\alpha(G-u) \leq 2$. Also, $G-u$ is connected as G has no cut-vertex by the assumption. So, by the induction hypothesis,

$$\pi(G-u, x) \leq (x)_{\downarrow \chi(G-u)} (x-1)^{n-1-\chi(G-u)}.$$

Now replacing x with $x-1$ in the latter, we get

$$\pi(G-u, x-1) \leq (x-1)_{\downarrow \chi(G-u)} (x-2)^{n-1-\chi(G-u)}.$$

Note that $k-1 \leq \chi(G-u) \leq k$. Also, $(x-1)_{\downarrow k}(x-2)^{n-1-k} < (x-1)_{\downarrow k-1}(x-2)^{n-k}$. Therefore,

$$\pi(G-u, x-1) \leq (x-1)_{\downarrow k-1}(x-2)^{n-k}.$$

Since $(x)_{\downarrow k} = x(x-1)_{\downarrow k-1}$, by (4) we obtain that

$$\pi(G_t, x) \leq (x)_{\downarrow k}(x-2)^{n-k}. \tag{5}$$

Now we shall give an upper bound for $\pi(H_i, x)$ for all i. Observe that

$$H_i \cong K_1 \vee (G - \{u, v_i\})$$

because $\alpha(G) = 2$ and hence every vertex in $G - \{u, v_i\}$ is adjacent to either u or v_i in G. Therefore,

$$\pi(H_i, x) = x\,\pi(G - \{u, v_i\}, x - 1). \tag{6}$$

It is clear that $\alpha(G - \{u, v_i\}) \leq 2$. Since G has no stable cut-set of size 2, the graph $G - \{u, v_i\}$ is connected. Also, $k - 1 \leq \chi(G - \{u, v_i\}) \leq k$, as u and v_i are nonadjacent in G and $\chi(G) = k$. By the induction hypothesis,

$$\pi(G - \{u, v_i\}, x) \leq (x)_{\downarrow \chi(G-\{u,v_i\})}(x - 1)^{n-2-\chi(G-\{u,v_i\})}.$$

Now replacing x with $x - 1$ in the latter, we get

$$\pi(G - \{u, v_i\}, x - 1) \leq (x - 1)_{\downarrow \chi(G-\{u,v_i\})}(x - 2)^{n-2-\chi(G-\{u,v_i\})}.$$

Observe that $(x - 1)_{\downarrow k}(x - 2)^{n-k-2} < (x - 1)_{\downarrow k-1}(x - 2)^{n-k-1}$. Thus,

$$\pi(G - \{u, v_i\}, x - 1) \leq (x - 1)_{\downarrow k-1}(x - 2)^{n-k-1}.$$

Since $(x)_{\downarrow k} = x(x - 1)_{\downarrow k-1}$, by (6) we obtain that

$$\pi(H_i, x) \leq (x)_{\downarrow k}(x - 2)^{n-k-1}. \tag{7}$$

By (3), (5) and (7), we get

$$\begin{aligned} \pi(G, x) &\leq (x)_{\downarrow k}(x - 2)^{n-k} + (n - 1 - \Delta(G))(x)_{\downarrow k}(x - 2)^{n-k-1} \\ &= (x)_{\downarrow k}(x - 2)^{n-k-1}(x - 3 + n - \Delta(G)). \end{aligned}$$

Now, it suffices to show that $(x - 2)^{n-k-1}(x - 3 + n - \Delta(G)) \leq (x - 1)^{n-k}$. The graph G is neither a complete graph nor an odd cycle, so $\Delta(G) \geq k$ by Brook's Theorem. Hence, $n - \Delta(G) \leq n - k$. Now,

$$\begin{aligned} &(x - 3 + n - \Delta(G))(x - 2)^{n-k-1} \\ &\quad \leq (x - 3 + n - k)\,(x - 2)^{n-k-1} \\ &\quad = (x - 2 - 1 + n - k)\,(x - 2)^{n-k-1} \\ &\quad = (x - 2)^{n-k} - (x - 2)^{n-k-1} + (n - k)(x - 2)^{n-k-1} \\ &\quad < (x - 2)^{n-k} + (n - k)(x - 2)^{n-k-1} \\ &\quad \leq (x - 2 + 1)^{n-k} \\ &\quad = (x - 1)^{n-k} \end{aligned}$$

where the last inequality holds, as

$$(x-2+1)^{n-k} = (x-2)^{n-k} + (n-k)(x-2)^{n-k-1} + \binom{n-k}{2}(x-2)^{n-k-2} + \cdots .$$

Thus, $\pi(G,x) \leq (x)_{\downarrow k}(x-1)^{n-k}$ and the result follows. $\square$

Acknowledgements

I would like to thank an anonymous referee for his/her helpful comments and careful reading.

References

[1] J. Brown, A. Erey, New Bounds for Chromatic Polynomials and Chromatic Roots, *Discrete Math.* **388**(11) (2015) 1938–1946. MR3357779

[2] F. M. Dong, K. M. Koh, and K. L. Teo, *Chromatic Polynomials And Chromaticity Of Graphs*, World Scientific, London, (2005). MR2159409

[3] I. Tomescu, Le nombre des graphes connexes k-chromatiques minimaux aux sommets étiquetés, *C. R. Acad. Sci. Paris* **273** (1971) 1124–1126. MR0291027

[4] I. Tomescu, Maximal Chromatic Polynomials of Connected Planar Graphs, *J. Graph Theory* **14** (1990) 101–110. MR1037425

Aysel Erey
Department of Mathematics
University of Denver
Denver, CO 80208
USA
E-mail address: aysel.erey@gmail.com

Received 4 July 2016