

Shifted dual equivalence and Schur P -positivity

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By considering type B analogs of permutations and tableaux, we extend abstract dual equivalence to type B in two directions. In one direction, we define involutions on shifted tableaux that give a dual equivalence, thereby giving a new combinatorial proof of the Schur positivity of Schur Q - and P -functions. In another direction, we define an abstract shifted dual equivalence parallel to dual equivalence and prove that it can be used to establish Schur P -positivity of a function expressed as a sum of shifted fundamental quasisymmetric functions. As a first application, we give a new combinatorial proof that the product of Schur P -functions is Schur P -positive.

1. Introduction

Symmetric function theory can be harnessed by other areas of mathematics to answer fundamental enumerative questions. For example, multiplicities of irreducible components, dimensions of algebraic varieties, and various other algebraic constructions that require the computation of certain integers may often be translated to the computation of the coefficients of a given function in a particular basis. Often the chosen basis is the Schur functions, which arise as Frobenius characters of irreducible representations of the symmetric group and as Schubert polynomials for the cohomology ring of the Grassmannian. Thus a quintessential problem in symmetric functions is to prove that a given function has nonnegative integer coefficients when expressed as a sum of Schur functions.

In [Ass15], the author introduced dual equivalence graphs as a universal tool by which one can approach such problems. This tool has been applied to various important classes of symmetric functions, including LLT and Macdonald polynomials [Ass], k -Schur functions [AB12], and products of Schubert polynomials [ABS].

In this paper, we give a further application of dual equivalence to Schur Q - and P -functions [Sch11]. These functions arise in the study of projec-

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tive representation of the symmetric group [Ste89] as well as the cohomology classes dual to Schubert cycles in isotropic Grassmannians [Józ91, Pra91]. These functions enjoy many nice properties parallel to Schur functions [Mac95]. In particular, they form dual bases for an important subspace of symmetric functions. While they have long been known to be Schur positive [Sag87] and to have positive structure constants [Ste89], the new proofs we provide lay the foundation for a stronger extension of dual equivalence to type B. We define an abstract notion of shifted dual equivalence that offers a tool by which one can show that a given function has nonnegative coefficients when expanded in terms of Schur P -functions. As a first application, we consider the Schur P -expansion of a product Schur P -functions. Related work by [BHR] holds further applications.

This paper is organized as follows. In Section 2, we introduce the classic combinatorial objects and their type B analogs. We connect the combinatorics with symmetric and quasisymmetric functions in Sections 3 and 4. In Section 5, we review abstract dual equivalence, and we give an application to type B combinatorial objects in Section 6 proving that Schur P -functions are Schur positive. In Section 7, we generalize the definitions and theorems of dual equivalence to the type B setting and define an abstract notion of shifted dual equivalence. Our main result, Theorem 7.5, is that this provides a universal tool for establishing Schur P -positivity. In Section 8, we apply this new theory to give a new proof that the product of Schur P -functions is Schur P -positive.

2. Partitions and tableaux

The main combinatorial objects we study are partitions, tableaux, and permutations, with their type B analogs being strict partitions, shifted tableaux, and signed permutations.

A *partition* λ is a non-increasing sequence of positive integers, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$. A *strict partition* γ is a partition whose parts are strictly decreasing, i.e. $\gamma_1 > \gamma_2 > \dots > \gamma_\ell > 0$. The *size* of a partition is the sum of its parts, i.e. $\lambda_1 + \lambda_2 + \dots + \lambda_\ell$.

We identify a partition λ with its Young diagram, the collection of left-justified cells with λ_i cells in row i . For a strict partition γ , the shifted Young diagram is the Young diagram with row i shifted $\ell(\gamma) - i$ cells to the left. It will often be useful to consider the shifted symmetric diagram for γ , which is obtained by adjoining the reflection of the shifted diagram. For examples, see Figure 1.

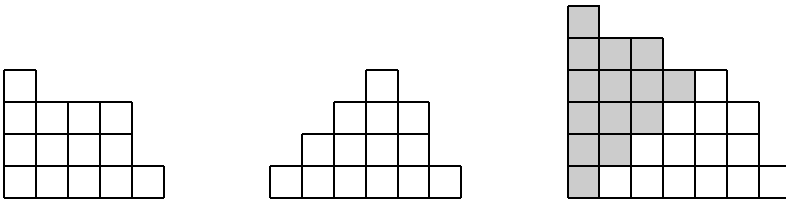


Figure 1: The Young diagram for $(5, 4, 4, 1)$, and the shifted Young diagram and shifted symmetric diagram for $(6, 4, 3, 1)$.

A *semi-standard Young tableau* of shape λ is a filling of the Young diagram for λ with positive integers such that entries weakly increase along rows and strictly increase up columns. For example, see Figure 2.

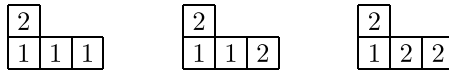


Figure 2: The semi-standard Young tableaux of shape $(3, 1)$ with entries in $\{1, 2\}$.

A *semi-standard shifted tableau* of shape γ is a filling of the shifted Young diagram for γ with marked or unmarked positive integers such that entries weakly increase along rows and columns according to the ordering $1' < 1 < 2' < 2 < \dots$, each row has at most one marked entry i' for each i and each column has at most one unmarked entry i for each i . For examples, see Figure 3.



Figure 3: The semi-standard shifted tableaux of shape $(3, 1)$ with entries in $\{1', 1, 2', 2\}$ and no marked entries on the main diagonal.

Note that these latter conditions allow any entry along the main diagonal to be marked or unmarked without changing the allowed entries for other positions. If one considers instead the shifted symmetric diagram and leaves unmarked letters in place while reflecting the marked letters, then the conditions for a semi-standard tableau are the same for straight and for shifted shapes. For example, see Figure 4.

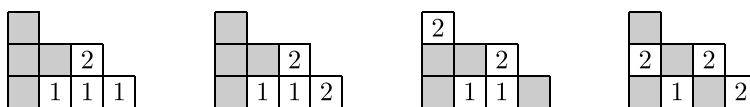


Figure 4: The semi-standard shifted symmetric tableaux of shape $(3, 1)$ with entries in $\{1, 2\}$ and no reflected entries on the main diagonal.

The *reading word* of a semi-standard tableau T , denoted $w(T)$, is the word obtained by reading the rows of T left to right, from top to bottom. For example, the reading words for the tableaux in Figure 2 from left to right are 2111, 2112, 2122, and the reading words for the shifted symmetric tableaux in Figure 4 from left to right are 2111, 2112, 2211, 2212.

A *permutation* of n is an ordering of the numbers $\{1, 2, \dots, n\}$. A semi-standard tableau T is *standard* if its reading word is a permutation. For example, see Figures 5 and 6.

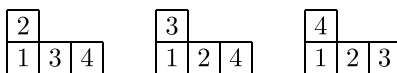


Figure 5: The standard Young tableaux of shape $(3, 1)$.

A semi-standard shifted symmetric tableau is a *standard marked tableau* if its reading word is a permutation and it has entries in the reflected side. For example, each of the two standard shifted tableaux in Figure 6 has 2^4 marked analogs of which 2^2 have no signs on the main diagonal; see Figure 9.

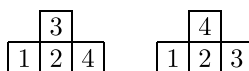


Figure 6: The standard shifted tableaux of shape $(3, 1)$.

The *descent set* of a permutation is given by

$$(2.1) \quad \text{Des}(w) = \{i \mid i \text{ right of } i + 1\}.$$

When w is a permutation of length n , we have $\text{Des}(w) \subseteq \{1, 2, \dots, n - 1\}$. When we wish to emphasize n , we write Des_n . Note that there are 2^{n-1} possible descent sets for permutations of length n . The *descent set* of a standard tableau or a marked standard tableau is the descent set of its

reading word. For the tableaux in Figure 5, the descent sets from left to right are $\{1\}, \{2\}, \{3\}$, and for the tableaux in Figure 6, the descent sets from left to right are $\{2\}, \{3\}$.

In addition to the descent set, we will often be interested in the *peak set* and the *spike set*, which can be derived directly from the descent set. For a set D , we have

$$(2.2) \quad \text{Spike}(D) = \{i \mid i - 1 \notin D \text{ and } i \in D \text{ or } i - 1 \in D \text{ and } i \notin D\},$$

$$(2.3) \quad \text{Peak}(D) = \{i \mid i - 1 \notin D \text{ and } i \in D\}.$$

Note that if $D \subseteq \{1, 2, \dots, n - 1\}$, then $\text{Spike}(D), \text{Peak}(D) \subseteq \{2, 3, \dots, n - 1\}$. Furthermore, peak sets are characterized as subsets containing no consecutive entries. Thus there are 2^{n-2} possible spike sets and F_n , the n th Fibonacci number, possible peak sets for permutations of length n . As with descents, when we wish to emphasize n , we write Spike_n or Peak_n .

3. Symmetric functions

We follow notation from [Mac95] for the classic bases for Λ , the ring of symmetric functions. The space Λ_n of symmetric functions homogeneous of degree n has dimension equal to the number of partitions of n , and so bases for Λ_n are naturally indexed by partitions of n . The most fundamental basis for Λ_n is the Schur function basis, which may be defined by

$$(3.1) \quad s_\lambda(X) = \sum_{T \in \text{SSYT}(\lambda)} X^T,$$

where $\text{SSYT}(\lambda)$ denotes the set of all semi-standard Young tableaux of shape λ , and X^T is the monomial where x_i occurs in X^T with the same multiplicity with which i occurs in T . For example, the three tableaux in Figure 2 contribute $x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3$ to the Schur function $s_{(3,1)}(X)$.

The irreducible characters of the symmetric group map under the Frobenius isomorphism to Schur functions. Therefore Schur functions are fundamental to understanding representations of the symmetric group.

Schur Q -functions, indexed by strict partitions, are given by

$$(3.2) \quad Q_\gamma(X) = \sum_{S \in \text{SSShT}(\gamma)} X^{|S|},$$

where $\text{SSShT}(\gamma)$ denotes the set of all semi-standard shifted tableaux of shifted shape γ , and $X^{|S|}$ is the monomial where x_i occurs in $X^{|S|}$ with

the same multiplicity with which i and i' occur in S . For example, the four tableaux in Figure 3 contribute $x_1^3x_2 + 2x_1^2x_2^2 + x_1x_2^3$ to the Schur Q -function $Q_{(3,1)}(X)$.

Schur P -functions indexed by strict partitions are given by

$$(3.3) \quad P_\gamma(X) = 2^{-\ell(\gamma)}Q_\gamma(X) = \sum_{S \in \text{SSShT}^*(\gamma)} X^{|S|},$$

where $\text{SSShT}^*(\gamma)$ denotes the set of all semi-standard shifted tableaux of shifted shape γ where the main diagonal has no marked entries, and $X^{|S|}$ is again the monomial where x_i occurs in $X^{|S|}$ with the multiplicity with which i and i' occur in S . The second equality follows easily from the first if one notes that the rules for which entries may be marked never precludes a marked entry along the main diagonal.

Schur P -functions are fundamental to understanding projective representations of the symmetric group [Ste89] similar to the role of Schur functions for linear representations.

Let $\Gamma \subset \Lambda$ denote the subspace of symmetric functions generated by the odd power sum symmetric functions. The graded component $\Gamma_n = \Gamma \cap \Lambda_n$ has dimension equal to the number of strict partitions of n . The Schur Q - and P -functions may be realized as specializations of Hall-Littlewood functions at $t = -1$ [Mac95]. For λ a nonstrict partition, the specialization $Q_\lambda(X; -1)$ vanishes, but for λ strict these specializations form dual bases for Γ_n .

Since the Schur Q - and P -functions are symmetric, they can be expanded in the Schur basis. Since the Schur P -functions form a basis for Γ , the product of two Schur P -functions may be expanded in the Schur P -function basis. The machinery we develop in this paper reproves the following positivity results.

Theorem 3.1. *For γ, δ strict partitions, if*

$$(3.4) \quad P_\gamma(X) = \sum_\lambda g_{\gamma,\lambda} s_\lambda(X) \quad \text{and} \quad P_\gamma(X)P_\delta(X) = \sum_\varepsilon f_{\gamma,\delta}^\varepsilon P_\varepsilon(X),$$

then $g_{\gamma,\lambda}, f_{\gamma,\delta}^\varepsilon$ are nonnegative integers.

Stanley conjectured the positivity of $g_{\gamma,\lambda}$, which follows as a corollary to Sagan’s shifted insertion [Sag87] independently developed by Worley [Wor84]. These ideas were extended by Stembridge [Ste89] in his study of projective representations of the symmetric group to give a proof of the positivity of $f_{\gamma,\delta}^\varepsilon$. More recently, Cho built on work of Serrano [Ser10] to give another positivity proof.

One of the main results of this paper is to give another combinatorial proof of Theorem 3.1 using dual equivalence and shifted dual equivalence, respectively.

4. Quasisymmetric functions

The space of quasisymmetric functions contains Λ and provides nice intermediate bases for (3.1) and for (3.2), (3.3). The subspace of quasisymmetric functions homogeneous of degree n has dimension 2^{n-1} and, as such, is naturally indexed by subsets of $\{1, 2, \dots, n - 1\}$. Gessel’s fundamental basis for quasisymmetric functions [Ges84] is given by

$$(4.1) \quad F_D(X) = \sum_{\substack{i_1 \leq \dots \leq i_n \\ j \in D \Rightarrow i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}.$$

Implicit in our notation is that D is regarded as a descent set for objects of size n . When we wish to make this explicit, we write $F_{n,D}$ or F_{D_n} .

One great advantage of quasisymmetric functions is that they facilitate the use of standard in place of semi-standard objects, allowing us to use a finite number of terms in the expression even when there is an infinite number of variables. For example, we have the following expansion for Schur functions due to Gessel[Ges84].

Theorem 4.1 ([Ges84]). *For λ a partition of n , we have*

$$(4.2) \quad s_\lambda(X) = \sum_{T \in \text{SYT}(\lambda)} F_{\text{Des}(T)}(X),$$

where $\text{SYT}(\lambda)$ denotes the set of all standard Young tableaux of shape λ .

For example, for $n = 4$, we have

$$\begin{aligned} s_{(3,1)} &= F_{\{1\}} + F_{\{2\}} + F_{\{3\}} \\ s_{(2,2)} &= F_{\{1,3\}} + F_{\{2\}} \\ s_{(2,1,1)} &= F_{\{1,2\}} + F_{\{1,3\}} + F_{\{2,3\}} \end{aligned}$$

Analogously, we may express the Schur P -functions in terms of the fundamental basis.

Proposition 4.2. *For γ a strict partition of n , we have*

$$(4.3) \quad Q_\gamma(X) = \sum_{S \in \text{SShT}_\pm(\gamma)} F_{\text{Des}(S)}(X) = 2^{\ell(\gamma)} \sum_{S \in \text{SShT}'_\pm(\gamma)} F_{\text{Des}(S)}(X)$$

$$(4.4) \quad P_\gamma(X) = \sum_{S \in \text{SShT}'_{\pm}(\gamma)} F_{\text{Des}(S)}(X),$$

where $\text{SShT}_{\pm}(\gamma)$ denotes the set of all marked standard tableaux of shape γ , and $\text{SShT}'_{\pm}(\gamma)$ denotes the subset where no entry on the main diagonal is marked.

Proof. The formula follows from (3.3) by standardizing the reading word while maintaining the positions of the marked letters. This is well-defined given the chosen total order. □

For example, signing the entries in Figure 6, in all possible ways gives

$$P_{(3,1)} = F_{\{1\}} + 2F_{\{2\}} + F_{\{3\}} + F_{\{1,2\}} + 2F_{\{1,3\}} + F_{\{2,3\}}.$$

The similarity between (4.2) and (4.4) is the key to our proof of Theorem 3.1. In particular, it is easy to compute from the above expansion that

$$P_{(3,1)} = s_{(3,1)} + s_{(2,2)} + s_{(2,1,1)}.$$

However, notice that the summation in (4.4) is not over *standard* objects for type B. For this, we need a new family of quasisymmetric functions.

For $P \subseteq \{2, 3, \dots, n - 1\}$ with no consecutive entries, define the shifted fundamental quasisymmetric function $G_P(X)$ by

$$(4.5) \quad G_P(X) = \sum_{P \subseteq \text{Spike}(D)} F_D(X),$$

where the sum is over all subsets $D \subseteq \{1, 2, \dots, n - 1\}$ for which $\text{Spike}(D)$ contains P . Again, when we wish to emphasize n , we may write $G_{n,P}$ or G_{P_n} . For example, for $n = 4$, we have

$$\begin{aligned} G_{\{2\}}(X) &= F_{\{1\}}(X) + F_{\{2\}}(X) + F_{\{1,3\}}(X) + F_{\{2,3\}}(X), \\ G_{\{3\}}(X) &= F_{\{2\}}(X) + F_{\{3\}}(X) + F_{\{1,2\}}(X) + F_{\{1,3\}}(X). \end{aligned}$$

The shifted fundamental quasisymmetric functions of degree n form a basis for a subspace of quasisymmetric functions of degree n of dimension the n th Fibonacci number. The shifted fundamental quasisymmetric functions allow us to rewrite the Schur P -functions as follows.

Theorem 4.3. *For γ a strict partition of n , we have*

$$(4.6) \quad Q_\gamma(X) = \sum_{S \in \text{SShT}(\gamma)} 2^{|\text{Peak}(S)|+1} G_{\text{Peak}(S)}(X)$$

$$(4.7) \quad P_\gamma(X) = 2^{-\ell(\gamma)} \sum_{S \in \text{SShT}(\gamma)} 2^{|\text{Peak}(S)|+1} G_{\text{Peak}(S)}(X),$$

where $\text{SShT}(\gamma)$ denotes the set of all standard shifted tableaux of shape γ , and $\text{Peak}(S) = \text{Peak}(\text{Des}(S))$.

Note that for any $S \in \text{SShT}(\gamma)$, $|\text{Peak}(S)| \geq \ell(\gamma) - 1$, so the expansion of P_γ in terms of the shifted fundamental quasisymmetric functions is integral.

Proof. Fix $S \in \text{SShT}(\gamma)$, and suppose $T \in \text{SShT}_\pm(\gamma)$ is such that removing the markings on T gives S . If $i - 1 \notin \text{Des}(S)$, then $i - 1 \in \text{Des}(T)$ if and only if i is marked, and if $i - 1 \in \text{Des}(S)$, then $i - 1 \in \text{Des}(T)$ if and only if $i - 1$ is unmarked.

First we claim that any $T \in \text{SShT}_\pm(\gamma)$ that gives S when the markings are removed satisfies $\text{Peak}(S) \subseteq \text{Spike}(\text{Des}(T))$. To see this, note that $i \in \text{Peak}(S)$ if and only if both $i - 1 \notin \text{Des}(S)$ and $i \in \text{Des}(S)$. By the previous analysis, if i is marked in T , then $i - 1 \in \text{Des}(T)$ and $i \notin \text{Des}(T)$, and if i is unmarked in T , then $i - 1 \notin \text{Des}(T)$ and $i \in \text{Des}(T)$. Therefore $i \in \text{Spike}(\text{Des}(T))$.

Next we claim that for any $D \subset [n - 1]$ for which $\text{Peak}(S) \subseteq \text{Spike}(D)$, there are exactly $2^{|\text{Peak}(S)|+1}$ standard marked tableaux T that give S when the markings are removed for which $\text{Des}(T) = D$. Indeed, for $i \in \text{Peak}(S)$, set $h = \min\{k \mid k < i \text{ and } k \notin \text{Des}(S)\}$ and set $j = \max\{k \mid k > i \text{ and } k \in \text{Des}(S)\}$. Then, by the analysis above, D determines the markings for all $h < k \leq i$ and all $i \leq k < j$, but toggling the marking for h or j does not change D . If $i < i'$ are consecutive entries of $\text{Peak}(S)$, then the j for i and the h for i' coincide. Thus there are exactly $|\text{Peak}(S)| + 1$ letters that can be marked or unmarked without affecting D .

These two claims prove the expansion for Q_γ , and the result for P_γ follows by (3.3). \square

For example, using Figure 6, we compute

$$P_{(3,1)} = G_{\{2\}} + G_{\{3\}}.$$

5. Dual equivalence and Schur positivity

Haiman [Hai92] defined *elementary dual equivalence involutions* on permutations as follows. If a, b are two consecutive letters of the word w , and c

is also consecutive with a, b and appears between a and b in w , then interchanging a and b is an elementary dual equivalence move. In this case, we refer to c as the *witness* for the dual equivalence interchanging a and b . When $\{a, b, c\} = \{i - 1, i, i + 1\}$, we denote this involution by d_i , and we regard words with c not between a and b as fixed points for d_i . For examples, see Figure 7.

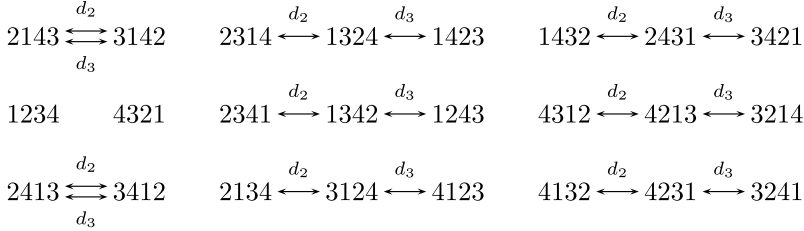


Figure 7: The dual equivalence classes of permutations of length 4.

Two permutations w and u are *dual equivalent* if there exists a sequence i_1, \dots, i_k such that $u = d_{i_k} \cdots d_{i_1}(w)$. Haiman [Hai92] showed that the dual equivalence involutions extend to standard Young tableaux via their reading words and that dual equivalence classes correspond precisely to all standard Young tableaux of a given shape, e.g. see Figure 8.

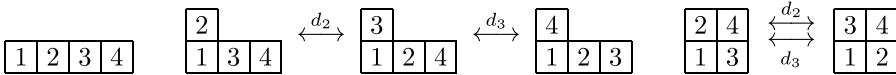


Figure 8: Three dual equivalence classes of SYT of size 4.

Given this, we may rewrite (4.2) in terms of dual equivalence classes as

$$(5.1) \quad s_\lambda(X) = \sum_{T \in [T_\lambda]} F_{\text{Des}(T)}(X),$$

where $[T_\lambda]$ denotes the dual equivalence class of some fixed $T_\lambda \in \text{SYT}(\lambda)$.

This paradigm shift to summing over objects in a dual equivalence class is the basis for the universal method for proving that a quasisymmetric generating function is symmetric and Schur positive [Ass15]. Motivated by (5.1), we have the abstract notion of dual equivalence for any set of objects endowed with a descent set.

Given involutions $\varphi_2, \dots, \varphi_{n-1}$ on a set \mathcal{A} of combinatorial objects, for $1 < h < i < n$ we consider the *restricted dual equivalence class* $[T]_{(h-1, i+1)}$

generated by $\varphi_h, \dots, \varphi_i$. In addition, we consider the *restricted and shifted descent set* $\text{Des}_{(h,i)}(T)$ obtained by intersecting $\text{Des}(T)$ with $\{h - 1, \dots, i\}$ and subtracting $h - 2$ from each element so that $\text{Des}_{(h,i)}(T) \subseteq [i - h + 2]$.

Definition 5.1 ([Ass15]). Let \mathcal{A} be a finite set, and let Des be a map on \mathcal{A} such that $\text{Des}(T) \subseteq [n - 1]$ for all $T \in \mathcal{A}$. A *dual equivalence* for $(\mathcal{A}, \text{Des})$ is a family of involutions $\{\varphi_i\}_{1 < i < n}$ on \mathcal{A} such that

- i. For all $i - h \leq 3$ and all $T \in \mathcal{A}$, there exists a partition λ of $i - h + 3$ such that

$$\sum_{U \in [T]_{(h,i)}} F_{\text{Des}_{(h,i)}(U)}(X) = s_\lambda(X).$$

- ii. For all $|i - j| \geq 3$ and all $T \in \mathcal{A}$, we have

$$\varphi_j \varphi_i(T) = \varphi_i \varphi_j(T).$$

By (5.1), dual equivalence classes of tableaux precisely correspond to Schur functions. Definition 5.1 was formulated so that the same property holds true for dual equivalence classes for any pair $(\mathcal{A}, \text{Des})$.

Theorem 5.2 ([Ass15]). *If $\{\varphi_i\}$ is a dual equivalence for $(\mathcal{A}, \text{Des})$, and $U \in \mathcal{A}$, then*

$$(5.2) \quad \sum_{T \in [U]} F_{\text{Des}(T)}(X) = s_\lambda(X)$$

for some partition λ . In particular, the quasisymmetric generating function for \mathcal{A} is symmetric and Schur positive.

Our first goal, undertaken in Section 6, is to construct dual equivalence involutions for the set $\text{SShT}_\pm(\gamma)$ of marked standard shifted tableaux of shifted shape γ with descent function given by Des . Our second goal, undertaken in Section 7, is to give a version of Theorem 5.2 that proves Schur P -positivity.

6. Dual equivalence for marked tableaux

The objects for which we will construct a dual equivalence are the marked standard tableaux, identified as shifted symmetric tableaux, with the associated descent function given by applying (2.1) to the reading word. Define the *diagonal* of a cell to be its row index minus its column index. In particular, the main diagonal of the unreflected shifted shape is zero, with the cells below positive and the reflected cells above negative.

Definition 6.1. Let S be a marked standard tableau. For $1 < i < n$, let $a \leq b \leq c$ be the diagonals on which $i - 1, i, i + 1$ reside. Then $\psi_i(w)$ is given by the following rule:

- if i lies on diagonal b , then $\psi_i(S) = S$;
- else if $a = b$ (respectively $b = c$), then reflect the occupant on diagonal c (respectively a) to lie on diagonal $-c$ (respectively $-a$);
- else if $||a| - |c|| = 1$, then reflect the occupant on diagonals a and c to lie on diagonal $-a$ and $-c$, respectively;
- else swap the occupants of diagonals a and c .

For examples of ψ on marked standard tableaux, see Figure 9.

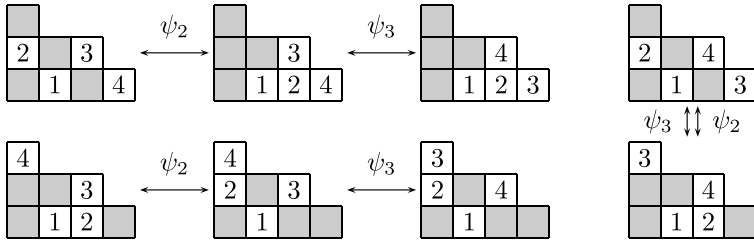


Figure 9: Dual equivalence for $\text{SShT}_{\pm}(3, 1)$.

Proposition 6.2. *The maps $\{\psi_i\}$ give well-defined involutions on the set of marked standard tableaux. Furthermore, if S has no marked entries along the main diagonal, then neither does $\psi_i(S)$ for any i .*

Proof. Clearly when S is a fixed point for ψ_i , it remains so after doing nothing. Suppose, then, that this is not the case. There are three cases to consider based on how ψ_i acts. A key observation in what follows is that two of a, b, c are equal in absolute value if and only if both $i - 1$ and $i + 1$ have diagonal 0. Indeed, if this is not the case then there is a cell southeast of the diagonal in question containing an entry x that lies strictly between two of $i - 1, i, i + 1$. This forces $x = i$, so the entries of the two cells in question are $i - 1$ and $i + 1$. Moreover, if this is not the main diagonal, then there exists a cell northwest of the diagonal in question containing an entry $y \neq i$ with $i - 1 < y < i + 1$, which is not possible.

First, suppose that $a = b < c$. From the observation above, we must have $0 = a = b < c$, in which case $-c < a = b < c$. Therefore ψ_i toggles the marking on the entry in c , and so is an involution for this case and the related case $a < b = c$.

Second, suppose $a < b < c$ and $||a| - |c|| = 1$. The latter constraint means that in the standard shifted tableau obtained by removing markings, the entries corresponding to a and c form a vertical or horizontal domino. In particular, the former assumption ensures that neither a nor c can be 0. Therefore we have $a < 0 < c$, and so $-c < b < -a$. Again, ψ_i preserves the case and acts as an involution.

Finally, suppose $a < b < c$ and $||a| - |c|| > 1$. In this case, ψ_i clearly preserves the case, so we need only show that the result is a marked standard tableau. Since i does not have diagonal b , the entries being swapped are consecutive, implying that the only potential row or column violation is with the two swapped entries. The latter condition above ensures that, in the standard shifted tableau obtained by removing the markings, the swapped entries do not lie in the same row or column, and so no violations can result. □

Theorem 6.3. *The maps $\{\psi_i\}_{1 < i < n}$ give a dual equivalence for marked standard tableaux. In particular, the Schur Q - and P -functions are Schur positive.*

Proof. The definition of ψ_i depends only on the positions of the entries $i - 1, i, i + 1$. If $|i - j| \geq 3$, then the relevant entries are disjoint, and so ψ_i and ψ_j must commute. Therefore by Definition 5.1, it is enough to show that the equivalence classes are single Schur functions for all marked standard tableaux of skew shape $\gamma \setminus \delta$, where $\delta \subset \gamma$ and $|\gamma| - |\delta| \leq 6$. From the definition of ψ , if two consecutive diagonals have no cells, then we may collapse them to one empty diagonal without changing the structure of the equivalence class. In particular, we may assume $\gamma \subset (11, 9, 7, 5, 3, 1)$. More precisely, the number of skew shapes to check for $n = 3, 4, 5, 6$ is 10, 31, 98, 307, respectively. These cases can be (and have been) checked by a patient hand or by a computer. □

The first half of Theorem 3.1 now follows from Theorems 5.2 and 6.3.

Corollary 6.4. *For γ a strict partition, we have*

$$(6.1) \quad P_\gamma(X) = \sum_\lambda g_{\gamma,\lambda} s_\lambda(X),$$

where $g_{\gamma,\lambda}$ is the number of dual equivalence classes of marked standard tableaux of shape γ under the action of $\{\psi_i\}$ that are isomorphic to $\text{SYT}(\lambda)$. In particular, $g_{\gamma,\lambda} \in \mathbb{N}$.

7. Shifted dual equivalence graphs

Haiman [Hai92] also defined *elementary shifted dual equivalence involutions* on permutations as follows. If a, b are two consecutive letters of the word w , c is also consecutive with a, b and appears between a and b in w , and d is also consecutive with a, b, c and appears left of c in w , then interchanging a and b is an elementary shifted dual equivalence move. In this case, we again refer to c as the *witness*, and we refer to d as the *bystander* for the shifted dual equivalence interchanging a and b . When $\{a, b, c, d\} = \{i - 1, i, i + 1, i + 2\}$, we denote this involution by b_i , and we regard words with c not between a and b or with d right of c as fixed points for b_i . This rule is illustrated in Figure 10.

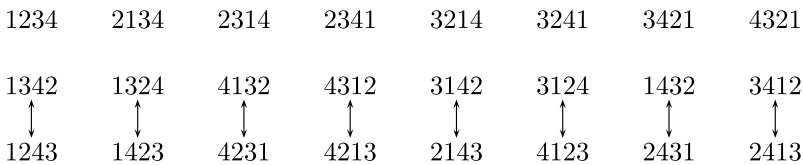


Figure 10: The shifted dual equivalence classes of permutations of length 4.

Haiman [Hai92] showed that the shifted dual equivalence involutions extend to standard shifted tableaux via their reading words and that dual equivalence classes correspond precisely to all standard shifted tableaux of a given shape. For examples, see Figures 11 and 12.

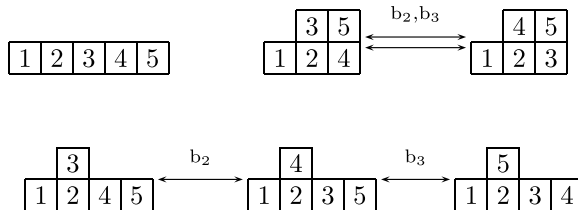


Figure 11: The shifted dual equivalence classes of SSHT of size 5.

Comparing Figure 11 with Figure 8, it might seem that shifted dual equivalence classes are the same as dual equivalence classes. However, shifted classes can have triple edges, whereas dual equivalence classes can have at most double edges, so the equality is an artifact of small numbers. To make this statement precise, we introduce the notion of a morphism between dual equivalences.

Definition 7.1. Let \mathcal{A}, \mathcal{B} be two sets of combinatorial objects, and let $\text{Des}_{\mathcal{A}}$ and $\text{Des}_{\mathcal{B}}$ be descent maps on each. Given involutions α_i on \mathcal{A} and β_i on \mathcal{B} , a *morphism* from $(\mathcal{A}, \text{Des}_{\mathcal{A}}, \alpha)$ to $(\mathcal{B}, \text{Des}_{\mathcal{B}}, \beta)$ is a map $\phi : \mathcal{A} \rightarrow \mathcal{B}$ such that for every $a \in \mathcal{A}$, we have $\text{Des}_{\mathcal{A}}(a) = \text{Des}_{\mathcal{B}}(\phi(a))$ and $\phi(\alpha_i(a)) = \beta_i(\phi(a))$. A morphism is an *isomorphism* if it is a bijection from \mathcal{A} to \mathcal{B} .

To avoid cumbersome notation, we omit the subscript for Des when it is clear from context.

Since dual equivalence pertains to descent sets and shifted dual equivalence pertains to peak sets, we also need to shift a peak set as follows. Given a subset D of positive integers greater than 1, let $D - 1$ be the subset obtained by subtracting 1 from each element of D .

Proposition 7.2. *For nonnegative integers $r > s$, the shifted dual equivalence for $(\text{SShT}((r, s)), \text{Peak} - 1)$ given by $\{b_i\}$ is isomorphic to the dual equivalence on $(\text{SYT}((r - 1, s)), \text{Des})$ given by $\{d_i\}$. For γ a strict partition with more than 2 parts, the shifted dual equivalence on $(\text{SShT}(\gamma), \text{Peak} - 1)$ given by $\{b_i\}$ is not isomorphic to $(\text{SYT}(\lambda), \text{Des})$ given by $\{d_i\}$ for any partition λ .*

Proof. Consider the map ϕ from $\text{SShT}((r, s))$ to $\text{SYT}((r - 1, s))$ given by removing the cell containing 1, subtracting 1 from each entry. On the level of sets, ϕ is clearly a bijection. One easily checks that, in addition, $\text{Peak}(T) - 1 = \text{Des}(\phi(T))$, and $\phi(b_{i+1}(T)) = d_i(\phi(T))$. Therefore ϕ is an isomorphism of dual equivalences.

The shifted tableau T of shape $\gamma = (3, 2, 1)$ with reading word 645123 has $b_2 = b_3 = b_4$. Any strict partition with at least 3 parts must contain γ , and so it contains an element that restricts to T . In particular, such an element has $b_2 = b_3 = b_4$, and so the equivalence cannot be isomorphic to any dual equivalence on standard Young tableaux. \square

Completely analogous to the unshifted case, for γ a strict partition, we may rewrite (4.7) in terms of dual equivalence classes as

$$(7.1) \quad Q_{\gamma}(X) = \sum_{T \in [T_{\gamma}]} 2^{|\text{Peak}(T)|+1} G_{\text{Peak}(T)}(X)$$

$$(7.2) \quad P_{\gamma}(X) = 2^{-\ell(\gamma)} \sum_{T \in [T_{\gamma}]} 2^{|\text{Peak}(T)|+1} G_{\text{Peak}(T)}(X),$$

where $[T_{\gamma}]$ denotes the shifted dual equivalence class of some fixed $T_{\gamma} \in \text{SShT}(\gamma)$.

By (7.2), shifted dual equivalence classes of standard shifted tableaux precisely correspond to Schur Q -functions or Schur P -functions, depending on the chosen scaling. Following the analogy, our goal is to use this paradigm shift to summing over objects in a shifted dual equivalence class to give a universal method for proving that a quasisymmetric generating function is symmetric and Schur Q -positive or Schur P -positive.

Since the subset statistic for this case is the peak set instead of the descent set, we make the following notation. Given involutions $\varphi_2, \dots, \varphi_{n-2}$, for $1 < j < i < n - 1$ on a set \mathcal{A} of combinatorial objects, we consider the restricted shifted dual equivalence classes $[T]_{(j,i)}$ generated by $\varphi_j, \dots, \varphi_i$. In addition, we consider the restricted and shifted peak sets $\text{Peak}_{(j,i)}(T)$ obtained by intersecting $\text{Peak}(T)$ with $\{j - 1, \dots, i + 1\}$ and subtracting $j - 2$ from each element so that $\text{Peak}_{(j,i)}(T) \subseteq [i - j + 1]$.

Definition 7.3. Let \mathcal{A} be a finite set, and let Peak be a peak map on \mathcal{A} such that $\text{Peak}(T) \subseteq \{2, \dots, n - 1\}$ with no consecutive entries for all $T \in \mathcal{A}$. A *shifted dual equivalence for $(\mathcal{A}, \text{Peak})$* is a family of involutions $\{\varphi_i\}_{1 < i < n-1}$ on \mathcal{A} such that

- i. For all $|i - j| \leq 5$ and all $T \in \mathcal{A}$, there exists a strict partition γ of $|i - j| + 4$ such that

$$\sum_{U \in [T]_{(j,i)}} 2^{|\text{Peak}_{(j,i)}(U)|+1} G_{\text{Peak}_{(j,i)}(U)}(X) = Q_\gamma(X).$$

- ii. For all $|i - j| \geq 4$ and all $T \in \mathcal{A}$, we have

$$\varphi_j \varphi_i(T) = \varphi_i \varphi_j(T).$$

Definition 7.3 completely characterizes shifted dual equivalence classes in the same sense that Definition 5.1 characterizes dual equivalence classes.

Proposition 7.4. For γ a strict partition of n , the involutions $\{b_i\}_{1 < i < n-1}$ give a shifted dual equivalence for $\text{SShT}(\gamma)$ with peak function given by (2.3).

Proof. Condition (i) holds by (7.2), and condition (ii) follows from the fact that b_i depends only on the relative positions of $i - 1, i, i + 1, i + 2$ and for $|i - j| \geq 4$ these sets are disjoint. □

The real purpose of Definition 7.3 is to establish the following analog of Theorem 5.2.

Theorem 7.5. *If $\{\varphi_i\}$ is a shifted dual equivalence for $(\mathcal{A}, \text{Des})$ and $U \in \mathcal{A}$, then*

$$(7.3) \quad \sum_{T \in [U]} 2^{|\text{Peak}(T)|+1} G_{\text{Peak}(T)}(X) = Q_\gamma(X)$$

for some strict partition γ . In particular, the quasisymmetric generating function for \mathcal{A} is symmetric and Schur Q -positive.

We prove Theorem 7.5 along the same lines as the structure theorem for dual equivalence given in [Ass15]. To begin, we show that the shifted dual equivalences for standard shifted tableaux are pairwise nonisomorphic with no nontrivial automorphisms. This is the shifted analog of [Ass15][Proposition 3.9].

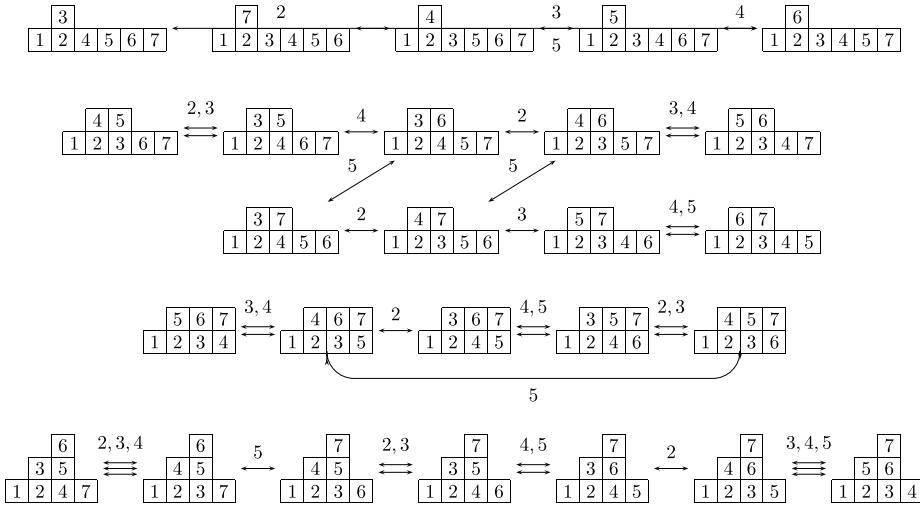


Figure 12: The standard shifted dual equivalences of size 7.

Proposition 7.6. *For γ, δ strict partitions, if $\phi : \text{SShT}(\gamma) \rightarrow \text{SShT}(\delta)$ is an isomorphism of shifted dual equivalences, then $\gamma = \delta$ and $\phi = \text{id}$.*

Proof. Let T_γ be the standard shifted tableau obtained by filling the numbers 1 through n into the rows of γ from left to right, bottom to top. For any standard shifted tableau T such that $\text{Peak}(T) = \text{Peak}(T_\gamma)$, the numbers 1 through λ_1 must form a horizontal strip since $1 \notin \text{Des}(T_\gamma)$ and λ_1 is the smallest element of $\text{Peak}(T_\gamma)$. Similarly, $\lambda_1 + 1$ through $\lambda_1 + \lambda_2$ must form a horizontal strip, and so on. In particular, if $\text{Peak}(T) = \text{Peak}(T_\gamma)$ for some

T of shape δ , then $\gamma \leq \delta$ in dominance order with equality if and only if $T = T_\gamma$.

Suppose $\phi : \text{SShT}(\gamma) \rightarrow \text{SShT}(\delta)$ is an isomorphism. Let T_γ be as above for γ , and let T_δ be the corresponding tableau for δ . Then since $\text{Peak}(\phi(T_\gamma)) = \text{Peak}(T_\gamma)$, $\gamma \leq \delta$. Conversely, since $\text{Peak}(\phi^{-1}(T_\delta)) = \text{Peak}(T_\delta)$, $\delta \leq \gamma$. Therefore $\gamma = \delta$. Furthermore, $\phi(T_\gamma) = T_\gamma$. For $T \in \text{SShT}(\gamma)$ such that $b_i(T_\gamma) = T$, we have $\phi(T) = b_i(T_\gamma) = T$. Extending this, every shifted tableau connected to a fixed point by some sequence of elementary shifted dual equivalences is also a fixed point for ϕ , hence $\phi = \text{id}$ on each $\text{SShT}(\gamma)$. \square

Thus far we have avoided using the language of signed, colored graphs to describe shifted dual equivalence. The following result is characterizing the local structure of shifted dual equivalence classes analogous to the original definition of dual equivalence graphs in [Ass15].

Lemma 7.7. *Let $\{\varphi_i\}_{1 < i < n-1}$ be a shifted dual equivalence for $(\text{SShT}(\gamma), \text{Peak})$ for some strict partition γ of n with $n \leq 7$. Then $\varphi_i = b_i$. In particular, there is a unique shifted dual equivalence for standard shifted tableaux of size at most 7.*

Proof. Given γ a strict partition of size $n \leq 7$, no two standard shifted tableaux of shape γ have the same peak set. This is easy to observe from Figure 12, for example. The result is trivial for $\gamma = (n)$ or for $n \leq 3$ since there are no nontrivial involutions, so we have four cases to consider, and we may assume $\gamma \neq (n)$.

For $n = 4$ (see Figure 6), the only case left to consider is $\gamma = (3, 1)$ which has two standard shifted tableaux which must necessarily be paired by φ_2 to ensure they lie in a single equivalence class. By condition (i) for degree 4, this is enough to characterize fixed points for φ_i as those elements T with $i, i + 1 \notin \text{Peak}(T)$ (cf. [Ass15][Definition 3.2, axiom 1]). Moreover, $i \in \text{Peak}(T)$ if and only if $i + 1 \in \text{Peak}(\varphi_i(T))$ (cf. [Ass15][Definition 3.2, axiom 2]). Armed with this, the result now follows for $\gamma = (n - 1, 1)$ since each standard shifted tableaux has a unique peak.

For $n = 5$ (see Figure 11), the only case left to consider is $\gamma = (3, 2)$ which has two standard shifted tableaux which must necessarily be paired by φ_2 and by φ_3 to ensure they lie in a single equivalence class. In particular, condition (i) for degree 5 now implies that when $\varphi_i(T) = \varphi_{i+1}(T)$ the cardinality of $\text{Peak}(T) \cap \{i, i + 1, i + 2\}$ changes (cf. [Ass15][Definition 3.2, axiom 3]).

For $n = 6$, there are two cases to consider. For $\gamma = (3, 2, 1)$, there are two standard shifted tableaux, neither of which is a fixed point for φ_i for $i =$

2, 3, 4. Therefore they must be connected by a triple edge. For $\gamma = (4, 2)$, the five standard shifted tableaux have peak sets $\{3\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{4\}$. Thus for each $i = 2, 3, 4$, exactly one of the five standard shifted tableaux is a fixed point for φ_i , so given the toggling condition on peak sets, there are two possible pairing for each $\varphi_i, i = 2, 3, 4$. If φ_3 pairs the tableaux with peak sets $\{3\}$ and $\{4\}$, then it must also pair the tableaux with peak sets $\{3, 5\}$ and $\{2, 4\}$. By the analysis for $n = 5$ and condition (i) for degree 5, this means that both φ_2 and φ_4 pair the tableaux with peak sets $\{3, 5\}$ and $\{2, 4\}$ as well. But then these two tableaux are in an equivalence class of their own, contradicting that classes are (locally) Schur Q -positive. Thus φ_3 pairs tableaux with peak sets $\{3\}$ and $\{2, 4\}$ and also tableaux with peak sets $\{3, 5\}$ and $\{4\}$. By the same logic, the former case is also paired by φ_2 , and the latter by φ_4 . This correctly constrains the structure.

Finally, for $n = 7$ (see Figure 12), there are three cases to consider, and the analysis is analogous to the case for $n = 6$, where now we may appeal to the case $n = 6$ as well. □

Lemma 7.7 states that the shifted dual equivalence on standard shifted tableaux of size up to 7 is unique. This result is tight since there exist standard shifted tableaux of size 8 with the same shape and the same peak set.

Given strict partitions γ, δ with $\gamma \subset \delta$, fix a filling of the cells of $\delta \setminus \gamma$ with $|\gamma| + 1, \dots, |\delta|$, say B . Let $\text{SShT}(\gamma, B) \subset \text{SShT}(\delta)$ be subset of shifted standard tableaux that restrict to B when skewed by γ . The resulting shifted dual equivalence on $\text{SShT}(\gamma, B)$ has the same involutions as $\text{SShT}(\gamma)$, but the Peak function has now been extended. With this in mind, we show that for γ a partition of n , any extension of the peak function for $\text{SShT}(\gamma)$ can be modeled by $\text{SShT}(\gamma, B)$ for some augmenting tableau B . The following result is the shifted analog of [Ass15][Lemma 3.11].

Lemma 7.8. *Let $\{\varphi_i\}_{1 < i < n}$ be a shifted dual equivalence for $(\mathcal{A}, \text{Peak})$. Let $T \in \mathcal{A}$ such that there exists an isomorphism ϕ from $([T]_{(2, n-2)}, \text{Peak} \cap \{2, 3, \dots, n-1\})$ to $(\text{SShT}(\gamma), \text{Peak})$ for some strict partition γ of n . Then there exists a unique standard shifted tableau B of shape $\delta/\gamma, \delta$ a strict partition of size $n+1$, such that ϕ gives an isomorphism from $([T]_{(2, n-2)}, \text{Peak})$ to $(\text{SShT}(\gamma, B), \text{Peak})$.*

Proof. The result follows for $n \leq 6$ by Lemma 7.7, so assume $n \geq 6$. The restricted equivalence classes of $\text{SShT}(\gamma)$ under b_2, \dots, b_{n-3} may each be identified with a strict partition of $n - 1$ contained in γ , or, equivalently, with the unique outer corner of γ containing the entry n for each tableau of the restricted equivalence class. Therefore the isomorphism from

$n - 1$ and n , then $n - 3$ and $n - 2$ would both occupy outer corners. Given this, we may place $n - 4$ below $n - 3$ and left of $n - 2$. Finally, since γ is still strict, there is at least one cell left of $n - 4$, so we may place $n - 5$ there. See Figure 14.

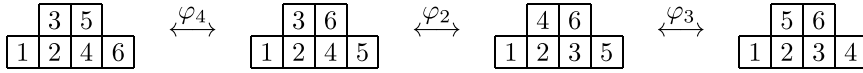


Figure 14: Relative positions for $n - 5, \dots, n$ in D (far left) and C (far right).

With D as described, b_{n-2} acts on D by interchanging $n - 1$ and n with witness $n - 2$ and bystander $n - 3$. Then b_{n-4} acts on $b_{n-2}(D)$ by interchanging $n - 3$ and $n - 2$ with witness $n - 4$ and bystander $n - 5$. Set $C = b_{n-3}b_{n-4}b_{n-2}(D)$, where b_{n-3} acts by interchanging $n - 2$ and $n - 1$ with witness $n - 3$ and bystander $n - 4$. Thus the element of \mathcal{D} that maps to D , say d , and the element of \mathcal{C} that maps to C , say c , lie in the same class under $\varphi_{n-4}, \varphi_{n-3}, \varphi_{n-2}$. Finally, we use the assumption on peak sets and Lemma 7.7 for $[c]_{(n-4, n-1)} = [d]_{(n-4, n-1)}$ to see that d must correspond to degree 7 shape $(5, 2)$ whereas c must correspond to degree 7 shape $(4, 2, 1)$, which are not the same. \square

The main result that will establish Theorem 7.5 is the following shifted analog of [Ass15][Theorem 3.12].

Theorem 7.9. *Let $\{\varphi_i\}_{1 < i < n}$ be a shifted dual equivalence for $(\mathcal{A}, \text{Peak})$ with a single equivalence class, and suppose that for each $T \in \mathcal{A}$ there exists an isomorphism from $([T]_{(2, n-2)}, \text{Peak} \cap \{2, 3, \dots, n-1\})$ to $(\text{SShT}(\gamma), \text{Peak})$ for some strict partition γ of n . Then there exists a morphism from $(\mathcal{A}, \text{Peak})$ to $(\text{SShT}(\varepsilon), \text{Peak})$ for a unique strict partition ε of $n + 1$ containing γ .*

Proof. For $n < 7$, the result follows from Lemmas 7.8 and 7.7. If φ_{n-1} acts trivially on the entire equivalence class, then $\text{Peak} \equiv \emptyset$ and $\varepsilon = (n + 1)$. Therefore we proceed by induction on n and assuming that \mathcal{A} has at least one element (thus two elements) not fixed by φ_{n-1} .

By Lemma 7.8, the isomorphism from $([T]_{(2, n-2)}, \text{Peak} \cap \{2, 3, \dots, n-1\})$ to $(\text{SShT}(\gamma), \text{Peak})$ extends to an isomorphism from $([T]_{(2, n-2)}, \text{Peak})$ to $(\text{SShT}(\gamma, B), \text{Peak})$ for a unique augmenting shifted tableau B , say with shape ε/γ . We will show that for any $[T]_{(2, n-2)}$, the shape ε is the same and that we may glue these isomorphisms together to obtain a morphism from $(\mathcal{A}, \text{Peak})$ to $(\text{SShT}(\varepsilon), \text{Peak})$.

Suppose $d = \varphi_{n-1}(c)$. Let $\mathcal{C} = [c]_{(2, n-2)}$ and $\mathcal{D} = [d]_{(2, n-2)}$. Let ϕ (resp. ψ) be the isomorphism from \mathcal{C} (resp. \mathcal{D}) to $\text{SShT}(\gamma)$ (resp. $\text{SShT}(\delta)$), and

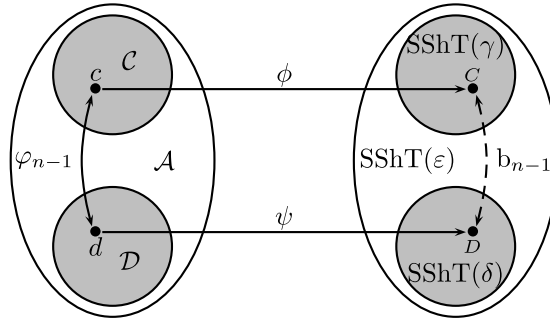


Figure 15: An illustration of the gluing process.

set $C = \phi(c)$; see Figure 15. We will show that $\psi(d) = b_{n-1}(C)$, and hence if γ, B has shape ε , then so does δ, B , and the maps ϕ and ψ glue together to give a morphism from $C \cup D$ to $\text{SShT}(\varepsilon)$ that commutes with φ_{n-1} and b_{n-1} . There are two cases to consider, based on the relative positions of $n - 1, n, n + 1$ in C , regarded as a shifted tableau of shape ε .

First suppose that b_{n-1} fixes $n + 1$. We will show that, in this case, $C = D$. There are three subcases to consider based on how b_{n-1} acts on C and D . For each, the idea is to use b_2, \dots, b_{n-5} to move to a tableau C' for which $b_{n-1}(C') = b_{n-2}(C') = D'$, and then move from D' back to D by reversing the sequence of b_2, \dots, b_{n-5} using condition (ii) of Definition 7.3; see Figure 16.

If b_{n-1} interchanges $n - 2$ and $n - 1$, then n must be the witness and $n + 1$ the bystander, so there must exist a cell between $n - 2$ and $n - 1$ containing an entry at most $n - 3$. Let C' be the shifted tableau with $n - 2, n - 1, n, n + 1$ in the same position as in C and with $n - 3$ between $n - 2$ and $n - 1$. Then φ_{n-2} also acts on C' by exchanging $n - 2$ and $n - 1$, where $n - 3$ and n are the witness and bystander in some order.

If b_{n-1} interchanges $n - 1$ and n with $n - 2$ as the witness and $n + 1$ the bystander, then there must be a cell between $n + 1$ and $n - 2$ that contains an entry at most $n - 3$. Let C' be the shifted tableau with $n - 2, n - 1, n, n + 1$ in the same position as in C and with $n - 3$ between $n - 2$ and $n + 1$. Then φ_{n-2} also acts on C' by exchanging $n - 1$ and n , where $n - 2$ is again the witness and $n - 3$ is the bystander.

If b_{n-1} interchanges $n - 1$ and n with $n + 1$ as the witness and $n - 2$ the bystander, then again there must be a cell between $n + 1$ and $n - 2$ that contains an entry at most $n - 3$. Let C'' be the shifted tableau with $n - 2, n - 1, n, n + 1$ in the same position as in C and with $n - 3$ between $n - 2$

and $n + 1$. Now we go further, noting that there must be a cell between $n - 2$ and $n - 3$ containing an entry at most $n - 4$, and we insist that $n - 4$ lie in this cell in C'' as well. Finally, there must be a cell left of this that contains an entry at most $n - 5$, and we insist that $n - 5$ lie in this cell in C'' as well. Observe that the positions of $n - 5, \dots, n + 1$ ensure that $[C'']_{(n-4, n-1)}$ is isomorphic to $\text{SShT}(4, 3)$; see Figure 12. In particular, in this case b_{n-4} commutes with b_{n-1} . Thus we may set $C' = b_{n-4}(C'')$, which interchanges $n - 3$ and $n - 2$. In this case, φ_{n-2} acts on C' by exchanging $n - 1$ and n , where $n - 2$ has been positioned as the witness and $n - 3$ as the bystander.

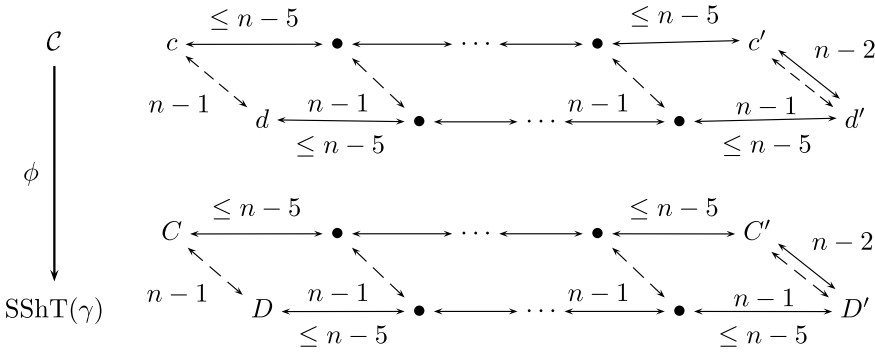


Figure 16: The path from C to D in $\text{SShT}(\gamma)$ and its lift in C .

In all cases, we have shown that $D \in [C]_{2, n-2}$. Therefore we can lift this to \mathcal{A} to conclude that $d \in [c]_{2, n-2} = \mathcal{C}$. Set $c' = \phi^{-1}(C')$ and $d' = \psi^{-1}(D')$. By Lemma 7.7, there is a unique shifted dual equivalence structure on $[c']_{n-4, n-1}$ and on $[C']_{n-4, n-1}$, which forces $\varphi_{n-1}(c') = \phi^{-1}(b_{n-1}(C')) = \phi^{-1}(D') = d'$. Thus we may invoke condition (ii) of Definition 7.3 to conclude that $d = \varphi_{n-1}(c) = \phi^{-1}(b_{n-1}(C)) = \phi^{-1}(D)$. In this case $\mathcal{C} = \mathcal{D}$ and, by Proposition 7.6, $\psi = \phi$.

For the second case, we may assume b_{n-1} acts on C by interchanging n and $n + 1$ with $n - 1$ as the witness and $n - 2$ as the bystander. Consider the subset of $\text{SShT}(\gamma, B)$ with n and $n + 1$ fixed in the same position as in C and $n - 1$ lying anywhere between them and $n - 2$ anywhere left of $n - 1$. In terms of the equivalences, for $n - 1$ in a fixed position, these are all tableaux reachable using b_h with $h \leq n - 5$ and certain instances of b_{n-4} . We will return soon to the question of which instances these are. For now, let $\overline{[C]}$ denote the union of the equivalence classes $[C']_{(2, n-5)}$, where C' has $n - 1$ between n and $n + 1$ and $n - 2$ somewhere left of $n - 1$. The range of positions for $n - 2$ determines the range of positions for $n - 1$, which in

turn uniquely determines the cells containing n and $n + 1$, and so this set uniquely determines ε . Furthermore, which of $n, n + 1$ occupies which cell is determined by whether or not n is a peak. Therefore ϕ^{-1} lifts $\overline{[C]}$ to a subset of \mathcal{C} that completely determines ε as well as the positions of n and $n + 1$ in the image of this set under ϕ . We will show that the corresponding lifted subset for \mathcal{D} is isomorphic but with n toggled into or out of the peak set.

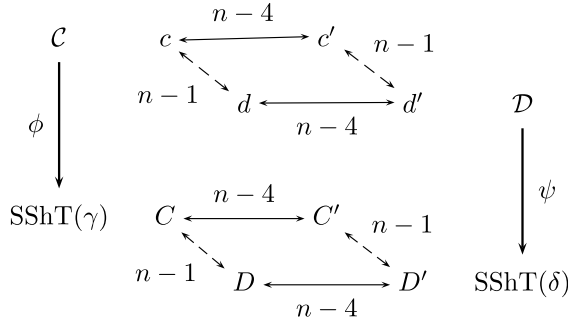


Figure 17: Illustration of b_{n-4} on $\overline{[C]} \cup \overline{[D]}$ and their lifts in $\mathcal{C} \cup \mathcal{D}$.

To prove the assertion, we return to the question of which instances of φ_{n-4} are allowed in generating $\overline{[C]}$. For this, we may consider the structure of the degree 7 equivalence class generated by $\varphi_{n-4}, \dots, \varphi_{n-1}$ which, by Lemma 7.7, have the structure of one of the components depicted in Figure 12. Examining the possibilities, it follows that whenever φ_{n-4} keeps $n - 2$ left of $n - 1$, the map also commutes with φ_{n-1} , as depicted in Figure 17. By condition (ii) of shifted dual equivalences, φ_h also commutes with φ_{n-1} for $h \leq n - 5$. Therefore all involutions used to generate $\overline{[C]}$ commute with φ_{n-1} . Thus φ_{n-1} gives an isomorphism from $\phi^{-1}(\overline{[C]})$ to a subset of \mathcal{D} . Let $D = \psi(d)$. Then $\overline{[D]} = \psi(\varphi_{n-1}(\phi^{-1}(\overline{[C]})))$. Since ϕ, ψ and φ_{n-1} are isomorphisms, $\overline{[D]}$ is isomorphic to $\overline{[C]}$, though exactly one has n as a peak. By the earlier characterization of $\overline{[C]}$, this implies that the tableaux in $\overline{[D]}$ have shape ε , with the cells containing n and $n + 1$ reversed from that in $\overline{[C]}$. In particular, $\overline{[D]} = b_{n-1}(\overline{[C]})$, that is to say, ϕ and ψ glue to give a morphism from $\mathcal{C} \cup \mathcal{D} \subset \mathcal{A}$ to $\text{SShT}(\gamma, A) \cup \text{SShT}(\delta, B) \subset \text{SShT}(\varepsilon)$ that respects φ_{n-1} . Therefore this map lifts to a morphism from \mathcal{A} to $\text{SShT}(\varepsilon)$. \square

Thus far we have used condition (i) of shifted dual equivalence only for restricted equivalence classes of degree up to 7. If this weaker condition is all that is used, then Theorem 7.9 proves that there is a morphism from

$(\mathcal{A}, \text{Peak})$ to $\text{SShT}(\varepsilon)$. While this morphism is always surjective, in order to show that it is injective we must invoke condition (i) for degree 9.

Theorem 7.10. *Given any shifted dual equivalence $\{\varphi_i\}_{1 < i < n}$ for $(\mathcal{A}, \text{Peak})$ with a single equivalence class, there exists a unique strict partition γ of size $n + 1$ such that there is an isomorphism of shifted dual equivalences between $(\mathcal{A}, \text{Peak})$ under $\{\varphi_i\}_{1 < i < n}$ and $(\text{SShT}(\gamma), \text{Peak})$ under $\{b_i\}_{1 < i < n}$.*

Proof. We proceed by induction on $n + 1$, noting that the result follows from Lemma 7.7 for $n + 1 \leq 7$. The involutions $\{\varphi_i\}_{1 < i < n-1}$ give a shifted dual equivalence for any restricted equivalence class $[T]_{(2, n-2)}$, and so, by induction, each such class is isomorphic to $(\text{SShT}(\delta), \text{Peak})$ for a unique strict partition δ of n . By Theorem 7.9, there exists a morphism, say ϕ , from \mathcal{A} to $\text{SShT}(\gamma)$ for a unique strict partition γ of $n + 1$. Surjectivity follows from the fact that \mathcal{A} has a single equivalence class under $\{\varphi_i\}_{1 < i < n}$ and that ϕ commutes with φ_i and b_i . To prove that ϕ is injective, we first claim that the fiber over each standard shifted tableau has the same cardinality.

To prove the claim, we show that for any restricted equivalence class \mathcal{C} under $\{\varphi_i\}_{1 < i < n-1}$, say with $\phi(\mathcal{C}) = \text{SShT}(\delta)$, and any strict partition $\varepsilon \subset \gamma$ of size n , there is a unique restricted equivalence class \mathcal{D} under $\{\varphi_i\}_{1 < i < n-1}$ with $\phi(\mathcal{D}) = \text{SShT}(\varepsilon)$ such that $d = \varphi_{n-1}(c)$ for some $c \in \mathcal{C}$ and some $d \in \mathcal{D}$. Once established, this gives a bijective correspondence between equivalence classes in $\phi^{-1}(\text{SShT}(\delta))$ and in $\phi^{-1}(\text{SShT}(\varepsilon))$, thus proving the result.

To prove existence, if $\varepsilon \neq \delta$, let C be a shifted standard tableau of shape γ with $n + 1$ in position γ/δ , n in position γ/ε , $n - 1$ lying between, and $n - 2$ lying left of $n - 1$. Otherwise let C be a standard shifted tableau with $n + 1$ in position γ/δ and n and $n - 1$ lying on opposite sides, again with $n - 2$ left of $n - 1$. Let c be the unique element in $\phi^{-1}(C) \cap \mathcal{C}$. Then $\phi(\varphi_{n-1}(c)) = \varphi_{n-1}(\phi(c)) \in \text{SShT}(\varepsilon)$.

To prove uniqueness, let $d = \varphi_{n-1}(c)$ with $c \in \mathcal{C}$ and $d \in \mathcal{D}$. If $n + 1$ lies between n and $n - 1$ in $\phi(C)$, then $\delta = \varepsilon$, and just as in the proof of Theorem 7.9, we concluded that $\mathcal{D} = \mathcal{C}$ as desired. Alternately, assume $n - 1$ lies between n and $n + 1$ in $\phi(c)$, and suppose $\varphi_{n-1}(c') = d'$ with $C' \in \mathcal{C}$ and $d' \in \mathcal{D}'$ where $\phi(D') = \text{SShT}(\varepsilon)$. Since $\phi(c)$ and $\phi(c')$ have the same shape, and $\varphi_{n-1}(\phi(c)) = \phi(\varphi_{n-1}(c)) = \phi(d)$ and $\varphi_{n-1}(\phi(c')) = \phi(\varphi_{n-1}(c')) = \phi(d')$ have the same shape as well. Just as in the proof of Theorem 7.9, $\phi(c)$ and $\phi(c')$ must lie in the same equivalence class under b_2, \dots, b_{n-5} and those instances of b_{n-4} that commute with b_{n-1} . Lifting this class via ϕ , c and c' must lie in the same equivalence class under $\varphi_2, \dots, \varphi_{n-5}$ and those instances of φ_{n-4} that commute with φ_{n-1} . Therefore applying φ_{n-1} , d and

d' lie in the same equivalence class under $\varphi_2, \dots, \varphi_{n-5}$ and those instances of φ_{n-4} that commute with φ_{n-1} . In particular, $d' \in \mathcal{D}$, and so $\mathcal{D} = \mathcal{D}'$.

Since each fiber has the same cardinality, say k , the generating function for \mathcal{A} is kQ_λ . When $n \leq 9$, condition (i) ensures $k = 1$ and the map to $\text{SShT}(\gamma)$ is an isomorphism. In particular, we may assume the restricted equivalence classes under $\varphi_{n-6}, \dots, \varphi_{n-1}$ are isomorphic to some $\text{SShT}(\delta)$. Given two standard shifted tableaux C, D of the same shape, there exist tableaux C', D' such that $C' \in [C]_{(2,n-2)}$, $D' \in [D]_{(2,n-2)}$, and $C' = \varphi_{n-1}(D')$ (cf. axiom 6 for dual equivalence graphs in [Ass15][Definition 3.2]).

Suppose $T, U, V, X \in \mathcal{A}$, with $U = \varphi_{n-1}(T)$, $V = \varphi_{n-1}(X)$, and U and V lie in the same restricted equivalence class under $\varphi_2, \dots, \varphi_{n-2}$. We will show that there exist T', X' in the same restricted equivalence classes as T, X , respectively, such that T' and X' lie in the same degree 9 equivalence class under $\varphi_{n-6}, \dots, \varphi_{n-1}$; see Figure 18. By the earlier remark and the result for degree up to 9, this implies that there exist T'', X'' such that $T'' \in [T']_{(n-6,n-2)} \subseteq [T]_{(2,n-2)}$, $X'' \in [X']_{(n-6,n-2)} \subseteq [X]_{(2,n-2)}$, and $T'' = \varphi_{n-1}(X'')$. In particular, we must have $k = 1$.

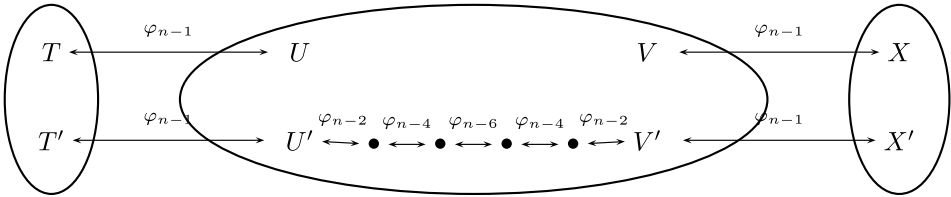


Figure 18: Establishing injectivity from degree 9.

By the inductive hypothesis and Theorem 7.9, we may identify T, U, V, X with shifted tableaux of shape γ , $|\gamma| = n + 1$, and, when restricted to entries up to n , T, U, V, X have shapes $\delta, \varepsilon, \varepsilon, \zeta$, respectively, with $\delta, \varepsilon, \zeta$ distinct strict partitions contained in γ . Then γ/ρ must be a corner (end of row, top of column) for $\rho = \delta, \varepsilon, \zeta$. Assume these cells appear with γ/δ northeast of γ/ζ northeast of γ/ε , noting that the other orders can be resolved in a similar way. Let T' be any shifted standard tableaux of shape γ with $n + 1$ in position γ/δ , n in position γ/ε , $n - 1$ in position γ/ζ , $n - 2$ between $n + 1$ and $n - 1$, $n - 3$ left of $n - 2$ (which is possible since γ is a shifted shape), $n - 4$ between $n - 1$ and n , $n - 5$ between $n - 2$ and $n - 4$, $n - 6$ between $n - 3$ and $n - 5$, and $n - 7$ left of $n - 6$ (which is possible since γ is a shifted shape). See Figure 19 for an illustration.

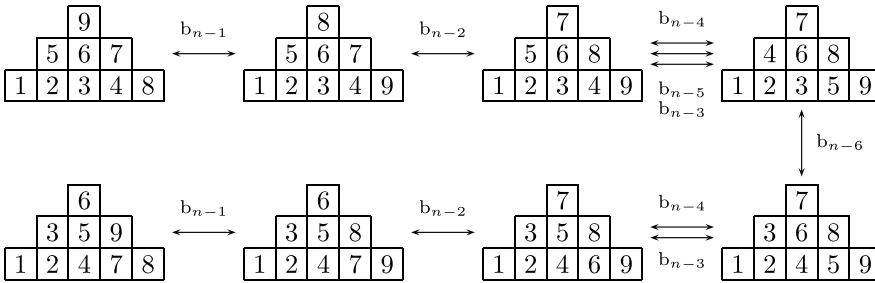


Figure 19: Relative positions for $n - 7, \dots, n + 1$ in T' (top left) and X' (bottom left).

Set $U' = b_{n-1}(T')$. Since the shape of U' restricted to entries up to n is ε , we have $U' \in [U]_{(2,n-2)}$ where $U = b_{n-1}(T)$, and $V' \in [V]_{(2,n-2)}$ where $V = b_{n-2}b_{n-4}b_{n-6}b_{n-4}b_{n-2}(U')$; again, see Figure 19. Set $X' = b_{n-1}(V')$. Since the shape of X' restricted to entries up to n is ζ , we have $X' \in [X]_{(2,n-2)}$ where $X = b_{n-1}(V)$. Moreover, since $X' = b_{n-1}b_{n-2}b_{n-4}b_{n-6}b_{n-4}b_{n-2} \times b_{n-1}(T')$, we have that $X' \in [T']_{(n-6,n-1)}$ as desired. \square

Theorem 7.5 now follows as a corollary to Theorem 7.10 and (7.2).

8. Products of Schur P -functions

We now present a first application of shifted dual equivalence. For $\gamma \subseteq \varepsilon$ strict partitions, we define the shifted skew diagram ε/γ to be the set theoretic difference between ε and γ . For example, the shifted skew diagram for $(6, 4, 3, 1)/(5, 2)$ is given in Figure 20.

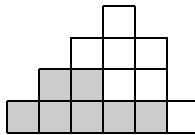


Figure 20: The shifted Young diagram and shifted symmetric diagram for $(6, 4, 3, 1)$.

The combinatorial definitions for Schur Q -functions extend to skew shifted diagrams [Mac95], and the quasisymmetric expansions in Section 4 hold for the shifted case as well. Precisely, we have

$$(8.1) \quad Q_{\varepsilon/\gamma}(X) = \sum_{S \in \text{SShT}(\varepsilon/\gamma)} 2^{|\text{Peak}(S)|+1} G_{\text{Peak}(S)}(X)$$

where $\text{SShT}(\varepsilon/\gamma)$ denotes the set of all standard shifted tableaux of skew shifted shape ε/γ .

Schur Q and P -functions have the same relationship as before, though one must take care to track how many cells now live on the main diagonal. The relation in (3.3) becomes

$$(8.2) \quad P_{\varepsilon/\gamma}(X) = 2^{\ell(\varepsilon)-\ell(\gamma)} Q_{\varepsilon/\gamma}(X).$$

Recall that the product of two Schur P -functions may be expanded in the Schur P -function basis, so we may define integers $f_{\gamma,\delta}^\varepsilon$ by

$$(8.3) \quad P_\gamma(X)P_\delta(X) = \sum_{\varepsilon} f_{\gamma,\delta}^\varepsilon P_\varepsilon(X).$$

Since the Schur Q - and P -functions form dual bases and the operation of skewing is adjoint to multiplication [Mac95](I.5), these integers may also be defined by

$$(8.4) \quad Q_{\varepsilon/\gamma}(X) = \sum_{\delta} f_{\gamma,\delta}^\varepsilon Q_\delta(X).$$

Using the machinery of shifted dual equivalence, the second half of Theorem 3.1 now follows.

Corollary 8.1. *For $\gamma \subseteq \varepsilon$ strict partitions, $f_{\gamma,\delta}^\varepsilon$ is the number of shifted dual equivalence classes of standard shifted tableaux of skew shifted shape ε/γ under the action of $\{\mathbf{b}_i\}_{1 < i < |\varepsilon| - |\gamma| - 1}$ that are isomorphic to $\text{SShT}(\delta)$. In particular, $f_{\gamma,\delta}^\varepsilon$ is a nonnegative integer.*

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