Combinatorics of symmetric plabic graphs

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A *plabic graph* is a planar bicolored graph embedded in a disk, which satisfies some combinatorial conditions. Postnikov's *boundary measurement map* takes the space of positive edge weights of a plabic graph G to a *positroid cell* in a totally nonnegative Grassmannian. In this note, we investigate plabic graphs which are symmetric about a line of reflection, up to reversing the colors of vertices. These *symmetric plabic graphs* arise naturally in the study of total positivity for the Lagrangian Grassmannian. We characterize various combinatorial objects associated with symmetric plabic graphs, and describe the subset of a Grassmannian which can be realized by *symmetric weightings* of symmetric plabic graphs.

1. Introduction

A plabic graph is a planar graph embedded in a disk, with vertices colored black or white. Postnikov introduced plabic graphs as a tool for studying the *positroid stratification* of the *totally nonnegative Grassmannian* $\text{Gr}_{\geq 0}(k, n)$ [7].

The totally nonnegative Grassmannian is the region of the Grassmannian $\text{Gr}(k, n)$ where all Plücker coordinates are nonnegative real numbers, up to multiplication by a common scalar. Postnikov defined a stratification of $\text{Gr}_{\geq 0}(k, n)$ by *positroid cells*, each defined as the locus in $\text{Gr}_{\geq 0}(k, n)$ where some set of Plücker coordinates vanish. The resulting *positroid stratification* of $\text{Gr}_{\geq 0}(k, n)$ has a rich geometric and combinatorial structure. There are numerous combinatorial objects which index positroid cells, including *bounded affine permutations*, *Grassmann necklaces*, and a class of matroids called *positroids* [7, 3].

Given a plabic graph G, Postnikov's *boundary measurement map* takes the space of positive real edge weights of G surjectively to a positroid cell Π_G in $\text{Gr}_{\geq 0}(k, n)$ for some values of k and n. Moreover, the bounded affine permutation, Grassmann necklace, and matroid of Π_G are encoded in the structure of G. Note that plabic graphs are not in bijection with positroid cells; rather, for each positroid cell, we have a family of plabic graphs.

In this note, we study plabic graphs which satisfy a symmetry condition. See Figure 1 for an example. These *symmetric plabic graphs* arise in the theory of total positivity for the Lagrangian Grassmannian $\Lambda(2n)$, the moduli space of maximal isotropic subspaces with respect to a symplectic bilinear form. The connection with $\Lambda(2n)$, as well as additional results on the combinatorics of symmetric plabic graphs, will be discussed in a forthcoming paper by the first author [2].

Let G denote a plabic graph, and let N denote the underlying uncolored network. Suppose N is symmetric with respect to reflection through a distinguished diameter d of the disc. Let V be the vertex set of E, and let $r : V \to V$ map each vertex $v \in V$ to its mirror image across the line d. Then G is a *symmetric plabic graph* if the following conditions hold:

1. G has no vertices on the line d, although edges may cross d
2. For each $v \in V$, the vertices v and $r(v)$ have opposite colors.

We briefly summarize our results. In Theorem 3.1, we characterize the positroids, Grassmann necklaces, and bounded affine permutations associated with symmetric plabic graphs. We then consider *symmetric weightings* of symmetric plabic graphs; that is, weightings where each edge (u, v) has the same weight as its reflection $(r(u), r(v))$. Theorem 4.2 and Corollary 4.3
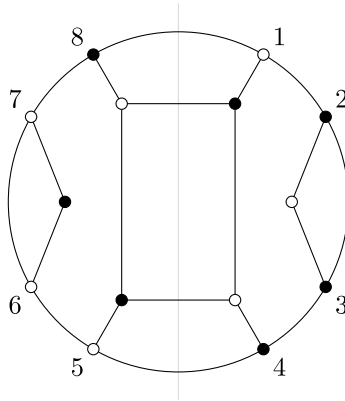
Figure 1: A symmetric plabic graph.

give a complete description of the set of points in $\text{Gr}_{\geq 0}(k, n)$ corresponding to such weightings of symmetric plabic graphs. This is a subvariety, cut out set-theoretically by linear equations, which we call the *symmetric part* of $Gr(k, n)$. With the right choice of conventions, the symmetric part of $\text{Gr}_{\geq 0}(k, n)$ is precisely the totally nonnegative part of the Lagrangian Grassmannian [2]. Finally, Theorem 4.4 gives an explicit construction which yields a symmetric weighting of a symmetric plabic graph for each point in the symmetric part of $\text{Gr}_{\geq 0}(k, n)$. In particular, we construct a symmetric *bridge graph* for each point in $\text{Gr}_{\geq 0}(k, n)$. Bridge graphs are a special class of plabic graphs which appear as a computational tool in particle physics [1].

2. Background

2.1. Notation

For natural numbers $k \leq n$, let $[n]$ denote the set $\{1, 2, \ldots, n\}$, and let $\binom{[n]}{k}$ denote the set of all k-element subsets of $[n]$. For $a \in [n]$, let $\leq_a$ denote the cyclic shift of the usual linear order on n given by

$$(1) \qquad a < a + 1 < \ldots < n < 1 < \ldots < a - 1.$$

Note that $\leq_1$ is the usual order $\leq$. We extend this to a partial order on $\binom{[n]}{k}$, by setting $I \leq_a J$ if we have $i_\ell \leq_a j_\ell$ for all $\ell \in [k]$, where

$$(2) \qquad I = \{i_1 <_a i_2 <_a \ldots <_a i_k\} \text{ and } J = \{j_1 <_a j_2 <_a \ldots <_a j_k\}.$$

For $a, b \in [n]$, we define the *cyclic interval* $[a, b]^{cyc}$ by

$$(3) \qquad [a, b]^{cyc} = \begin{cases} \{a, a+1, \ldots, b\} & a \leq b \\ \{a, a+1, \ldots, n-1, n, 1 \ldots, b\} & a > b \end{cases}.$$

(Note that this differs slightly from Postnikov's convention.) If we arrange the elements of $[n]$ clockwise around a circle, then $[a, b]^{cyc}$ represents a sequence of consecutive numbers.

Let S_n be the symmetric group in n letters, and let s_i denote the simple transposition $(i, i+1)$ which switches i and $i+1$. Let $\mathbb{R}_{>0}$ denote the positive reals.

2.2. Grassmannians and Plücker coordinates

Let $\mathrm{Gr}(k, n)$ denote the Grassmannian of k-dimensional linear subspaces of the vector space $\mathbb{C}^n$. We may realize $\mathrm{Gr}(k, n)$ as the space of full-rank $k \times n$ matrices modulo the left action of $\mathrm{GL}(k)$, the group of invertible $k \times k$ matrices; a matrix M represents the space spanned by its rows.

The *Plücker embedding*, which we denote p, maps $\mathrm{Gr}(k, n)$ into the projective space $\mathbb{P}^{\binom{n}{k}-1}$ with homogeneous coordinates x_J indexed by the elements of $\binom{[n]}{k}$. For $J \in \binom{[n]}{k}$ let Δ_J denote the minor with columns indexed by J. Let V be a k-dimensional subspace of $\mathbb{C}^n$ with representative matrix M. Then $p(V)$ is the point defined by $x_J = \Delta_J(M)$. This map embeds $\mathrm{Gr}(k, n)$ as a smooth projective variety in $\mathbb{P}^{\binom{n}{k}-1}$. The homogeneous coordinates Δ_J are known as *Plücker coordinates* on $\mathrm{Gr}(k, n)$. The *totally nonnegative Grassmannian*, denoted $\mathrm{Gr}_{\geq 0}(k, n)$, is the subset of $\mathrm{Gr}(k, n)$ whose Plücker coordinates are all nonnegative real numbers, up to multiplication by a common scalar.

Let $V \in \mathrm{Gr}_{\geq 0}(k, n)$. The indices of the non-vanishing Plücker coordinates of V give a set $\mathcal{J} \subseteq \binom{[n]}{k}$ called the *matroid* of V. We define the *matroid cell* $\mathcal{M}_{\mathcal{J}}$ as the locus of points $V \in \mathrm{Gr}_{\geq 0}(k, n)$ with matroid $\mathcal{J}$. The matroids $\mathcal{J}$ for which $\mathcal{M}_{\mathcal{J}}$ is nonempty are called *positroids*, and the corresponding matroid cells are called positroid cells. Positroid cells form a *stratification* of $\mathrm{Gr}_{\geq 0}(k, n)$. That is, the closure of a positroid cell Π in $\mathrm{Gr}_{\geq 0}(k, n)$ is the union of Π and some lower-dimensional positroid cells [7].

2.3. Plabic graphs

A *plabic graph* is a planar graph embedded in a disk, with each vertex colored black or white. The boundary vertices are numbered $1, 2, \ldots, n$ in
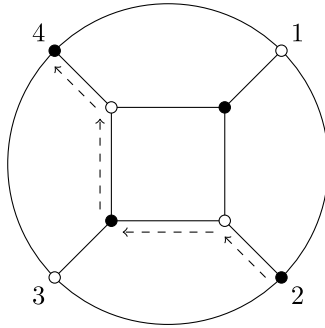
Figure 2: A trip in a plabic graph G. We have $\bar{f}_G(2) = 4$.

clockwise order, and all boundary vertices have a degree of one. We call the edges adjacent to boundary vertices *legs* of the graph, and a leaf adjacent to a boundary vertex a *lollipop*. A *white lollipop* is a white leaf adjacent to a black boundary vertex, while a *black* lollipop is the opposite.

Postnikov introduced plabic graphs in [7, Section 11.5]. We follow the conventions of [5], which are more restrictive than Postnikov's. In particular, we require a plabic graph to be bipartite, with the black and white vertices forming the partite sets. An *almost perfect matching* on a plabic graph is a subset of its edges which covers each interior vertex exactly once; boundary vertices may or may not be covered. We consider only plabic graphs which admit an almost perfect matching. Finally, we require that no edge in a plabic graph connects two boundary vertices.

We define a collection of paths and cycles in G, called *trips*, as follows. We begin by traversing an edge $\{u, v\}$ of G, from u to v. We then proceed according to the *rules of the road*: turn (maximally) left at every white internal vertex, and (maximally) right at every black internal vertex. Continuing in this fashion, we eventually reach a boundary vertex. The resulting directed path is a *trip* in G. See Figure 2 for an example. We repeat this process for every boundary vertex. If the resulting collection of trips covers every edge of G twice, once in each direction, we are done.

Otherwise, we find an internal edge $e = \{u, v\}$ such that no trip covers e in the direction $u \to v$. We begin by tracing e in this direction, and proceed according to the rules of the road, until we find ourselves once again about to trace the edge $u \to v$. The resulting directed cycle is a trip. Repeat this process until each edge of G is covered twice by trips, once in each direction.

Given a plabic graph G with n boundary vertices, we define the *trip permutation* $\bar{f}_G \in S_n$ of G by setting $\bar{f}_G(a) = b$ if the trip that starts at boundary vertex a ends at boundary vertex b.

The boundary measurement map is defined only for *reduced* plabic graphs. Postnikov defined reduced plabic graphs in terms of certain local transformations of plabic graphs, and gave a criterion for a plabic graph G to be reduced [7, Section 13]. We take this criterion as the *definition* of a reduced graph.

Definition 2.1. *A plabic graph G is* reduced *if it satisfies the following criteria:*

1. *G has no trips which are cycles.*
2. *G has no leaves, except perhaps some which are adjacent to boundary vertices.*
3. *No trip uses the same edge twice, once in each direction, unless that trip starts (and ends) at a boundary vertex connected to a leaf.*
4. *No trips T_1 and T_2 share two edges e_1, e_2 such that e_1 comes before e_2 in both trips.*

If G is a reduced graph, each fixed point of $\bar{f}_G$ corresponds to a lollipop [7].

2.4. The boundary measurement map

Let G be a reduced plabic network with e edges, and assign weights $t_1, \ldots, t_e$ to the edges of G. Postnikov defined a surjective map from the space of positive real edge weights of G to some positroid cell Π_G in $\mathrm{Gr}_{\geq 0}(k, n)$, called the *boundary measurement map* [7, Section 11.5]. Postnikov, Speyer and Williams re-cast this construction in terms of almost perfect matchings [8, Section 4-5], an approach Lam developed further in [5]. We use the definition of the boundary measurement map found in [5].

As mentioned above, an *almost perfect matching* of a plabic graph G is a subset of the edge set of G which covers each internal vertex exactly once. (Boundary vertices may or may not be covered.) For P an almost perfect matching on a plabic graph G with e edges, let

$$(4) \qquad \begin{aligned} \partial(P) = &\{\text{black boundary vertices used in } P\} \\ &\cup \{\text{white boundary vertices } not \text{ used in } P\} \end{aligned}$$

We define the *boundary measurement map*

$$(5) \qquad\qquad \partial_G : (\mathbb{R}_{>0})^e \to \mathbb{P}^{\binom{n}{k}-1}$$

to be the map which sends $(t_1, \ldots, t_e)$ to the point with homogeneous coordinates

$$(6) \qquad \Delta_J = \sum_{\partial(P)=J} t^P$$

where the sum is over all matchings P of G, and t^P is the product of the weights of all edges used in P.

The boundary measurement map ∂_G is surjective onto the totally non-negative cell Π_G. However, it is almost never injective, due to the existence of *gauge transformations*. Let v be a vertex of G, and let ω be a weighting of G with $\partial_G(\omega) = X$. Multiplying the weights of all edges incident at v by $\lambda \in \mathbb{R}_{>0}$ simply multiplies all Plücker coordinates by λ, and hence yields a new weighting ω' with $\partial_G(\omega') = X$. Conversely, if two weightings of G map to the same point in $\mathrm{Gr}_{\geq 0}(k, n)$, then they are related by some sequence of gauge transformations [7].

2.5. Bounded affine permutations

Bounded affine permutations are one of many families of combinatorial objects which index positroid cells. There is a natural bijection between bounded affine permutations and Postnikov's *decorated permutations*, so Postnikov's results about the latter translate easily to statements about the former [7, 3].

Definition 2.2. *An **affine permutation** of order n is a bijection $f : \mathbb{Z} \to \mathbb{Z}$ which satisfies the condition*

$$(7) \qquad f(i + n) = f(i) + n$$

for all $i \in \mathbb{Z}$. The affine permutations of order n form a group, which we denote $\widetilde{S}_n$.

We may embed S_n in $\widetilde{S}_n$ by extending each permutation periodically, in accordance with (7). Conversely, for f an affine permutation, there is a unique permutation $\bar{f} \in S_n$ such that for all $i \in [n]$, we have $\bar{f}(i) \cong f(i)$ (mod n).

Definition 2.3. *An affine permutation f is a* bounded affine permutation *of type (k, n) if it satisfies the following conditions*

1. $\dfrac{1}{n} \sum_{i=1}^{n} (f(i) - i) = k$.
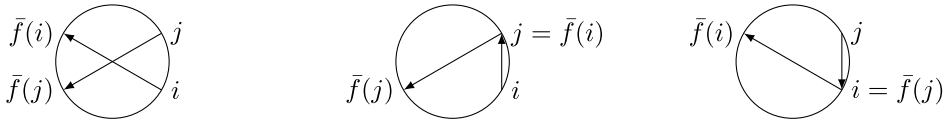*2. $i \leq \bar{f}(i) \leq i + n$ for all $i \in \mathbb{Z}$.*

Figure 3: Crossings in a chord diagram.

We write $\text{Bound}(k, n)$ *for the set of all bounded affine permutations of type* (k, n).

 Let G be a reduced plabic graph with n boundary vertices, and trip permutation $\bar{f}_G$. Suppose we have

$$(8) \qquad k = |\{i \in [n] : \bar{f}_G(i) < i \text{ or } i \text{ is a white lollipop of } G\}|.$$

Then G has an associated bounded affine permutation f_G of type (k, n) defined by setting

$$(9) \qquad f_G(i) = \begin{cases} \bar{f}_G(i) & \bar{f}_G(i) > i \text{ or } G \text{ has a black lollipop at } i \\ \bar{f}_G(i) + n & \bar{f}_G(i) < i \text{ or } G \text{ has a white lollipop at } i \end{cases}$$

for $i \in [n]$, and extending periodically using (7). For f a bounded affine permutation, we say f has a *white fixed point* at i if $f(i) = i + n$, and that f has a *black fixed point* at i if $f(i) = i$.

 Bounded affine permutations of type (k, n) are in bijection with positroid cells in $\text{Gr}_{\geq 0}(k, n)$ [7]. Let G be as above, and weight the edges of G with indeterminates. The *boundary measurement map* carries the space of positive real edge weights of G to the positroid cell Π_G corresponding to $\bar{f}_G$ [7]. There is a family of reduced graphs for each bounded affine permutation, and hence for each positroid cell.

 We represent bounded affine permutations visually using *chord diagrams*, introduced in [7, Section 16]. A chord diagram for a bounded affine permutation f is a circle with vertices labeled $1, 2, \ldots, n$ in clockwise order. We draw arrows from vertex i to vertex $\bar{f}(i)$ for all $i \in [n]$. By convention, if i is a white fixed point of f, we draw a clockwise loop at i; if i is a black fixed point, we draw a counter-clockwise loop.

 Let $(i, \bar{f}(i))$ and $(j, \bar{f}(j))$ be a pair of chords in the chord diagram of a bounded affine permutation f. This pair represents a crossing if $\bar{f}(j) \in [i, \bar{f}(i)]^{cyc}$ and $j \in [\bar{f}(i), i]^{cyc}$, so the two chords intersect. See Figure 3.

 Let Π be a positroid cell in $\text{Gr}_{\geq 0}(k, n)$ with bounded affine permutation f, and let V be a matrix representing a point in Π. Let $v_1, \ldots, v_n$ be the

columns of G. We extend this periodically to a sequence of vectors in $v_i \in \mathbb{C}^k$ by setting $v_i = v_j$ if $i \equiv j \pmod{n}$. The following lemma, which is implicit in [3], gives a characterization of f in terms of the v_i.

Lemma 2.4. *With the notation above, $f(i)$ is the smallest $r \geq i$ with $v_i \in \operatorname{Span}(v_{i+1}, v_{i+2}, \ldots, v_r)$. Similarly $f^{-1}(i)$ is the largest $r \leq i$ such that $v_i \in \operatorname{Span}(v_r, v_{r+1}, \ldots, v_{i-1})$.*

2.6. Grassmann and reverse Grassmann necklaces

We now introduce a final pair of indexing sets for positroid cells: Grassmann necklaces and dual Grassmann necklaces. The combinatorics of Grassmann necklaces are vital to the proof of Theorem 4.4, which shows that we can construct a symmetric plabic graph with symmetric weights for any point in the symmetric part of $\operatorname{Gr}_{\geq 0}(k, n)$.

Definition 2.5. *A Grassmann necklace $\mathcal{I} = (I_1, \ldots, I_n)$ of type (k, n) is a sequence of k-element subsets of $[n]$ such that the following hold, with indices taken modulo n:*

   1. If $i \in I_i$ then $I_{i+1} = (I_i \cup \{j\}) - \{i\}$ for some $j \in [n]$.
   2. If $i \notin I_i$, then $I_{i+1} = I_i$.

Postnikov defined a combinatorial bijection between Grassmann necklaces of type (k, n) and bounded affine permutations. Let $\mathcal{I}$ be a Grassmann necklace of type (k, n). We construct the bounded affine permutation f corresponding to $\mathcal{I}$ as follows, with indices taken modulo n:

   1. If $I_{i+1} = (I_i \cup \{j\}) - \{i\}$ for some $j \neq i$, then $\bar{f}(i) = j$.
   2. If $I_{i+1} = I_i$ and $i \notin I_i$, then i is a black fixed point of f.
   3. If $I_{i+1} = I_i$ and $i \in I_i$, then i is a white fixed point of f.

Next, we describe how to recover the Grassmann necklace $\mathcal{I}$ from the bounded affine permutation f. We say $i \in [n]$ is an *anti-exceedance* of f if either $\bar{f}^{-1}(i) > i$ or i is a white fixed point. We say i is an *a-anti-exceedance* if we have $\bar{f}^{-1}(i) >_a i$ or i is a white fixed point. The Grassmann necklace $\mathcal{I}$ corresponding to f is given by setting

(10) $$I_a = \{i \in [n] \mid i \text{ is an } a\text{-anti-exceedance of } f\}.$$

Let $\mathcal{M}$ be a positroid of type (k, n). Then $\mathcal{M}$ is a collection of k-element subsets of $[n]$. For each $1 \leq i \leq n$, let I_i be the minimal element of $\mathcal{M}$ with respect to the shifted linear order $\leq_i$. Then $\mathcal{M}$ is a Grassmann necklace

of type (k, n). This procedure gives a bijection between Grassmann necklaces and positroids [7]. For the inverse bijection, let $\mathcal{I} = (I_1, \ldots, I_n)$ be a Grassmann necklace of type (k, n), and let $\mathcal{M}$ be the set of all k-element subsets $J \in \binom{[n]}{k}$ such that $I_i \leq_i J$ for all $i \in [n]$. Then $\mathcal{M}$ is the positroid corresponding to $\mathcal{I}$ [6].

Let $\mathcal{M}$ and $\mathcal{I}$ be as above. If V is a matrix representing some point in the positroid cell Π corresponding to $\mathcal{M}$, with columns $v_1, \ldots, v_n$, then the columns indexed by I_i represent the minimal basis for $\mathbb{C}^k$ among the columns of V with respect to the cyclic order $\leq_i$. We will use this fact frequently in the proof of Theorem 4.4.

We also recall the dual notion, defined in [4, Section 3.6].

Definition 2.6. *A* dual Grassmann necklace *of type (k, n) is a sequence $\mathcal{J} = (J_1, \ldots, J_n)$ of k-element subsets of n such that the following hold, with indices taken modulo n:*

1. *If $i \in J_{i+1}$, then $J_i = (J_{i+1} \cup \{j\}) - \{i\}$ for some j*
2. *If $i \notin J_{i+1}$, then $J_i = J_{i+1}$.*

We have bijections between reverse Grassmann necklaces, decorated permutations, and positroids, which commute with the bijections given above for Grassmann necklaces. We describe these briefly below.

For $\mathcal{J}$ a dual Grassmann necklace, we define the corresponding bounded affine permutation f by setting

1. If $J_i = (J_{i+1} \cup \{j\}) - \{i\}$ for some $j \neq i$, then $\bar{f}^{-1}(i) = j$.
2. If $J_i = J_{i+1}$ and $i \notin J_{i+1}$, then i is a black fixed point of f.
3. If $J_i = J_{i+1}$ and $i \in J_{i+1}$, then i is a white fixed point of f

where again indices are taken modulo n.

If Π is a positroid cell with positroid $\mathcal{M}$ and dual Grassmann necklace $\mathcal{J}$, then J_i gives the maximal element of $\mathcal{M}$ with respect to the shifted order $\leq_i$.

2.7. Bridge graphs

We now describe a way to build up plabic graphs inductively. For more details on this construction, see [4].

Let G be a reduced plabic graph with bounded affine permutation f of type (k, n), corresponding to a positroid cell Π. If $f(i) > f(i + 1)$ for some $i \in [n]$, then $f s_i$ is a bounded affine permutation of type (k, n), with corresponding positroid cell Π^*. Moreover, we have $\dim(\Pi^*) = \dim(\Pi) + 1$.

We may add an edge, called a *bridge*, between the two edges of G incident at i and $i + 1$, to produce a graph G^* corresponding to Π^*. The bridge has a white vertex adjacent to i and a black vertex adjacent to $i + 1$. If G has a boundary leaf, or lollipop, at i or $i + 1$, that leaf becomes one endpoint of the bridge. Note that after adding the bridge, we add degree-two vertices as needed to make the graph bipartite, as in Figure 4.

For $i \in [n]$, let $x_i(c)$ be the elementary matrix with 1's along the diagonal, a nonzero entry c at position $(i, i+1)$, and 0's everywhere else. Let G be as above, and suppose $f(i) < f(i+1)$. Assign positive real weights to the edges of G, and let M be a matrix representing the corresponding point in $\mathrm{Gr}_{\geq 0}(k, n)$. Applying gauge transformations, we can assume the edges adjacent to boundary vertices i and $i + 1$ have weight 1. Adding a bridge at $(i, i+1)$ with weight c corresponds to multiplying M on the right by $x_i(c)$. Let X denote the point in $\mathrm{Gr}_{\geq 0}(k, n)$ corresponding to G with its original weighting, and X^* denote the point corresponding to the weighted graph obtained by adding the bridge. The boundary measurements change as follows:

$$(11) \quad \Delta_I(X^*) = \begin{cases} \Delta_I(X) + c\Delta_{(I \cup \{i\}) - \{i+1\}}(X) & \text{if } i+1 \in I \text{ but } i \notin I \\ \Delta_I(X) & \text{otherwise} \end{cases}.$$

Note that we are abusing notation slightly, since the Δ_I represent homogeneous coordinates rather than functions.
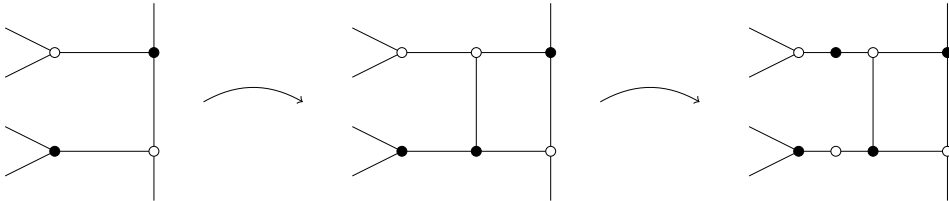

Figure 4: Adding a bridge to a plabic graph.

The following proposition, from [5], shows a way to "undo" the operation of adding a bridge, at least on the level of matrix representatives.

Proposition 2.7. *Let $X \in \mathrm{Gr}_{\geq 0}(k, n)$, and suppose X is contained in the positroid cell Π with bounded affine permutation f. Suppose $i < f(i) < f(i+1) < i+n+1$. Then*

$$(12) \qquad\qquad c = \frac{\Delta_{I_{i+1}}(X)}{\Delta_{(I_{i+1} \cup \{i\}) - \{i+1\}}(X)}$$

is positive and well defined, and $X^* = X \cdot x_i(-c)$ is in Π^*, where $\Pi^* \subset \mathrm{Gr}_{\geq 0}(k, n)$ is the positroid with bounded affine permutation fs_i and $\dim(\Pi^*) = \dim(\Pi) - 1$.

Note that we have

$$(13) \qquad\qquad i < f(i) < f(i+1) < i + n + 1$$

if and only if the chords in the chord diagram for f which start at i and $i+1$ respectively form a crossing as in Figure 3. If this occurs, we say that f *has a bridge* at $(i, i+1)$.

3. Characterizing symmetric plabic graphs

Let G be a plabic graph with $2n$ boundary vertices, corresponding to a positroid cell Π. Label the boundary vertices of G with the numbers $1, 2, \ldots, 2n$ in clockwise order, and fix a distinguished diameter d of G with one end between vertices $2n$ and 1, and the other between vertices n and $n+1$. For each $i \in 2n$, let $i' = 2n + 1 - i$.

For $I = \{i_1, i_2, \ldots, i_k\}$, we define $R(I) = [2n] \setminus \{i' \mid i \in I\}$. Let G be a symmetric plabic graph, and let $I \subseteq [2n]$. The reflection through d gives a bijection between the set of almost perfect matchings P of G with $\partial(P) = I$ and the set of almost perfect matchings P' of G with $\partial(P') = R(I)$. One immediate consequence is that if $I = \partial(P)$ for some matching P of G then $|I| = n$, so Π lies in $Gr_{\geq 0}(n, 2n)$.

Theorem 3.1. *Let Π be a positroid cell in $Gr_{\geq 0}(n, 2n)$ with positroid $\mathcal{M}$ and bounded affine permutation f. Let $\mathcal{I} = (I_1, \ldots, I_{2n})$ be the Grassmann necklace of Π, and let $\mathcal{J} = (J_1, \ldots, J_{2n})$ be the dual Grassmann necklace. The following are equivalent:*

- *(i) Π can be represented by a symmetric plabic graph.*
- *(ii) $I \in \mathcal{M}$ if and only if $R(I) \in \mathcal{M}$.*
- *(iii) For $a \in [2n]$, if $f(a) = b$, then $f(2n + 1 - a) = 4n + 1 - b$.*
- *(iv) $R(I_i) = I_{i'+1}$ for all $i \in [2n]$ with indices taken modulo $2n$.*
- *(v) $R(J_i) = J_{i'+1}$ for all $i \in [2n]$ with indices taken modulo $2n$.*

Proof. The above discussion shows that (i) implies (ii).

Next, we show that (ii) and (iii) are equivalent. Consider a reduced plabic graph H corresponding to Π, not necessarily symmetric, but with distinguished diameter d as above. Reflect H about d, and reverse the colors of all vertices. Let H' be the resulting plabic graph. We note that H' is reduced. Indeed, suppose H has a trip from boundary vertex a to boundary
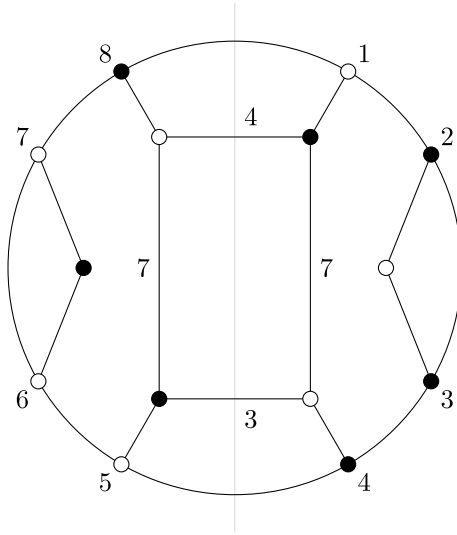
Figure 5: A symmetric weighting of a symmetric plabic graph. The distinguished diameter d is shown in gray. Unlabeled edges have weight 1.

vertex t. Then this trip corresponds, under reflection though d and reversing colors of vertices, to a trip from a' to t' in H'. Hence, H' satisfies the criterion for being reduced, which is phrased entirely in terms of forbidden intersections between trips.

Let $\bar{f}$ be the trip permutation of H, and $\bar{f}'$ the decorated permutation of H'. Then $\bar{f}(a) = t$ implies $\bar{f}'(a') = t'$. Note also that $a < t$ if and only if $a' > t'$. Moreover, a is a black fixed point of f if and only if a' is a white fixed point of f', and vice versa. It follows that $f(a) = b$ implies

$$(14) \qquad f'(2n + 1 - a) = 4n + 1 - b.$$

The reduced plabic graph H' corresponds to some positroid cell Π', with positroid $\mathcal{M}'$. It is clear that $\mathcal{M}' = \{I \mid R(I) \in \mathcal{M}\}$. Hence $\mathcal{M}' = \mathcal{M}$ if and only if $\mathcal{M}$ satisfies condition (ii) above, while $\bar{f} = f'$ if and only if f satisfies condition (iii). But the statement $\mathcal{M}' = \mathcal{M}$ and $\bar{f} = f'$ are both, in turn, equivalent to $\Pi = \Pi'$. Hence (iii) and (ii) are equivalent as desired.

Next, we show (iii) implies (i). Suppose $f(a) = b$ implies

$$f(2n + 1 - a) = 4n + 1 - b.$$

We claim that we can construct a symmetric plabic graph with bounded affine permutation f, using the bridge graph construction given above. We

proceed by induction on the dimension of the positroid Π corresponding to G. Note first that if a is a black fixed point of f, then (iii) implies that a' is a white fixed point, and vice versa. We may therefore assume that f has no fixed points. Hence there is some pair $(i, i+1)$ with $1 \leq i \leq 2n$ such that the bounded affine permutation f has a bridge at $(i, i+1)$. If $i = n$, let $g = fs_i$. Otherwise, let $g = fs_{i'-1}s_i$. In each case, g is a bounded affine permutation which satisfies (iii). By induction, we can build a symmetric plabic graph corresponding to g. Adding a bridge at $(n, n+1)$ or a pair of bridges at $(i, i+1)$ and $(i'-1, i')$, respectively, we obtain a symmetric plabic graph for f.

Next, we show that (iii) implies (iv) and (v). Indeed, if (iii) holds, then the chord diagram for f has an arrow from a to t if and only if it has an arrow from a' to t'. Hence for each $i \in [2n]$, we have $\bar{f}^{-1}(a) >_i a$ if and only if $\bar{f}^{-1}(a') <_{(i'+1)} a'$, and so $I_i = R(I_{i'+1})$. The argument for the dual Grassmann necklace is analogous.

Finally, we show that (iv) and (v) each imply (iii). Suppose (iv) holds. Note first that for $a \in [2n]$, we have $a \in I_i$ if and only if $a' \notin I_{i'+1}$. It follows in particular that if $a \in I_i$ for all $i \in [2n]$ (that is, a corresponds to a "white" fixed point of f) then $a' \notin I_i$ for all i, and a' corresponds to a black fixed point. Hence $f(a') = 4n+1-f(a)$ as desired. The analogous argument holds if a is a black fixed point.

Suppose a is not a fixed point of f. Then we have

$$(15) \qquad I_a = (I_{a+1} \cup \{a\}) - \{\bar{f}(a)\}$$

$$(16) \qquad I_{a'+1} = (I_{a'} \cup \{\bar{f}(a')\}) - \{a'\}$$

Applying R to the first line above, we have

$$(17) \qquad R(I_a) = (R(I_{a+1}) \cup \{(\bar{f}(a))'\}) - \{a'\}$$

By (iv), this implies

$$(18) \qquad I_{a'+1} = (I_{a'} \cup \{(\bar{f}(a))'\}) - \{a'\}$$

Comparing this with our previous expression for $I_{a'+1}$ above, we have

$$(19) \qquad \bar{f}(a') = (\bar{f}(a))'$$

It follows that f is a bounded affine permutation of type $(n, 2n)$ which satisfies (iii). Again, note that we could make an analogous argument using the dual Grassmann necklace. This completes the proof. $\qquad \square$

4. Network realizations of symmetric points

Let G be a symmetric plabic graph. Suppose we assign weights to the edges of G such that each edge (u, v) has the same weight as its reflection $(r(u), r(v))$ about the distinguished diameter d. For $I \in \binom{[n]}{k}$, reflection across d then gives a weight-preserving bijection between the almost perfect matchings P with $\partial(P) = I$ and the almost perfect matchings P' with $\partial(P') = R(I)$.

Definition 4.1. *A point $X \in \mathrm{Gr}_{\geq 0}(n, 2n)$ is a* symmetric point *if $\Delta_I(P) = \Delta_{R(I)}(P)$ for all I. The subvariety of symmetric points in $\mathrm{Gr}_{\geq 0}(n, 2n)$ is the* symmetric part *of $\mathrm{Gr}_{\geq 0}(n, 2n)$. Similarly, for a positiod cell Π, the* symmetric part *of Π is the intersection of Π with the symmetric part of $\mathrm{Gr}_{\geq 0}(n, 2n)$.*

Clearly, the image of every symmetric weighting of a symmetric plabic graph is a symmetric point. We show the converse.

Theorem 4.2. *Let $X \in \mathrm{Gr}_{\geq 0}(n, 2n)$ be a symmetric point. Then X may be represented by a symmetric plabic graph with symmetric weights.*

Proof. Since X is a symmetric point, the positroid of X satisfies condition (ii) from Theorem 3.1. Hence X may be represented by some weighting ω of a symmetric plabic graph G. Choose a collection F of edges of G which has all of the following properties:

1. F is symmetric about the diameter d. That is, an edge (u, v) of G is in F if and only if $(r(u), r(v))$ is also in F.
2. Every vertex of G is covered by F at least once.
3. F consists of a disjoint collection of trees, each of which has exactly one vertex on the boundary of G.

It is not hard to show that such a collection of edges exists, since F is symmetric. Since F is a disjoint union of trees, each with exactly one vertex on the boundary, we may gauge fix all edges in F to 1. Let ν be the resulting weighting of G, and let ν' be the weighting of G obtained by swapping the weights of (u, v) and $(r(u), r(v))$ for each edge (u, v) of G.

By symmetry, ν' is another weighting of G with $\partial_G(\nu') = X$. It follows that ν and ν' are the same up to gauge transformations. Moreover, ν' assigns the weight 1 to each edge of F. Since F satisfies conditions 2 and 3 above, a sequence of gauge transformations which fixes all edges in F must be trivial. Hence $\nu = \nu'$ and ν is a symmetric weighting. This completes the proof. $\square$

The following is immediate.

Corollary 4.3. *Let Π be a positroid cell. Then the symmetric part of Π is nonempty if and only if Π can be realized as the image of a symmetric plabic graph G, where the symmetric part of Π is precisely the image of all symmetric weightings of G under the boundary measurement map.*

A *lollipop graph* is a plabic graph which has a lollipop at each boundary vertex. A *bridge graph* is a plabic graph obtained by starting with a lollipop graph, and repeatedly adding bridges adjacent to the boundary. See Figure 6 for an example. Given $X \in \mathrm{Gr}_{\geq 0}(k, n)$, we can apply Proposition 2.7 repeatedly to construct a weighted plabic graph whose image under the boundary measurement map is the point X [5]. We now give a symmetric version of this result.
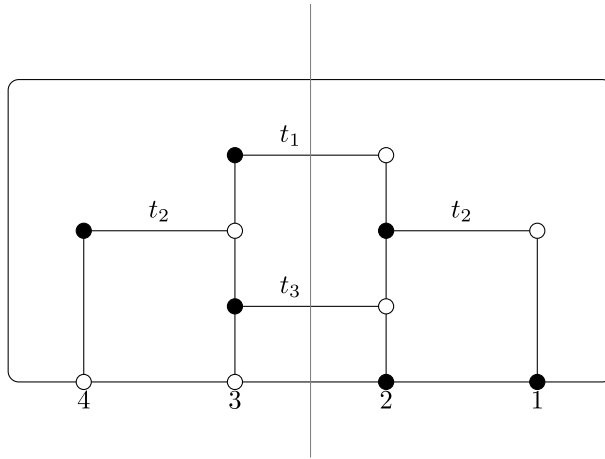

Figure 6: A symmetric bridge graph with symmetric weights.

Theorem 4.4. *Let $X \in \mathrm{Gr}_{\geq 0}(k, n)$ be a symmetric point. Then we can iteratively construct a weighted bridge graph corresponding to X which is symmetric and has symmetric weights.*

Proof. Let Π be the positroid cell containing X, and let f be the decorated permutation of Π. We induct on the dimension of Π. If the dimension is 0, then X may be represented by a lollipop graph G [7]. In this case, X has a single non-zero Plücker coordinate I, corresponding to the positions of the white lollipops, which necessarily satisfies $I = R(I)$. Thus G has a white lollipop at i if and only if G has a black lollipop at i', and G is symmetric.

Suppose $\dim(\Pi) > 0$. We note first that Π satisfies condition (ii) from Theorem 3.1. Hence i is a white fixed point of f if and only if i' is a black

fixed point of f. We may therefore reduce to the case where f has no fixed points. It follows that f has a bridge at $(i, i+1)$ for some $1 \le i \le 2n - 1$.

First, suppose f has a bridge at $(n, n+1)$. Let $(I_1, \ldots, I_{2n})$ denote the Grassmann necklace of X. By Proposition 2.7, the quantity

$$(20) \qquad c = \Delta_{I_{n+1}}(X)/\Delta_{(I_{n+1} \cup \{n\})-\{n+1\}}(X)$$

is positive and well-defined, and the point $X^* = X \cdot x_n(-c)$ lies in $\mathrm{Gr}_{\ge 0}(n, 2n)$.

The action of $x_n(-c)$ fixes the Plücker coordinates of X, except for those Δ_I with $n + 1 \in I$ and $n \notin I$. Note that $I \in \binom{[2n]}{n}$ has this property if and only if $R(I)$ does. For each I with $n + 1 \in I$ and $n \notin I$, we have

$$(21) \qquad \Delta_{R(I)}(X^*) = \Delta_{R(I)}(X) - c\Delta_{(R(I) \cup \{n\})-\{n+1\}}(X)$$
$$(22) \qquad = \Delta_I(X) - c\Delta_{R((I \cup \{n\})-\{n+1\})}(X)$$
$$(23) \qquad = \Delta_I(X) - c\Delta_{(I \cup \{n\})-\{n+1\}}(X)$$
$$(24) \qquad = \Delta_I(X^*)$$

So X^* is a symmetric point. By induction, we may build a symmetric bridge graph representing the point X^*, which has symmetric weights. Adding a bridge $(n, n+1)$ with edge weight c gives the desired weighted graph.

Next, suppose f does not have a bridge at $(n, n+1)$. Then f has a bridge $(i, i+1)$ for some $1 \le i \le 2n-1$ with $i \ne n$. Hence by symmetry, f has a pair of commuting bridges at $(i, i+1)$ and $(i'-1, i')$, respectively. Without loss of generality, assume $1 \le i \le n-1$. By Theorem 2.1, we have $I_{i'} = R(I_{i+1})$, which implies

$$(25) \qquad (I_{i'} \cup \{i'-1\}) - \{i'\} = R((I_{i+1} \cup \{i\}) - \{i+1\}).$$

Since X satisfies $\Delta_{R(I)}(X) = \Delta_I(X)$ for all I, we have

$$(26) \quad c := \Delta_{I_{i+1}}(X)/\Delta_{(I_{i+1} \cup \{i\})-\{i+1\}}(X) = \Delta_{I_{i'}}(X)/\Delta_{(I_{i'} \cup \{i'-1\})-\{i'\}}(X)$$

where both quantities are positive and well-defined, as above.

Once again, let $X^* = X \cdot x_i(-c)$. Let $(I_1^*, \ldots, I_{2n}^*)$ denote the Grassmann necklace of X^*, and let

$$(27) \qquad c^* = \Delta_{I_{i'}^*}(X^*)/\Delta_{(I_{i'}^* \cup \{i'-1\})-\{i'\}}(X^*).$$

We claim that $c^* = c$. First, note that $I_j^* = I_j$ unless $j = i + 1$, so $I_{i'}^* = I_{i'}$. If either $i \in I_{i'}$ or $i + 1 \notin I_{i'}$, then we have

$$(28) \qquad \Delta_{I_{i'}}(X) = \Delta_{I_{i'}}(X^*)$$

(29) $\Delta_{(I_{i'} \cup \{i'-1\})-\{i'\}}(X) = \Delta_{(I_{i'} \cup \{i'-1\})-\{i'\}}(X^*),$

and so $c^* = c$, as desired.

Suppose $i \notin I_{i'}$ and $i + 1 \in I_{i'}$. Let M be a matrix representative for X with columns $v_1, \ldots, v_{2n}$. Suppose $f(i' - 1) = i'$. Then $v_{i'}$ is a scalar multiple of $v_{i'-1}$, and in particular we have $v_{i'} = cv_{i'-1}$. Let $v_1^*, \ldots, v_{2n}^*$ be the columns of $M^* = M \cdot x_i(-c)$. Then $v_{i'}^* = v_{i'}$ and $v_{i'-1}^* = v_{i'-1}$. It follows that we have $c^* = c$ as desired.

Next, suppose $f(i' - 1) \neq i'$, and consider the submatrix S of M with columns indexed by $I_{i'}$. Since $i \notin I_{i'}$, it follows that v_i is in the span of the columns of S indexed by elements of $[i', i-1]^{cyc}$. Hence adding a multiple of v_i to the column v_{i+1} does not change the determinant of S, and $\Delta_{I_{i'}}(X^*) = \Delta_{I_{i'}}(X)$. Thus the ratios c^* and c have the same numerator.

We must now show that c^* and c have the same denominator. Since $f(i' - 1) \neq i'$, we have $i' \in I_{i'-1} \cap I_{i'}$. It follows that

(30) $(I_{i'} \cup \{i'-1\}) - \{i'\} = (I_{i'-1} \cup \{\bar{f}(i'-1)\}) - \{i'\}.$

Let S^* be the submatrix of M with columns indexed by this set. There are two cases to consider, depending on whether $\bar{f}(i')$ lies in the cyclic interval $[i', i]^{cyc}$. Throughout this proof, we write $v_a, \ldots, v_b$ to denote the columns of M indexed by elements of $[a, b]^{cyc}$.

For the first case, suppose $\bar{f}(i') \in [i', i]^{cyc}$. Since G has a bridge at $(i' - 1, i')$ and $\bar{f}(i' - 1) \neq i'$, this means $\bar{f}(i' - 1)$ must be in the cyclic interval $[i' + 1, i - 1]^{cyc}$. Let $y = \bar{f}(i' - 1)$. Then we may express v_y as a linear combination of the columns indexed by $I_{i'-1}$ which form a basis of $\langle v_{i'-1}, \ldots, v_{y-1} \rangle$. Note that we must have a nonzero coefficient of $v_{i'}$ in this linear combination, since the columns of S^* are linearly independent.

Hence $v_{i'}$ lies in the the span of the columns of S^* indexed by elements of $[i'-1, y]^{cyc}$. In particular, $v_{i'}$ lies in the span of the columns of S^* indexed by elements of $[i'-1, i-1]^{cyc}$, which therefore span $\langle v_{i'-1}, v_{i'}, \ldots, v_{i-1} \rangle$. Thus v_i lies in the span of the columns of S^* indexed by elements of $[i'-1, y]^{cyc}$.

For the second case, suppose $\bar{f}(i')$ does not lie in the cyclic interval $[i', i]^{cyc}$. Then $v_{i'}$ is linearly independent of the columns $v_{i'+1}, \ldots, v_i$, so $(I_{i'} \cup \{i'-1\}) - \{i'\}$ contains a basis for $\langle v_{i'+1}, \ldots, v_{i-1} \rangle$. Since $v_i \notin I_{i'}$, the corresponding columns of S^* contain v_i in their span.

Hence $c^* = c$, and so $X^{**} = X \cdot x_i(-c)x_{i'-1}(-c)$ lies in a positroid cell of dimension two less than Π. It is straightforward to check that X^{**} is a symmetric point. By induction, we may build a symmetric plabic graph for X^{**} with symmetric edge weights. Adding two bridges of weight c at

$(i' - 1, i')$ and $(i, i + 1)$ respectively gives a symmetric plabic network for X. $\qquad\square$

Note that the sequence of bridges added above depends only on the bounded affine permutation of Π, and the fact that X is symmetric. Hence, the proof of Theorem 4.4 yields a method for constructing a symmetric bridge graph G corresponding to Π, and explicitly realizing each point in the symmetric part of Π with a symmetric weighting of G. We have thus demonstrated a symmetric analog of the bridge-graph construction for $\mathrm{Gr}_{\geq 0}(k, n)$ found in [5].

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