Journal of Combinatorics
Volume 9, Number 1, 119–161, 2018

Revisiting the Hamiltonian theme in the square of a block: the case of DT-graphs

Gek L. Chia*, Jan Ekstein†, and Herbert Fleischner‡

The *square* of a graph G, denoted G^2, is the graph obtained from G by joining by an edge any two nonadjacent vertices which have a common neighbor. A graph G is said to have the $\mathcal{F}_k$ *property* if for any set of k distinct vertices $\{x_1, x_2, \ldots, x_k\}$ in G, there is a hamiltonian path from x_1 to x_2 in G^2 containing $k-2$ distinct edges of G of the form $x_i z_i$, $i = 3, \ldots, k$. In [7], it was proved that every 2-connected graph has the $\mathcal{F}_3$ property. In the first part of this work, we extend this result by proving that every 2-connected DT-graph has the $\mathcal{F}_4$ property (Theorem 2) and will show in the second part that this generalization holds for arbitrary 2-connected graphs, and that there exist 2-connected graphs which do not have the $\mathcal{F}_k$ property for any natural number $k \geq 5$. Altogether, this answers the second problem raised in [4] in the affirmative.

AMS 2000 subject classifications: 05C38, 05C45.
Keywords and phrases: Hamiltonian cycles and paths, square of a block.

1 Introduction and history **119**

2 Preliminary discussion **121**

3 DT-graphs **127**

Acknowledgements **159**

References **159**

1. Introduction and history

All concepts not defined in this paper can be found in the book by Bondy and Murty, [1], or in the other references. However, we prefer definitions

arXiv: 1706.04414

*The author was supported by the FRGS Grant (FP036-2013B).

†The author was supported by P202/12/G061 of the Grant Agency of the Czech Republic.

‡The author was supported by FWF-grant P27615-N25.

as given in Fleischner's papers if they differ from the ones given in [1]. In particular, we define a graph to be *eulerian* if its vertices have even degree only; that is, it is not necessarily connected. This is in line with D. König's original definition of an Eulerian graph, [12], and this is how eulerian graphs have been defined in Fleischner's papers quoted below (many authors call such graphs even graphs, whereas they consider a graph to be eulerian if it is a connected even graph). In any case, we consider finite loopless graphs only, but allow for multiple edges which may arise in certain constructions.

The study of hamiltonian cycles and hamiltonian paths in powers of graphs goes back to the late 1950s/early 1960s and was initiated by M. Sekanina who studied certain orderings of the vertices of a given graph. In fact, he showed in [17] that the vertices of a connected graph G of order n can be written as a sequence $a = a_1, a_2, \ldots, a_n = b$ for any given $a, b \in V(G)$, such that the distance $d_G(a_i, a_{i+1}) \leq 3$, $i = 1, \ldots, n-1$. This led to the general definition of the *k-th power of a graph G*, denoted by G^k, as the graph with $V(G^k) = V(G)$ and $xy \in E(G^k)$ if and only if $d_G(x, y) \leq k$. Thus Sekanina's result says that G^3 is hamiltonian connected for every connected graph G.

Unfortunately, this result cannot be generalized to hold for G^2, the *square* of an arbitrary connected graph G (the square of the subdivision graph of $K_{1,3}$ is not hamiltonian). Thus Sekanina asked in 1963 at the Graph Theory Symposium in Smolenice, which graphs have a hamiltonian square, [18]. In 1964, Neuman, [15], showed, however, that a tree has a hamiltonian square if and only if it is a caterpillar. On the other hand, it wasn't until 1978 when it was shown in ([19]), that Sekanina's question was too general, for it was tantamount to asking which graphs are hamiltonian (that is, an NP-complete problem).

However, in 1966 at the Graph Theory Colloquium in Tihany, Hungary, C. St. J. A. Nash-Williams asked whether it is true that G^2 is hamiltonian if G is 2-connected, [14], and noted that L.W. Beineke and M.D. Plummer had thought of this problem independently as well.

By the end of 1970, the third author of this paper answered Nash-Williams' question in the affirmative; the corresponding papers [5, 6] were published in 1974. In the same year, it was shown that this result implied that G^2 is hamiltonian connected for a 2-connected graph G, [2].

Further related research was triggered by Bondy's question (asked in 1971 at the Graph Theory Conference in Baton Rouge), whether hamiltonicity in G^2 implies that G^2 is *vertex pancyclic* (i.e., for every $v \in V(G)$ there are cycles of any length from 3 through $|V(G)|$). In fact, Hobbs showed in 1976, [11], that Bondy's question has an affirmative answer for the square of 2-connected graphs and connected bridgeless DT-graphs (the latter type of graphs in which every edge is incident to a vertex of *d*egree *t*wo, was essential for answering Nash-Williams' question – and it is essential for the main proofs of the current paper as well). The same issue of JCT B contains, however, a paper by Faudree and Schelp, [9], in which they proved for the same classes of graphs, that since G^2 is hamiltonian connected, there are paths joining v and w of arbitrary length from $d_{G^2}(v,w)$ through $|V(G)|-1$ for any $v,w \in V(G)$ (that is, G^2 is panconnected). They asked, however, whether this is a general phenomenon in the square of graphs (i.e., hamiltonian connectedness in G^2 implies panconnectedness in G^2). Bondy's question and the question by Faudree and Schelp were answered in full in [7].

Already in 1973 (and published in 1975) the most general block-cutvertex structure was determined such that every graph within this structure has a hamiltonian total graph, [8].

In the second part of the current work we establish in [3] the strongest possible results in some sense ($\mathcal{F}_k$-property), for the square of a block to be hamiltonian connected. As for hamiltonicity in the square of a block, the strongest possible result is cited Theorem E ([7, Theorem 3]). Altogether, these results will enable us to establish (in joint work with others) the most general block-cutvertex structure such that if G satisfies this structure then G^2 is hamiltonian connected or at least hamiltonian. That is, what has been achieved for total graphs, [8], will be achieved for general graphs correspondingly. Here, but also in the papers [5, 6, 7, 8] the concept of EPS-graphs plays a central role; and some of the theorems in the subsequent paper [3] require intricate proofs involving explicitly or implicitly EPS-graphs.

We are fully aware that there are shorter proofs on the existence of hamiltonian cycles in the square of a block; one has been found by Říha, [16]; and more recently, a still shorter proof was found by Georgakopoulos, [10]. Moreover, a short proof of Theorem E (cited below) has been found by Müttel and Rautenbach, [13]. Unfortunately, their methods of proof do not seem to yield the special results which we can achieve with the help of EPS-graphs. This is not entirely surprising: [8, Theorem 1] states that for a graph G, the total graph $T(G)$ is hamiltonian if and only if G has an EPS-graph (note that the *total graph* of G is the square of the subdivision graph of G).

2. Preliminary discussion

By a *uv-path* we mean a path from u to v. If a uv-path is hamiltonian, we call it a *uv-hamiltonian path*.

Definition 1. *Let G be a graph and let $A = \{x_1, x_2, \dots, x_k\}$ be a set of $k \geq 3$ distinct vertices in G. An x_1x_2-hamiltonian path in G^2 which contains $k-2$ distinct edges $x_iy_i \in E(G)$, $i = 3, \dots, k$ is said to be $\mathcal{F}_k$. Hence we speak of an $\mathcal{F}_k$ x_1x_2-hamiltonian path. If x_i is adjacent to x_j, we insist that x_iy_i and x_jy_j are distinct edges. A graph G is said to have the $\mathcal{F}_k$ property if for any set $A = \{x_1, \dots, x_k\} \subseteq V(G)$, there is an $\mathcal{F}_k$ x_1x_2-hamiltonian path in G^2.*

Let G be a graph. By an *EPS-graph*, *$JEPS$-graph* respectively, of G, denoted $S = E \cup P$, $S = J \cup E \cup P$ respectively, we mean a spanning connected subgraph S of G which is the edge-disjoint union of an eulerian graph E (which may be disconnected) and a linear forest P, respectively a linear forest P together with an open trail J. For $S = E \cup P$, let $d_E(v)$ and $d_P(v)$ denote the degree of v in E and P, respectively. In the ensuing discussion we need, however, special types of EPS-graphs: thus a $[v; w]$-EPS-graph $S = E \cup P$ of G with $v, w \in V(G)$, satisfies $d_P(v) = 0$ and $d_P(w) \leq 1$. For $k \geq 2$, $[v; w_1, \dots, w_k]$-EPS-graphs are defined analogously, whereas in $[w_1 \dots, w_k]$-EPS-graphs only $d_P(w_i) \leq 1$, $i = 1, \dots, k$, needs to be satisfied.

Let $bc(G)$ denote the block-cutvertex graph of the graph G. If $bc(G)$ is a path, we call G a *block chain.* A block chain G is called *trivial* if $E(bc(G)) = \emptyset$; otherwise it is called *non-trivial.* A block of G is an endblock of G if it contains at most one cutvertex of G.

In [5, Lemma 2], it was shown that if G is a block chain whose endblocks B_1, B_2 are 2-connected and $v \in B_1$ and $w \in B_2$ are not cutvertices of G, then G has an EPS-graph $S = E \cup P$ such that $d_P(v) = 0 = d_P(w)$. A more refined statement is now given below. In Lemma 1 we apply [5, Lemma 2, Theorem 3] and in Theorem 1 we apply Theorem D (stated explicitly below) several times to the blocks of G, respectively to G itself, to obtain EPS-graphs of the required type.

Lemma 1. *Suppose G is a block chain with a cutvertex, v and w are vertices in different endblocks of G and are not cutvertices. Then*

(i) there exists an EPS-graph $E \cup P \subseteq G$ such that $d_P(v),\ d_P(w) \leq 1$. If the endblock which contains v is 2-connected, then we have $d_P(v) = 0$ and $d_P(w) \leq 1$; and

(ii) there exists a $JEPS$-graph $J \cup E \cup P \subseteq G$ such that $d_P(v) = 0 = d_P(w)$. Moreover, v, w are the only odd vertices of J. Also, we have $d_P(c) = 2$ for at most one cutvertex c of G (and hence $d_P(c') \leq 1$ for all other cutvertices c' of G).

Proof. If G is a path, the result is trivially true.

So assume that G is not a path. If G has a suspended path (i.e., a maximal path whose internal vertices are 2-valent in G) starting at the endvertex v of G, then let P_v denote this path and let v_1 denote the other endvertex of P_v. Note that v_1 is a cutvertex of G. If there is no such suspended path, then define P_v to be an empty path. Likewise, P_w is defined similarly with w (respectively w_1) taking the place of v (respectively v_1).

(i) By [5, Lemma 2], $G' = G-(P_v \cup P_w)$ has an EPS-graph $S' = E' \cup P'$ with $d_{P'}(v_1) = 0$ and $d_{P'}(w_1) \leq 1$. But this means that G has an EPS-graph $S = E \cup P$ with $d_P(v) \leq 1$ and $d_P(w) \leq 1$ if we set $E = E'$ and $P = P' \cup P_v \cup P_w$. Clearly, in the case that P_v is an empty path, then $v = v_1$ and we have $d_P(v) = 0$ and $d_P(w) \leq 1$.

(ii) Let B be a block of G. Let $c_1, c_2 \in V(B)$. If B is not an endblock, then let $c_1, c_2 \in B$ be the cutvertices of G in B. If B is an endblock of G, then let only one of c_1, c_2, say c_2, to be a cutvertex of G, and let $c_1 = v$, $c_1 = w$ respectively, depending on the endblock c_1 belongs to. By [5, Theorem 3], B has a $JEPS$-graph $S_B = J_B \cup E_B \cup P_B$ with $d_{P_B}(c_1) = 0$, $d_{P_B}(c_2) \leq 1$, and c_1, c_2 are the only odd vertices of J_B. If B is not an endblock, then we may interchange c_1 and c_2. Thus we can ensure that for at most two blocks of G, B' and B'' say, satisfying $B' \cap B'' = c_2$, we have $d_{P_{B'}}(c_2) = d_{P_{B''}}(c_2) = 1$.

Note that if B is not a 2-connected block, then $E_B = \emptyset = P_B$ so that $S_B = J_B$. In this case, $d_{P_B}(c_1) = 0 = d_{P_B}(c_2)$.

By taking $S = \bigcup_B S_B$, where the union is taken over all blocks B of G, we have a $JEPS$-graph that satisfies the conclusion of (ii).

This completes the proof. □

Theorem 1. *Suppose G is a 2-connected graph and v, w are two distinct vertices in G. Then either*

(i) there exists an EPS-graph $S = E \cup P \subseteq G$ with $d_P(v) = 0 = d_P(w)$; or

(ii) there exists a $JEPS$-graph $S = J \cup E \cup P \subseteq G$ with v, w being the only odd vertices of J, and $d_P(v) = 0 = d_P(w)$.

Proof. If G is a cycle, then clearly the result is true. Hence assume that G is not a cycle.

Let K' be a cycle in G containing v, w. If $d_G(v) = 2$, then we take a $[w; v]$-EPS-graph with $K' \subseteq E$. If $d_G(w) = 2$, then we take a $[v; w]$-EPS-graph with $K' \subseteq E$. In either case, Theorem D (stated below) guarantees the existence of such EPS-graphs. Thus conclusion (i) of the theorem is satisfied.

Hence we assume that $d_G(v), d_G(w) \geq 3$. We proceed by contradiction, letting G be a counterexample with minimum $|E(G)|$.

Let $G' = G - K'$ denote the graph obtained from G by deleting all edges of K' (including all possibly resulting isolated vertices).

(a) Suppose G' is 2-connected. G' either has an EPS-graph $S' = E' \cup P'$ or a $JEPS$-graph $S' = J' \cup E' \cup P'$ satisfying the additional property (i) or (ii), respectively.

Suppose $S' = E' \cup P'$. Then set $E = K' \cup E'$, $P = P'$ to obtain an EPS-graph $S = E \cup P$ of G satisfying property (i). If G' has a $JEPS$-graph $S' = J' \cup E' \cup P'$ satisfying property (ii), then set $E = E'$, $P = P'$ and $J = J' \cup K'$, to obtain a $JEPS$-graph $S = J \cup E \cup P$ as required. Whence G' is not 2-connected.

(b) Suppose G' has an endblock B' with $(B' - \gamma c') \cap \{v, w\} = \emptyset$ where $\gamma c' = c'$ if B' contains a cutvertex c' of G', and $\gamma c' = \emptyset$ otherwise (in this latter case, B' is a component of G' having at least two vertices with K' in common). It follows that $G' \supseteq H'$ where H' is a block chain with $B' \subseteq H'$ and $G^* := G - H'$ is 2-connected. Suppose H' is chosen in such a way that G^* is as large as possible.

It follows that if H' is not 2-connected then $|V(G^*) \cap V(H' - V(B'))| = 1$. Denote the corresponding vertex with c^* and observe that c^* is a cutvertex if $c^* \in V(G')$. Also, by the choice of B' and the maximality of G^* we have

$$(H' - c^*) \cap \{v, w\} = \emptyset$$

and c^* is not a cutvertex of H'. Let $u' \in V(B') - \gamma c'$ be chosen arbitrarily. We set $\delta c^* = c^*$ if c^* is a pendant vertex of H', and $\delta c^* = \emptyset$ otherwise. By repeated application of Theorem D (see below) we obtain an EPS-graph $S' = E' \cup P'$ of $H' - \delta c^*$ with $d_{P'}(\delta c^*) = 0$ (setting $d_{P'}(\emptyset) = 0$) and $d_{P'}(u') \leq 1$.

If however, H' is 2-connected, i.e. $H' = B'$, then we let $c^* = (G' - B') \cap B'$, if B' contains a cutvertex of G', otherwise $c^* \in V(B') \cap V(K')$ arbitrarily. Futhermore we choose $u' \in V(B') - c^*$ arbitrarily. By Theorem D, $B' = H'$ has a $[c^*; u']$-EPS-graph $S' = E' \cup P'$.

Also, G^* has an EPS-graph $S^* = E^* \cup P^*$ or a $JEPS$-graph $S^* = J^* \cup E^* \cup P^*$ with $d_{P^*}(v) = d_{P^*}(w) = 0$; and $K' \subset E^*$, $K' \subset J^* \cup E^*$ respectively.

Observing that $P^* \cap P' = \emptyset$ and that S^* and S' are edge-disjoint, we conclude that $E = E^* \cup E'$ and $P = P^* \cup P'$ together with $J = J^*$ yield $S = E \cup P$, $S = J \cup E \cup P$ respectively, a spanning subgraph of G as claimed by the theorem (observe that $d_P(c^*) = d_{P^*}(c^*)$ because $d_{P'}(c^*) = 0$, and $d_{P^*}(c^*) = 0$ if $c^* \in \{v, w\}$).

(c) Because of the cases solved already, we now show that G' is connected and for every endblock B' of G', $V(B') \cap \{v, w\} \neq \emptyset$. For, if G' is disconnected and because of case (b) already solved, G' could be written as

$$G' = G'_1 \,\dot{\cup}\, G'_2$$

where G'_i is a component of G'; and

$$G'_i \cap \{v, w\} \neq \emptyset, \quad i = 1, 2.$$

Without loss of generality $v \in G'_1$, $w \in G'_2$. Consequently, $G_i := G'_i \cup K'$, $i = 1, 2$, is 2-connected with $d_{G_1}(w) = 2$, $d_{G_2}(v) = 2$. Arguing as at the very beginning of the proof of this theorem (where we considered the case $d_G(v) = 2$ or $d_G(w) = 2$) we conclude that the corresponding EPS-graphs $S_i = E_i \cup P_i$ with $K' \subseteq E_i$, $i = 1, 2$, satisfy conclusion (i) of the theorem, and so does $S = E \cup P$ where $E = E_1 \cup (E_2 - K')$ and $P = P_1 \cup P_2$.

Because of case (a) already solved, we thus have that G' is a non-trivial block chain with v, w belonging to different endblocks B_v, B_w respectively, of G' and they are not cutvertices of G'. Let c_v and c_w be the respective cutvertices of B_v and B_w (possibly $c_v = c_w$). If B_v is not a bridge of G' we use a $[v; c_v]$-EPS-graph S_v of B_v and a $[w; c_w]$-EPS-graph S_w of B_w if B_w is also not a bridge, or $S_w = \emptyset$ if B_w is a bridge. Proceeding similarly for every block B of $G' - (B_v \cup B_w)$ we conclude that G' has an EPS-graph $S' = E' \cup P'$ with $d_{P'}(v) = d_{P_v}(v) = 0$ and $d_{P'}(w) = d_{P_w}(w) = 0$, where $P_v \subseteq S_v$, $P_w \subseteq S_w$ (defining $d_{P_w}(w) = 0$ if $P_w = \emptyset$). Thus in either case $S' \cup K'$ is an EPS-graph of G satisfying conclusion (i). However, if both B_v and B_w are bridges, i.e., $d_{G'}(v) = d_{G'}(w) = 1$, we introduce $z \notin V(G')$ and form $G_z := G' \cup \{z, zv, zw\}$. G_z contains a cycle K_z through z, v, w since $\kappa(G_z) \geq 2$, so it contains a $[v, w]$-EPS-graph $S_z = E_z \cup P_z$ with $K_z \subseteq E_z$. Trivially, $d_{P_z}(v) = d_{P_z}(w) = 0$, and for the component $E_0 \subseteq E_z$ with $z \in E_0$ we have $J := (E_0 - z) \cup K$ being an open trail joining v and w. Setting $E = E_z - E_0$ and $P = P_z$ we conclude that $S = J \cup E \cup P$ is a $JEPS$-graph satisfying conclusion (ii) of the theorem. Theorem 1 now follows. □

The following results from [8], [5], and [7] will be used quite frequently in the proof of Theorem 2.

Let G be a graph and let W be a set of vertices in G. A cycle K in G is said to be W-*maximal* if $|V(K') \cap W| \leq |V(K) \cap W|$ for any cycle K' of G. Moreover, we say that the W-maximal K *is* W-*sound* if $|V(K) \cap W| \geq 4$.

The following Theorems A and B are special cases of the theorems quoted.

Theorem A. *([8, Theorem 4]) Let G be a 2-connected graph and let W be a set of five distinct vertices in G. Suppose K is a W-sound cycle in G. Then there is an EPS-graph $S = E \cup P$ of G such that $K \subseteq E$ and $d_P(w) \leq 1$ for every $w \in W$.*

An EPS-graph which satisfies the conclusion of Theorem A is also called a *W-EPS-graph.*

Theorem B. *([8, Theorem 3]) Let G be a 2-connected graph and let v, w_1, w_2, w_3 be four distinct vertices of G. Suppose K is a cycle in G such that $\{v, w_1, w_2, w_3\} \subseteq K$. Then G has a $[v; w_1, w_2, w_3]$-EPS-graph $S = E \cup P$ such that $K \subseteq E$.*

Suppose G is a 2-connected graph and v, w_1, w_2 are distinct vertices in G. A cycle K in G is a $[v; w_1, w_2]$-*maximal cycle* in G if $\{v, w_1\} \subseteq V(K)$, and $w_2 \in V(K)$ unless G has no cycle containing all of $\{v, w_1, w_2\}$.

Theorem C. *([8, Theorem 2]) Let G be a 2-connected graph and let v, w_1, w_2 be three distinct vertices of G. Suppose K is a $[v; w_1, w_2]$-maximal cycle in G. Then G has a $[v; w_1, w_2]$-EPS-graph $S = E \cup P$ such that $K \subseteq E$.*

Theorem D. *([5, Theorem 2]) Let G be a 2-connected graph and let v, w be two distinct vertices of G. Let K be a cycle through v, w. Then G has a $[v; w]$-EPS-graph $S = E \cup P$ with $K \subseteq E$.*

Theorem E. *([7, Theorem 3]). Suppose v and w are two arbitrarily chosen vertices of a 2-connected graph G. Then G^2 contains a hamiltonian cycle C such that the edges of C incident to v are in G and at least one of the edges of C incident to w is in G. Further, if v and w are adjacent in G, then these are three different edges.*

A hamiltonian cycle in G^2 satisfying the conclusion of Theorem E is also called a $[v; w]$-hamiltonian cycle. More generally, a hamiltonian cycle C in G^2 which contains two edges of G incident to v, and at least one edge of G incident to each w_i, $i = 1, \ldots, k$, is called a $[v; w_1, \ldots, w_k]$-hamiltonian cycle, provided the edges in question are all different.

Theorem F. *([7, Theorem 4]). Let G be a 2-connected graph. Then the following hold.*

(i) G has the $\mathcal{F}_3$ property.

(ii) For a given $q \in \{x, y\}$, G^2 has an xy-hamiltonian path containing an edge of G incident to q.

By applying Theorems E and F to each block of a block chain B, we have the following.

Corollary 1. *Suppose B is a non-trivial block chain with $|V(B)| \geq 3$ and v and w are vertices in different endblocks of G. Assume further that v, w are not cutvertices of B. Then*

(i) B^2 has a hamiltonian cycle which contains an edge of B incident to v and an edge of B incident to w. In the case that the endblock which contains v is 2-connected, then B^2 has a hamiltonian cycle which contains two edges of B incident to v and an edge of B incident to w. Also,

(ii) B^2 has a vw-hamiltonian path containing an edge of B incident to v and an edge of B incident to w.

3. *DT*-graphs

Recall that a graph is called a *DT-graph* if every edge is incident to a 2-valent vertex. If G is a graph, we denote by $V_2(G)$ the set of all vertices of degree 2 in G.

The following result which is interesting in itself, is obtained by applying Theorem 1 and the construction in [5] of a hamiltonian cycle/path in the corresponding spanning subgraph.

Corollary 2. *Let G be a DT-block and $x_1, x_2 \in V(G)$ satisfying $N(x_1) \cup N(x_2) \subseteq V_2(G)$ and $x_1x_2 \notin E(G)$. Then either (i) there exists a hamiltonian cycle in $G^2 - x_2$ whose edges incident to x_1 are in G, or else (ii) there exists an x_1x_2-hamiltonian path in G^2 whose first and final edges are in G.*

Theorem 2. *Every 2-connected DT-graph has the $\mathcal{F}_4$ property.*

The proof of Theorem 2 is rather involved. We first give an outline of the general strategy used in the proof.

Let G be a 2-connected DT-graph and let $A = \{x_1, x_2, x_3, x_4\}$ be a set of four distinct vertices in G. Let G^+ denote the 2-connected graph obtained from G by adding a new vertex y which joins x_1 and x_2. Then G^+ is a DT-graph unless $N_G(x_i) \nsubseteq V_2(G)$ for some $i \in \{1, 2\}$. We shall show that $(G^+)^2$ contains a hamiltonian cycle C containing edges of G^+ of the form $yx_1, yx_2, x_3z_3, x_4z_4$ where x_3z_3, x_4z_4 are edges of G. Then clearly C gives rise to an $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2 when we delete the vertex y from $(G^+)^2$.

In order to show the existence of such hamiltonian cycle C in $(G^+)^2$, we shall apply induction or show that G^+ admits an EPS-graph $S = E \cup P$ with some additional properties. In particular, in almost all cases, E will contain a prescribed cycle K^+ passing through y. K^+ will also contain as many elements of $\{x_3, x_4\}$ as possible. Note that G^+ is 2-connected and hence contains a cycle through y and x_i, $i \in \{3,4\}$, which automatically contains x_1, x_2.

Note that in [5] it was shown that if a 2-connected DT-graph H admits an EPS-graph, then H^2 has a hamiltonian cycle. We refer the reader to [5] for the method of constructing such hamiltonian cycle and to see how edges of H can be included in such hamiltonian cycle. Also, we may automatically assume that in an EPS-graph $S = E \cup P$ the edges of P are the bridges of S (otherwise, we could delete step-by-step P-edges (i.e., edges of P) until such situation is achieved).

However, G^+ may not be a DT-graph and/or some elements in A may be 2-valent and (at least) one of its neighbors may not be 2-valent. In such cases, the existence of the various types of EPS-graphs S in G^+ may not be sufficient to guarantee a hamiltonian cycle to begin with in S^2. Even if we can derive the existence of a hamiltonian cycle from these EPS-graphs, they may not suffice to guarantee a hamiltonian cycle with the additional properties. Thus we need to consider neighbors of elements of A to assure that they are incident to less than two P-edges. This applies, in particular, to $z_i \in N_G(x_i)$ with $z_i x_i \in E(K^+)$, $i \in \{1,2,3,4\}$.

The following observations will be used quite frequently (sometimes implicitly) in the proof of Theorem 2.

Observation (*): *Suppose $S = E \cup P$ is an EPS-graph of G^+ such that $d_P(x_i) \leq 1$ for $i = 1,2$. Let x be a 2-valent vertex of G belonging to E.*

(i) Suppose $N(x) = \{u_1, u_2\}$. Then S^2 has a hamiltonian cycle which contains the edges yx_1, yx_2 and $u_i x$ for some $i \in \{1,2\}$ unless $x_j \in N(u_j) \cup \{u_j\}$ and $d_P(x_j) = 1, d_S(u_j) > 2$ for $j = 1,2$; or for some $j \in \{1,2\}$, $d_P(x_j) = 1, d_P(z_j) = 2$ and $z_j \in N(x_j) \cap V(K^+)$; or $d_P(u_1) = d_P(u_2) = 2$ – in all three cases $N_G(x_j) \not\subseteq V_2(G)$.

(ii) We further note that any pendant edge in S will always be contained in any hamiltonian cycle of S^2.

(iii) Consider $W \subseteq V(G^+)$ with $|W| = 5$ and $K^+ \subset G^+$. Suppose $|W \cap V(K^+)| \geq 4$. If K^+ is W-sound, then Theorem A applies. If, however, K^+ is not W-sound, then there is a W-sound cycle K^* with $W \subseteq K^*$ and we operate with K^* in place of K^+. This follows from the definition of W-soundness (see the discussion immediately preceding Theorem A).

The observations (i) and (ii) follow directly from the degree of freedom inherent in the construction of a hamiltonian cycle in S^2 as given in [5].

The proof of Theorem 2 is divided into several cases depending on whether $N(x_i) \subseteq V_2(G)$ or not, $i = 1,2,3,4$. Note that if $N(x_i) \not\subseteq V_2(G)$, then $d_G(x_i) = 2$. If $d_G(x_i) = 2$, we let $N(x_i) = \{u_i, v_i\}$ throughout the proof. Also, we define $x_i^* = x_i$ if $d_G(x_i) > 2$; and $x_i^* = z_i$ otherwise.

Lemma 2. *Let G^+ be defined as before with $N(x_3) \not\subseteq V_2(G)$ and $N(x_4) \not\subseteq V_2(G)$. Suppose $N(x_i) \subseteq V_2(G)$ for some $i \in \{1,2\}$. Assume further that every proper 2-connected subgraph of G has the $\mathcal{F}_4$ property. Then $(G^+)^2$ has a hamiltonian cycle containing the edges $x_1y, x_2y, x_3z_3, x_4z_4$ where x_3z_3, x_4z_4 are different edges of G.*

Proof. By the hypotheses, $d_G(x_3) = d_G(x_4) = 2$. Assume without loss of generality that $N(x_1) \subseteq V_2(G)$.

(1) Suppose $\{u_i, v_i\} \neq \{x_1, x_2\}$ for $i = 3, 4$.
Let K^+ be a cycle containing the vertices $y, x_1, x_2, x_4, u_4, v_4$.

(1.1) Assume that K^+ also contains the vertex x_3.
We may assume that

$$K^+ = yx_1z_1 \ldots u_4x_4v_4 \ldots u_3x_3v_3 \ldots z_2x_2y.$$

(a) Assume that $u_4 \neq x_1$.

Since $\{x_1, x_2, x_3, x_4, u_3, u_4, z_2\} \subseteq V(K^+)$, Theorem B ensures the existence of a $[u_4; x_1, z_2, x_2]$-EPS-graph $S_4 = E_4 \cup P_4$ of G^+ with $K^+ \subseteq E_4$ in the case $x_3x_4 \in E(G)$. Likewise, we obtain a $[u_4; x_1, u_3, x_2^*]$-EPS-graph $S_3 = E_3 \cup P_3$ of G^+ with $K^+ \subseteq E_3$ if $x_3x_4 \notin E(G)$ where $x_2^* = x_2$ if $d_G(x_2) > 2$, and $x_2^* = z_2 = V(K^+) \cap N_G(x_2)$ otherwise. It is straightforward to see that in both cases, the EPS-graph yields a hamiltonian cycle in $(G^+)^2$ as required by the lemma (see Observation (*)(i)).

(b) Assume that $u_4 = x_1$ and $v_3 = x_2$.

(b1) Suppose x_3 and x_4 are adjacent or $N(x_3) \cap N(x_4) \neq \emptyset$.

(i) x_3 and x_4 are adjacent. Let $G^- = G - \{x_3, x_4\}$. If G^- is not 2-connected, then it is a non-trivial block chain with x_1, x_2 belonging to different endblocks, and x_1, x_2 are not cutvertices of G^-. Hence $(G^-)^2$ has a hamiltonian path $P(x_1, x_2)$ starting with an edge x_1w_1 of G and ending with an edge x_2w_2 of G (see Corollary 1(ii)). Then

$$(P(x_1, x_2) - \{x_1w_1, x_2w_2\}) \cup \{x_1x_4, x_2x_3, x_4w_1, x_3w_2\}$$

defines a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

If G^- is 2-connected, then $(G^-)^2$ has a hamiltonian cycle C^- containing x_1w_1, x_1t_1, x_2w_2 which are edges of G. Then

$$(C^- - \{x_1w_1, x_1t_1, x_2w_2\}) \cup \{w_1t_1, x_1x_4x_3w_2\}$$

is a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

(ii) Suppose $N(x_3) \cap N(x_4) = \{u\}$.

If $d_G(u) = 2$, then let $G^- = G - \{x_3, x_4, u\}$ and proceed similarly as before to obtain a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2. Hence we assume that $d_G(u) > 2$. Suppose further that $G - x_i$ is 2-connected for some $i \in \{3, 4\}$. Then $G - x_i$ has the $\mathcal{F}_4$ property with u taking the place of x_i; and any such $\mathcal{F}_4$ x_1x_2-hamiltonian path in $(G - x_i)^2$ can be extended to a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2. Thus we have to consider the case $\kappa(G - x_i) < 2$ for $i \in \{3, 4\}$.

Consider $G - x_4$. Since $d_G(x_4) = 2$, $G' = G - x_4$ is a non-trivial block chain with x_1, u belonging to different endblocks of G' and are not cutvertices of G'. The endblock B_u of G' with $u \in V(B_u)$ also contains x_3, x_2 because $d_{G'}(u) \geq 2$ and $d_G(x_3) = d_{G'}(x_3) = 2$. Hence B_u is 2-connected. Let c be the cutvertex of G' belonging to B_u.

Suppose first $c \neq x_2$. Because of the hypothesis of the lemma, B_u has the $\mathcal{F}_4$ property. Correspondingly, there is a hamiltonian path $P(c, x_2)$ in $(B_u)^2$ containing x_3w_3, uu' with $w_3 \in \{u, x_2\}$, which are different edges of B_u. Likewise, there is a hamiltonian path $P(x_1, c)$ in $(G' - B_u)^2$ by Theorem F, Corollary 1(ii), respectively. Then

$$P(x_1, c) \cup (P(c, x_2) - uu') \cup \{u'x_4, x_4u\}$$

is a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2.

Finally suppose $c = x_2$. By Theorem F(ii) or Corollary 1(ii), $(G' - B_u)^2$ has a x_1x_2-hamiltonian path $P_{1,2}$ ending with an edge w_2x_2 of G. By Theorem E, $(B_u)^2$ has a hamiltonian cycle C_u with $\{ux_3, x_2x_3, z_2x_2\} \subset E(B_u)$.

$$(P_{1,2} \cup C_u - \{w_2x_2, z_2x_2, ux_3\}) \cup \{w_2z_2, ux_4, x_4x_3\}$$

defines a hamiltonian path as required.

(b2) Suppose x_3 and x_4 are not adjacent and $N(x_3) \cap N(x_4) = \emptyset$.

Let $W = \{y, x_1, x_2, u_3, v_4\}$. Then K^+ is W-sound. By Theorem A, G^+ has an EPS-graph $S = E \cup P$ with $K^+ \subseteq E$ and $d_P(w) \leq 1$ for every $w \in W$; and $d_P(x_3) = d_P(x_4) = 0$. Because of the hypothesis of this case a required hamiltonian cycle can be constructed in $(G^+)^2$ (see Observation (*)(i)). In particular, the hamiltonian cycle contains x_4v_4 and u_3x_3.

(c) Assume that $u_4 = x_1$ and $v_3 \neq x_2$.

If $x_3x_4 \notin E(K^+)$, then Theorem B ensures the existence of an $[x_2; x_1, v_4, v_3]$-EPS-graph $S_3 = E_3 \cup P_3$ of G^+ with $K^+ \subseteq E_3$. By construction, S^2

contains a hamiltonian cycle C with $x_4v_4, x_3v_3 \in E(C)$. (see Observation (*)(i)). Hence we assume that $x_3x_4 \in E(K^+)$.

If $v_3x_2 \notin E(K^+)$, or $v_3x_2 \in E(K^+)$ and $d_G(x_2) > 2$, then we invoke Theorem C to obtain a $[v_3; x_1, x_2^*]$-EPS-graph $S_3 = E_3 \cup P_3$ of G^+ with $K^+ \subseteq E_3$. If, however, $v_3x_2 \in E(K^+)$ and $d_G(x_2) = 2$, then Theorem C ensures the existence of an $[x_2; x_1, v_3]$-EPS-graph $S_3 = E_3 \cup P_3$ of G^+ with $K^+ \subseteq E_3$. Note that K^+ contains all these special vertices. In all these cases, $(S_3)^2$ contains a hamiltonian cycle C with $x_3x_4, x_3v_3 \in E(C)$ (see Observation (*)(i)).

(1.2) In view of case (1.1), we may assume that G^+ has no cycle containing y, x_4, x_3, and that

$$K^+ = yx_1z_1 \cdots u_4x_4v_4 \cdots z_2x_2y,$$

and $G^+ - x_3$ is 2-connected if $G^+ - x_i$ is 2-connected for some $i \in \{3, 4\}$.

Without loss of generality, assume that $u_3 \notin \{x_1, x_2\}$.

(a) Consider first the case that $G^* = G^+ - x_3$ is 2-connected.

Define $W^* = \{y, x_1, x_2^*, u_4, u_3\}$ if $x_1 \neq u_4$ and $W^* = \{y, x_1, x_2^*, v_4, u_3\}$ otherwise. Abbreviate $W^* = \{y, x_1, x_2^*, t_4, u_3\}$ with $t_4 \in \{u_4, v_4\}$.

(a1) We first deal with the case $|W^*| = 5$.

In view of Observation (*)(iii), set $K^* = K^+$ if K^+ is W^*-sound in G^*, or else there exists $K^* \supset W^*$ in G^* (note $|K^+ \cap W^*| \geq 4$).

(a1.1) Assume that $x_4 \in K^*$. In this case we may assume that $K^+ = K^*$. By Theorem A, there exists a W^*-EPS-graph $S^* = E^* \cup P^*$ of G^* with $K^* \subseteq E^*$. Noting that $d_{P^*}(u_3) \leq 1$, we set $E = E^*$ and $P = P^* \cup \{u_3x_3\}$. Then $S = E \cup P$ is an EPS-graph of G^+ whose structure implies that $(G^+)^2$ has a hamiltonian cycle containing the edges u_3x_3 and t_4x (because x_3 is a pendant vertex in S – see Observation (*)(i)–(ii)).

(a1.2) Assume that $x_4 \notin K^*$. Then $u_3 \in K^*$ (hence $K^+ \neq K^*$). Since $x_i \notin K^*$, for $i = 3, 4$, $d_G(t_4) > 2$, $d_G(u_3) > 2$. We define x_2^{**} as x_2^* with respect to K^*.

First suppose $x_2^{**} = x_2^*$. By Theorem B, there exists a $[u_4; x_1, u_3, x_2^*]$-EPS-graph $S^* = E^* \cup P^*$ of G^* with $K^* \subseteq E^*$ if $x_1 \neq u_4$. By the same token, there is a $[u_4; v_4, u_3, x_2^*]$-EPS-graph $S^* = E^* \cup P^*$ of G^* with $K^* \subseteq E^*$ if $x_1 = u_4$. In both cases, we set $E = E^*$, $P = P^* \cup \{x_3u_3\}$. Then $S = E \cup P$ is an EPS-graph of G^+ which yields a hamiltonian cycle in $(G^+)^2$ containing

u_3x_3 and x_4z for some $z \in N(x_4)$. If x_4 is a pendant vertex in S^*, then it is adjacent to v_4 (see Observation (*)(i)–(ii)).

If $x_2^{**} \neq x_2^*$, then we proceed analogously as before using x_2^{**} instead of x_2^*. Note that $u_3 = x_2^{**}$ is not an obstacle (we use Theorem C) because of $d_S(x_2) = 2$ since $d_G(x_2) = 2$ and x_2^* is also in K^* (thus $x_2x_2^* \notin E(S)$) in this case.

(a2) Assume that $|W^*| = 4$.

(a2.1) $W^* = \{y, x_1, x_2^*, t_4\}$ where $t_4 \in \{u_4, v_4\}$. If $u_3 = t_4$, then we operate with a $[t_4; x_1, x_2^*]$-EPS graph $S^* = E^* \cup P^*$ of G^* with $K^+ \subseteq E^*$, which exists by Theorem C. If $u_3 = x_2^*$, then we operate with a $[x_2^*; x_1, t_4]$-EPS graph $S^* = E^* \cup P^*$ of G^* with $K^+ \subseteq E^*$, which exists by Theorem C (note that $x_2^* \neq x_2$ in this case using $u_3 \neq x_2$).

In either case, set $E = E^*$ and $P = P^* \cup \{x_3u_3\}$. Then $S = E \cup P$ is an EPS-graph of G^+ which yields a hamiltonian cycle C in $(G^+)^2$ containing either $x_3t_4x_4$ or $x_2^*x_3$, x_4t_4 (see Observation (*)(i)).

(a2.2) $W^* = \{y, x_1, x_2^*, u_3\}$. Then either (i) $u_4 \neq x_1$, or (ii) $u_4 = x_1$ and $v_4 \neq x_2^*$ or (iii) $u_4 = x_1$ and $v_4 = x_2^*$.

In cases (i) and (ii) we are back to case (a2.1) with $u_3 = t_4 \neq x_2^*$.

In case (iii) we have $x_2^* \neq x_2$ because $N(x_4) \neq \{x_1, x_2\}$. We consider $G' = G^+ - \{x_4, \delta x_2^*\}$; again, $\delta x_2^* = x_2^*$ if x_2^* is a pendant vertex in $G^+ - x_4$ and $\delta x_2^* = \emptyset$ otherwise. Set $x_2' = x_2^*$ if $x_2^* \in V(G')$ and $x_2' = x_2$ otherwise. Suppose $\kappa(G') = 1$. In any case, G' has different endblocks B_1' and B_2'; they are 2-connected with $x_1 \in B_1'$ and $x_2' \in B_2'$ not being cutvertices of G'. Since G' is homeomorphic to G if $x_2' = x_2$ (a contradiction to $\kappa(G') = 1$), it follows that $x_2^* \in B_2'$ and that x_2 is a cutvertex of G' since $\{x_2\} = B_1' \cap B_2'$. However, $3 = d_{G^+}(x_2) = d_{B_1'}(x_2) + d_{B_2'}(x_2) \geq 2+2$, an obvious contradiction. Thus G' is 2-connected in any case. Starting with a cycle $K' \subseteq G'$ with $y, x_1, x_2, x_3 \in V(K')$ we apply Theorem C to obtain an $[x_1; u_3, x_2^{**}]$-EPS-graph $S' = E' \cup P'$ of G' with $K' \subseteq E'$, where $x_2^{**} = N_G(x_2) - x_2^*$. Setting $E = E'$, $P = P' \cup \{x_1x_4, \delta(x_4x_2^*)\}$, where $\delta(x_4x_2^*) = x_4x_2^*$ if $x_2^* \notin V(G')$ and $\delta(x_4x_2^*) = \emptyset$ otherwise, we obtain $S = E \cup P$ of G^+ with $K' \subseteq E$ and $d_P(x_1) = 1$ and $d_P(u_3) \leq 1$. It is clear that S^2 yields a hamiltonian cycle of $(G^+)^2$ as required (see Observation (*)(i)).

(a3) Assume that $|W^*| = 3$.

Then $W^* = \{y, x_1, x_2^*\}$.

Hence $u_3 \notin \{x_1, y\}$, therefore $u_3 = x_2^*$. Analogously $t_4 \notin \{x_1, y\}$, therefore $t_4 = v_4 = x_2^*$. That is, $u_3 = x_2^* = t_4 = v_4$ and $x_1 = u_4$. $G' = G^+ - x_4$ is

2-connected since there is a cycle K' in G' containing y and x_3 and hence also x_2^*, x_1, v_3. If $v_3 \neq x_1$, we operate with an $[x_2^*; x_1, v_3]$-EPS-graph $S' = E' \cup P'$ of G' with $K' \subseteq E^*$ (by Theorem C). Setting $E = E^*$ and $P = P^* \cup \{x_4 x_2^*\}$, we obtain an EPS-graph $S = E \cup P$ of G^+ which will yield a hamiltonian cycle in $(G^+)^2$ containing x_3v_3, x_4v_4 (see Observation (*)(i)). If $v_3 = x_1$, then $G - x_4$ is 2-connected (since $N(x_3) = N(x_4)$). Hence $G - x_4$ has the $\mathcal{F}_4$ property with v_4 taking the place of x_4; and any such $\mathcal{F}_4$ x_1x_2-hamiltonian path in $(G - x_4)^2$ can be extended to a required $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2. This finishes the proof of case **(a)**.

(b) Now consider the case where $G^* = G^+ - x_3$ has a cutvertex and hence $G^+ - x_4$ has also a cuvertex, because of the assumptions of case **(1.2)**. Thus G^* is a non-trivial block chain since $d_G(x_3) = 2$. Note that K^+ is contained in some endblock B_y of G^*.

Let $W = \{y, x_1, x_2^*, x_3, t_4\}$ where we define t_4 as follows:

- $t_4 = u_4$ if $u_4 \neq x_1$;
- $t_4 = v_4$ if $u_4 = x_1$ and either $x_2^* = x_2$ or $v_4 \neq x_2^* \neq x_2$;
- $t_4 = x_2$ if $u_4 = x_1$ and $v_4 = x_2^* \neq x_2$.

Note that by this definition of t_4, $|W| = 5$.

Assume first that the cycle K^+ (which passes through $y, x_1, x_2, x_2^*, u_4, x_4, v_4$) is W-sound in G^+. Let $\widehat{G}$ denote the subgraph of G^+ which is a non-trivial block chain containing u_3, x_3, v_3 such that $G^+ - \widehat{G} = B_y$. Suppose w_3 is the vertex in one of the endblocks of $\widehat{G}$ and w_3' the vertex in the other endblock of $\widehat{G}$ such that $\widehat{G} \cap B_y = \{w_3, w_3'\}$. Possibly $\{w_3, w_3'\} \cap \{u_3, v_3\} \neq \emptyset$, but $\{w_3, w_3'\} \neq \{u_3, v_3\}$.

We replace $\widehat{G}$ in G^+ by a path $P_4 = a_1a_2x_3a_3a_4$ (where a_1, a_3 are identified with w_3, w_3' respectively, and $\{a_2, a_3\} = \{u_3, v_3\}$) to obtain the graph G''. Note that $K^+ \subseteq G''$. Set $W = \{y, x_1, x_2^*, x_3, t_4\}$ as above. Then K^+ is W-sound (by assumption), and by Theorem A, G'' has an EPS-graph $S'' = E'' \cup P''$ such that $K^+ \subseteq E''$ and $d_{P''}(z) \leq 1$ for every $z \in W$.

(b1) Suppose $E(P_4) \cap E(P'') = \emptyset$. Then $P_4 \subseteq E''$. Since $\widehat{G}$ is a non-trivial block chain, by Lemma 1(ii), $\widehat{G}$ contains a $JEPS$-graph $\widehat{S} = \widehat{J} \cup \widehat{E} \cup \widehat{P}$ such that $d_{\widehat{P}}(w_3) = 0 = d_{\widehat{P}}(w_3')$, and w_3, w_3' are the odd vertices of $\hat{J}$; hence $d_{\widehat{J}}(x_3) = 2$ and $d_{\widehat{P}}(x_3) = 0$. Note that by the second part of Lemma 1(ii) we can make sure that $\min\{d_{\widehat{P}}(u_3), d_{\widehat{P}}(v_3)\} \leq 1$. In this case, we obtain an EPS-graph $S = E \cup P$ of G^+ by setting $E = (E'' - P_4) \cup \widehat{J} \cup \widehat{E}$ and $P = P'' \cup \widehat{P}$. Here $d_P(x_3) = 0$ and $d_P(w) \leq 1$ for every $w \in W - x_3$.

(b2) Suppose $E(P_4) \cap E(P'') \neq \emptyset$. That is, $V(P_4) \subseteq V(P'')$ (so that $E(P_4) \cap E(E'') = \emptyset$) and $d_{P''}(x_3) = 1$. This means that either $a_2x_3 \notin E(P'')$ or $x_3a_3 \notin E(P'')$. Suppose $x_3a_3 \notin E(P'')$ (so that $a_3a_4 \in E(P'')$). In this case, we delete x_3v_3 from $\widehat{G}$ and thus split $\widehat{G}$ into two block chains $\widehat{G_1}$ and $\widehat{G_2}$ with $x_3, w_3 \in \widehat{G_1}$ and $v_3, w'_3 \in \widehat{G_2}$. If $\widehat{G_j}$ is an edge only, then $\widehat{S_j} = \widehat{G_j}$. If $\widehat{G_2} = \emptyset$, then $\widehat{S_2} = \emptyset$. Otherwise by Lemma 1(i) (or by Theorem D if $\widehat{G_2}$ is 2-connected), $\widehat{G_j}$ has an EPS-graph $\widehat{S_j} = \widehat{E_j} \cup \widehat{P_j}$ where $d_{\widehat{P_1}}(w_3) \leq 1$, $d_{\widehat{P_1}}(x_3) = 1$, $d_{\widehat{P_2}}(v_3) \leq 1$, $d_{\widehat{P_2}}(w'_3) \leq 1$, $j = 1, 2$. Now, if we take $E = \widehat{E_1} \cup \widehat{E_2} \cup E''$ and $P = \widehat{P_1} \cup \widehat{P_2} \cup (P'' - \{a_2, a_3\})$, we have an EPS-graph $S = E \cup P$ of G^+ with $d_P(w) \leq 1$ for every $w \in W$ (note that $w_3a_2, w'_3a_3 \in P''$), x_3 is a pendant vertex in S, and it works also if $\widehat{G}$ is a path on at least 4 vertices.

In both cases **(b1)** and **(b2)**, a required hamiltonian cycle in $(G^+)^2$ can be constructed from S (see Observation (*)(i)–(ii)). Note that $G^+ - x_4$ is 2-connected if $K^+ = yx_1x_4x_2^*x_2y$ (hence $d_G(x_2) = 2$) and if $d_G(x_2^*) > 2$. Here we have a contradiction to the assumption of this case **(1.2)(b)**.

Now assume that the cycle K^+ is not W-sound. Since $y, x_1, x_2^*, t_4 \in K^+$ and $|W| = 5$, there exists a cycle $K^* \subseteq G^+$ containing all of W and not x_4.

(i) Suppose $t_4 = v_4$ or $t_4 = x_2$. In both cases, K^* contains $u_4 = x_1$ and $v_4 = t_4$, $v_4 = x_2^*$, respectively, but not x_4. Hence $G^+ - x_4$ is 2-connected, a contradiction with assumptions of case **(1.2)(b)**.

(ii) Suppose $t_4 = u_4$. Because $G^+ - x_3$ has a cutvertex, without loss of generality suppose that $u_3 \notin K^+$ but clearly $u_3 \in K^*$. Hence $u_3 \notin \{x_1, x_2^*, u_4\}$. We define x_2^{**} as x_2^* with respect to K^*.

First suppose $x_2^{**} = x_2^*$. By Theorem B, G^+ has a $[u_4; x_1, x_2^*, u_3]$-EPS-graph $S = E \cup P$. Note that either x_4 is a pendant vertex in S, or else x_4 is a vertex in E. It is clear that S^2 yields a hamiltonian cycle of $(G^+)^2$ as required (see Observation (*)(i)–(ii)).

If $x_2^{**} \neq x_2^*$, then we proceed analogously as before using x_2^{**} instead of x_2^*. Note that $x_2^{**} = u_3$ or $x_2^{**} = u_4$ is not an obstacle (we use Theorem C) because of $d_S(x_2) = 2$ since $d_G(x_2) = 2$ and x_2^* is also in K^* (thus $x_2x_2^* \notin E(S)$); and $x_2^{**} = u_3 = u_4$ is not possible in this case.

(2) Suppose $\{u_3, v_3\} = \{x_1, x_2\}$.

Note that, in G, there exists a cycle containing x_3 and x_4 (and hence also the vertices u_3, v_3, u_4, v_4).

Let $G^* = G^+ - x_3$ which is homeomorphic to G and thus G^* is 2-connected. Note that there exists a cycle $K^* = K^+$ (see above) in G^* containing the vertices y, x_1, x_2, x_4.

(2.1) Suppose $w \in N(x_4) - \{x_1, x_2\}$ exists; let $x_2z_2 \in E(K^*)$. Note that $d_{G^*}(x_2) = 2$ if $d_G(z_2) > 2$.

By Theorem B, there exists an $[x_1; x_2, z_2, w]$-EPS-graph, an $[x_1; x_2, z_2]$-EPS-graph by Theorem C respectively, if $w = z_2$; in both cases we denote $S^* = E^* \cup P^* \subset G^*$ with $K^* \subseteq E^*$ and $d_{P^*}(x_1) = 0$. Note that K^* is $[x_1; x_2, z_2]$-maximal if $z_2 = w$. Set $E = E^*$ and $P = P^* \cup \{x_1x_3\}$; thus $d_P(x_1) = 1$. Also, $d_P(x_2) + d_P(z_2) \leq 1$ since $d_P(z_2) > 0$ implies $d_P(x_2) = 0$ since $d_{G^*}(x_2) = 2$. Then $S = E \cup P$ is an EPS-graph of G^+ and a hamiltonian cycle in $(G^+)^2$ can be constructed (using S) which starts with yx_1, x_1x_3, ends with yx_2 and traverses wx_4 even if $w = z_2$ (see Observation (*)(i)–(ii) for x_1x_3).

(2.2) Next assume that $\{u_4, v_4\} = \{x_1, x_2\}$.

Note that, in this case, $d_G(x_2) > 2$ since $d_G(x_1) > 2$ can be assumed and x_3, x_4 are 2-valent (note that the lemma is trivially true if G is a 4-cycle).

Consider the graph $G' = G - \{x_3, x_4\}$.

(a) Suppose G' is 2-connected. We shall apply Theorem 1 to G' with x_1, x_2 in place of v, w.

(i) Suppose G' has an EPS-graph $S' = E' \cup P'$ with $d_{P'}(x_i) = 0$ for $i = 1, 2$. Let $E = E' \cup \{yx_1x_4x_2y\}$ and $P = P' \cup \{x_1x_3\}$; this yields an EPS-graph $S = E \cup P$ of G^+ with $d_P(x_1) = 1$, $d_P(x_2) = 0$, $d_P(x_3) = 1$ and $d_P(x_4) = 0$. Hence we may construct a hamiltonian cycle in $(G^+)^2$ containing the edges x_1x_3 and x_2x_4 apart from yx_1, yx_2.

(ii) Suppose G' has a $JEPS$-graph $S' = J' \cup E' \cup P'$ with x_1, x_2 being the only odd vertices of J' and $d_{P'}(x_1) = 0 = d_{P'}(x_2)$. Let $E = E' \cup (J' \cup \{x_1yx_2\})$ and $P = P' \cup \{x_1x_3, x_2x_4\}$. Then $S = E \cup P$ is an EPS-graph of G^+ with $d_P(x_1) = d_P(x_2) = d_P(x_3) = d_P(x_4) = 1$. Hence a hamiltonian cycle in $(G^+)^2$ containing the edges yx_1, yx_2, x_1x_3 and x_2x_4 can be constructed.

(b) Finally assume that G' is not 2-connected. Then G' is a non-trivial block chain. By Lemma 1(ii) with $x_1 = v$ and $x_2 = w$, G' has a $JEPS$-graph $S' = J' \cup E' \cup P'$ with $d_{P'}(x_1) = 0 = d_{P'}(x_2)$. As before, take $E = E' \cup (J' \cup \{x_1yx_2\})$ and $P = P' \cup \{x_1x_3, x_2x_4\}$. Then $S = E \cup P$ is an EPS-graph of G^+ with $d_P(x_1) = d_P(x_2) = d_P(x_3) = d_P(x_4) = 1$. Hence a hamiltonian cycle in $(G^+)^2$ containing the edges yx_1, yx_2, x_1x_3 and x_2x_4 can be constructed.

This completes the proof of the lemma. □

Proof of Theorem 2. Let G be a 2-connected DT-graph and $A = \{x_1, x_2, x_3, x_4\}$ be a set of four distinct vertices in G. It is easy to see that the theorem holds if G is a cycle. Hence we also apply induction, apart from direct construction at the given graph. However, in general let G^+ be defined as before.

Case (A): $N(x_i) \subseteq V_2(G)$, $i = 1, 2, 3, 4$.

There exists a cycle K^+ in G^+ containing the vertices y, x_1, x_2, x_4 (and possibly x_3), assuming that K^+ is at least as long as any cycle containing y, x_1, x_2, x_3. Assume K^+ is W-sound for $W = \{y, x_1, x_2, x_3, x_4\}$. By Theorem A, there exists a W-EPS-graph $S = E \cup P$ in G^+ with $K^+ \subseteq E$ (that is, $d_P(w) \leq 1$ for every vertex w in W). Moreover $d_P(y) = 0$ (since y is 2-valent in G^+ and $K^+ \subseteq E$).

Since $N(x_i) \subseteq V_2(G)$ for $i = 1, 2, 3, 4$, a hamiltonian cycle C in $(G^+)^2$ can be constructed, and C will contain yx_1, yx_2 and at least one edge of G incident to x_j for $j = 3, 4$. That is, G^2 contains a hamiltonian path as required (see Observation (*)(i)).

Case (B): $N(x_i) \subseteq V_2(G)$, $i = 1, 2, 3$ and $N(x_4) \nsubseteq V_2(G)$; i.e., $d_G(x_4) = 2$.

Let K^+ be a cycle in G^+ containing y, x_1, x_2, x_4 and possibly x_3.

(B)(1) Suppose x_3 is not in K^+ (so, no cycle of G^+ contains y and x_i, $i = 1, 2, 3, 4$).

(a) Suppose $\{u_4, v_4\} \neq \{x_1, x_2\}$.

Then we may assume that $u_4 \notin \{x_1, x_2\}$. Let $G' = G^+ - x_4$ and let $K' \subseteq G'$ be a cycle containing y, x_1, x_2, x_3.

(a1) Suppose G' is 2-connected.

Set $W' = \{y, x_1, x_2, x_3, u_4\}$ and suppose without loss of generality that K' is W'-sound (i.e., $u_4 \in V(K')$ if G' has a cycle containing all of W'). By Theorem A, G' has a W'-EPS-graph $S' = E' \cup P'$ with $K' \subseteq E'$ such that $d_{P'}(w) \leq 1$ for all $w \in W'$ with $d_{P'}(y) = 0$. Take $E = E'$ and $P = P' \cup \{u_4x_4\}$. Then $S = E \cup P$ is an EPS-graph of G^+ with $K' \subseteq E$, $d_P(y) = 0, d_P(x_4) = 1$ and $d_P(w) \leq 1$ for $w \in W' - \{u_4\}$; and $d_P(u_4) \leq 2$. A careful examination of this case and Observation (*)(i)–(ii) show that a required hamiltonian cycle in $(G^+)^2$ can be constructed (note that u_4x_4 is a pendant edge of S).

(a2) Suppose G' is not 2-connected.

Then G' is a non-trivial block chain. Let B_y denote the block in G' containing K'. Note that u_4, v_4 belong to different endblocks of G'. Let

$z_4 \in \{u_4, v_4\}$ be a vertex in an endblock B_1 of G' where $B_1 \neq B_y$. Further let $\widehat{G}$ denote the maximal block chain in G' containing B_1 but no edges of B_y. Let $c_0 \in V(B_y) \cap V(\widehat{G})$ be a cutvertex of G' (which is not a cutvertex of $\widehat{G}$).

Now replace $\widehat{G}$ in G^+ with a path $P_2 = z_4 z c_0$ of length 2 joining z_4 and c_0 and call the resulting graph G^*; $z \notin V(G)$. In so doing the cycle K^+ is transformed into the cycle K^* in G^* containing $P_2 \cup \{y, x_1, x_2, x_4\}$. Observe that $x_3 \notin V(K^*)$; otherwise K^* could be extended to become a cycle in G^+ containing $y, x_1, \ldots, x_4$ contrary to the supposition of this case. Set $W = \{y, x_1, x_2, x_3, x_4\}$. K^* is W-sound in G^*; by Theorem A, G^* contains a W-EPS-graph $S^* = E^* \cup P^*$ with $K^* \subseteq E^*$, $d_{P^*}(y) = 0 = d_{P^*}(x_4) = d_{P^*}(z_4)$ and $d_{P^*}(w) \leq 1$ for all $w \in W - \{y, x_4\}$.

Let $H = \widehat{G} \cup P_2$. Then H is a 2-connected graph and hence has a $[c_0; z_4]$-EPS-graph $S_H = E_H \cup P_H$ with $K_H \subseteq E_H$ where $K_H = (K^+ \cap \widehat{G}) \cup P_2$ (see Theorem D).

By taking $E = ((E^* \cup E_H) - (K^* \cup K_H)) \cup K^+$ and $P = P^* \cup P_H$ we have $S = E \cup P$ being a W-EPS-graph of G^+ with $K^+ \subseteq E$, $d_P(y) = 0 = d_P(x_4)$, $d_P(w) \leq 1$ for all vertices $w \in W - \{y, x_4\}$, $d_P(z_4) \leq 1$ and $d_P(y_4) \leq 2$ where $y_4 \in N(x_4) - z_4$. Hence a required hamiltonian cycle H in $(G^+)^2$ can be constructed (as $\{u_4 x_4, x_4 v_4\} \subseteq K^+$); in particular $z_4 x_4 \in E(H)$ (see Observation (*)(i)).

(b) Suppose $\{u_4, v_4\} = \{x_1, x_2\}$.

Let $G' = G^+ - x_4$ (which is 2-connected since G is 2-connected) and let K' be a cycle in G' containing y, x_1, x_2, x_3. By Theorem C, there exists an $[x_1; x_2, x_3]$-EPS-graph $S' = E' \cup P'$ in G' with $K' \subseteq E'$, $d_{P'}(w) \leq 1$ for $w \in \{x_2, x_3\}$ and $d_{P'}(x_1) = 0$. Let $E = E'$ and $P = P' \cup \{x_1 x_4\}$. Then we have an EPS-graph $S = E \cup P$ of G^+ with $K' \subseteq E$ and $d_P(x_i) \leq 1$ for $i = 1, 2, 3$, and x_4 is a pendant vertex in S. Hence we can can construct a hamiltonian cycle in $(G^+)^2$ containing the edges $yx_1, yx_2, x_1 x_4$ and $x_3 t_3$ where $t_3 \in N(x_3)$ since $N(x_3) \subseteq V_2(G)$ (see Observation (*)(i)–(ii)).

(B)(2) Suppose also x_3 is in K^+.

Assume without loss of generality that $K^+ = y x_1 z_1 \cdots z_3 x_3 w_3 \cdots u_4 x_4 v_4 \cdots z_2 x_2 y$.

(a) Suppose $u_4 \neq x_3$.

Set $W = \{y, x_1, x_2, x_3, u_4\}$. Then $W \subseteq K^+$ and hence K^+ is W-sound. By Theorem A, there is a W-EPS-graph $S = E \cup P$ in G^+ such that $K^+ \subseteq E$ and $d_P(w) \leq 1$ for every $w \in W$. Then it is possible to construct in $(G^+)^2$ a hamiltonian cycle C containing the edges x_3w_3 and u_4x_4 (recall that x_4, w_3 are 2-valent vertices in G) (see Observation (*)(i)).

(b) Suppose $u_4 = x_3$.

(i) Suppose $v_4 \neq x_2$. We apply Theorem B to G^+ to obtain an $[x_3; x_1, x_2, v_4]$-EPS-graph $S = E \cup P$ with $K^+ \subseteq E$ and $d_P(x_3) = 0$, $d_P(x_i) \leq 1$ for $i = 1, 2$, and $d_P(v_4) \leq 1$. Since $K^+ \subseteq E$ and $x_4 \in K^+$, we have $d_P(x_4) = 0$. We can construct a hamiltonian cycle C in $(G^+)^2$ whose two edges incident to x_i are edges of G for $i = 3$ or $i = 4$, one of which is (without loss of generality) x_3x_4 (see Observation (*)(i)).

(ii) Suppose $v_4 = x_2$. We operate analogously as in case (i) with an $[x_3; x_1, x_2, y]$-EPS-graph S provided $x_3 \notin N(x_1)$. However S^2 does not yield a hamiltonian cycle as required if $x_3 \in N(x_1)$. That is, $d_G(x_3) = 2$; $d_G(x_4) = 2$, and $N(x_1) \subseteq V_2(G)$ by the assumptions. This is a special case of Lemma 2. This finishes the proof of **Case (B)**.

Case (C): $N(x_i) \subseteq V_2(G)$, $i = 1, 2$ and $d_G(x_3) = 2 = d_G(x_4)$.
The proof of this case follows from Lemma 2.

Case (D): $N(x_1) \subseteq V_2(G)$ and $N(x_2) \not\subseteq V_2(G)$; $d_G(x_2) = 2$ follows.

(D)(1) $N(x_4) \subseteq V_2(G)$**.**

There is a cycle K^+ in G^+ containing y, x_1, x_2, x_3 and also x_4 if such a cycle exists. Recall that $x_3^* = x_3$ if $d_G(x_3) > 2$ and $x_3^* = u_3 = z_3$ if $d_G(x_3) = 2$, and $N(x_2) = \{u_2, v_2\}$ and assume that v_2 is in K^+. Let x_3^-, x_3^+ denote the predecessor, successor respectively, of x_3 in K^+, where we start the traversal of K^+ with the edge yx_1. We also note that $x_3^* = u_3 = x_3^-$ and $v_3 = x_3^+$ if $x_3 \in V_2(G)$.

(1.1) Assume that $v_2 \notin \{x_3, x_4\}$.

(a) $N(x_3) \subseteq V_2(G)$.

Let $W = \{y, x_1, v_2, x_3, x_4\}$. Without loss of generality let K^+ be chosen such that it is W-sound, since $\{y, x_1, v_2, x_3\} \subseteq K^+$ anyway, and possibly $x_4 \in K^+$. Let $S = E \cup P$ be a W-EPS-graph of G^+ with $K^+ \subseteq E$ (by Theorem A). Observe that if $x_4 \notin E$, then it is a pendant vertex in S; also $d_P(x_2) \leq 1$ automatically since $N(x_2) \not\subseteq V_2(G)$ and $x_2 \in K^+$. Now it is easy to construct a required hamiltonian cycle C in S^2 having the required properties; we may assume that $x_3x_3^+ \in E(C)$ and $x_4w_4 \in E(G) \cap E(C)$, since $d_P(x_4) \leq 1$ and $N(x_4) \subseteq V_2(G)$. This is even true if $x_1 = x_3^-$ since both x_3 and x_3^+ are 2-valent in G in this case (see Observation (*)(i)–(ii)).

(b) $N(x_3) \nsubseteq V_2(G)$; $d_G(x_3) = 2$ follows.

Set $W^+ = \{y, x_1, v_2, x_3^+, x_4\}$.

(b1) Suppose $x_4 = x_3^+$. Since $K^+ \supset \{y, x_1, v_2, x_3, x_3^+, x_4\}$, by Theorem B, G^+ contains an $[x_4; y, x_1, v_2]$-EPS-graph $S = E \cup P$ with $K^+ \subseteq E$ such that $d_P(x_4) = d_P(x_3) = 0$, $d_P(v_2) \leq 1$, $d_P(x_1) \leq 1$, but also $d_P(x_2) \leq 1$. We obtain a hamiltonian cycle $C \subset S^2$ as required with $x_3x_3^+, x_4w_4 \in E(G) \cap E(C)$ ($w_4 \notin V(K^+)$ may hold, if $d_S(x_4) > 2$). This covers also the case $x_1x_3 \in E(K^+)$.

(b2) Suppose $x_4 \neq x_3^+$.

(b2.1) Now assume that K^+ is W^+-sound.

(i) Suppose $x_3^+ \neq v_2$. Let $S = E \cup P$ be a W^+-EPS-graph of G^+ with $K^+ \subseteq E$, by Theorem A. Then S^2 contains a hamiltonian cycle C of $(G^+)^2$ as required, even if $x_1x_3 \in E(K^+)$ and $d_G(x_3^+) > 2$. In any case, also here C can be constructed from S such that $x_3x_3^+, x_4w_4 \in E(G) \cap E(C)$.

(ii) Suppose $x_3^+ = v_2$. Hence $x_4 \in K^+$, otherwise $|V(K^+) \cap W^+| = 3$ and K^+ is not W^+-sound, a contradiction. Then G^+ contains an $[x_2; x_1, x_3^+, x_4]$-EPS-graph $S = E \cup P$ with $K^+ \subseteq E$, by Theorem B. Hence we obtain a hamiltonian cycle $C \subset S^2$ as required with $x_3x_3^+, x_4w_4 \in E(G) \cap E(C)$.

(b2.2) Assume that K^+ is not W^+-sound.

(i) Suppose $|V(K^+) \cap W^+| > 3$. Then there exists a cycle K^* in G^+ containing y, x_1, v_2, x_3^+, x_4 but $x_3 \notin K^*$; otherwise, we should have chosen $K^+ = K^*$ which is W^+-sound, a contradiction.

First suppose $x_2v_2 \in E(K^*)$. By Theorem B (if $x_3^+ \neq v_2$), Theorem C (if $x_3^+ = v_2$), there is an $[x_3^+; x_1, v_2, x_4]$-EPS-graph, $[x_3^+; x_1, x_4]$-EPS-graph, respectively, $S = E \cup P$ of G^+ with $K^* \subseteq E$. Note that either x_3 is a vertex in E, or else it is a pendant vertex in S. Also take note that $d_P(x_2) \leq 1$ and $d_P(v_2) \leq 1$. By Observation (*) (i)–(ii), S^2 has a hamiltonian cycle with the required properties.

If $x_2v_2 \notin E(K^*)$, then $x_2u_2 \in E(K^*)$ and we proceed analogously as before using u_2 instead of v_2. Note that $u_2 = x_4$ is not an obstacle (we use Theorem C) because of $d_S(x_2) = 2$ since $d_G(x_2) = 2$ and v_2 is also in K^* (thus $x_2v_2 \notin E(S)$); and $v_2 = x_3^+$ is not possible in this case.

(ii) Suppose $|V(K^+) \cap W^+| = 3$.

Hence $x_4 \notin K^+$ and $v_2 = x_3^+$. If $x_1x_3 \notin E(K^+)$, then we set $W^* = \{y, x_1, x_3^-, x_3^+, x_4\}$. If $x_1x_3 \in E(K^+)$ and $d_G(v_2) = 2$, then we set $W^* = \{y, x_1, x_2, x_3^+, x_4\}$. In both cases K^+ is W^*-sound. By Theorem A, G^+ contains a W^*-EPS-graph $S = E \cup P$ with $K^+ \subseteq E$. Observe that if $x_4 \notin E$, then it is a pendant vertex in S. Now it is easy to construct a required hamiltonian cycle C in S^2 having the required properties (see Observation (*)(i)–(ii)).

If $x_1x_3 \in E(K^+)$ and $d_G(v_2) > 2$, then we consider $G - x_3$.

If $G - x_3$ is 2-connected, then we apply induction and get an $\mathcal{F}_4$ x_1x_2-hamiltonian path P_1 in $(G - x_3)^2$ containing edges $x_4w_4, x_3^+w_3^+ \in E(G)$. Then

$$P = P_1 \cup \{w_3^+x_3, x_3x_3^+\} - \{x_3^+w_3^+\}$$

defines a hamiltonian path in G^2 as required.

If $G - x_3$ is not 2-connected, then x_1 belongs to one endblock and x_3^+, x_2 to the other endblock of a non-trivial block chain $G - x_3$ because of the degree condition of x_3, x_2. Moreover x_1, x_3^+, and x_2 are not cutvertices of $G - x_3$. Depending on the position of x_4 in $G - x_3$ we construct a hamiltonian path P in G^2 as in the preceding case applying either induction, or Theorem F, proceeding block after block. Since this procedure is straightforward we do not work out the details.

(1.2) $v_2 \in \{x_3, x_4\}$.

(1.2.1) Assume that $v_2 = x_3$.

$G^+ - x_2u_2$ is a trivial or non-trivial block chain. Let B_y denote the endblock (in $G^+ - x_2u_2$) containing the cycle K^+ and let $\widehat{G} = (G^+ - x_2u_2) - B_y$.

Suppose first that $\widehat{G} \neq \emptyset$. Hence $\widehat{G}$ is a block chain in $G^+ - x_2u_2$ containing u_2 and $N(u_2) - x_2$. Let t be the cutvertex of $G^+ - x_2u_2$ belonging to B_y. That is, $\widehat{G} \cap B_y = \{t\}$.

(a) Suppose $x_4 \notin \widehat{G} - t$.

Then x_4 is in B_y. Observe that $G - x_2$ is a block chain with $\widehat{G}$ being an induced subgraph of $G - x_2$ (note that $d_G(x_2) = 2$). Since B_y is 2-connected, it contains a path $P(x_3, x_1)$ through x_4. It follows that $x_2, y \notin P(x_3, x_1)$. Thus $P(x_3, x_1) \subseteq G - x_2$ with $x_4 \in P(x_3, x_1)$. Now, $P(x_2, x_1) = x_2x_3P(x_3, x_1)$ is a path in $G - \widehat{G}$. Thus we may assume that $K^+ = yx_2P(x_2, x_1)x_1y \subseteq B_y$ and thus passes through y, x_1, x_2, x_3^*, x_4. By Theorem C, B_y has an $[x_3^*; x_1, x_4]$-EPS-graph $S_y = E_y \cup P_y$ with $K^+ \subseteq E_y$ and $d_{P_y}(x_2) = 0$ (note that $d_{B_y}(x_2) = 2$). Let the same S_y denote an $[x_3^*; x_1]$-EPS-graph of B_y if $x_4 = x_3^*$ (i.e., $x_3x_4 \in E(K^+)$) (see Theorem D).

Since $x_4 \notin \widehat{G} - t$, by Lemma 1(i) or Theorem D if $\widehat{G}$ is 2-connected, $\widehat{G}$ contains an EPS-graph $\widehat{S} = \widehat{E} \cup \widehat{P}$ with $d_{\widehat{P}}(t) = 0$ and $d_{\widehat{P}}(u_2) \leq 1$, provided

$d_{\widehat{G}}(t) > 1$. If $d_{\widehat{G}}(t) = 1$, then either $\widehat{S} = \emptyset$ if $\widehat{G} = u_2t$ or by Lemma 1(i), $\widehat{G} - t$ contains an EPS-graph $\widehat{S} = \widehat{E} \cup \widehat{P}$ with $d_{\widehat{P}}(t_1) \leq 1$ and $d_{\widehat{P}}(u_2) \leq 1$, where $t_1 \in N_{\widehat{G}}(t)$.

Since $P_y \cap \widehat{P} = \emptyset$, by setting $E = E_y \cup \widehat{E}$ and $P = P_y \cup \widehat{P} \cup \{u_2x_2\}$, we obtain an EPS-graph $S = E \cup P$ of G^+ with $K^+ \subseteq E$, $d_P(x_2) = 1$, $d_P(x_3^*) = 0$, $d_P(x_i) \leq 1$ for $i = 1, 4$ and $d_P(u_2) \leq 2$.

If $|V(K^+)| \geq 6$, a required hamiltonian cycle in S^2 can be constructed (note that the cases $x_4 = t$ and $x_4 \neq t$ are treated simultaneously) (see Observation (*)(i)).

If, however, $|V(K^+)| < 6$, i.e., $|V(K^+)| = 5$, then $d_G(x_1) = 2$ since $N(x_4) \subseteq V_2(G)$ and $N(x_1) \subseteq V_2(G)$, which in turn implies $d_G(x_4) = d_G(x_3) = 2$. Hence $G - \{x_3, x_4\}$ is a block chain G^- with x_1, x_2 being pendant vertices of G^-. It follows that $(G^-)^2$ has a hamiltonian path HP^- starting with $x_1w_1 \in E(G)$ and ending with $x_2u_2 \in E(G)$. Clearly,

$$HP = (HP^- - \{x_1w_1, x_2u_2\}) \cup \{x_1x_4w_1, x_2x_3u_2\}$$

defines a required hamiltonian path in G^2.

(b) Suppose $x_4 \in \widehat{G} - t$.

In this case, we note that in B_y, the cycle K^+ can be assumed to traverse y, x_1, t, x_3, x_2 in this order; it also contains x_3^* if x_3 is 2-valent. As for $t \in V(K^+)$, see the preceding observation at the beginning of (a), with t assuming the role of x_4.

Suppose $t \neq x_1$. By Theorem C, B_y has an $[x_3^*; x_1, t]$-EPS-graph $S_y = E_y \cup P_y$ with $K^+ \subseteq E_y$ if $x_3^* \neq t$; by Theorem D, B_y has an $[x_3^*; x_1]$-EPS-graph $S_y = E_y \cup P_y$ with $K^+ \subseteq E_y$ if $x_3^* = t$; and $d_{P_y}(x_2) = 0$ since $d_G(x_2) = 2$.

Suppose $t = x_1$. We let $S_y = E_y \cup P_y$ be an $[x_1; x_3^*]$-EPS-graph in B_y with $K^+ \subseteq E_y$ by Theorem D. Note that we set $x_3^* = x_2$ if $x_1x_3 \in E(K^+)$.

(b1) Assume that x_4 is a cutvertex in $\widehat{G}$.

(i) Consider the case x_4 is not incident to a bridge of $\widehat{G}$. Let $\widehat{G_1}$ and $\widehat{G_2}$ be defined by $\widehat{G} = \widehat{G_1} \cup \widehat{G_2}$ with $t, x_4 \in V(\widehat{G_1})$, $x_4, u_2 \in V(\widehat{G_2})$ and $\widehat{G_1} \cap \widehat{G_2} = \{x_4\}$.

By Lemma 1(i) or Theorem D, $\widehat{G_i}$ has an EPS-graph $\widehat{S_i} = \widehat{E_i} \cup \widehat{P_i}$ with $d_{\widehat{P_i}}(x_4) = 0$ for $i = 1, 2$, $d_{\widehat{P_1}}(t) \leq 1$ and $d_{\widehat{P_2}}(u_2) \leq 1$.

By taking $E = E_y \cup \widehat{E_1} \cup \widehat{E_2}$, $P = P_y \cup \widehat{P_1} \cup \widehat{P_2}$, we have an EPS-graph $S = E \cup P$ of G^+ with $d_P(x_2) = 0 = d_P(x_4)$, and $d_P(x_1) \leq 1$ and $d_P(x_3^*) \leq 1$;

$d_P(t) \leq 2$ by construction, provided $t \neq x_1$. Moreover, if $t = x_3^*$, we have $d_{P_y}(x_3^*) = 0$ and hence $d_P(x_3^*) \leq 1$ because of $d_{\widehat{P_1}}(x_3^*) \leq 1$; and $d_P(x_1) \leq 1$. Also if $t = x_1$, we have $d_{P_y}(x_1) = 0$ and hence $d_P(x_1) \leq 1$ because of $d_{\widehat{P_1}}(t) \leq 1$; and $d_P(x_3^*) \leq 1$. Hence a required hamiltonian cycle in S^2 can be constructed (the various construction details are straightforward and are thus omitted).

(ii) Now suppose x_4 is incident to a bridge f of $\widehat{G}$ and $|V(K^+)| > 4$. In this case, we delete f and thus split $\widehat{G}$ into two block chains $\widehat{G_1}$ and $\widehat{G_2}$ with $t \in \widehat{G_1}$, $u_2 \in \widehat{G_2}$ and x_4 is either in $\widehat{G_1}$ or in $\widehat{G_2}$. By Lemma 1(i) or Theorem D, $\widehat{G_i}$ has an EPS-graph $\widehat{S_i} = \widehat{E_i} \cup \widehat{P_i}$ with $d_{\widehat{P_1}}(t) \leq 1$, $d_{\widehat{P_2}}(u_2) \leq 1$ and $d_{\widehat{P_i}}(x_4) \leq 1$ for some $i \in \{1, 2\}$. Note that $\widehat{S_i} = G_2$ if $G_i = K_2$; or $\widehat{S_i} = \emptyset$ if $G_i = t$ or $G_i = u_2$. Proceeding similarly to case (i) let $E = E_y \cup \widehat{E_1} \cup \widehat{E_2}$ and $P = P_y \cup \widehat{P_1} \cup (\widehat{P_2} \cup \{u_2x_2\})$. Then we have an EPS-graph $S = E \cup P$ of G^+.

Because of the choice of S_y in the cases $t \notin \{x_1, x_3^*\}$, $t = x_3^*$, and $t = x_1$, we have in any case, $d_P(x_1) \leq 1$, $d_P(x_2) = 1$, $d_P(x_3^*) \leq 1$ and x_4 is either a pendant vertex in S or $d_P(x_4) = 0$ (which occurs when $d_G(x_4) > 2$).

By a similar argument as in case (i), we conclude that in all cases S^2 contains a hamiltonian cycle with the required properties unless $d_P(x_2) = 1$, $d_P(x_3^*) = 1$ and $x_3^* = x_3 = t$. In this case there exists a cycle containing y, x_1, x_2, x_3, x_4, a contradiction to the choice of K^+.

(iii) Now suppose x_4 is incident to a bridge f of $\widehat{G}$ and $|V(K^+)| = 4$. It follows that $t = x_1$ and therefore $G' = G - x_3$ is a non-trivial block chain containing f as a bridge. Hence $(G - x_3)^2$ contains a hamiltonian path HP' starting with an edge $x_1w_1 \in E(G)$ and ending with $u_2x_2 \in E(G)$ and containing f. It follows that

$$HP = (HP' - x_1w_1) \cup \{x_1x_3w_1\}$$

yields a hamiltonian path in G^2 as required if $f \neq x_1x_4$. However, if $f = x_1x_4$, then we set

$$HP = (HP' - x_2u_2) \cup \{x_2x_3u_2\}.$$

(b2) Hence assume that x_4 is not a cutvertex in $\widehat{G}$.

Suppose first that x_4 is contained in a 2-connected block B in $\widehat{G}$. Further, let c_1, c_2 be two vertices in B which are also cutvertices of $\widehat{G}$ if B is not an endblock of $\widehat{G}$. If, however, B is an endblock of $\widehat{G}$, then let c_1 be the unique cutvertex of $\widehat{G}$ in B, and let $c_2 \in \{t, u_2\}$ depending on which of the endblocks of $\widehat{G}$ is B. If $x_4 \neq c_2$, we apply Theorem C to B to obtain an $[x_4; c_1, c_2]$-EPS-graph of B; if $x_4 = c_2$ (which means $x_4 = u_2$), then we apply Theorem D to B to obtain an $[x_4; c_1]$-EPS-graph of B. In both cases by using Lemma 1(i), extend these EPS-graphs to an EPS-graph $\widehat{S} = \widehat{E} \cup \widehat{P}$ of $\widehat{G}$ with $d_{\widehat{P}}(t) \leq 1$, $d_{\widehat{P}}(u_2) \leq 1$, and $d_{\widehat{P}}(x_4) = 0$.

Setting $E = E_y \cup \widehat{E}$ and $P = P_y \cup \widehat{P}$, we obtain an EPS-graph $S = E \cup P$ of G^+ with $d_P(x_2) = 0 = d_P(x_4)$, $d_P(x_3^*) \leq 1$ and $d_P(x_1) \leq 1$.

Hence assume that x_4 is not contained in a 2-connected block. That is, x_4 is a pendant vertex in $\widehat{G}$. In this case, $x_4 = u_2$. We apply Lemma 1(i) to obtain an EPS-graph $\widehat{S} = \widehat{E} \cup \widehat{P}$ of $\widehat{G}$ with $d_{\widehat{P}}(t) \leq 1$, and $d_{\widehat{P}}(x_4) \leq 1$ if $\widehat{G} \neq x_4 t$. If $\widehat{G} = x_4 t$, then $\widehat{S} = \widehat{G}$. Setting $E = E_y \cup \widehat{E}$ and $P = P_y \cup \widehat{P}$, we obtain an EPS-graph $S = E \cup P$ of G^+ with $d_P(x_2) = 0$ and $d_P(x_3^*) \leq 1$, $d_P(x_1) \leq 1$, $d_P(x_4) = 1$ and x_4 is a pendant vertex in S.

In any of these cases, S^2 contains a hamiltonian cycle C with the required properties (note that $x_3 x_2 \in E(C)$ because $d_E(x_2) = d_{G^+}(x_2) - 1 = 2$); see Observation(*)(i)–(ii).

Finally if $\widehat{G} = \emptyset$, we find S_y as in Case **(1.2.1)(a)** and construct a hamiltonian cycle as required using S_y only.

(1.2.2) Assume that $v_2 = x_4$.

Recall that the cycle K^+ in G^+ contains y, x_1, x_2, x_3, v_2. Therefore

$$K^+ = y x_1 \ldots z_3 x_3 w_3 \ldots z_4 x_4 x_2 y.$$

Consider the graph $G' = G^+ - x_2 u_2$.

Case (a) G' is 2-connected.

Suppose x_3, x_4 are adjacent in K^+. Then apply Theorem D to obtain an $[x_4; x_1]$-EPS-graph $S = E \cup P$ of G' with $K^+ \subseteq E$. Suppose x_3, x_4 are not adjacent in K^+. Then apply Theorem C to obtain an $[x_1; x_3^*, x_4]$-EPS-graph $S = E \cup P$ of G' with $K^+ \subseteq E$. In either case, a required hamiltonian cycle in S^2 can be constructed (setting $x_3^* = x_3^+ = w_3$ if $x_1 x_3 \in E(K^+)$).

Case (b) G' is not 2-connected.

Then G' is a non-trivial block chain. As before, let B_y denote the end-block in G' containing y (and hence containing the cycle K^+). Set $\widehat{G} = G' - B_y$ which is a trivial or non-trivial block chain; $\widehat{G} \neq \emptyset$ in any case. It follows that $B_y \cap \widehat{G} = \{t\}$ and t is a cutvertex of G'. By Theorem D or Lemma 1(i), $\widehat{G}$ has an EPS-graph $\widehat{S} = \widehat{E} \cup \widehat{P}$ with $d_{\widehat{P}}(t) \leq 1$ if $\widehat{G} \neq u_2 t$. If $\widehat{G} = u_2 t$, then $\widehat{S} = \widehat{G}$.

(i) Suppose $t = x_4$.

Then $G'' = G^+ - x_2x_4$ is 2-connected. Replace in K^+ the edge x_4x_2 with a path $P(x_4, x_2)$ in $\widehat{G} \cup \{u_2x_2\}$ to obtain the cycle K''. Since $\{y, x_1, x_3, x_4, u_2, x_2\} \subseteq V(K'')$, we may apply Theorem B to obtain an $[x_4; x_1, u_2, x_2]$-EPS-graph $S'' = E'' \cup P'' \subseteq G''$ if x_3 and x_4 are adjacent in K'', or to obtain an $[x_1; x_3^*, x_4, u_2]$-EPS-graph $S'' = E'' \cup P'' \subseteq G''$ if x_3 and x_4 are not adjacent in K'' (setting $x_3^* = x_3^+ = w_3$ if $x_1x_3 \in E(K'')$). In both cases, $K'' \subseteq E''$. A required hamiltonian cycle in $(S'')^2$ can be constructed (since the situation is similar to Case **(a)** above); see Observation (*)(i).

(ii) Suppose $t = x_3$.

We apply Theorem C to B_y to obtain an $[x_3; x_1, x_4]$-EPS-graph $S_y = E_y \cup P_y$ of G' with $K^+ \subseteq E_y$. Note that $N(x_3) \subseteq V_2(G)$ in this case.

(iii) Suppose $t = x_1$ and $x_1x_3x_4 \not\subseteq K^+$.

We set $x_3^* = x_3^+ = w_3$, if $x_1x_3 \in E(K^+)$. We apply Theorem C to B_y again to obtain an $[x_1; x_3^*, x_4]$-EPS-graph $S_y = E_y \cup P_y$ of G' with $K^+ \subseteq E_y$.

In the cases (ii) and (iii), we let $E = \widehat{E} \cup E_y$, $P = \widehat{P} \cup P_y$ and obtain an EPS-graph $S = E \cup P$ of $G^+ - x_2u_2$ with $K^+ \subseteq E$, $d_P(x_2) = 0$ and $d_P(w) \leq 1$ for $w \in \{x_1, x_3^*, x_4\}$. Hence a required hamiltonian cycle in S^2 can be constructed; see Observation (*)(i).

(iv) Suppose $t \notin \{x_1, x_3, x_4\}$.

We set $x_3^* = x_3^+ = w_3$ if $x_1x_3 \in E(K^+)$. Note that $d_{B_y}(x_2) = 2$. Let $W = \{y, x_1, x_3^*, x_4, t\}$; $W - \{t\} \subseteq V(K^+)$.

First suppose that $|W| = 5$. If K^+ is not W-sound, then there is a cycle $K' \subseteq B_y$ containing all vertices of W, in which case we apply Theorem B to B_y to obtain an $[x_3^*; x_1, x_4, t]$-EPS-graph $S_y = E_y \cup P_y$ with $K' \subseteq E_y$. Note that, if $x_3 \notin K'$, then either x_3 is a pendant vertex in S_y or $d_{P_y}(x_3) = 0$ and $x_3^*x_3 \in E(E_y)$. If K^+ is W-sound, then we set $K' = K^+$ and apply Theorem A to B_y to obtain a W-EPS-graph $S_y = E_y \cup P_y$ with $K' \subseteq E_y$.

Suppose $|W| = 4$. Hence $x_3^* \neq x_3$.

If $x_3^* = x_4 \neq t$, then $N_G(x_3) = \{x_1, x_4\}$ and $B_y - x_3$ is 2-connected: for, there are two internally disjoint paths from t to K^+, and the endvertices of these paths in K^+ are x_1 and x_4 since $x_3, x_2 \in V_2(G)$. Thus $B_y - x_3$ contains a cycle K' containing y, x_1, x_2, x_4, t. Hence we apply Theorem B to $B_y - x_3$ to obtain an $[x_1; x_2, x_4, t]$-EPS-graph $S'_y = E'_y \cup P'_y$ with $K' \subseteq E'_y$. Let $E_y = E'_y$ and $P_y = P'_y \cup \{x_1x_3\}$. Thus we have an EPS-graph $S_y = E_y \cup P_y$ of B_y with $K' \subseteq E_y$. Moreover $d_{P_y}(x_1) = 1$, $d_{P_y}(x_2) = 0$, $d_{P_y}(x_3) = 1$, $d_{P_y}(x_4) \leq 1$, $d_{P_y}(t) \leq 1$, and x_3 is a pendant vertex in S_y.

If $x_3^* = t \notin \{x_1, x_3, x_4\}$, then we set $K' = K^+$ and apply Theorem C to B_y to obtain an $[x_3^*; x_1, x_4]$-EPS-graph $S_y = E_y \cup P_y$ with $K' \subseteq E_y$.

In all cases, we let $E = \widehat{E} \cup E_y$, $P = \widehat{P} \cup P_y$ and obtain an EPS-graph $S = E \cup P$ of $G^+ - x_2u_2$ with $K' \subseteq E$, $d_P(x_2) = 0$ and $d_P(w) \leq 1$ for $w \in \{x_1, x_3^*, x_4\}$ (even if $x_1x_3 \in E(K^+)$ or $x_3^* = x_3^+ = t \neq x_3$). Hence a required hamiltonian cycle in S^2 can be constructed; see Observation (*)(i)–(ii).

(v) Suppose $t = x_1$ and $x_1x_3x_4 \subseteq K^+$.

Note that $K^+ = yx_1x_3x_4x_2y$. Let $G_3 = B_y - \{y, x_2\}$; it is 2-connected if $d_G(x_4) > 2$, or else it is a path $x_1x_3x_4$. We have $G - x_2x_4 = G_3 \cup (\widehat{G} \cup \{u_2x_2\})$ with $t = G_3 \cap \widehat{G}$. Consequently,

$$\widetilde{G} := \widehat{G} \cup \{u_2x_2\} = G - (\{x_2x_4\} \cup G_3).$$

By Corollary 1(ii), $\widetilde{G}^2$ has a hamiltonian path $P_{1,2}$ starting with $x_1w_1 \in E(G)$ and ending with u_2x_2. If G_3 is 2-connected, then $G_3 - x_3$ is a (trivial or non-trivial) block chain and thus $(G_3 - x_3)^2$ has a hamiltonian path $P_{4,1}$ starting in $x_4 = v_2$ and ending with an edge $s_1x_1 \in E(G)$ (using Theorem F(ii) if $G_3 - x_3$ is 2-connected, Corollary 1(ii) if $G_3 - x_3$ is a non-trivial block chain, and $P_{4,1} = G_3 - x_3$ if $G_3 - x_3 = x_1x_4$). Set

$$P(x_1, x_2) = x_1x_3w_1(P_{1,2} - \{x_1, x_2\})u_2x_4x_2$$

if $G_3 = x_1x_3x_4$; and

$$P(x_1, x_2) = x_1x_3x_4(P_{4,1} - x_1)s_1w_1(P_{1,2} - x_1)$$

if G_3 is 2-connected. In both cases, $P(x_1, x_2)$ is a $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2. This finishes the proof of Case **(D)(1)**.

Since the case $N(x_3) \subseteq V_2(G)$ is analogous to the Case **(D)(1)**, we are left with the following case.

(D)(2) $N(x_i) \nsubseteq V_2(G)$ for $i = 3, 4$.

However, the proof of this case follows from Lemma 2. This finishes the proof of Case **(D)**.

Case (E): $N(x_1) \nsubseteq V_2(G)$ and $N(x_2) \nsubseteq V_2(G)$.

Then $d_G(x_1) = 2 = d_G(x_2)$.

Let K^+ be a cycle containing the vertices y, u_1, u_2, x_3 and possibly x_4 where we assume that

$$K^+ = yx_1u_1 \cdots x_3 \cdots u_2x_2y.$$

(E)(1) Suppose x_4 is not in any cycle containing y and x_3.

(1.1) $d_G(x_3) > 2,\ d_G(x_4) > 2$.

(a) Suppose $x_3 \notin \{u_1, u_2\}$.

Set $W = \{y, u_1, u_2, x_3, x_4\}$. By supposition, K^+ is W-sound. By Theorem A, we have an EPS-graph $S = E \cup P$ with $K^+ \subseteq E$ and $d_P(w) \leq 1$ for every $w \in W$. In this case a required hamiltonian cycle C in S^2 can be constructed (taking note that x_1, x_2 are 3-valent in G^+, and that $x_i x_4 \in E(P)$, $i \in \{1, 2\}$, does not constitute an obstruction in the construction of C).

(b) Suppose $x_3 = u_1$.

Note that if $x_4 \notin N(x_1) \cup N(x_2)$, then we are back to case (a) with x_3 and x_4 changing roles. Hence we have $x_4 \in \{v_1, v_2\}$. Also, $x_4 = v_1 = v_2$ cannot hold; otherwise, $d_G(x_4) > 2$ and $x_i \in N(x_4)$, $d_G(x_i) = 2$, $i = 1, 2$ imply the existence of an x_4q_3-path $P(x_4, q_3) \subset G$ with $q_3 \in V(K^+)$ and $(P(x_4, q_3) - q_3) \cap K^+ = \emptyset$, yielding in turn a cycle containing y, x_3, x_4 contradicting **E(1)**. By the same token, $x_3 = u_1 = u_2$ cannot hold.

(b1) $x_4 = v_2$.

Consider $G^- = G - \{x_1u_1, x_2u_2\}$.

Note that x_3, x_4 belong to different components of G^-; otherwise there is a path P_0 in G^- joining x_3 and x_4 implying that $C_0 = P_0 x_4 x_2 y x_1 x_3$ is a cycle in G^+ with $y, x_3, x_4 \in V(C_0)$, a contradiction to the supposition. Since G is 2-connected, G^- contains precisely two components $G_3^- \neq K_1$ and G_4^- containing x_3, x_4, respectively. Clearly $x_2 \in V(G_4^-)$. We also have $x_1 \in V(G_4^-)$ because P_0 as above does not exist.

Observe that G_4^- and G_3^- are (trivial or non-trivial) block chains in which $x_1, x_2 \in V(G_4^-)$ and $x_3, u_2 \in V(G_3^-)$ are not cutvertices. Thus $G^+ - \{x_1u_1, x_2u_2\}$ is a disconnected graph with two components $G_3 = G_3^-$ (which contains $x_3 = u_1$ and u_2) and G_4 (which contains $y, x_1, v_1, x_4 = v_2$ and x_2).

Note that in G_4, there is a cycle C^+ containing y, x_1, v_1, x_4, x_2, implying that G_4 is 2-connected, whereas G_3 is a block chain. By Theorem D, let $S_4 = E_4 \cup P_4$ be a $[v_1; x_4]$-EPS-graph in G_4 with $C^+ \subseteq E_4$, $d_{P_4}(v_1) = 0 = d_{P_4}(x_1) = d_{P_4}(x_2)$ and $d_{P_4}(x_4) \leq 1$. By Lemma 1(i) or Theorem D (respectively depending on whether G_3 has a cutvertex or G_3 is 2-connected), there is an EPS-graph $S_3 = E_3 \cup P_3$ in G_3 such that $d_{P_3}(x_3) = 0$ and $d_{P_3}(u_2) \leq 1$. Taking $E = E_3 \cup E_4$ and $P = P_3 \cup \{x_1x_3\} \cup P_4$, we have an EPS-graph $S = E \cup P$ of G^+ with $C^+ \subseteq E$ and $d_P(v_1) = 0 = d_P(x_2)$, $d_P(x_1) = 1 = d_P(x_3)$ and $d_P(x_4) \leq 1$.

Note that in this case, since $d_G(x_3),\ d_G(x_4) > 2$, $d_P(x_2) = 0$, $d_P(x_3) = 1$ and $d_P(x_4) \leq 1$, it is straightforward that one can obtain a required hamiltonian cycle of $(G^+)^2$.

(b2) $x_4 = v_1$.

Let $G' = G^+ - x_1x_3$, and we may assume that a cycle $K' = yx_1x_4 \cdots v_2x_2y \subseteq G'$ exists. Note that in G we have two internally disjoint paths $x_1x_3 \cdots u_2x_2$ and $x_1x_4 \cdots v_2x_2$. This is in line with the notation of K^+ above.

(b2.1) Suppose G' is 2-connected.

Take $W = \{y, x_3, x_4, v_2, x_2\}$. Then K' is W-sound in G' since $v_2 \neq x_4$ (see the observation in **(b)**). Let $S = E \cup P$ be a W-EPS-graph of G' (and hence a W-EPS-graph of G^+) with $K' \subseteq E$ and $d_P(w) \leq 1$ for every $w \in W$. Since $d_G(x_3) > 2$, $d_G(x_4) > 2$, a hamiltonian cycle in S^2 can be constructed containing x_1y, x_2y and x_iz_i where $z_i \in N_G(x_i)$, $i = 3, 4$.

(b2.2) Suppose G' is not 2-connected.

By symmetry, $G^+ - x_1x_4$ is also not 2-connected. Then G' is a block chain with endblocks B_3, B', with $x_3 \in B_3$ and $K' \subset B'$ and x_1 and x_3 are not cutvertices of G'. Furthermore, let c denote the cutvertex of G' which belongs to B'; $c \neq x_4$ (otherwise, G^+ contains a cycle through y, x_3, x_4).

Set $G_0 = G' - B'$. Note that x_3, c are vertices in G_0 and are not cutvertices of G_0. By Lemma 1(i) or Theorem D (depending on whether G_0 has a cutvertex or not), G_0 contains an EPS-graph $S_0 = E_0 \cup P_0$ with $d_{P_0}(c) \leq 1$ and $d_{P_0}(x_3) = 0$ (B_3 is 2-connected because $d_{B_3}(x_3) > 1$).

(i) Suppose $c \notin \{v_2, x_2\}$. Let $W' = \{y, x_4, c, v_2, x_2\}$. $B' \supseteq K' \supset (W' - c)$ in any case. So, K' is W'-sound, or there is a cycle $K'' \supset W'$ with $B' \supseteq K''$, in which case K'' is W'-sound in B'.

(ii) Now suppose $c = x_2$. Set $W' = \{y, x_1, x_2, v_2, x_4\}$ and observe that K' is W'-sound in B' again.

In both cases, we obtain by Theorem A an EPS-graph $S' = E' \cup P'$ of B' with $K' \subseteq E'$ or $K'' \subseteq E'$, and $d_{P'}(w) \leq 1$ for every $w \in W'$. Note that if $c \notin \{v_2, x_2\}$, $c \notin K'$ and $x_2v_2 \notin E(K'')$, or if $c = x_2$, then $d_{P'}(x_2) = 0$ because $d_{B'}(x_2) = 2$.

Set $E = E_0 \cup E'$, $P = P_0 \cup P'$ to obtain an EPS-graph $S = E \cup P$ of G^+ with $K^* \subseteq E$ where $K^* \in \{K', K''\}$, $d_P(x_3) = 0$, $d_P(z) \leq 1$ for every $z \in \{y, x_4, v_2, x_2\}$, and $d_P(c) \leq 2$ if $c \notin \{v_2, x_2\}$, and $d_P(c) \leq 1$ if $c = x_2$. Also, $d_P(x_1) = 0$ since $x_1x_3 \notin E(S)$. Since $N(x_i) \subseteq V_2(G)$, $i = 3, 4$ and $d_P(x_3) = 0$, $d_P(x_4) \leq 1$, a hamiltonian cycle in S^2 containing the edges incident to y and containing edges x_iz_i, can be constructed, where

$z_i \in N_G(x_i)$, $i = 3, 4$. Observe that $d_P(v_2) = d_P(x_2) = 1$ does not create any obstacle.

(iii) Suppose $c = v_2$. In this case, by Theorem C we take in B' a $[v_2; x_2, x_4]$-EPS-graph and proceed as in case (i).

(1.2) $d_G(x_3) > 2, d_G(x_4) = 2.$

Let K' be a cycle in G^+ containing the vertices $y, x_1, w_1, u_4, x_4, v_4, w_2, x_2$ in this order where $w_i \in \{u_i, v_i\}$, $i = 1, 2$.

(a) $x_4 \notin \{w_1, w_2\}$

(a1) Suppose $v_4 \notin N(x_2)$. Note that in this case $|V(K')| > 6$.

Set $W = \{y, w_1, w_2, x_3, v_4\}$ and observe that $|W| = 5$ and $|K' \cap W| \geq 4$.

Suppose K' is W-sound in G^+. Then by Theorem A, G^+ has a W-EPS-graph $S = E \cup P$ with $K' \subseteq E$ and $d_P(y) = 0 = d_P(x_4)$. Moreover, for $i = 1, 2$, we have $d_P(x_i) \leq 1$ since $d_G(x_i) = 2$. Hence we can construct a hamiltonian cycle in S^2 having the required properties.

Now we assume that K' is not W-sound. Then there is a cycle K^* in G^+ containing all of W but not containing x_4. Consider $G' = G^+ - x_4$.

(i) Suppose G' is 2-connected. By Theorem B, G' has a $[v_4; x_3, w_1, w_2]$-EPS-graph $S' = E' \cup P'$ with $K^* \subseteq E'$. Set $E = E'$ and $P = P' \cup \{v_4x_4\}$ to obtain an EPS-graph of G^+ with $K^* \subseteq E$ and v_4x_4 is a pendant edge in S. Hence a hamiltonian cycle in S^2 with the required properties can be constructed. For $i = 1, 2$, note that if $w_ix_i \notin E(K^*)$, then $d_P(x_i) = 0$ since $d_G(x_i) = 2$ and $w_i \in K^*$. Observe also that $v_4x_i \in E(K^*)$ and $x_3x_i \in E(K^*)$ do not constitute any obstacle in this case.

(ii) Suppose G' is not 2-connected. Let B_y be the endblock in (the non-trivial block chain) G' containing K^*, and let t_4 be the cutvertex of G' belonging to B_y. Set $\widehat{G} = (G' - B_y) \cup \{u_4x_4\}$. Note that $\widehat{G}$ is a non-trivial block chain and $\widehat{G} = (G^+ - B_y) - x_4v_4$.

Set $W^* = \{y, w_1, w_2, x_3, t_4\}$ and observe that $x_3 \notin \{w_1, w_2\}$; otherwise, G^+ has a cycle containing y, x_3, x_4 (contradicting **E(1)**). In any case, $\widehat{G}$ has an EPS-graph $\widehat{S} = \widehat{E} \cup \widehat{P}$ with $d_{\widehat{P}}(t_4) \leq 1$ and $d_{\widehat{P}}(x_4) = 1$ by Lemma 1(i).

Now if $t_4 \in \{w_1, w_2, x_3\}$, let $S_y = E_y \cup P_y$ be a $[t_4; r_4, s_4, y]$-EPS-graph of B_y with $K^* \subseteq E_y$ where $\{r_4, s_4, t_4\} = \{w_1, w_2, x_3\}$, by Theorem B.

If, however, $t_4 \notin \{w_1, w_2, x_3\}$, we may assume without loss of generality that K^* is W^*-sound (since $|W^*| = 5$ and $K^* \supset W^* - t_4$). Consequently, let in this case $S_y = E_y \cup P_y$ be a W^*-EPS-graph of B_y with $K^* \subseteq E_y$.

In all cases, let an EPS-graph $S = E \cup P$ of G^+ be defined by $E = E_y \cup \widehat{E}$, $P = P_y \cup \widehat{P}$. We have $K^* \subseteq E$ and note that $d_P(w) \leq 1$ for every $w \in W^* - t_4$, and $d_P(t_4) \leq 2$ but $d_P(t_4) \leq 1$ if $t_4 \in \{w_1, w_2, x_3\}$. It is now straightforward to see that in each of the cases in question, S^2 contains a hamiltonian cycle as required (see the argument at the end of case (i); moreover, $t_4x_i \in E(K^*)$ does not constitute an obstacle, $i = 1, 2$). This finishes case **(a1)**.

Since the case $u_4 \notin N(x_1)$ can be treated analogously, we are led to the following case.

(a2) $u_4 = w_1$ and $v_4 = w_2$. Then $|V(K')| = 6$. In view of case **(a1)**, we may assume that any cycle in G^+ containing $y, x_1, x_2, u_4, x_4, v_4$ has length 6.

Suppose $H = G^+ - x_1u_4$ is 2-connected. Then H has a cycle C containing the edges $u_4x_4, x_4v_4, yx_1, yx_2, x_1w_1'$ (where $w_1' \neq u_4$). But this means that $|V(C)| > 6$ (because at least 2 more edges are required to form the cycle C), a contradiction.

Thus H is not 2-connected, and let B_y and B_4 denote the endblocks of H containing y and x_4, respectively.

Suppose x_2 is not a cutvertex of H. Since $\kappa(B_y) \geq 2$, it follows that $\{x_2, u_2, v_2\} \subset V(B_y)$. Now, we have a path $P = P(v_2, u_2)$ in B_y with $x_2 \notin V(P)$. Since $d_G(x_4) = 2$, $x_4 \notin V(P)$; otherwise $u_4 \in V(P)$ as well and hence $x_4u_4 \in E(B_y \cap B_4)$ which is impossible. Thus we obtain for $\{r_2, w_2\} = \{u_2, v_2\}$ a cycle

$$K^* = (K' - w_2x_2) \cup P \cup \{r_2x_2\}$$

in G^+ containing $V(K')$ and $|V(K^*)| > 6$, contradicting the assumption at the beginning of this case. Thus x_2 is a cutvertex of H.

Observing that $d_{G^+}(x_2) = 3$ and $\kappa(B_y) \geq 2$, we conclude $d_{B_y}(x_2) = 2$ and thus $x_2w_2 \in E(H) - E(B_y)$ is the other block of H containing the cutvertex x_2. It now follows that $B_y \cap B_4 = \emptyset$ since $x_2w_2 \notin E(B_4)$. Without loss of generality $w_2 = v_2$; hence $u_2x_2 \in E(B_y)$.

It now follows that $H - B_y$ is either a path of length 3, or it is a block chain with B_4 being 2-connected and x_2v_2 being a block.

(a2.1) Suppose $x_3 \in V(B_y)$. Let K_y be a cycle in B_y containing y, x_1, x_2, x_3 where we may assume that

$$K_y = yx_1w_1' \cdots x_3 \cdots w_2'x_2y.$$

Note that $x_3 = w_1' = w_2'$ is impossible because of $d_G(x_3) > 2$. If $x_3 \neq w_1'$ and $x_3 \neq w_2'$, then B_y has an $[x_3; y, w_1', w_2']$-EPS-graph $S_y = E_y \cup P_y$ with $K_y \subseteq E_y$ by Theorem B. If $x_3 = w_1'$ or $x_3 = w_2'$, then B_y has an $[x_3; y, w_2']$-EPS-graph or an $[x_3; y, w_1']$-EPS-graph $S_y = E_y \cup P_y$ with $K_y \subseteq E_y$ by

Theorem C, respectively. Likewise, if $d_G(u_4) > 2$, then B_4 has a $[u_4; v_4]$-EPS-graph $S_4 = E_4 \cup P_4$ with $K^{(4)} \subseteq E_4$ where $K^{(4)}$ is a cycle in B_4 containing u_4, x_4, v_4, by Theorem D. If, however, B_4 is a bridge of H, then the path $P_4 = u_4x_4v_4$ has the only EPS-graph $S_4 = E_4 \cup P_4$ with $E_4 = \emptyset$.

Setting $E = E_y \cup E_4$ and $P = P_y \cup P_4 \cup \{x_1u_4\}$, we have an EPS-graph $S = E \cup P$ of G^+ with $d_P(x_1) = 1$, $d_P(x_2) = d_P(x_3) = d_P(y) = 0$, $d_P(w_1') \leq 1$, $d_P(w_2') \leq 1$, $d_P(x_4) \in \{0, 2\}$, $d_P(u_4) \leq 2$ and $d_P(v_4) \leq 1$. However, $d_P(x_4) = 2$ implies $d_P(v_4) = 1$ and thus x_4v_4 is a pendant edge. Hence a hamiltonian cycle in S^2 with the required properties can be constructed.

(a2.2) Suppose $x_3 \in V(B_4)$; thus B_4 is 2-connected. Let K_y be a cycle in B_y containing y, x_1, x_2 where we may assume that

$$K_y = yx_1w_1' \cdots w_2'x_2y.$$

Note that if $w_1' = w_2'$, then $d_{B_y}(w_1') = 2$. If $w_1' \neq w_2'$, then B_y has an $[x_1; y, w_1', w_2']$-EPS-graph $S_y = E_y \cup P_y$ with $K_y \subseteq E_y$ by Theorem B. If $w_1' = w_2'$, then B_y has an $[x_1; y, w_1']$-EPS-graph $S_y = E_y \cup P_y$ with $K_y \subseteq E_y$ by Theorem C. Likewise, B_4 has a $[u_4; x_3, v_4]$-EPS-graph $S_4 = E_4 \cup P_4$ with $K^{(4)} \subseteq E_4$ where $K^{(4)}$ is a cycle in B_4 containing x_3, u_4, x_4, v_4. Setting $E = E_y \cup E_4$ and $P = P_y \cup P_4 \cup \{x_1u_4\}$, we have an EPS-graph $S = E \cup P$ of G^+ and S^2 contains a hamiltonian cycle as required.

(b) $x_4 \in \{w_1, w_2\}$ but $w_1 \neq w_2$.

Without loss of generality assume $x_4 = w_1$ and hence $x_1x_4 \in E(G)$ (the case $x_4 = w_2$, $w_1 \neq w_2$, can be solved by a symmetrical argument). Note that $x_3 = u_1 = u_2$ cannot hold (see the argument in case **(1.1)(b)**).

(b1) Suppose $v_4 \notin N(x_2)$; i.e., $v_4 \neq w_2$. Let K' be a cycle in G^+ containing $y, x_1, x_4, v_4, w_2, x_2$ in this order and let $W = \{y, x_4, v_4, w_2, x_3\}$. Then K' is W-sound because of the supposition at the beginning of **(E)(1)**. By Theorem A, G^+ has a W-EPS-graph $S = E \cup P$ with $K' \subseteq E$ and hence a hamiltonian cycle in S^2 with the required properties can be constructed.

(b2) Suppose $v_4 = w_2$. Assume first that $d_G(v_4) = 2$. Let K' be the cycle $yx_1x_4w_2x_2y$ and let $W = \{y, x_1, x_2, x_3, x_4\}$. Then K' is W-sound. By Theorem A, G^+ has a W-EPS-graph with $K' \subseteq E$.

Now assume that $d_G(v_4) > 2$. Let $z \in N(v_4) - \{x_4, x_2\}$. There is a path $P(v_4, x_1)$ in G from v_4 to x_1 via the vertex z since G is 2-connected; $x_2 \notin P(v_4, x_1)$ since $d_G(x_2) = 2$. Now $K^* = P(v_4, x_1)x_1yx_2v_4$ is a cycle in G^+ containing $N(x_4)$ but not x_4 itself. Hence $G'' = G^+ - x_4$ is 2-connected.

We may assume that K^+ is also a cycle in G'' containing $y, x_1, u_1, x_3, u_2, x_2$ in this order. If $x_3 \neq u_1$ and $x_3 \neq u_2$, then by Theorem C, G'' has an $[x_3; u_1, u_2]$-EPS-graph $S'' = E'' \cup P''$ with $K^+ \subseteq E''$. If $x_3 = u_1$ or $x_3 = u_2$, then by Theorem D, G'' has an $[x_3; u_2]$-EPS-graph or an $[x_3; u_1]$-EPS-graph $S'' = E'' \cup P''$ with $K^+ \subseteq E''$, respectively.

Set $E = E''$ and $P = P'' \cup \{x_1x_4\}$. Then $S = E \cup P$ is an EPS-graph of G^+ such that $d_P(x_1) = 1$, $d_P(x_3) = 0 = d_P(y)$ and $d_P(w) \leq 1$ for $w \in \{x_2, u_1, u_2\}$ and x_1x_4 is a pendant edge in S. In either case, a hamiltonian cycle in S^2 with the required properties can be constructed.

(c) $N(x_4) = \{x_1, x_2\}$.

Clearly $G'' = G^+ - x_4$ is 2-connected. Let K'' be a cycle in G'' containing y, x_1, x_2, x_3, and let $u_1 \in V(K'') \cap N_G(x_1)$. Without loss of generality, $u_1 \neq x_3$: for $d_G(x_3) > 2$ implies $\{x_1, x_2\} \not\subseteq N(x_3)$.

Then G'' has an $[x_3; u_1]$-EPS-graph $S'' = E'' \cup P''$ with $K'' \subseteq E''$. Set $E = E''$ and $P = P'' \cup \{x_1x_4\}$. Then $S = E \cup P$ is an EPS-graph of G^+ with $d_P(y) = 0 = d_P(x_3) = d_P(x_2)$ and x_1x_4 being a pendant edge in S. Hence a hamiltonian cycle in S^2 with the required properties can be constructed.

(1.3) $d_G(x_3) = 2,\ d_G(x_4) = 2$.

Recall that x_3, x_4 are not on the same cycle containing y, x_1, x_2. For each $i = 3, 4$, let l_i denote the length of a longest cycle in G^+ containing y, x_1, x_2, x_i.

(a) Suppose $l_3 \geq 7$ or $l_4 \geq 7$; without loss of generality assume that $l_3 \geq 7$. Recall that

$$K^+ = yx_1u_1 \cdots u_3x_3v_3 \cdots u_2x_2y$$

Then either $u_1 \notin \{u_3, x_3\}$ or $u_2 \notin \{v_3, x_3\}$. Without loss of generality, assume that $u_1 \notin \{u_3, x_3\}$.

(a1) Assume that $G' = G^+ - x_4$ is 2-connected.

Set $W = \{y, u_1, u_2, u_3, q_4\}$, where $q_4 \in \{u_4, v_4\}$. Note that $|\{y, u_1, u_2, u_3\}| = 4$.

Suppose q_4 exists such that $|W| = 4$, say for $q_4 = u_4$. Then $u_4 \in \{u_1, u_2, u_3\}$ and G' has a $[u_4; w_1, w_2]$-EPS-graph $S' = E' \cup P'$ with $K^+ \subseteq E'$, where $\{u_4, w_1, w_2\} = \{u_1, u_2, u_3\}$, by Theorem C.

Now suppose that $|W| = 5$ and K^+ is W-sound in G' for some choice of q_4, say for $q_4 = u_4$. Then by Theorem A there is a W-EPS-graph $S' = E' \cup P'$ of G' with $K^+ \subseteq E'$.

In both cases, taking $E = E'$ and $P = P' \cup \{x_4u_4\}$, we have an EPS-graph $S = E \cup P$ of G^+ such that $d_P(w) \leq 1$ for all $w \in W - \{u_4\}$, $d_P(x_4) = 1$ and $d_P(u_4) \leq 2$. Hence a required hamiltonian cycle in S^2 can be constructed; it can be made to contain x_4u_4 and u_3x_3.

Hence we assume that $|W| = 5$ and K^+ is not W-sound in G' for any choice of $q_4 \in \{u_4, v_4\}$. Then there is another cycle K' in G' such that $V(K') \supseteq W$. We may assume that $q_4 = u_4$ and $x_3 \notin K'$. Then by Theorem B, G' contains a $[u_3; u_1, u_2, u_4]$-EPS-graph $S' = E' \cup P'$ with $K' \subseteq E'$. Taking $E = E'$ and $P = P' \cup \{x_4u_4\}$, we have an EPS-graph $S = E \cup P$ of G^+. Note that x_4 is a pendant vertex in S and either x_3 is a vertex in E, or else it is a pendant vertex in S. Hence a required hamiltonian cycle in S^2 can be constructed. For $i = 1, 2$, also note that if $u_ix_i \notin E(K')$, then $d_P(x_i) = 0$ since $d_G(x_i) = 2$ and $u_i \in K'$. Observe also that $u_3x_i \in E(K')$ and $u_4x_i \in E(K')$ do not constitute any obstacle in this case.

(a2) Assume that $G' = G^+ - x_4$ is not 2-connected.

In view of case **(a1)**, we may assume, by symmetry, that $G^+ - x_3$ is also not 2-connected.

Let $K^{(i)}$ denote a cycle containing y, x_1, x_2, x_i where $i \in \{3, 4\}$. Let B_i be the block of $G^+ - x_i$ with $K^{(7-i)} \subset B_i$. Let G_i, G_i' denote the block chains in $G^+ - x_i - B_i$ (possibly $G_i = \emptyset$ or $G_i' = \emptyset$) which contain $\{u_i, c_i\}$ and $\{v_i, c_i'\}$ respectively, where c_i, c_i' denote the cutvertices of $G^+ - x_i$ belonging to B_i, provided $G_i \neq \emptyset$, $G_i' \neq \emptyset$. If $G_i = \emptyset$, then $u_i = c_i$ and is not a cutvertex, and likewise $v_i = c_i'$ if $G_i' = \emptyset$.

We observe that $K^{(7-i)}$ is edge-disjoint from $G_i \cup G_i'$, $i = 3, 4$ and that $G_3 \cup G_3'$ and $G_4 \cup G_4'$ are edge-disjoint (since every block of $G_i \cup G_i'$ contains an edge of $K^{(i)}$). Finally, if C_y (in G^+) is a cycle containing y, then $E(C_y \cap (G_i \cup G_i')) = \emptyset$ for at least one $i \in \{3, 4\}$; otherwise, $C_y \supset \{x_3, x_4\}$, contrary to **(E)(1)**. Without loss of generality C_y is one of the cycles $K^{(3)}$, and we may also assume that $K^{(3)} = K^+$ (see the beginning of **(a)**).

Set $W = \{y, u_1, u_2, u_3, x_4\}$. The definition of W together with the last sentences of the preceding paragraph ensure that $|W| = 5$ and $K^{(3)} = K^+$ is W-sound in G^+.

Set $G_0 = G_4 \cup G_4' \cup \{u_4x_4v_4\}$; it is a block chain.

(a2.1) Suppose G_0 is a path with $3 \leq l(G_0) \leq 4$.

Then by Theorem A, G^+ has a W-EPS-graph $S = E \cup P$ with $K^{(3)} \subseteq E$ and $d_P(x_4) \leq 1$. If $d_P(x_4) = 0$, then x_4 is in E, and one of its neighbors is 2-valent because $l(G_0) \geq 3$. If $d_p(x_4) = 1$, then x_4 is a pendant vertex in S. In either case, a required hamiltonian cycle in $(G^+)^2$ can be constructed.

(a2.2) Suppose G_0 is a path with $l(G_0) \geq 5$, or G_0 is a block chain having non-trivial blocks.

Replace G_0 in G^+ by a path $P_4 = c_4u_4x_4v_4c_4'$ to obtain the graph G^* (note that $|E(G_0)| \geq 5$). Observe that the cycle $K^{(3)}$ in G^* passes through the vertices y, x_1, x_2, x_3. Then as in case **(a2.1)**, G^* has a W-EPS-graph $S^* = E^* \cup P^*$ with $d_{P^*}(x_4) = 0$ or $d_{P^*}(x_4) = 1$.

(i) If $d_{P^*}(x_4) = 0$, then $P_4 \subset E^*$. Since G_0 is a block chain, by Lemma 1(ii), G_0 contains a $JEPS$-graph $S_0 = J_0 \cup E_0 \cup P_0$ such that $d_{P_0}(c_4) = 0 = d_{P_0}(c_4')$. Moreover, in constructing S_0 by proceeding block by block, one can achieve $d_{P_0}(u_4) \leq 1$, $d_{P_0}(v_4) \leq 1$. In this case, we obtain a W-EPS-graph $S = E \cup P$ of G^+ by setting $E = (E^* - P_4) \cup J_0 \cup E_0$ and $P = P^* \cup P_0$. Here $d_P(x_4) = 0$, $d_P(u_4) \leq 1, d_P(v_4) \leq 1, d_P(c_4) \leq 2, d_P(c_4') \leq 2$ and a required hamiltonian cycle in $(G^+)^2$ can be constructed.

(ii) If $d_{P^*}(x_4) = 1$, then $V(P_4) \subseteq V(P^*)$. Hence either $u_4x_4 \notin E(P^*)$ or $v_4x_4 \notin E(P^*)$. Suppose $v_4x_4 \notin E(P^*)$ (so that $u_4x_4 \in E(P^*)$). By Lemma 1(i), $G_4 \cup \{u_4x_4\}$ (respectively G_4') has an EPS-graph $S^{(4)} = E^{(4)} \cup P^{(4)}$ (respectively $S'^{(4)} = E'^{(4)} \cup P'^{(4)}$) such that $d_{P^{(4)}}(c_4) \leq 1$, $d_{P^{(4)}}(u_4) \leq 2$, $d_{P^{(4)}}(x_4) = 1$ with u_4x_4 being a pendant edge in $S^{(4)}$ and $d_{P'^{(4)}}(c_4') \leq 1$, $d_{P'^{(4)}}(v_4) \leq 1$. Now, if we take $E = E^* \cup E^{(4)} \cup E'^{(4)}$ and $P = (P^* - \{u_4, v_4\}) \cup P^{(4)} \cup P'^{(4)}$, we have an EPS-graph $S = E \cup P$ of G^+ with $d_P(w) \leq 1$ for every $w \in W$ from which a required hamiltonian cycle in $(G^+)^2$ can be constructed (take note that $c_4u_4, v_4c_4' \in E(P^*)$ resulting in $d_P(c_4) \leq 2$ and $d_P(c_4') \leq 2$; and $d_P(x_i) \leq 1$ is guaranteed by the assumption $d_G(x_i) = 2$, $i = 1, 2$).

In view of case **(1.3)(a)** solved, we may assume from now on that $l_3 \leq l_4$ and hence we are left with the following case.

(b) Suppose $4 \leq l_3 \leq l_4 \leq 6$.

(b1) Suppose $l_3 = 6$.

(b1.1) Suppose $u_3 = u_4 = u_1$ and $v_3 = v_4 = v_2$. Set $G^* = G - x_4$.

$\kappa(G^*) = 2$ since $N(x_4) = N(x_3)$. By induction, G^* has the $\mathcal{F}_4$-property; that is, there exists an x_1x_2-hamiltonian path $P(x_1, x_2)$ in $(G^*)^2$ containing different edges $x_3z_3, u_4z_4 \in E(G^*)$. We may write

$$P(x_1, x_2) = x_1 \cdots st \cdots x_2$$

where $\{s, t\} = \{u_4, z_4\}$. Then

$$x_1 \cdots sx_4t \cdots x_2$$

is a required hamiltonian path in G^2; it contains x_3z_3 because $P(x_1, x_2)$ does.

(b1.2) Suppose $u_3 = u_1,\ v_3 = v_2$ and $u_4 = v_1,\ v_4 = u_2$.

Consider $G^- = G^+ - \{x_1v_1, x_2u_2\}$. If there is a path $P(s,t)$ from $s \in \{v_1, u_2\}$ to $t \in \{u_1, v_2\}$ in G^-, then either $l_3 > 6$ or $l_4 > 6$, or G^+ has a cycle containing both x_3 and x_4. Thus x_3 and x_4 belong to different components of G^-. Let G_i denote the component of G^- containing the vertices u_i, x_i, v_i, $i \in \{3, 4\}$. We reach the same conclusion when considering $G^+ - \{x_1u_1, x_2v_2\}$ instead of G^-. Since $N_G(x_1) \nsubseteq V_2(G)$, $N_G(x_2) \nsubseteq V_2(G)$, we may assume without loss of generality that $d(v_1) > 2$ or $d(u_2) > 2$ (otherwise, x_3 and x_4 switch their roles) and hence both v_1, u_2 are not 2-valent (otherwise, v_1 or u_2 would be a cutvertex of G). It follows that G_4 is 2-connected. Likewise, G_3 is also 2-connected.

There is a cycle $C^{(4)}$ in G_4 containing u_4, x_4, v_4 and there is a cycle $C^{(3)}$ in G_3 containing $y, x_1, x_2, u_3, x_3, v_3$. By Theorem D, G_i has a $[u_i; v_i]$-EPS-graph $S_i = E_i \cup P_i$ with $C^{(i)} \subseteq E_i$, $i = 3, 4$. Note that $d_{P_3}(z) = 0$ for $z \in \{y, x_1, x_2, x_3\}$.

Now set $E = E_3 \cup E_4$ and $P = P_3 \cup P_4 \cup \{x_1v_1\}$. Then $S = E \cup P$ is an EPS-graph of G^+ with $C^{(3)} \cup C^{(4)} \subseteq E$ and a required hamiltonian cycle in $(G^+)^2$ containing x_4v_1, x_3v_2 can be constructed.

(b1.3) Suppose $u_3 = u_1 = u_4, v_3 = v_2$ and $v_4 = u_2$ (the case $u_3 = u_1$, $u_4 = v_1$ and $v_2 = v_3 = v_4$ is symmetric).

This subcase is impossible; otherwise, it gives rise to a cycle containing y, x_3, x_4, a contradiction to the assumption (just consider in G a path from x_1 to u_2 avoiding u_1).

It is straightforward to see that $x_i \notin N(x_j)$ for $i = 3, 4$ and $j = 1, 2$ for all choices of i and j; otherwise, $l_i > 6$ or there exists a cycle containing y, x_3, x_4. Therefore, subcase **(b1)** is finished.

(b2) Suppose $l_3 = 5$.

We may assume without loss of generality that $u_3 = x_1,\ x_3 = u_1$ and $v_3 = v_2$.

Suppose $d_G(v_3) = 2$. Consider $G' = G - \{x_3, v_3\}$; it is a non-trivial block chain with pendant edges x_1v_1, x_2u_2. By Corollary 1(ii), there exists a hamiltonian path $P(x_1, x_2) \subseteq (G')^2$ starting with x_1v_1 and ending with u_2x_2. We proceed block by block to construct $P(x_1, x_2)$ such that $x_4z_4 \in E(G) \cap P(x_1, x_2)$ and $x_4z_4 \notin \{x_1v_1, u_2x_2\}$: this is clear if x_4 is a cutvertex of G'; and if $x_4 \in V(B_4)$ where $B_4 \subseteq G'$ is a 2-connected block containing the cutvertices c_4, c_4' of G', one uses a hamiltonian path $P(c_4, c_4')$ in $(B_4)^2$ containing an edge incident to x_4 (Theorem F(i)). Then

$$(P(x_1, x_2) - u_2x_2)u_2v_3x_3x_2$$

is a required hamiltonian path in G^2.

If $d_G(v_3) > 2$, then $G^{(0)} = G - \{x_1, x_2, x_3\}$ is connected (or else v_3 is a cutvertex of G). Any v_3u_2-path $P(v_3, u_2) \subset G^{(0)}$ can be extended to a cycle $yx_1x_3P(v_3, u_2)x_2y$ of length ≥ 6, contradicting the assumption of this subcase.

(b3) Suppose $l_3 = 4$.

In this case, let $G' = G - x_3$. Operating with $P(x_1, x_2) \subseteq (G')^2$ as in case **(b2)**, we obtain an $\mathcal{F}_4$ x_1x_2-hamiltonian path $(P(x_1, x_2) - u_2x_2)u_2x_3x_2$ in G^2.

(1.4) $d_G(x_3) = 2, d_G(x_4) > 2$.

This case is symmetrical to the case **(1.2)**.

(E)(2) Suppose x_3 and x_4 are in K^+.

Without loss of generality, assume that

$$K^+ = yx_1u_1 \cdots z_3x_3 \cdots x_4z_4 \cdots u_2x_2y.$$

As for the definition of x_3^*, x_4^* see the paragraph preceding the statement of Lemma 2.

(2.1) $x_3 \neq u_1$ and $x_4 \neq u_2$.

(a) Suppose either $u_{i-2} \notin N_G(x_i)$, or $u_{i-2} \in\in N_G(x_i)$ and $d_G(x_i) > 2$ for some $i \in \{3, 4\}$. Without loss of generality, assume that $i = 4$.

If $u_1 \neq x_3^*$, set $W = \{y, u_1, u_2, x_3^*, x_4^*\}$. Then $|W| = 5$ and K^+ is W-sound, so by Theorem A, G^+ has a W-EPS-graph $S = E \cup P$ with $K^+ \subseteq E$.

If $u_1 = x_3^*$, then $d_G(x_3) = 2$ since $x_3 \neq u_1$ by supposition. Now, let $S = E \cup P$ be an $[x_4^*; u_1, u_2]$-EPS-graph of G^+ with $K^+ \subseteq E$ by Theorem C.

In either case, a required hamiltonian cycle in $(G^+)^2$ can be constructed.

(b) Suppose $u_{i-2} \in N_G(x_i)$ and $d_G(x_i) = 2$ for $i = 3, 4$.

If w_4 is the predecessor of x_4 in K^+ and $w_4 \neq x_3$, then let $S = E \cup P$ be a $[x_1; u_1, u_2, w_4]$-EPS-graph with $K^+ \subseteq E$ by Theorem B. If $w_4 = x_3$, then let $S = E \cup P$ be an $[x_1; u_1, u_2]$-EPS-graph with $K^+ \subseteq E$ by Theorem C. Hence a required hamiltonian cycle in $(G^+)^2$ can be constructed from S.

(2.2) $x_3 = u_1$ and $x_4 \neq u_2$.

(a) Suppose either $u_2 \notin N_G(x_4)$, or $u_2 \in N_G(x_4)$ and $d_G(x_4) > 2$.

(a1) $x_3x_4 \in E(G)$.

If $d_G(x_4) > 2$, then $d_G(x_3) = 2$ and we choose an $[x_1; x_4, u_2]$-EPS-graph $S = E \cup P$ of G^+ with $K^+ \subseteq E$ by Theorem C. If, however $d_G(x_4) = 2$, we choose an $[x_1; x_3, z_4, u_2]$-EPS-graph $S = E \cup P$ of G^+ with $K^+ \subseteq E$ by Theorem B. In either case, S^2 contains a required hamiltonian cycle.

(a2) $x_3x_4 \notin E(G)$.

Here w_3 is the successor of x_3 in K^+. Let $S = E \cup P$ be an $[x_3; u_2, w_3, x_4^*]$-EPS-graph with $K^+ \subseteq E$ by Theorem B. Also here, S^2 contains a required hamiltonian cycle; it contains $x_3v \in E(G)$ which is consecutive to x_1x_3 in the eulerian trail of the component of E containing K^+ (possibly $v = w_3$) and it contains x_4z_4.

(b) Suppose $u_2 \in N(x_4)$ and $d_G(x_4) = 2$.

(b1) $x_3x_4 \in E(G)$.

Let $H = G - x_4$. Suppose H is 2-connected. Then by induction, H has an $\mathcal{F}_4$ x_1x_2-hamiltonian path $P(x_1, x_2)$ in H^2 containing x_3w_3 and u_2w_2 which are edges of H. By deleting u_2w_2 from $P(x_1, x_2)$ and joining x_4 to u_2, w_2, we obtain an $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2 containing x_3w_3, x_4u_2 which are edges of G.

Suppose H is not 2-connected. Then H is a non-trivial block chain with endblock B_i containing u_i; u_i is not a cutvertex of H, $i = 1, 2$. Let c_i denote the cutvertex of H which is contained in B_i, $i = 1, 2$. Set $B_{1,2} = H - (B_1 \cup B_2)$. If $c_1 = c_2$, then set $B_{1,2} = c_1$. In any case, c_1 and c_2 are not cutvertices of $B_{1,2}$.

By supposing $x_i \neq c_i$ (and thus B_i is 2-connected) we apply Theorem F to conclude that $(B_i)^2$ has an $\mathcal{F}_3$ x_ic_i-hamiltonian path $P(x_i, c_i)$, $i = 1, 2$ containing x_3w_3, u_2w_2 respectively, which are edges of G. Let $P(c_1, c_2)$ denote a c_1c_2-hamiltonian path in $(B_{1,2})^2$. By deleting the edge u_2w_2 from the x_1x_2-hamiltonian path $P(x_1, c_1)P(c_1, c_2)P(x_2, c_2)$ in $(G - x_4)^2$ and joining x_4 to u_2, w_2, we obtain an $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2 containing x_3w_3, x_4u_2 which are edges of G. Now suppose $x_1 = c_1$ or $x_2 = c_2$; i.e., $d_G(u_1) = 2$ or $d_G(u_2) = 2$. In this case we consider G^+ and choose an

$[x_1; u_1, u_2]$-EPS-graph $S = E \cup P$ of G^+ with $K^+ \subseteq E$ by Theorem C. Hence S^2 contains a hamiltonian cycle as required.

(b2) $x_3x_4 \notin E(G)$.

If $w_3 \neq w_4$, then we set $S = E \cup P$ to be an $[x_3; u_2, w_3, w_4]$-EPS-graph of G^+ with $K^+ \subseteq E$ by Theorem B. If $w_3 = w_4$, then we set $S = E \cup P$ to be an $[x_3; u_2, w_3]$-EPS-graph of G^+ with $K^+ \subseteq E$ by Theorem C. Here w_3 is the successor of x_3 and w_4 is the predecessor of x_4 in K^+. Hence S^2 yields a required hamiltonian cycle unless $w_3 = w_4$ and $d_G(w_3) > 2$, in which case $d_G(x_3) = 2$ holds, and we operate with an $[x_1; w_3, u_2]$-EPS-graph by Theorem C. This settles case **(2.2)**.

Since the case $x_3 \neq u_1$ and $x_4 = u_2$ is symmetrical to the case **(2.2)** just dealt with, we are left with the following case.

(2.3) $x_3 = u_1$ and $x_4 = u_2$.

(a) $d_G(x_3) = 2$.

(a1) $x_3x_4 \notin E(G)$.

Choose an $[x_4; u_3]$-EPS-graph $S = E \cup P$ of G^+ with $K^+ \subseteq E$ by Theorem C if $u_3 \neq u_4$, and an $[x_4; u_3, u_4]$-EPS-graph $S = E \cup P$ of G^+ with $K^+ \subseteq E$ by Theorem D if $u_3 = u_4$; here u_3 is taken to be the successor of x_3 and u_4 the predecessor of x_4 in K^+. Then S^2 yields a required hamiltonian cycle unless $u_3 = u_4$ and $d_G(u_3) > 2$. In this case $d_G(x_4) = 2$ and we may operate with an $[x_2; u_3]$-EPS-graph to obtain a required hamiltonian cycle in S^2 by Theorem D.

(a2) $x_3x_4 \in E(G)$.

(i) Suppose $d_G(x_4) > 2$.

$G - x_3$ is a block chain in which x_1 and x_4 are not cutvertices and belong to different endblocks. However, the endblock containing x_4 is 2-connected since $d_G(x_4) > 2$; and it contains x_2 as well which is not a cutvertex of $G - x_3$ either. Therefore, $G^+ - x_3$ is 2-connected. Set

$$H = (G^+ - \{y, x_1, x_3\}) \cup \{x, xv_1, xx_2\}.$$

H is 2-connected since $G^+ - x_3$ is 2-connected. By Theorem E, H^2 has a hamiltonian cycle C containing v_1x, xx_2, x_4w_4 which are edges of H. Now $(C - x) \cup \{v_1x_3x_1yx_2\}$ is a hamiltonian cycle in $(G^+)^2$ with the required properties.

(ii) Suppose $d_G(x_4) = 2$.

Let H be the graph obtained from G^+ by deleting y, x_2, x_3, x_4. Then H is a non-trivial block chain containing x_1 which is not a cutvertex of H. By Corollary 1(i), H^2 has a hamiltonian cycle C containing the edge x_1v_1 (which is an edge of G). This implies that the cycle $yx_1(C - x_1v_1)v_1x_3x_4x_2y$ is a hamiltonian cycle in $(G^+)^2$ having the required properties.

(b) $d_G(x_3) > 2$, hence $d_G(x_4) > 2$; otherwise we are back to **(a)** above, by symmetry. Then $x_3x_4 \notin E(G)$.

Suppose $G' = G - x_1$ is 2-connected. Then by induction, G' has an $\mathcal{F}_4$ v_1x_2-hamiltonian path $P(v_1, x_2)$ in $(G')^2$ containing x_3w_3 and x_4w_4 which are edges of G'. Now $\{x_1v_1\} \cup P(v_1, x_2)$ is an $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2 containing x_3w_3, x_4w_4 which are edges of G.

Now suppose $G' = G - x_1$ is not 2-connected. Then G' is a non-trivial block chain with x_3, v_1 in different endblocks and not cutvertices. Note that the block containing x_3 is 2-connected and at least one block contaning x_4 is 2-connected, since $d_G(x_3) > 2$ and $d_G(x_4) > 2$.

(b1) Suppose x_2 is a cutvertex of G'. Let G_1 and G_2 be the components of $G' - x_2$ with either $x_3, x_4 \in V(G_1)$ and $v_2, v_1 \in V(G_2)$, or $x_3, v_2 \in V(G_1)$ and $x_4, v_1 \in V(G_2)$ (note $d_{G'}(x_2) = 2$). Observe that in the first case $v_2 = v_1$ is possible. However, $v_1 = x_4$ is impossible because of the assumptions of this case **(b)**; i.e., $d_G(x_4) > 2$. By the same token $v_2 = x = 3$ is impossible.

Suppose $x_3, x_4 \in V(G_1)$ and $v_2, v_1 \in V(G_2)$. Then by Theorem F(ii) or Corollary 1(ii), respectively, $(G_1)^2$ has an x_3x_4-hamiltonian path P_1 containing an edge $x_3w_3 \in E(G)$. If $G_2 = K_1 = v_1$, then we set $P = P_1 \cup \{x_2x_4, x_3v_1, v_1x_1\}$. If $G_2 = K_2 = v_2v_1$, then we set $P = P_1 \cup \{x_2x_4, x_3v_1, v_1v_2, v_2x_1\}$. Otherwise, by Theorem E or Corollary 1(i), respectively, $(G_2)^2$ has a hamiltonian cycle C_2 containing an edge $t_1v_1 \in E(G)$. Then we set $P = P_1 \cup C_2 \cup \{x_2x_4, x_3v_1, t_1x_1\} - \{t_1v_1\}$. In all cases P is an $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2 containing x_3w_3, x_4x_2 which are edges of G as required.

Suppose $x_3, v_2 \in V(G_1)$ and $x_4, v_1 \in V(G_2)$. Then we apply an analogous strategy as in the preceding case using Theorems E, F and Corollary 1, but considering G_1 instead of G_2 and vice versa.

(b2) Suppose x_2 is not a cutvertex of G'. Let B_2 be the 2-connected block containing x_2.

(i) Suppose $x_3 \in V(B_2)$. Let t be the cutvertex of G' in B_2; possibly $t = x_4$, $t \notin \{x_2, x_3\}$ in any case. We define the block chain G_1 such that

$G' = B_2 \cup G_1$ and $B_2 \cap G_1 = \{t\}$. If $t = x_4$, then $(B_2)^2$ has an x_2t-hamiltonian path P_2 containing $x_3w_3 \in E(G)$ by Theorem F(i). If $t \neq x_4$, then by induction $(B_2)^2$ has an x_2t-hamiltonian path P_2 containing x_3w_3, x_4w_4 which are different edges of G. In both cases $(G_1)^2$ has a tv_1-hamiltonian path starting with $tw \in E(G)$, by Theorem F(ii) or Corollary 1(ii), respectively. Then $P = P_2 \cup P_1 \cup \{v_1x_1\}$ is an $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2 containing x_3w_3, x_4w_4 which are edges of G as required. Note that if $t = x_4$, then $x_4w_4 = tw$.

(ii) Suppose $x_3 \notin V(B_2)$. If B_2 is not an endblock, then t, t' denote the cutvertices of G' in B_2 and we define block chains G_0, G_1 such that $G' = G_1 \cup B_2 \cup G_0$, $x_3 \in V(G_1), v_1 \in V(G_0)$ and $G_1 \cap B_2 = t$, $B_2 \cap G_0 = t'$. If B_2 is an endblock, then we proceed analogously: we set $G_0 = \emptyset$ and $t' = v_1$ in this case. Note that $t = x_4$ ot $t' = x_4$ is possible.

If $t' \neq x_4$, then by Theorem F(i) $(B_2)^2$ has an x_2t-hamiltonian path P_2 containing $t'w' \in E(G)$ for $t = x_4$ and by induction $(B_2)^2$ has an $\mathcal{F}_4$ x_2t-hamiltonian path P_2 containing $t'w', x_4w_4$ which are different edges of G for $t \neq x_4$. By the same token $(G_1)^2$ has an tx_3-hamiltonian path P_1 containing $tw \in E(G)$. If $G_0 = \emptyset$, then we set $P = P_2 \cup P_1 \cup \{x_3x_1\}$. If $G_0 = t'v_1$, then we set $P = P_2 \cup P_1 \cup \{x_3x_1, w'v_1, v_1t'\} - \{t'w'\}$. Otherwise $(G_0)^2$ has a hamiltonian cycle C_0 containing $t'w^* \in E(G)$ by Theorem E or Corollary 1(i), respectively, and we set $P = P_2 \cup C_0 \cup P_1 \cup \{x_3x_1, w'w^*\} - \{t'w', t'w^*\}$. In all cases P is an $\mathcal{F}_4$ x_1x_2-hamiltonian path in G^2 containing x_3x_1, x_4w_4 which are edges of G as required. Note that if $t = x_4$, then $x_4w_4 = tw$.

If $t' = x_4$, we proceed analogously as in the previous case with G_1 and G_0 switching roles.

This completes the proof of Theorem 2. □

Acknowledgements

This publication was partly supported by the project LO1506 of the Czech Ministry of Education, Youth and Sports.

References

[1] Bondy, J.A., and Murty, U.S.R. *Graph Theory*, Graduate Texts in Mathematics, **244**. Springer, New York 2008. MR2368647

[2] Chartrand, G., Hobbs, A.M., Jung, H.A., Kapoor, S.F., and Nash-Williams, C.St.J.A. The square of a block is Hamiltonian connected, *J. Combinat. Theory Ser. B* **16** (1974) 290–292. MR0345865

[3] Chia, G.L., and Fleischner, H. Revisiting the Hamiltonian Theme in the Square of a Block: The General Case, (in preparation).

[4] Chia, G.L., Ong S.-H., and Tan, L.Y. On graphs whose square have strong hamiltonian properties, *Discrete Math.* **309** (2009) 4608–4613. MR2519200

[5] Fleischner, H. On spanning subgraphs of a connected bridgeless graph and their application to *DT*-graphs, *J. Combinat. Theory Ser. B* **16** (1974) 17–28. MR0332572

[6] Fleischner, H. The square of every two-connected graph is Hamiltonian, *J. Combinat. Theory Ser. B* **16** (1974) 29–34. MR0332573

[7] Fleischner, H. In the square of graphs, Hamiltonicity and pancyclicity, hamiltonian connectedness and panconnectedness are equivalent concepts, *Monatsh. Math.* **82** (1976) 125–149. MR0427135

[8] Fleischner, H., and Hobbs, A.M. Hamiltonian total graphs, *Math. Nachr.* **68** (1975) 59–82. MR0384601

[9] Faudree, R.J., and Schelp, R.H. The square of a block is strongly path connected, *J. Combinat. Theory Ser. B* **20** (1976) 47–61. MR0424609

[10] Georgakopoulos, A. A short proof of Fleischner's theorem, *Discrete Math.* **309** (2009) 6632–6634. MR2558627

[11] Hobbs, A.M. The square of a block is vertex pancyclic, *J. Combinat. Theory Ser. B* **20** (1976) 1–4. MR0416980

[12] König D., Theorie der endlichen und unendlichen Graphen, Chelsea Publ. Comp., NY 1950; first publ. by Akad. Verlagsges., Leipzig, 1936. MR0036989

[13] Müttel, J., and Rautenbach, D. A short proof of the versatile version of Fleischner's theorem, *Discrete Math.* **313** (2013) 1929–1933. MR3073122

[14] Nash-Williams, C.St.J.A. Problem No. 48, *Theory of Graphs* (P. Erdös and G. Katona, Eds.), Academic Press, New York 1968. MR0232693

[15] Neuman, F. On a certain ordering of the set of vertices of a tree, *Časopis Pěst. Mat.* **89** (1964) 323–339. MR0181587

[16] Říha, S. A new proof of the theorem of Fleischner, *J. Combinat. Theory Ser. B* **52** (1991) 117–123. MR1109427

[17] Sekanina, M. On an ordering of the set of vertices of a connected graph, *Publ. Fac. Sci. Univ. Brno* **412** (1960) 137–142. MR0140095

[18] Sekanina, M. Problem No. 28, *Theory of Graphs and its Applications*, (M. Fiedler, Ed.), Academic Press, New York 1964. MR0172259

[19] Underground, P. On graphs with hamiltonian squares, *Discrete Math.* **21** (1978) 323. MR0522906

Gek L. Chia
Department of Mathematical and Actuarial Sciences
Universiti Tunku Abdul Rahman
Jalan Sungai Long, Bandar Sungai Long
Cheras 43000 Kajang
Selangor
Malaysia
Institute of Mathematical Sciences
University of Malaya
50603 Kuala Lumpur
Malaysia
E-mail address: glchia@um.edu.my

Jan Ekstein
Department of Mathematics
Institute for Theoretical Computer Science
European Centre of Excellence NTIS
New Technologies for the Information Society
Faculty of Applied Sciences
University of West Bohemia
Pilsen
Technická 8, 306 14 Plzeň
Czech Republic
E-mail address: ekstein@kma.zcu.cz

Herbert Fleischner
Institut für Computergraphik und Algorithmen 186/1
Technical University of Vienna
Favoritenstrasse 9–11, 1040 Wien
Austria
E-mail address: herbtravel@yahoo.com

Received 1 June 2015