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3-dimensional polygons determined by permutations

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In this paper we introduce the notion of *d-dimensional permupolygons* on $\mathbb{Z}^d$, with $d \geq 2$. $2-$dimensional permupolygons, also called permutominides, where introduced by Incitti et al. [12]. By using an encoding of permupolygons inspired by the encoding given for convex polyominoes by Bousquet-Mélou and Guttmann [6], we easily recover enumerative results about 2-dimensional parallelogram, unimodal, column convex and convex permupolygons. Moreover, we extend these results for dimension $d = 3$. Finally, we study combinatorial characterizations of permutations defining 3-dimensional permupolygons. We show some necessary and sufficient conditions for a triple of 2-dimensional permutations (π_1, π_2, π_3) to define a 3-dimensional permupolygon.

1. Introduction

The goal of this paper is to extend the notions of permutomino/permutominide to the 3-dimensional case, by introducing the notion of 3-*dimensional permupolygon*, and to study some subclasses of these objects defined in terms of convexity constraints in order to determine their combinatorial properties and enumeration. We thus start by briefly reviewing definitions and results on permutominoes and permutominides before defining permupolygons and discussing our results.

1.1. Permutominoes and permutominides

We start this section by briefly recalling a few basic definitions of *polyominoes.*

Definition 1. In the plane $\mathbb{Z} \times \mathbb{Z}$ a *polyomino* is a finite union P of unit squares (called *cells*) whose interior is connected.

Polyominoes are defined up to a translation (see Fig. 1). Polyominoes have been extensively studied in the literature. In particular we will be concerned with some subclasses which can be defined using the geometrical notions of *directedness* and *convexity.*

Definition 2. A polyomino is said to be:
i) *parallelogram* if its boundary can be decomposed in two paths, the upper and the lower paths, which are made of north and east unit steps and meet only at their starting and final points (see Fig. 1 (a));
ii) *directed* when each of its cells can be reached from a distinguished cell, called *the root*, by a path which is contained in the polyomino and uses only north and east unit steps (see Fig. 1 (b));
iii) *column-convex* or *vertically-convex* (resp. *row-convex* or *horizontally-convex*) if all its columns (resp. rows) are connected (see Fig. 1 (c), (d));
iv) *convex*, if it is both horizontally and vertically-convex (see Fig. 1 (e)).

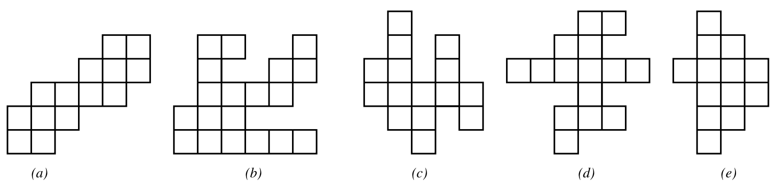

Figure 1: (a) A parallelogram polyomino; (b) a directed polyomino; (c) a column-convex polyomino; (d) a row-convex polyomino; (e) a convex polyomino.

For more details we refer to the vast literature on the subject, in particular to [5]. Now, let us recall some basics about *permutominoes* and *permutominides.*

Definition 3. Let P be a polyomino, then a *vertex* of P is an intersection point of two consecutive boundary edges with different direction, and a *side* of P is a segment joining two consecutive vertices.

Definition 4. Let P be a polyomino without "holes", i.e. a polyomino whose boundary is a single loop, and having n rows and n columns, $n \geq 1$; we assume without loss of generality that the south-west corner of its minimal bounding rectangle is placed at $(1, 1)$. The polyomino P is a *permutomino* if for each abscissa (resp. ordinate) between 1 and $n+1$ there is exactly one vertical (resp. horizontal) side in the boundary of P with that coordinate, and n is called the *size* of the permutomino.

A permutomino P can be equivalently defined by a pair of permutations of length $n + 1$. Indeed for each vertical (resp. horizontal) side of P there are exactly 2 vertices of P with the same abscissa (resp. ordinate).

Definition 5. Let P be a permutomino of size n, and let us denote by $r(P)$ its leftmost vertex with minimal ordinate. By following the boundary of P in clockwise order, starting from $r(P)$, we obtain a sequence $s(P)$ of $2n+2$ vertices of P. Then the *first* and the *second components* of P are the two permutations of length $n+1$, denoted by $\pi_1(P)$ and $\pi_2(P)$, whose graphical representation are respectively given by vertices with odd and even indices in $s(P)$.

Figure 2 shows an example of permutomino and its associated permutations. For more detailed definitions on permutominoes we address the reader to [3, 8, 11].

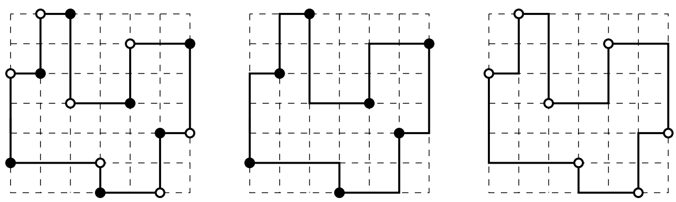

Figure 2: A permutomino of size 6 and its two components $\pi_1 = (2,5,7,1,4,3,6)$ and $\pi_2 = (5,7,4,2,6,1,3)$.

Permutominoes were introduced in [13] in an algebraic context and then they were considered by F. Incitti in studying *permutation diagrams*, determined by a pair (π_1, π_2) of permutations [12]. Permutominoes are closely related to another class of combinatorial objects uniquely defined by pairs of permutations, the family of *permutominides* [9].

Definition 6. A *polyominide* is a finite connected union of cells.

As for polyominoes, a polyominide is said to be *vertically-convex* (resp. *horizontally-convex*) if all its columns (resp. rows) are connected. Finally, a polyominide is said to be *convex*, if it is both horizontally and vertically-convex.

Definition 7. A *permutominide* of size n is a polyominide whose boundary can be drawn as a single (possibly self-intersecting) loop that has exactly one side for every abscissa (resp. every ordinate) between 1 and $n+1$.

A permutominide can be equivalently represented by a pair of permutations of length $n+1$, defined in the same way as for permutominoes. Figure 3 shows a permutominide of size 6 and its associated permutations $\pi_1 = (3,6,7,5,2,4,1)$ and $\pi_2 = (5,2,6,7,1,3,4)$.

A *vertically-convex* (resp. *horizontally-convex*, *convex*) permutominide is a vertically-convex (resp. horizontally-convex, convex) polyominide, which is also a permutominide. A convex permutominide of size n is said to be *directed-convex* if it contains the cell with lowest leftmost vertex in position $(1, 1)$. A directed-convex permutominide of size n is said to be *parallelogram* if it contains the cell with upper rightmost vertex in position $(n + 1, n + 1)$.

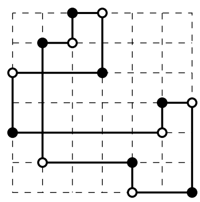

Figure 3: A permutominide of size 6 and its two components.

Here we would like to recall the main results on permutominoes and permutominides, which can be roughly grouped into two main research guidelines:

1. *Enumeration of restricted classes of permutominoes/permutominides.* The enumeration of permutominoes according to the size has been considered and solved for restricted classes defined by imposing convexity or directedness constraints. The largest class which has been studied is that of *vertically-convex permutominoes*: in [2] the authors determine a direct recursive construction for vertically-convex permutominoes of a given size, which leads to a functional equation. However, they are not able to solve the equation in order to obtain the generating function of vertically-convex permutominoes. By using numerical analysis they were led to conjecture that the number f_n of vertically-convex permutominoes of size n has the following asymptotic behavior:

   (1) $$f_n \sim k\,(n+1)!\,h^n\,, \text{ where } k = 0.3419111 \text{ and } h = 1.385933.$$

   Exact results have been obtained for some classes of convex permutominoes [3, 8, 11], and in particular it was proved that the number of *convex permutominoes* of size n is:

   (2) $$2\,(n+3)\,4^{n-2} - \frac{n}{2}\binom{2n}{n} \qquad n \geq 1.$$

We point out that (2) was proved independently in [4] and in [8], by using analytical techniques. Recently, a bijective proof of (2) was given in [9] by encoding convex permutominoes in terms of lattice paths.
On the other side, concerning the enumeration of permutominides, some restricted classes defined by similar convexity constraints have been considered. In particular Disanto et al. [9] enumerated convex permutominides of size n:

$$2\,(n+1)4^{n-2}\,, \tag{3}$$

as well as parallelogram and directed convex permutominides, while in Beaton et al. [2] enumerated vertically convex permutominides of size n:

$$2^{n-2}\,(n+1)!\,, \tag{4}$$

2. *Study of permutation pairs defining restricted classes of permutominoes.*
The second research line concerns the study of permutations defining restricted classes of permutominoes, and has been studied in [3, 11]. This topic shows connections with the vast literature on pattern avoiding permutations. So, let $\mathcal{P}$ be a restricted class of permutominoes (defined as usual by imposing certain convexity constraints), and let $\mathcal{P}_n$ be the set of permutominoes of $\mathcal{P}$ with size n. Researchers have investigated the problem of giving a characterization of the following sets:
   (i) $\{(\pi_1, \pi_2) : \exists P \in \mathcal{P}$ such that $\pi_1(P) = \pi_1,\ \pi_2(P) = \pi_2\,\}$.
   (ii) $\Sigma^1_n = \{\pi_1(P) : P \in \mathcal{P}_n\}$, $\Sigma^2_n = \{\pi_2(P) : P \in \mathcal{P}_n\}$;
   (iii) $\Sigma^{12}_n = \Sigma^1_n \cup \Sigma^2_n$.
In [3], a characterization of the sets Σ^1_n and Σ^{12}_n for the classes of parallelogram, directed-convex and convex permutominoes has been given. In particular the characterization of the sets Σ^1_n and Σ^{12}_n for convex permutominoes has been given in terms of square permutations, which have been recently investigated by several authors [1, 10, 14].

1.2. Permupolygons

In this section we introduce a different definition of permutominoes and permutominides, which relies on the paper [6] by Bousquet-Mélou and

Guttmann. In this paper the authors, with the aim of extending the notion of polyomino to the 3-dimensional case, observed that the property of being a polyomino without holes and the convexity constraint can be thought as properties of the self avoiding polygon which forms the boundary of the polyomino rather than properties of the set of cells forming the interior of the object.

Similarly, the definition of permutomino and permutominide of size n can be given using polygons: A *permutominide* of size n is a polygon with vertices on the integer lattice $[1, n+1]^2$ such that its intersection with any vertical (resp. horizontal) line with integer coordinate between 1 to $n+1$ has exactly one connected component different from an isolated point; a *permutomino* of size n is a permutominide which is self avoiding.

Let us generalize these definitions to higher dimensions: In the lattice $\mathbb{Z}^d$ with its canonical basis $(e_1, e_2, \ldots, e_d)$, a *permupolygon* P is a polygon such that, in each hyperplane orthogonal to e_i and with coordinate j, with $i = 1, 2, \ldots d$ and $j = 1 \ldots n+1$, there lie exactly $d-1$ sides of P; precisely one side for each direction e_j, with $j \neq i$. A more formal definition will be provided in Section 2. Following this approach, starting from Section 2, we will replace the words permutomino and permutominide, commonly used in the literature and in the 2-dimensional case, by the word *permupolygon*. The use of permupolygons turns out to be decisive since it provides a unified setting to represent these objects in the multidimensional case.

A remarkable property of d-dimensional permupolygons, directly following from their definition, is that for $d \geq 3$ they are self avoiding polygons. We point out that, while our definition is given for the d-dimensional case, in the paper we will deal only with $d = 2, 3$.

The main idea of the paper is to use a unique representation of each permupolygon P by means of a word $s(P)$, called the *path encoding* of P, which will allows us to deal with 3-dimensional permupolygons in a simple way. We will show that 3-dimensional permupolygons are an "appropriate" extension of permutominoes and we will enumerate some restricted classes of 3-dimensional permupolygons defined by convexity constraints.

We believe that this representation could be used to tackle some algebraic problems related to the classification and characterization of permutation diagrams, as those mentioned in [12]. In particular, extending what happens for permutominoes and permutominides, we will also see that each 3-dimensional permupolygon P is uniquely determined by a triple of 2-dimensional permutations (π_1, π_2, π_3), called the *first*, *second*, and *third components* of P, respectively.

1.3. Results of the paper

In Section 2 we provide a formal definition of a d-dimensional permupolygon, and present the most important subclasses of permupolygons, which we are going to study.

In Section 3 we return to the 2-dimensional case. Our representation of 2-dimensional permupolygons in terms of their path encoding allows us to re-obtain results already known for vertically-convex, convex, directed-convex and parallelogram permutominides, in a simpler and compact way.

In Section 4 we consider 3-dimensional permupolygons. Using the representation of these objects via their path encodings, we can handle these objects with no need of a graphical representation. Then, we provide enumeration of some restricted classes of convex permupolygons, including convex, directed-convex and parallelogram permupolygons. These results extend those proved in Section 3 for $d = 2$.

In Section 5 we study combinatorial characterizations of permutations defining 3-dimensional permupolygons. We show some necessary and sufficient conditions for a triple of 2-dimensional permutations (π_1, π_2, π_3) to define a 3-dimensional permupolygon.

2. Basic definitions

In this section we first recall some definitions given in [6] concerning d-dimensional polygons, then we introduce the notion of *d-dimensional permupolygon* on $\mathbb{Z}^d$. Finally we recall the notion of *multidimensional permutation*.

2.1. d-dimensional permupolygons

Definition 8. Let $d \geq 1$, and let us consider the lattice $\mathbb{Z}^d$, with its canonical basis $(e_1, ..., e_d)$. An *oriented rooted polygon* of perimeter $2n$ is a $2n$-tuple $(s_1, ..., s_{2n})$ of points of $\mathbb{Z}^d$ such that s_i and s_{i+1} are neighbors (namely, there exists a vector e_k of the canonical basis, such that s_{i+1} can be reached from s_i by taking a unit step along the direction of e_k) for $1 \leq i \leq 2n$ (with $s_{2n+1} = s_1$). The point s_1 is called the *root* of the polygon.

We can represent an oriented rooted polygon P of perimeter $2n$ as a word $u(P) = u_1\, u_2\, \dots\, u_{2n}$ on the alphabet $\mathcal{A} = \{1, 2, \dots, d\} \cup \{\overline{1}, \overline{2}, \dots, \overline{d}\}$, namely, if $u_i = k$ (resp. $\overline{k}$) it means that one goes from the point s_i to the point s_{i+1} by taking a unit step along the unit vector e_k (resp. $-e_k$). Remark that the number of occurrences of k in $u(P)$, denoted $|u|_k$, is equal to the number of occurrences of $\overline{k}$ in $u(P)$. Conversely, any word u on $\mathcal{A}$ such that $|u|_k = |u|_{\overline{k}}$ (for $1 \leq k \leq d$) defines an oriented rooted polygon.

Observe that, according to Definition 8, oriented rooted polygons are not necessarily self-avoiding.

Definition 9. Let P be an oriented rooted polygon with associated word $u = u(P)$. A *vertex* of P is any point s_i such that $u_{i-1} \neq u_i$, for $i > 0$, and a *side* of P is the segment joining two consecutive vertices.

The *dimension* of an oriented rooted polygon is the dimension of its minimal bounding hypercube. We would like to point out that, since we are interested in the shape of the object, an oriented rooted polygon is defined up to translation. Therefore, given an oriented rooted polygon, we can assume, without loss of generality, that the origin of its minimal bounding hypercube is placed at $(1, \ldots, 1)$ so that all vertices of the polygon have positive coordinates.

Definition 10. An *oriented polygon* is an oriented rooted polygon considered up to a cyclic permutation of its points.

Now we define some classes of oriented rooted polygons in terms of properties of the associated words. We would like to point out that, in the restricted case of *self-avoiding polygons*, these classes are a generalization in dimension $d \geq 3$ of some well-known classes of polyominoes studied in the literature.

Definition 11. An *oriented rooted parallelogram polygon* P is an oriented rooted polygon whose word $u = u(P)$ can be written as vw (i.e. $u = vw$), where v (resp. w) is a word on $\{1, ..., d\}$ (resp. $\{\overline{1}, ..., \overline{d}\}$).

Definition 12. An oriented polygon is *parallelogram* if there exists a cyclic permutation of its points such that the oriented rooted polygon associated with this permutation is parallelogram. Observe that such a permutation is unique if it exists.

We point out that the class of parallelogram self-avoiding polygons with $d = 2$ is identical to the class of parallelogram polyominoes.

Figure 4 (a) shows a 2-dimensional oriented parallelogram polygon, and its unique rooted representation is

$$2\,2\,1\,1\,1\,2\,2\,1\,1\,\overline{2}\,\overline{1}\,\overline{1}\,\overline{1}\,\overline{2}\,\overline{1}\,\overline{2}\,\overline{2}\,\overline{1}\,,$$

with $(1, 1)$ as root. In the graphical representation, the root of an oriented rooted polygon is represented using a black dot, while we use white dots for other vertices of the polygon. This convention is adopted also in the sequel.

Definition 13. An oriented rooted polygon P is said to be *unimodal in direction* k if its word $u = u(P)$ can be written as vw (i.e. $u = vw$), where v and w are words on $\{1, ..., d\} \cup \{\overline{1}, ..., \overline{d}\}$, with $|v|_{\overline{k}} = |w|_k = 0$. Moreover, P is *unimodal* if P is unimodal in all directions.

Definition 14. An oriented polygon is *unimodal* if there exists a cyclic permutation of its points such that the oriented rooted polygon associated with this permutation is unimodal. Observe that such a permutation is unique if it exists.

We point out that the class of unimodal self-avoiding polygons with $d = 2$ is identical to the class of directed convex polyominoes.

Definition 15. An oriented polygon is *unimodal in direction* k if there exists a cyclic permutation of its points such that the oriented rooted polygon associated with this permutation is unimodal in direction k. It is *convex* if it is unimodal in all directions.

We point out that the class of convex self-avoiding polygons with $d = 2$ is identical to the class of convex polyominoes.

Observe that, if an oriented polygon P is unimodal in direction k, it may have several different representations as an oriented rooted polygon unimodal in direction k. We will refer to each of these representations as a *k-unimodal representation* of P. On the other hand, for unimodal polygons there is a unique rooted polygon which is unimodal in all directions. For instance, Figure 4 (b) shows a 2-dimensional oriented rooted polygon P, unimodal in direction 1, with root in $(1, 3)$ and word

$$u(P) = 1\,1\,2\,1\,1\,\overline{2}\,1\,2\,2\,\overline{1}\,\overline{1}\,\overline{2}\,\overline{2}\,\overline{2}\,\overline{1}\,\overline{2}\,\overline{1}\,2\,\overline{1}\,2\,.$$

We observe that P is unimodal in direction 1 but not in direction 2. We can obtain another 1-unimodal representation for the same oriented polygon, by placing the root at $(1, 2)$ instead of $(1, 3)$. Figure 5(a) shows the unique rooted unimodal representation P of a 2-dimensional oriented unimodal polygon, where

$$u(P) = 2\,2\,1\,1\,2\,2\,1\,2\,2\,2\,1\,\overline{2}\,1\,\overline{2}\,1\,1\,\overline{1}\,\overline{1}\,\overline{2}\,\overline{1}\,\overline{2}\,\overline{1}\,\overline{1}\,\overline{1}\,\overline{2}\,\overline{2}\,\overline{2}\,\overline{1},$$

and the root is $(1, 1)$; it is unimodal in direction 1, as it can be encoded vw, with

$$v = 2\,2\,1\,1\,2\,2\,1\,2\,2\,2\,1\,\overline{2}\,1\,\overline{2}\,1\,1 \qquad \text{and} \qquad w = \overline{1}\,\overline{1}\,\overline{2}\,\overline{1}\,\overline{2}\,\overline{1}\,\overline{1}\,\overline{1}\,\overline{2}\,\overline{2}\,\overline{2}\,\overline{1},$$

and in direction 2 as shown by encoding

$$v = 2\,2\,1\,1\,2\,2\,1\,2\,2\,2\,1 \qquad \text{and} \qquad w = \overline{2}\,1\,\overline{2}\,1\,1\,\overline{1}\,\overline{1}\,\overline{2}\,\overline{1}\,\overline{2}\,\overline{1}\,\overline{1}\,\overline{1}\,\overline{2}\,\overline{2}\,\overline{2}\,\overline{1}.$$

Figure 5(b) shows an oriented convex polygon P. A 1-unimodal representation of P (with root in $(1,2)$) is

$$2\,1\,1\,2\,1\,1\,2\,2\,2\,2\,1\,\overline{2}\,1\,\overline{2}\,1\,\overline{1}\,\overline{1}\,\overline{2}\,\overline{1}\,\overline{1}\,\overline{2}\,\overline{2}\,\overline{2}\,\overline{1}\,\overline{2}\,\overline{1}\,2\,\overline{1};$$

which is not unimodal in direction 2. A 2-unimodal (non 1-unimodal) representation of P can be obtained by placing the root in $(2,1)$.

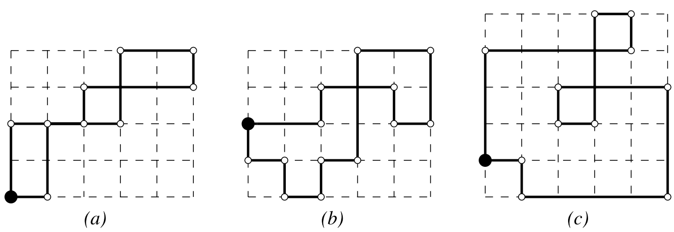

Figure 4: 2-dimensional polygons: (a) an oriented rooted parallelogram polygon; (b) an oriented rooted polygon unimodal in direction 1; (c) a permupolygon.

Now we are ready to give the definition of the main object of interest in the paper, i.e. the *permupolygon*.

Definition 16. An oriented rooted polygon P of dimension d and size n is an *oriented rooted permupolygon* if, in each hyperplane $h_{i,j} = \{x_i = j\}$, i.e. orthogonal to e_i and with coordinate j, with $i = 1, 2, \dots d$ and $j = 1 \dots n{+}1$, there are exactly $d-1$ sides of P in $h_{i,j}$, precisely one side for each direction e_k with $k \neq i$.

Remark that, by definition, in an oriented permupolygon of dimension d and size n, each hyperplane $h_{i,j}$ contains exactly d vertices. Therefore the total number of vertices is $d(n+1)$. Indeed we can choose a direction e_i, and consider all the hyperplanes $h_{i,j}$, $j = 1, \dots, n+1$, orthogonal e_i. Each vertex of P belongs to one of these $n+1$ hyperplanes, and each hyperlane contains exactly d vertices.

Given an oriented permupolygon P we choose a rooted representation of P, that we call *canonical*, by assuming that the *root* $r(P)$ of P is the minimal among the vertices of P, ordered according to the usual lexicographic order.

Figure 4 (c) shows a 2-dimensional oriented permupolygon P of size 5 (with root in $(1,2)$) and word

$$u(P) = 2\,2\,2\,1\,1\,1\,1\,2\,\overline{1}\overline{2}\,\overline{2}\,\overline{2}\,\overline{1}\,2\,1\,1\,1\,\overline{2}\,\overline{2}\,\overline{2}\,\overline{1}\,\overline{1}\,\overline{1}\,\overline{1}\,2\,\overline{1}\,.$$

An *oriented parallelogram* (resp. *unimodal*, *convex in direction* k, *convex*) *permupolygon* is an oriented parallelogram (resp. unimodal, convex in direction k, convex) polygon which is also an oriented permupolygon.

Given an oriented parallelogram (resp. unimodal) permupolygon P, the rooted parallelogram (resp. unimodal) representation of P obtained by assuming $r(P)$ as defined above is exactly the unique rooted representation of P considered as oriented polygon.

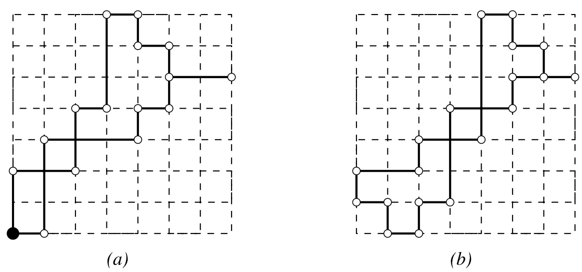

Figure 5: (a) An oriented rooted unimodal polygon; (b) an oriented convex polygon.

Proposition 1. Let $u = u(P)$ be the word associated with the canonical representation of an oriented permupolygon P of size n, and let $v_1, v_2, ..., v_{dn+d}$ be the sequence of vertices of P. Moreover, for $j = 1, \ldots, dn + d$, let $w_j = k$ if $u_j = k$ or $u_j = \bar{k}$ (equivalently w_j indicates the direction of the unit step starting from v_j, regardless of the orientation of this step). Then

i) initial part: $w_s \neq w_t$ for $s, t = 1, ..., d$, and $s \neq t$;
ii) periodicity: $w_{kd+j} = w_j$ for $j = 1, ..., d$ and $k = 1, ..., n$.

Proof. **i)** Let us consider the hyperplane H containing the first d vertices of P, i.e. containing v_j, with $j = 1, \ldots, d$. Then, by definition of permupolygon, it contains exactly $d - 1$ sides of P, one for each direction e_i that is not orthogonal to H. Then, the entries w_j of u, corresponding to the unit steps starting from v_j, for $j = 1, \ldots, d$, must all be different since they belong to d different sides.

ii) $w_{kd+j} = w_j$ for $j = 1, ..., d$ and $k = 1, ..., n$. Let us suppose that this is not true, then we take the leftmost value $w_{k'd+j'}$, with $k' \neq 1$, such that $w_{k'd+j'} \neq w_{j'}$. Let us say $w_{k'd+j'} = e_\ell$, we look for the first $k''d + j'' < k'd + j'$ (there is at least one since $k' \neq 1$) such that $w_{k''d+j''} = e_\ell$. Observe that the length of the subword $w_{k''d+j''+1} \dots w_{k'd+j'-1}$ is less than d. Indeed, if it was not the case, there would be at least d elements different from e_ℓ in the subword $w_{k''d+j''+1} \dots w_{k'd+j'-1}$. This means that there would be two identical values different from e_ℓ in the subword $w_{k''d+j''+1} \dots w_{k'd+j'-1}$, that is e_ℓ would not be the leftmost unit step such that $w_{k'd+j'} \neq w_{j'}$. Since the length of $w_{k''d+j''+1} \dots w_{k'd+j'-1}$ is less than d, the hyperplanes containing $v_{k''d+j''+1} \dots v_{k'd+j'-1}$ and orthogonal to one of the unit steps that is not in $w_{k''d+j''+1} \dots w_{k'd+j'-1}$ would not respect the definition of permupolygon. Therefore we have a contradiction. □

We will refer to the initial part $w_1 w_2 ... w_d$ of the sequence introduced in Proposition 1 as the *trigger* of the oriented permupolygon P.

Given a word u which is the canonical representation $u = u(P)$ of an oriented permupolygon P, observe that the word $\overline{u}^r$, obtained by taking the reverse of u and exchanging k with $\overline{k}$ for $1 \leq k \leq d$, is the canonical representation of the oriented permupolygon obtained from P by changing the orientation. From a geometric point of view it is natural to consider that these two words correspond to a same object, which we call an *unoriented permupolygon*.

Remark 2.1. By the previous considerations, it follows that, for each sequence $t_1 t_2 ... t_d$ such that $t_i \neq t_j$ (with $i \neq j$ and $i, j \in \{1, ..., d\}$), the class of oriented permupolygons with trigger $t_1 t_2 ... t_d$ and the class of oriented permupolygons with trigger $t_d t_{d-1} ... t_1$ coincide, if thought as unoriented permupolygons. We choose to represent unoriented permupolygons by means of the oriented permupolygons with trigger $t_1 t_2 ... t_d$, with $t_1 > t_d$.

For instance, the 2-dimensional permupolygon shown in Figure 4 (c) will be represented as a 2-dimensional oriented permupolygon with trigger 2 1.

2.2. Multi-dimensional permutations

In this section we introduce a notion of multidimensional permutations, that we will be useful to describe properties of permupolygons. A (*one-dimensional*) *permutation* $\pi = (\pi(1), \dots, \pi(n))$ of length n is a bijective function from the set $[n]$ to itself. A permutation π can be viewed as a 2-row array made of the two sequences $(1, \dots, n)$ and $(\pi(1), \dots, \pi(n))$.

A *d-dimensional permutation* (briefly, *d-permutation*) is a sequence of d unidimensional permutations $\pi_1, \ldots, \pi_d$ which can be viewed as a $(d+1)$-row array made of the identity permutation and the permutations $\pi_1, \ldots, \pi_d$. A d-dimensional permutation π is denoted by $\pi = (id, \pi_1, \ldots, \pi_d)$, where the identity permutation can be omitted. Clearly, the number of d-dimensional permutations of length n is equal to $(n!)^d$. A d-dimensional permutation of length n is naturally interpreted as a collection of n points with integer coordinates in $\mathbb{Z}^{d+1}$. Geometrically, in each hyperplane $h_{i,j}$, with $i = 1, \ldots, d$ and $j = 1, \ldots, n+1$ there is exactly one point of the permutation. Figure 6 shows a 2-dimensional permutation $\pi = ((1,2,3,4),(2,3,1,4),(4,1,3,2))$ of length 4.

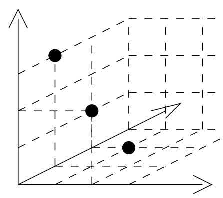

Figure 6: A 2-permutation of length 4.

3. Permupolygons on the square lattice

In this section we consider 2-dimensional unoriented permupolygons, and study their combinatorial properties, explaining their relations with the classes of permutominoes, and permutominides.

Referring to Remark 2.1, a 2-dimensional unoriented permupolygon is represented by means of a 2-dimensional oriented permupolygon with trigger 2 1, simply called permupolygon in this section.

3.1. Permupolygons and path encoding

Now we show the relation between these objects and permutations, which explains the name permupolygon.

Proposition 2. The class of 2-dimensional permupolygons is identical to the class of permutominides. Moreover, the class of 2-dimensional self avoiding permupolygons is equivalent to the class of permutominoes.

Proof. The first assertion follows from Definitions 7 and 16 and the second from Definitions 4 and 16. □

As we already saw in the Introduction, permutominoes and permutominides can be represented by a pair of permutations. Here we recall the definition of such a pair of permutations, given for 2-dimensional permupolygons.

Definition 17. Let P be a 2-dimensional permupolygon of size n. The *first* and the *second components* of P are two permutations of length $n+1$, denoted $\pi_1(P)$ and $\pi_2(P)$, respectively, defined by choosing alternatively the vertices of P, with the convention that the root $r(P)$ is an entry of $\pi_1(P)$. Besides, if the vertex with coordinate (i, j) is a point of $\pi_1(P)$ (resp. $\pi_2(P)$) it means that $\pi_1(i) = j$ (resp. $\pi_2(i) = j$).

Remark that $\pi_1(1) < \pi_2(1)$ and $\pi_1(i) \neq \pi_2(i)$ for all i, and that $\pi_1^{-1} \circ \pi_2$ has a unique cycle.

Now we give a unique representation of a permupolygon P in terms of a *path word* encoding the word $u(P)$.

Definition 18. Let P be a permupolygon of size n, we define the *path encoding* $s(P)$ of P (briefly encoding of P) as the word of the form $(Y\,X)^{n+1} = \underbrace{Y\,X \dots Y\,X}_{n+1}$, where each X and Y is endowed of an index according to the following rules:

i. the word starts with $Y_{\pi_1(1)}\,X_1$;
ii. the letter following Y_j is $X_{\pi_1^{-1}(j)}$;
iii. the letter following X_i (if there is one) is $Y_{\pi_2(i)}$.

For example, the path encoding of the permupolygon in Figure 4 (c) is $Y_2\,X_1\,Y_5\,X_5\,Y_6\,X_4\,Y_3\,X_3\,Y_4\,X_6\,Y_1\,X_2$.

Remark 3.1. The definition of permupolygon implies that for every index i, X_i and Y_i appear precisely once in $s(P)$.

Proposition 3. A permupolygon P of size n is uniquely determined by its path encoding $s(P)$.

Proof. This can simply be proved by showing that, using the path encoding $s(P)$, we can uniquely build a permupolygon P', and P' coincides with P. First we notice that the root $r(P)$ can be easily be recovered, since its coordinates are given by the indices of the first occurrence of Y and X. Then, starting from the root $r(P)$, we build a polygon P' as follows: if in the reconstruction of P' we are considering the point (i, j) and in $s(P)$ we read Y_s (resp. X_t) then we have to connect point (i, j) to point (i, s) (resp. (t, j)) using unit steps along the direction of e_2 (resp. of e_1). It is then clear that at the end of the reconstruction process we obtain a permupolygon P', and that P' coincides with P, by construction. □

The following statement clearly explains that permutominoes and permutominides can be viewed as special cases of permupolygons.

Proposition 4. The class of vertically-convex (resp. horizontally-convex, convex, directed-convex, parallelogram) permutominides is identical to the class of permupolygons that are convex in direction 1 (resp. convex in direction 2, convex, unimodal, parallelogram).

Proof. Assume the P is a vertically convex permutominide with $2k$ horizontal steps. By definition P has at most two horizontal edges per column, hence, starting from a leftmost point, the sequence of horizontal steps on its boundary consists of k horizontal steps to the right (in the direction e_1), followed by k horizontal steps to left (in the direction $-e_1$): the boundary path is thus a 2d-permupolygon that is unimodal in direction 1. Conversely given a 2d-permupolygon that is unimodal in direction 1 the horizontal steps visit abscissa 1 to $k+1$ and then $k+1$ to 1: there are at most two steps in each column, hence each column is connected.

The other cases (convex, unimodal, parallelogram permutominide) are proven in a similar way. □

We are going to enumerate the subclasses of parallelogram, directed-convex, convex and vertically-convex (convex in direction 1) permupolygons by using the encoding of a permupolygon. This leads us to recover, in a simpler way, Equations (3), (4) at page 61, respectively for convex and vertically-convex permutominides, as well as those for parallelogram and directed-convex permutominides [9].

3.2. Parallelogram permupolygons on the square lattice

Proposition 5. Let P be a parallelogram permupolygon of size n; its encoding $s(P)$ is a word $(Y\,X)^{n+1}$, such that:

1) each X and Y is endowed of an index such that for all $i \in \{1, \cdots, n+1\}$ X_i (resp. Y_i) appears exactly once;
2) it starts with $Y_1\,X_1$;
3) the indices of its subsequence of Y_i's form a unimodal sequence (Property U_Y);
4) the indices of its subsequence of X_i's form a unimodal sequence (Property U_X);
5) it contains the factor $Y_{n+1}\,X_{n+1}$ (case y)) or $X_{n+1}\,Y_{n+1}$ (case x)).

Proof. Properties **1)** − **5)** follow from the definition of parallelogram permupolygon and from the fact that the root of a parallelogram permupolygon is always in $(1,1)$. □

Figure 7 (y) shows a parallelogram permupolygon P of case y) with $\pi_1(P) = (1,2,4,5,3,7,6,8,9)$ and $\pi_2(P) = (3,1,2,4,6,5,9,7,8)$ whose encoding is

$$s(P) = Y_1\, X_1\, Y_3\, X_5\, Y_6\, X_7\, Y_9\, X_9\, Y_8\, X_8\, Y_7\, X_6\, Y_5\, X_4\, Y_4\, X_3\, Y_2\, X_2.$$

Similarly Figure 7 (x) shows a parallelogram permupolygon P of case x) with $\pi_1(P) = (1,2,3,4,6,5,8,9,7)$ and $\pi_2(P) = (4,1,2,5,3,7,6,8,9)$ whose encoding is

$$s(P) = Y_1\, X_1\, Y_4\, X_4\, Y_5\, X_6\, Y_7\, X_9\, Y_9\, X_8\, Y_8\, X_7\, Y_6\, X_5\, Y_3\, X_3\, Y_2\, X_2.$$

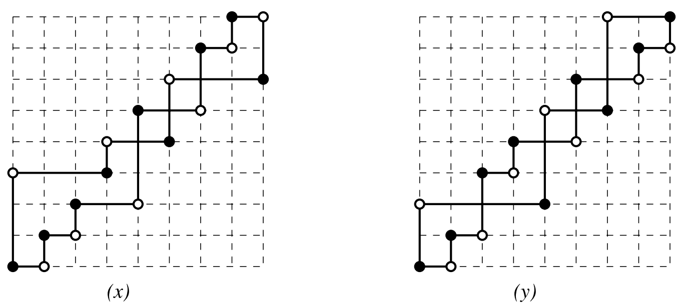

Figure 7: (x) A parallelogram permupolygon P of size 8, case x). (y) A parallelogram permupolygon P of size 8, case y).

Definition 19. Let P be a parallelogram permupolygon of size n. The *upper path encoding* $s_u(P)$ of P is the subsequence of $s(P)$ starting from Y_1 and ending in X_{n+1} (resp. Y_{n+1}) in case y) (resp. x)).

Proposition 6. Given a parallelogram permupolygon P of size n, we have that:

i) P is determined by its upper path encoding;
ii) referring to case y), $s_u(P)$ contains the same number of elements X_i and Y_j, as shown in Fig. 7 (y);
iii) referring to case x), in $s_u(P)$ the number of Y_j is equal to the number of X_i plus one, as shown in Fig. 7 (x).

Proof. **i)** By definition of permupolygon P there are exactly two vertices in each hyperplane $h_{i,j} = \{x_i = j\}$, $\{i = 1, 2\}$, $\{j = 1, \ldots, n+1\}$. Since the lower path encoding of P is made up only by south and west steps, then there is only one way to reconstruct the entire permupolygon: starting from the last vertex of $s_u(P)$ we go west (resp. south) if we are in the case $x)$ (resp. $y)$), until we cross the hyperplane $x_1 = j$ (resp. $x_1 = j$) which does not contain exactly two vertices of P. Then we go south (resp. west) until we cross the hyperplane $x_1 = j$ (resp. $x_2 = j$) which does not contain exactly two vertices of P, and so on.

ii) In this case $s_u(P)$ starts with Y_1 and ends X_{n+1}. Since Y_j and X_i alternate, then we have the same number.

iii) In this case $s_u(P)$ starts with Y_1 and ends Y_{n+1}. Since Y_j and X_i alternate then the number of Y_j is equal to the number of X_i plus one. □

Proposition 7. Let $\mathcal{P}_n$ be the class of parallelogram permupolygons of size n. For any $n \geq 1$, we have

$$|\mathcal{P}_n| = \frac{1}{2}\binom{2n}{n}.$$

Proof. To prove this statement we use Propositions 5 and 6. We are going to count the number of upper path encodings:

- referring to case $y)$: We observe that the set of upper encodings of parallelogram permupolygons of type $y)$ is exactly the set of alternating words starting with X_1Y_1 and ending with X_{n+1} and such that X_i and Y_j appear at most once for all $2 \leq i, j \leq n$.
  Hence to construct a parallelogram permupolygon of type $y)$ we choose h elements in the set $\{X_2, X_3, ..., X_n\}$ and h elements in the set $\{Y_2, Y_3, ..., Y_n\}$, so we have

$$\sum_{h=0}^{n-1} \binom{n-1}{h}\binom{n-1}{h}$$

  parallelogram permupolygons of type $y)$.
- referring to case $x)$: As above we observe that the set of upper encodings of parallelogram permupolygons of type $x)$ is exactly the set of alternating words starting with X_1Y_1 and ending with Y_{n+1} and such that X_i and Y_j appear at most once for all $2 \leq i, j \leq n$.
  Hence to construct a parallelogram permupolygon of type $x)$ we choose $h-1$ elements in the set $\{X_2, X_3, ..., X_n\}$ and h elements in the set

$\{Y_2, Y_3, ..., Y_n\}$ with $h \geq 1$, so we have

$$\sum_{h=1}^{n-1} \binom{n-1}{h-1}\binom{n-1}{h} = \sum_{h=0}^{n-1} \binom{n-1}{h-1}\binom{n-1}{h},$$

parallelogram permupolygons of type x).

It follows that:

$$\begin{aligned} |\mathcal{P}_n| &= \sum_{h=0}^{n-1} \left(\binom{n-1}{h} + \binom{n-1}{h-1}\right)\binom{n-1}{h} \\ &= \sum_{h=0}^{n-1} \binom{n}{h}\binom{n-1}{h} = \frac{1}{2}\binom{2n}{n}. \end{aligned}$$

□

3.3. Directed-convex permupolygons on the square lattice

In this paragraph we deal with directed-convex permupolygons, giving a characterization of their encoding $s(P)$, which leads to their enumeration.

Proposition 8. The encoding $s(P)$ of a directed-convex permupolygon P of size n is uniquely represented by a word $(Y\,X)^{n+1}$, such that:

1) each X and Y is endowed of an index such that for all $i \in \{1, \cdots, n+1\}$ X_i (resp. Y_i) appears exactly once,
2) $s(P)$ starts with $Y_1\,X_1$;
3) $s(P)$ satisfies Properties U_Y and U_X (in Proposition 5).

Proof. It follows from the definition of directed-convex permupolygon and from the fact that the root of a directed-convex permupolygon is placed in $(1, 1)$. □

Figure 8 (a) shows a directed-convex permupolygon P with $\pi_1(P) = (1, 2, 3, 4, 5, 7, 9, 8, 6)$ and $\pi_2(P) = (3, 1, 6, 2, 4, 5, 7, 9, 8)$ whose encoding is

$$s(P) = Y_1\,X_1\,Y_3\,X_3\,Y_6\,X_9\,Y_8\,X_8\,Y_9\,X_7\,Y_7\,X_6\,Y_5\,X_5\,Y_4\,X_4\,Y_2\,X_2.$$

Proposition 9. Let $\mathcal{D}_n$ be the class of directed-convex permupolygons of size n. For any $n \geq 1$, we have:

$$|\mathcal{D}_n| = 4^{n-1}.$$

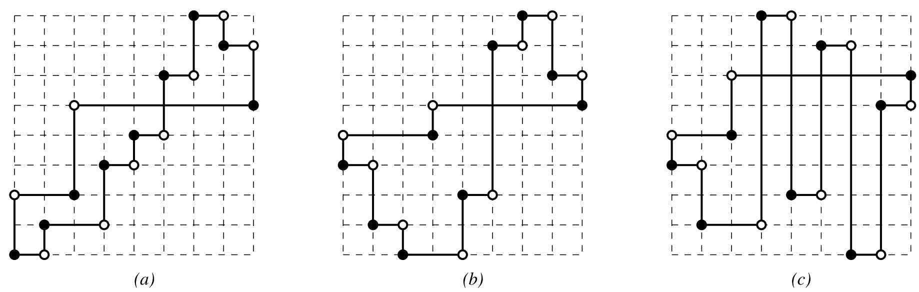

Figure 8: (a) A directed-convex permupolygon of size 8. (b) A convex permupolygon of size 8. (c) A vertically-convex permupolygon of size 8.

Proof. From Proposition 8 it follows that a directed-convex permupolygon is determined by two unimodal sequences that are completely independent: a sequence of X_i's and a sequence of Y_j's. Each unimodal sequence such that each element appears exactly one time, is uniquely determined by the choice of the elements put in increasing order. To obtain the number of sequences of X_i's, we consider each possible way to choose h elements in the set $\{X_2, X_3, ..., X_n\}$, which will be the elements put in increasing order in the sequence, so we obtain:

$$\sum_{h=0}^{n-1} \binom{n-1}{h} = 2^{n-1}.$$

The number of sequence of Y_j's is obviously the same as the number of sequences of X_i's. It follows that:

$$|\mathcal{D}_n| = \sum_{h=0}^{n-1} \binom{n-1}{h} \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1} \cdot 2^{n-1} = 4^{n-1}. \qquad \square$$

3.4. Convex permupolygons on the square lattice

In this paragraph we deal with the problem of enumerating convex permupolygons according to their size.

Proposition 10. The encoding $s(P)$ of a convex permupolygon P of size n is uniquely represented by a word $(Y\,X)^{n+1}$, such that

1) each symbol is endowed of an index such that for all $i \in \{1, \cdots, n+1\}$ X_i (resp. Y_i) appears exactly one time,

2) $s(P)$ satisfies Property U_X,
3) $s(P)$ starts with $Y_k\, X_1\, Y_j$ with $k < j$;
4) if we consider the cyclic shift $s_y(P)$ of $s(P)$ starting from Y_1, the indices of sequence (in $s_y(P)$) of Y_i's form an unimodal sequence (Property C_Y).

Proof. It follows from the definition of convex permupolygon and from the fact that the root of a convex permupolygon always has abscissa equal to 1. □

Figure 8 (b) shows a convex permupolygon P with $\pi_1(P) = (4, 2, 1, 5, 3, 8, 9, 7, 6)$ and $\pi_2(P) = (5, 4, 2, 6, 1, 3, 8, 9, 7)$ whose encoding is

$$s(P) = Y_4\, X_1\, Y_5\, X_4\, Y_6\, X_9\, Y_7\, X_8\, Y_9\, X_7\, Y_8\, X_6\, Y_3\, X_5\, Y_1\, X_3\, Y_2\, X_2.$$

Proposition 11. Let $\mathcal{C}_n$ be the class of convex permupolygons of size n. For any $n \geq 1$, we have

$$|\mathcal{C}_n| = 2(n+1)4^{n-2}\,.$$

Proof. From Proposition 10 it follows that a convex permupolygon is determined by two sequences that are completely independent: a sequence of X_i's which is unimodal, and a sequence of Y_j's which is a cyclic shift of a unimodal sequence. As usual the number of possible sequences of X_i's is 2^{n-1}. Concerning the sequence of Y_j's, as usual, to find the number of unimodal sequences it suffices to choose k elements in the set $\{Y_2, Y_3, ..., Y_n\}$ to arrange in increasing order; now, we need to consider all of cyclic shifts (satisfying Property **3** in Proposition 10) of these unimodal sequences. Clearly, the only allowed shifts are those starting with one of the k chosen elements or Y_1. So we have:

$$\sum_{k=0}^{n-1}(k+1)\binom{n-1}{k} = \sum_{k=0}^{n-1}k\binom{n-1}{k} + \sum_{k=0}^{n-1}\binom{n-1}{k} =$$

$$= \sum_{k=1}^{n-1}(n-1)\binom{n-2}{k-1} + 2^{n-1} = (n-1)2^{n-2} + 2^{n-1} = 2^{n-2}(n+1).$$

It follows that:

$$|\mathcal{C}_n| = 2^{n-1}2^{n-2}(n+1) = 2(n+1)4^{n-2}\,.$$ □

3.5. Vertically convex permupolygons

In this paragraph we consider the class of vertically convex permupolygons (i.e. permupolygons that are convex in direction 1), and we obtain their enumeration according their size. Trivially, the class of horizontally convex permupolygons (i.e. permupolygons that are convex in direction 2) is in bijection with the class of vertically convex permupolygons.

Proposition 12. The encoding $s(P)$ of a vertically convex permupolygon P of size n is uniquely represented by a word $(Y\,X)^{n+1}$, where each symbol is endowed of an index such that for all $i \in \{1, \cdots, n+1\}$ X_i (resp. Y_i) appears exactly one time, which has Properties U_X and the additional Property:

2) it starts with $Y_k\,X_1\,Y_j$ with $k < j$.

Proof. It follows from the definition of vertically convex permupolygon and from the fact that the root of a permupolygon has always abscissa equal to 1. Furthermore, since the coordinates of the root are given by the indices of the first Y and X in $s(P)$, the index of the second Y cannot be less than the ordinate of the root, otherwise we could find a vertex of the permupolygon different from the root which is less than the root according to the usual lexicographic order. □

Figure 8 (c) shows a vertically convex permupolygon P with $\pi_1(P) = (4, 2, 5, 9, 3, 8, 1, 6, 7)$ and $\pi_2(P) = (5, 4, 7, 2, 9, 3, 8, 1, 6)$ whose encoding is

$$s(P) = Y_4\,X_1\,Y_5\,X_3\,Y_7\,X_9\,Y_6\,X_8\,Y_1\,X_7\,Y_8\,X_6\,Y_3\,X_5\,Y_9\,X_4\,Y_2\,X_2.$$

Proposition 13. Let $\mathcal{V}_n$ be the class of vertically convex permupolygons of size n. For any $n \geq 1$, we have

$$|\mathcal{V}_n| = 2^{n-2}\,(n+1)!\,.$$

Proof. From Proposition 12 it follows that a vertically convex permupolygon is determined by two sequences, completely independent: a unimodal sequence of X_i's and a sequence of Y_j's which is a permutation, where the second entry is greater than the first entry. Then, the number of sequences of X_i's is 2^{n-1}. On the other side, the sequences of Y_j's are given by permutations of $\{Y_1, Y_2, ..., Y_{n+1}\}$, where the second entry is greater than the first entry, so precisely $\frac{(n+1)!}{2}$. It follows that:

$$|\mathcal{V}_n| = 2^{n-1}\,\frac{(n+1)!}{2} = 2^{n-2}\,(n+1)!\,. \qquad \square$$

4. Permupolygons on the cubic lattice

In this section we consider 3-dimensional permupolygons, thus providing an extension of the notion of permutomino to the 3-dimensional case. Since in this section we will work only with 3-dimensional permupolygons, the word 3-dimensional will be omitted.

The following statement shows that – differently from what happens in the 2-dimensional case – a 3-dimensional permupolygon is always a self avoiding polygon: then permupolygons are an appropriate extension of the notion of permutomino.

Proposition 14. A permupolygon is a self avoiding polygon.

Proof. If two sides of a permupolygon P intersect then we can find four vertices of P belonging to the same plane, precisely a plane orthogonal to e_i for some $i \in 1, 2, 3$. But this is not possible for a permupolygon, which has exactly three vertices in each plane orthogonal to e_j for each $j = 1, 2, 3$. □

Remark 4.1. We would like to emphasize one more time that Proposition 14 does not work in the 2-dimensional case, as shown in Fig. 3. This is the reason why in the literature there is an explicit distinction between permutominoes and permutominides.

Here, the trigger of an oriented rooted permupolygon is a permutation of length 3, which will be usually represented by means of a permutation of the symbols X, Y, Z, where the symbol X (resp. Y, Z) stands for 1 (resp. 2, 3).

As in the 2-dimensional case (see Remark 2.1), we need to remark that the class of oriented permupolygons with trigger $Z\,X\,Y$ (resp. $Z\,Y\,X$, $Y\,Z\,X$) and the class of oriented permupolygons with trigger $Y\,X\,Z$ (resp. $X\,Y\,Z$, $X\,Z\,Y$) coincide, if thought as unoriented objects.

It follows that the class of permupolygons of size n is given by the union of the classes of permupolygons with triggers $Z\,X\,Y$, $Z\,Y\,X$, and $Y\,Z\,X$.

We can associate to 3-dimensional permupolygons a triple of 2-permutations.

Definition 20. Let P be a permupolygon with trigger $t_1\,t_2\,t_3$ of size n. We consider the sequences of vertices $v_1(P)$, $v_2(P)$, $v_3(P)$ defined by choosing alternatively the vertices of P, with the convention that the root $r(P)$ is an entry of $v_1(P)$, the next two vertices following the root along the permupolygon P are respectively an entry of $v_2(P)$ and $v_3(P)$, and so on. Then we define three 2-permutations of length $n+1$, denoted by $\pi_1(P) = (\pi_{11}, \pi_{12})$, $\pi_2(P) = (\pi_{21}, \pi_{22})$ and $\pi_3(P) = (\pi_{31}, \pi_{32})$, as follows: $\pi_{11}(i) = j$ and

$\pi_{12}(i) = k$ (resp. $\pi_{21}(i) = j$, $\pi_{22}(i) = k$ and $\pi_{31}(i) = j$, $\pi_{32}(i) = k$) if and only if the vertex (i, j, k) belongs to the sequence $v_1(P)$ (resp. $v_2(P)$ and $v_3(P)$).

Remark 4.2. We would like to point out that $\pi_{\ell,m}$, $\ell \in \{1, 2, 3\}$ and $m \in \{1, 2\}$ are indeed permutations and, as a consequence, $\pi_\ell(P)$, $\ell \in \{1, 2, 3\}$, are 2-permutations. Indeed if we take two vertices (i, j, k) and (i', j', k') of a sequence $v_\ell(P)$, then we have that $i \neq i'$, $j \neq j'$, and $k \neq k'$ and consequently $\pi_{\ell,m}$, $m \in \{1, 2\}$ are permutations. If it was not true, let us say for example $j = j'$, then there would be an hyperplane orthogonal to e_j containing two vertices of the same permutation. This would contradict the definition of permupolygon.

Remark 4.3. Permutations $\pi_1(P)$, $\pi_2(P)$, $\pi_3(P)$ uniquely determine a permupolygon with trigger $t_1\, t_2\, t_3$. Indeed from permutations $\pi_1(P)$, $\pi_2(P)$, $\pi_3(P)$ we can easily obtain the sequence of vertices $v_1(P)$ $v_2(P)$, $v_3(P)$ of permupolygons. By convention the first vertex of $v_1(P)$, that is $v_1(1) = (1, \pi_{1,1}(1), \pi_{2,1}(1))$, is the root of the permupolygon. By knowing the trigger $t_1\, t_2\, t_3$ we can find all the other vertices. Indeed, in the permupolygon P, each element of the sequence v_1 (resp. v_2, v_3) must follow one of the sequence v_3 (resp. v_1, v_2) along the direction e_{t_3} (resp. e_{t_1}, e_{t_2}), this uniquely determines v_1, v_2 and v_3.

4.1. Enumeration of permupolygons with different triggers

Here we show that the choice of the minimal vertex in the lexicographic order as root of a permupolygon leads to an asymmetry in the enumeration of permupolygons with different triggers. By definition, for every permupolygon P there are exactly three vertices of P belonging to plane $x = 1$ and one of these vertices is the root of P. Let $A = (1, y_1, z_1)$, $B = (1, y_2, z_1)$ and $C = (1, y_2, z_2)$ be these three vertices lying on $x = 1$. Then, four situations may occur, depending on the relations between y_1, y_2 and z_1, z_2. Figure 9 shows these four cases, where the root is represented by a white dot.

More precisely:

1. if $y_1 < y_2$ and $z_1 < z_2$ (see Fig. 9 (a)), then A is the root, and the permupolygon has trigger $Y\, Z\, X$;
2. if $y_2 < y_1$ and $z_1 < z_2$ (see Fig. 9 (b)), then B is the root, and the permupolygon has trigger $Z\, X\, Y$;
3. if $y_2 < y_1$ and $z_2 < z_1$ (see Fig. 9 (c)), then C is the root, and the permupolygon has trigger $Z\, Y\, X$;

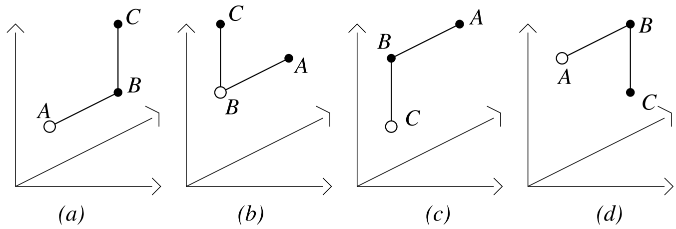

Figure 9: The four possible positions of the three vertices belonging to plane $x = 1$.

4. if $y_1 < y_2$ and $z_2 < z_1$ (see Fig. 9 (d)), then A is the root, and the permupolygon has trigger $Y\,Z\,X$.

Remark 4.4. By simple symmetry arguments it follows that the number of permupolygons with trigger $Z\,X\,Y$ is equal to the number of permupolygons with trigger $Z\,Y\,X$, while the number of permupolygons with trigger $Y\,Z\,X$ is twice the number of those with trigger $Z\,X\,Y$ (or $Z\,Y\,X$). Let us point out that the root of a directed-convex (parallelogram) permupolygon is placed at $(1, 1, 1)$, then case (d) in Fig. 9 cannot arise. This is why enumeration of directed-convex (resp. parallelogram) permupolygons does not depend on the choice of the trigger.

In the next sections we are going to provide enumeration of parallelogram (resp. directed-convex, convex) permupolygons with trigger $Z\,X\,Y$ and show that the number of convex permupolygons with trigger $Y\,Z\,X$ is effectively twice the number of convex permupolygons with trigger $Z\,X\,Y$. Finally we obtain the enumeration of the whole class of parallelogram (resp. directed-convex, convex) permupolygons.

4.2. Permupolygons with trigger $Z\,X\,Y$

Similarly to what we have done in the 2-dimensional case, we provide a *path encoding* $s(P)$ of a 3-dimensional permupolygon, which will be useful for enumeration.

Remark 4.5. We would like to point out that, as for the 2-dimensional case, $s(P)$ encodes the vertices of P, i.e. the 2-dimensional permutations $\pi_1(P)$, $\pi_2(P)$, and $\pi_3(P)$. In particular, a vertex $V = (i, j, k)$ in $\pi_2(P)$ (resp. $\pi_3(P)$, $\pi_1(P)$) corresponds to the subword $Z_k X_\ell$ (resp. $X_i Y_\ell$, $Y_j Z_\ell$) in $s(P)$. Observe that, in the case $Z_k X_\ell$, (resp. $X_i Y_\ell$, $Y_j Z_\ell$) ℓ is the first (resp. second, third) coordinate of the vertex $V' = (\ell, j, k)$ (resp. $V' = (i, \ell, k)$, $V' = (i, j, \ell)$) belonging to $\pi_3(P)$ (resp. $\pi_1(P)$, $\pi_2(P)$) and following $V = (i, j, k)$ in P.

Then we have the following definition.

Definition 21. The path encoding (briefly encoding) $s(P)$ of a permupolygon P with trigger $Z\,X\,Y$ and size n, is the word $(Z\,X\,Y)^{n+1}$, where each symbol is endowed of an index according to the following rules:

i. $s(P)$ starts with $Z_{\pi_{12}(1)}\,X_1\,Y_{\pi_{11}(1)}$;
ii. the letter following Z_k is $X_{\pi_{32}^{-1}(k)}$;
iii. the letter following X_i is $Y_{\pi_{11}(i)}$;
iv. the letter following Y_j (if there is one) is $Z_{\pi_{22}(\pi_{11}^{-1}(j))}$.

Remark 4.6. The definition of permupolygon implies that for every index i, X_i, Y_i, and Z_i appear precisely once in $s(P)$.

Proposition 15. A permupolygon P with trigger $Z\,X\,Y$ of size n is uniquely determined by its path encoding $s(P)$.

Proof. The proof is similar to that for the 2-dimensional case : given a path encoding $s(P)$, we can uniquely build a permupolygon P', and P' coincides with P. The root $r(P')$ is given by $(1, \pi_{11}(1), \pi_{12}(1))$, then we build a polygon P' as follows: if in the reconstruction of P' we are considering the point (i,j,k) and in $s(P)$ we read Z_s (resp. X_t, Y_ℓ) then we have to connect the point (i,j,k) to the point (i,j,s) (resp. (t,j,k), resp. (x,ℓ,k)) using unit steps along the direction of e_3 (resp. e_1, e_2). It is then clear that at the end of the reconstruction process we obtain a permupolygon P', and that P' coincides with P, by construction. $\square$

In the rest of this section, we will study some restricted classes of permupolygons with trigger $Z\,X\,Y$, precisely: parallelogram, directed-convex and convex permupolygons with trigger $Z\,X\,Y$.

4.2.1. Parallelogram permupolygons with trigger $Z\,X\,Y$

Proposition 16. Let P be a parallelogram permupolygon with trigger $Z\,X\,Y$ and size n. Its encoding $s(P)$ has the following properties:

1) it is a word $(Z\,X\,Y)^{n+1}$, where each symbol is endowed of an index such that for all $i \in \{1, \cdots, n+1\}$ Z_i (resp. X_i, Y_i) appears exactly one time;
2) it starts with $Z_1\,X_1\,Y_1$;
3) the indices of its sequence of X_i's form an unimodal sequence (Property U_X);
4) the indices of its sequence of Y_i's form an unimodal sequence (Property U_Y);

5) the indices of its sequence of Z_i's form an unimodal sequence (Property U_Z);

6) it contains the factor $X_{n+1}\,Y_{n+1}\,Z_{n+1}$ (case x)) or $Y_{n+1}\,Z_{n+1}\,X_{n+1}$ (case y)) or $Z_{n+1}\,X_{n+1}\,Y_{n+1}$ (case z)).

Proof. Property **1** follows from the fact that P is a permupolygon, Property **2** descends from the fact that the root of a parallelogram permupolygon is always $(1,1,1)$, Properties **3 − 5** follow from the fact that P is parallelogram, while **6** follows from the fact that P is a parallelogram permupolygon with trigger $Z\,X\,Y$. □

Figure 10 (x) shows a parallelogram permupolygon P with trigger $Z\,X\,Y$ of case x), with

$$\pi_1(P) = ((1,2,3,4),(1,2,3,4),(1,2,4,3)),$$
$$\pi_2(P) = ((1,2,3,4),(1,2,3,4),(2,3,1,4)),$$
$$\pi_3(P) = ((1,2,3,4),(3,1,4,2),(1,2,4,3))$$

whose encoding is

$$s(P) = Z_1\,X_1\,Y_1\,Z_2\,X_2\,Y_2\,Z_3\,X_4\,Y_4\,Z_4\,X_3\,Y_3.$$

Figure 10 (y) shows a parallelogram permupolygon P with trigger $Z\,X\,Y$ of case y), with

$$\pi_1(P) = ((1,2,3,4),(1,4,2,3),(1,2,3,4)),$$
$$\pi_2(P) = ((1,2,3,4),(1,4,2,3),(2,4,1,3)),$$
$$\pi_3(P) = ((1,2,3,4),(2,1,3,4),(1,2,3,4))$$

whose encoding is

$$s(P) = Z_1\,X_1\,Y_1\,Z_2\,X_2\,Y_4\,Z_4\,X_4\,Y_3\,Z_3\,X_3\,Y_2.$$

Figure 10 (z) shows a parallelogram permupolygon P with trigger $Z\,X\,Y$ of case z), with

$$\pi_1(P) = ((1,2,3,4),(1,2,3,4),(1,2,3,4)),$$
$$\pi_2(P) = ((1,2,3,4),(1,2,3,4),(2,4,1,3)),$$
$$\pi_3(P) = ((1,2,3,4),(3,1,4,2),(1,2,3,4))$$

whose encoding is

$$s(P) = Z_1\, X_1\, Y_1\, Z_2\, X_2\, Y_2\, Z_4\, X_4\, Y_4\, Z_3\, X_3\, Y_3.$$

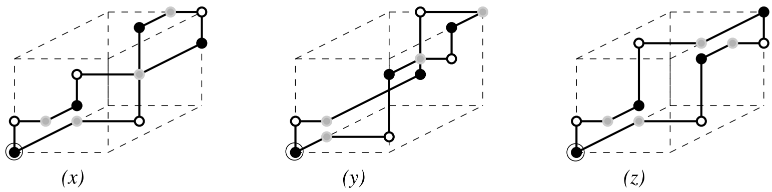

Figure 10: (x) (resp. (y), (z)) A parallelogram permupolygon of size 3 with trigger $Z\,X\,Y$, case x) (resp. y), z)). The root of the permupolygon is indicated.

Definition 22. Let P be a parallelogram permupolygon with trigger $Z\,X\,Y$ of size n, the upper path encoding $s_u(P)$ of P is the subsequence of $s(P)$ starting from Z_1 and ending in Z_{n+1} (resp. X_{n+1}, Y_{n+1}) in case x) (resp. y), z)).

Proposition 17. Given a parallelogram permupolygon P of size n, with trigger $Z\,X\,Y$, we have that:

i) P is determined by its upper path encoding;
ii) referring to case z), $s_u(P)$ contains the same number of X_i's, Y_j's and Z_k's (see Fig. 10 (z));
iii) referring to case x), $s_u(P)$ contains the same number of X_i's and Y_j's, and the number of Z_k's is equal to the number of X_i's plus one (see Fig. 10 (x));
iv) referring to case y), $s_u(P)$ contains the same number of X_i's and Z_k's, and the number of Y_j's is equal to the number of X_i's minus one (see Fig. 10 (y)).

Proof. **i)** By definition of 3-dimensional permupolygon P there are exactly three vertices in each hyperplane $h_{i,j} = \{x_i = j\}$, $\{i = 1, 2, 3\}$, $\{j = 1, \ldots, n+1\}$. Since the lower path encoding of P is made up only by south, west, and south-west steps, then there is only one way to reconstruct the entire permupolygon: starting from the last vertex of $s_u(P)$ we go west (resp. south-west, south) if we are in the case x), (resp. y), z)) until we cross the hyperplane $x_1 = j$ (resp. $x_2 = j$, $x_3 = j$) then we go south-west (resp. south, west) until we cross the hyperplane $x_2 = j$ (resp. $x_3 = j$, $x_1 = j$) and so on, according to the trigger ZXY.

ii) In this case $s_u(P)$ starts with Z_1 and ends with Y_{n+1}. Since Z_k, X_i, and Y_j alternate, then we have the same number.

ii) In this case $s_u(P)$ starts with Z_1 and ends with Z_{n+1}. Since Z_k, X_i, and Y_j alternate, then the number of Z_k is equal to the number of X_i (resp. Y_j) plus one.

iii) In this case $s_u(P)$ starts with Z_1 and ends with X_{n+1}. Since Z_k, X_i, and Y_j alternate then the number of Z_j (resp. X_i) is equal to the number of Y_j plus one. □

Proposition 18. Let $\overline{\mathcal{P}}'_n$ be the class of parallelogram permupolygons of size n with trigger $Z\,X\,Y$. For any $n \geq 1$, we have:

$$|\overline{\mathcal{P}}'_n| = \sum_{h=0}^{n-1} \binom{n-1}{h}^3 + \sum_{h=1}^{n-1} \binom{n-1}{h}^2 \binom{n-1}{h-1} + \sum_{h=1}^{n-1} \binom{n-1}{h} \binom{n-1}{h-1}^2.$$

Proof. To prove this statement we use Propositions 16 and 17. We are going to count the number of upper path encodings, in particular:

- referring to the case z) we can choose h elements in the set $\{X_2, X_3, ..., X_n\}$, h elements in the set $\{Y_2, Y_3, ..., Y_n\}$ and h elements in the set $\{Z_2, Z_3, ..., Z_n\}$, so we have
$$\sum_{h=0}^{n-1} \binom{n-1}{h}^3$$
objects of type z);
- referring to the case x) we can choose $h-1$ elements in the set $\{X_2, X_3, ..., X_n\}$, $h-1$ elements in the set of $\{Y_2, Y_3, ..., Y_n\}$ and h elements in the set $\{Z_2, Z_3, ..., Z_n\}$ with $h \geq 1$, so we have
$$\sum_{h=1}^{n-1} \binom{n-1}{h-1}^2 \binom{n-1}{h}$$
objects of type x);
- referring to y) we can choose $h-1$ elements in the set $\{Y_2, Y_3, ..., Y_n\}$, h elements in the set of $\{X_2, X_3, ..., X_n\}$ and h elements in the set $\{Z_2, Z_3, ..., Z_n\}$ with $h \geq 1$, so we have
$$\sum_{h=1}^{n-1} \binom{n-1}{h-1} \binom{n-1}{h}^2$$

objects of type y).

It follows that:

$$|\overline{\mathcal{P}}'_n| = \sum_{h=0}^{n-1} \binom{n-1}{h}^3 + \sum_{h=1}^{n-1} \binom{n-1}{h}^2 \binom{n-1}{h-1} + \sum_{h=1}^{n-1} \binom{n-1}{h} \binom{n-1}{h-1}^2 . \quad \square$$

4.2.2. Directed-convex permupolygons with trigger $\boldsymbol{Z\,X\,Y}$ In this paragraph we consider directed-convex permupolygons with trigger $Z\,X\,Y$ and we provide their enumeration according to size.

Proposition 19. Let P be a directed-convex permupolygon with trigger $Z\,X\,Y$ and size n. Its encoding $s(P)$ is a word $(Z\,X\,Y)^{n+1}$, where each symbol is endowed of an index such that

1) for all $i \in \{1, \cdots, n+1\}$ Z_i (resp. X_i, Y_i) appears exactly once,
2) it starts with $Z_1\,X_1\,Y_1$,
3) it satisfies Properties U_X, U_Y and U_Z.

Proof. It follows from the definition of directed-convex permupolygon and from the fact that the root of a directed-convex permupolygon is placed in $(1, 1, 1)$. $\square$

Figure 11 (a) shows a directed-convex permupolygon P with trigger $Z\,X\,Y$ where

$$\pi_1(P) = ((1,2,3,4),(1,2,3,4),(1,3,4,2)),$$
$$\pi_2(P) = ((1,2,3,4),(1,2,3,4),(3,4,2,1)),$$
$$\pi_3(P) = ((1,2,3,4),(4,1,2,3),(1,3,4,2)).$$

The encoding of P is

$$s(P) = Z_1\,X_1\,Y_1\,Z_3\,X_2\,Y_2\,Z_4\,X_3\,Y_3\,Z_2\,X_4\,Y_4.$$

Proposition 20. Let $\overline{\mathcal{D}}'_n$ be the class of directed-convex permupolygons of size n with trigger $Z\,X\,Y$. For any $n \geq 1$, we have:

$$|\overline{\mathcal{D}}'_n| = 8^{n-1}.$$

Proof. The proof is similar to that of Proposition 9 for the number of directed-convex permupolygons on the square lattice. From Proposition 19,

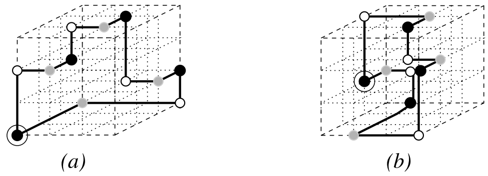

Figure 11: (a) A directed-convex permupolygon of size 3 with trigger $Z\,X\,Y$. (b) A convex permupolygon of size 3 with trigger $Z\,X\,Y$.

it follows that a directed-convex permupolygon is determined by three unimodal sequences, completely independent: one of X_i's, one of Y_j's and one of Z_k's. As observed before, the number of sequences of X_i's is 2^{n-1}. Furthermore, the number of sequences of Y_j's (resp. Z_k's) is clearly the same as the number of sequences of X_i's. It follows:

$$|\overline{\mathcal{D}}'_n| = 2^{n-1} \cdot 2^{n-1} \cdot 2^{n-1} = 8^{n-1}. \qquad \square$$

4.2.3. Convex permupolygons with trigger $\boldsymbol{Z\,X\,Y}$ This section is dedicated to the enumeration of convex permupolygons with trigger $Z\,X\,Y$, according to their size. We apply the same technique used in the previous section.

Proposition 21. The encoding $s(P)$ of a convex permupolygon P of size n with trigger $Z\,X\,Y$, is a word $(Z\,X\,Y)^{n+1}$, such that

1) each X, Y, Z is endowed of an index such that for all $i \in \{1, \cdots, n+1\}$ Z_i (resp. X_i, Y_i) appears exactly once,

2) it satisfies Property U_X,

3) if we consider the cyclic shift $s_z(P)$ of $s(P)$ starting from Z_1, the indices of sequence (in $s_z(P)$) of Z_i's form an unimodal sequence (Property C_Z);

4) if we consider the cyclic shift $s_y(P)$ of $s(P)$ starting from Y_1, the indices of sequence (in $s_y(P)$) of Y_i's form an unimodal sequence (Property C_Y);

5) it starts with $Z_k\,X_1\,Y_j$, where $k \neq n+1$ (resp. $j \neq n+1$) is an index of the increasing (resp. decreasing) subsequence of $s_z(P)$ (resp. $s_y(P)$) or $k = 1$ (resp. $j = 1$).

Proof. From the definition of convex permupolygon it follows that the encoding of P satisfies properties U_X, C_Y and C_Z. Instead, Property **5** is a consequence of the choice of the root as the minimal vertex in lexicographic order (see Fig. 9 (b)). $\square$

Figure 11 (b) shows a convex permupolygon P with trigger $Z\,X\,Y$ where

$$\begin{aligned}\pi_1(P) &= ((1,2,3,4),(3,4,2,1),(2,1,4,3)),\\ \pi_2(P) &= ((1,2,3,4),(3,4,2,1),(4,2,3,1)),\\ \pi_3(P) &= ((1,2,3,4),(4,1,3,2),(2,1,4,3))\end{aligned}$$

whose encoding is

$$s(P) = Z_2\,X_1\,Y_3\,Z_4\,X_3\,Y_2\,Z_3\,X_4\,Y_1\,Z_1\,X_2\,Y_4.$$

Proposition 22. Let $\overline{\mathcal{C}}'_n$ be the class of convex permupolygons of size n with trigger $Z\,X\,Y$. For any $n \geq 1$, we have

$$|\overline{\mathcal{C}}'_n| = 2^{3n-5}\,(n+1)^2\,.$$

Proof. From Proposition 21 it follows that a convex permupolygon with trigger $Z\,X\,Y$ is determined by three sequences, completely independent: one of X_i's which is unimodal, one of Y_j's and one of Z_k's which are both cyclic shifts of unimodal sequences. The number of sequences of X_i's is 2^{n-1}. Concerning the sequence of Y_j's, we choose k elements (for the increasing subsequence) in the set $\{Y_2, Y_3, ..., Y_n\}$. According to Proposition 21, we consider the only cyclic shifts starting with one of the remaining elements (namely those not chosen) different from Y_{n+1}, then the number of sequences of Y_i's is:

$$\begin{aligned}\sum_{k=0}^{n-1}(n+1-k-1)\binom{n-1}{k} &= n\,2^{n-1} - \sum_{k=0}^{n-1}k\binom{n-1}{k}\\ &= n\,2^{n-1} - (n-1)\,2^{n-2}\\ &= 2^{n-2}\,(n+1).\end{aligned}$$

The number of sequences of Z_k's is equal to the number of sequences of Y_j's. We have to observe that the chosen elements are used to form the decreasing subsequence of Z_k's in $s_z(P)$ (instead of the increasing one). It follows that:

$$|\overline{\mathcal{C}}'_n| = (2^{n-1})\,(2^{n-2}\,(n+1))^2 = 2^{3n-5}\,(n+1)^2\,. \qquad \square$$

4.3. Permupolygons with trigger $Y\ Z\ X$

As observed in Remark 4.4, we have that the number of permupolygons with trigger $Y\,Z\,X$ is twice the number of permupolygons with trigger $Z\,X\,Y$ (or

$Z\,Y\,X$). We can directly prove this fact by providing a path encoding of a permupolygon with trigger $Y\,Z\,X$.

Remark 4.7. We want to point out that a vertex $V = (i, j, k)$ in $\pi_2(P)$ (resp. $\pi_3(P)$, $\pi_1(P)$) corresponds to the subword $Y_j Z_\ell$ (resp. $Z_k X_\ell$, $X_i Y_\ell$) in $s(P)$. Observe that, in the case $Y_j Z_\ell$, (resp. $Z_k X_\ell$, $X_i Y_\ell$) ℓ is the third (resp. first, second) coordinate of the vertex $V' = (i, j, \ell)$, (resp. $V' = (\ell, j, k)$, $V' = (i, \ell, k)$) belonging to $\pi_3(P)$ (resp. $\pi_1(P)$, $\pi_2(P)$) and following $V = (i, j, k)$ in P.

Then we have the following definition.

Definition 23. The path encoding (briefly encoding) $s(P)$ of a permupolygon P with trigger $Y\,Z\,X$ of size n, is the word $(Y\,Z\,X)^{n+1}$, where each symbol is endowed of an index according to the following rules:

i. $s(P)$ starts with $Y_{\pi_{11}(1)}\,Z_{\pi_{12}(1)}\,X_1$;
ii. the right neighbor of Y_j is $Z_{\pi_{32}(\pi_{21}^{-1}(j))}$;
iii. the right neighbor of Z_k is $X_{\pi_{12}^{-1}(k)}$;
iv. the right neighbor of X_i (if there is one) is $Y_{\pi_{21}(i)}$.

Also in this case we can prove a statement similar to that of Remark 4.6 and the following is straightforward.

Proposition 23. A permupolygon with trigger $Z\,Y\,X$ is uniquely determined by its path encoding.

4.3.1. Convex permupolygons with trigger $Y\,Z\,X$

Proposition 24. The encoding $s(P)$ of a convex permupolygon P of size n with trigger $Y\,Z\,X$ is a word $(Y\,Z\,X)^{n+1}$, such that

1) each X, Y, Z is endowed of an index such that for all $i \in \{1, ..., n+1\}$ Y_i (resp. Z_i, X_i) appears exactly once,
2) it satisfies Property U_X, C_Y, C_Z
3) it starts with $Y_j\,Z_k\,X_1$, where $j \neq n+1$ is an index of the increasing subsequence of $s_y(P)$ or $j = 1$ and $k \in \{1, ..., n+1\}$.

Proof. From the definition of convex permupolygon it follows that the sequence of X_i's has Property U_X and the sequence of Y_i's (resp. Z_i's) has Property C_Y (resp. C_Z). Furthermore, Property **3** follows from the convention of the choice of the root (see Fig. 9 (a),(d)). □

Figure 12 (a) shows a convex permupolygon P with trigger $Y\,Z\,X$ (case (a) in Fig. 9) with

$$\pi_1(P) = ((1,2,3,4),(2,4,1,3),(3,4,1,2)),$$
$$\pi_2(P) = ((1,2,3,4),(4,3,2,1),(3,4,1,2)),$$
$$\pi_3(P) = ((1,2,3,4),(4,3,2,1),(4,2,3,1)),$$

whose encoding is

$$s(P) = Y_2\,Z_3\,X_1\,Y_4\,Z_4\,X_2\,Y_3\,Z_2\,X_4\,Y_1\,Z_1\,X_3.$$

Figure 12 (b) shows a convex permupolygon P with trigger $Y\,Z\,X$ (case (d) in Fig. 9) with

$$\pi_1(P) = ((1,2,3,4),(2,1,3,4),(4,1,2,3)),$$
$$\pi_2(P) = ((1,2,3,4),(4,2,1,3),(4,1,2,3)),$$
$$\pi_3(P) = ((1,2,3,4),(4,2,1,3),(3,4,1,2)),$$

whose encoding is

$$s(P) = Y_2\,Z_4\,X_1\,Y_4\,Z_3\,X_4\,Y_3\,Z_2\,X_3\,Y_1\,Z_1\,X_2.$$

Proposition 25. Let $\overline{\mathcal{C}}''_n$ be the class of convex permupolygons of size n with trigger $Y\,Z\,X$. For any $n \geq 1$, we have

$$|\overline{\mathcal{C}}''_n| = 2\,(2^{3n-5}\,(n+1)^2).$$

Proof. Using the same arguments as above, we have that the number of sequences of X_i's is 2^{n-1}, and the number of sequences of Y_i's is $2^{n-2}\,(n+1)$. Moreover, the sequences of Z_i's are cyclic shifts of unimodal sequences, then their number is equal to

$$\sum_{k=0}^{n-1}(n+1)\binom{n-1}{k} = (n+1)\,2^{n-1}.$$

It follows that:

$$|\overline{\mathcal{C}}''_n| = 2^{n-2}\,(2^{n-1}(n+1))^2 = 2\,(2^{3n-5}\,(n+1)^2). \qquad \square$$

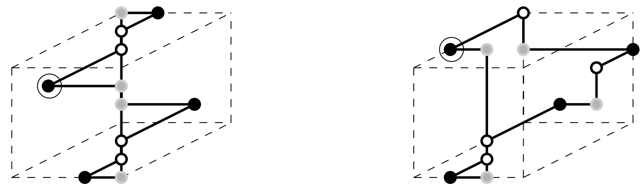

Figure 12: Two convex permupolygons of size 3 with trigger $Y\,Z\,X$.

4.4. Permupolygons enumeration

Now we are ready to provide the enumeration of the whole class of parallelogram (resp. directed-convex, convex) permupolygons according to size.

Proposition 26. Let $\overline{\mathcal{P}}_n$ be the class of parallelogram permupolygons of size n. For each $n \geq 1$, we have

$$|\overline{\mathcal{P}}_n| = 3\left(\sum_{h=0}^{n-1}\binom{n-1}{h}^3 + \sum_{h=1}^{n-1}\binom{n-1}{h}^2\binom{n-1}{h-1} + \sum_{h=1}^{n-1}\binom{n-1}{h}\binom{n-1}{h-1}^2\right).$$

Proof. It just follows from Proposition 18 and Remark 4.4. □

Proposition 27. Let $\overline{\mathcal{D}}_n$ be the class of directed-convex permupolygons of size n. For any $n \geq 1$, we have:

$$|\overline{\mathcal{D}}_n| = 3 \cdot 8^{n-1}.$$

Proof. Similarly to the case of parallelogram permupolygons, it just follows from Proposition 20 and Remark 4.4. □

Proposition 28. Let $\overline{\mathcal{C}}_n$ be the class of convex permupolygons of size n. For each $n \geq 1$, we have

$$|\overline{\mathcal{C}}_n| = 8^{n-1}\,(n+1)^2.$$

Proof. According to Proposition 22 and Remark 4.4, we have

$$|\overline{\mathcal{C}}_n| = 2\,(2^{3n-5}\,(n+1)^2) + 2\,(2^{3n-5}\,(n+1)^2) = 8^{n-1}\,(n+1)^2.$$ □

5. Permutations defining permupolygons

As we have mentioned in Section 4, a 3-dimensional permupolygon P of size n is uniquely determined by a triple of 2-dimensional permutations (the components of P) $\pi_1(P) = (\pi_{11}, \pi_{12})$, $\pi_2(P) = (\pi_{21}, \pi_{22})$ and $\pi_3(P) = (\pi_{31}, \pi_{32})$ of length $n + 1$. In this section we provide a characterization of the triples of 2-dimensional permutations which define a 3-dimensional permupolygon.

Proposition 29. Let P be a permupolygon of size n with trigger ZXY. The three components of P have the following properties:

(1) π_{11}, π_{12} and π_{22} satisfy:

- (1.1) $\pi_{22}(i) \neq \pi_{12}(i)$, for each $i = 1, \ldots, n + 1$,
- (1.2) $\pi_{12}(1) < \pi_{22}(1)$,
- (1.3) $\pi_{11}(1) < \pi_{11} \circ \pi_{22}^{-1} \circ \pi_{12}(1)$
- (1.4) the permutation $\pi_{12}^{-1} \circ \pi_{22}$ has a unique cycle (of length $n + 1$);

(2) $\pi_{21} = \pi_{11}$;
(3) $\pi_{32} = \pi_{12}$;
(4) $\pi_{31} = \pi_{11} \circ \pi_{22}^{-1} \circ \pi_{12}$.

Proof. Let us start from the root and follow the sequence of vertices of P. If we encounter a point of π_1, it is connected to its neighbor – which is a point of π_2 – using only unit steps in direction e_3; therefore, passing from a point of π_1 to its neighbor of π_2, only the third coordinate changes and so we have (2), namely $\pi_{21} = \pi_{11}$. Similarly, a point of π_3 is connected to a point of π_1 using only unit steps in direction e_2; in this case only the second coordinate changes and so we have (3), namely $\pi_{32} = \pi_{12}$. Finally, we pass from a point of π_2 to its neighbor (a point of π_3) using only unit steps in direction e_1, hence only the first coordinate changes and so we obtain (4). Since the vertices of P are distinct it follows that $\pi_{22}(i) \neq \pi_{12}(i)$ for each $i = 1, \ldots, n + 1$. Moreover, $r(P) = (1, \pi_{11}(1), \pi_{12}(1))$, then it follows that $\pi_{12}(1) < \pi_{22}(1)$, $\pi_{11}(1) < \pi_{31}(1) = \pi_{11} \circ \pi_{22}^{-1} \circ \pi_{12}(1)$. Furthermore, P is a single loop, then the permutation π', such that $\pi'(i) = j$ means that in $s(P)$ the leftmost X following X_i has index equal to j, must have a unique cycle of length $n+1$. Moreover, from Definition 21 and from (4) it follows exactly that $\pi' = \pi_{32}^{-1} \circ (\pi_{22} \circ \pi_{11}^{-1}) \circ \pi_{11} = \pi_{32}^{-1} \circ \pi_{22} = \pi_{12}^{-1} \circ \pi_{22}$. □

Let us prove that Condition (1) of Proposition 29 is a necessary and sufficient condition to determine a permupolygon with trigger ZXY. In particular we can prove:

Proposition 30. Given three permutations π, σ and τ of length $n+1$ such that:

1. $\tau(i) \neq \sigma(i)$ for each $i = 1, \dots, n+1$,
2. $\sigma(1) < \tau(1)$,
3. $\pi(1) < \pi \circ \tau^{-1} \circ \sigma(1)$,
4. the permutation $\sigma^{-1} \circ \tau$ has a unique cycle of length $n+1$.

Setting $\pi_{11} = \pi_{21} = \pi$, $\pi_{12} = \pi_{32} = \sigma$, $\pi_{22} = \tau$ and $\pi_{31} = \pi \circ \tau^{-1} \circ \sigma$, the three 2-permutations $\pi_1 = (\pi_{11}, \pi_{12})$, $\pi_2 = (\pi_{21}, \pi_{22})$ and $\pi_3 = (\pi_{31}, \pi_{32})$ uniquely determine a permupolygon with trigger $Z\,X\,Y$ of size n and root in $(1, \pi(1), \sigma(1))$.

Proof. Using π_1, π_2 and π_3 we can define the word s as in Definition 21. Then we build the unique polygon P such that $s(P) = s$ (as in the proof of Proposition 15). We can easily prove that P is a permupolygon with trigger $Z\,X\,Y$. In fact since $\tau(i) \neq \sigma(i)$ for each $i = 1, \dots, n+1$ it follows that the points of π_1, π_2 and π_3 are all distinct. Moreover, the two conditions $\sigma(1) < \tau(1)$ and $\pi(1) < \pi \circ \tau^{-1} \circ \sigma(1)$ ensure that the root is the point $(1, \pi(1), \sigma(1))$. Furthermore, the constraint that the permutation $\sigma^{-1} \circ \tau$ has a unique $n+1$-cycle, ensures that P is a single loop. □

According to Remark 4.4, the characterization in Proposition 30 is sufficient to find the cardinality of the whole class of permupolygons of given size.

Conclusion 1. We believe that is possible to extend the results of this paper for dimension $d > 3$, but this will be a further work.

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