Journal of Combinatorics
Volume 9, Number 1, 35–56, 2018

A hamilton cycle in which specified vertices are located in polar opposite

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Enomoto conjectured that if a graph G of order n has minimum degree at least $n/2+1$, then for any two vertices x and y, there is a hamilton cycle C such that $d_C(x,y) = \lfloor n/2 \rfloor$. In this paper, we show the existence of a hamilton cycle C in G such that $d_C(x,y) \geq (n-4)/3$.

AMS 2000 subject classifications: Primary 05C45.
Keywords and phrases: Hamilton cycle, Dirac condition, hamilton connectedness, panconnectivity.

1. Introduction

In this paper, we consider finite simple graphs. The order and the size, i.e., the number of edges, of a graph G are denoted by $|G|$ and $||G||$, respectively. The set of all neighbours of a vertex $x \in V(G)$ is denoted by $N(x) = N_G(x)$, and $d(x) = d_G(x) = |N(x)|$ is the degree of x. The minimum degree of G is denoted by $\delta(G)$. For both the vertex set, $V(G)$, and the edge set, $E(G)$, of G we will eventually use G whenever the context is clear. And we denote the order and the minimum degree of G by simply n and δ, respectively. The *distance* $d_G(x,y)$ of two vertices x and y in G is the length of a shortest path joining x and y. For terminology and notation not defined in this paper, we refer the readers to [3]. The following result is well known.

Theorem A (Dirac [4]). *If G is a 2-connected graph of n vertices with minimum degree at least δ, then there is a cycle C such that $|C| \geq \min\{2\delta, n\}$.*

This result immediately implies that a graph with $\delta \geq n/2$ is hamiltonian. Ore [14] improved this as follows: a graph with

$$\sigma_2(G) = \min\{d_G(u) + d_G(v) : uv \notin E(G)\} \geq n$$

is hamiltonian.

A graph is called *pancyclic* if the graph contains cycles of all lengths from 3 to n. Bondy suggested an interesting metaconjecture that any nontrivial condition which implies the graph is hamiltonian also implies the graph is pancyclic and showed that a graph with $\sigma_2(G) \geq n$ is pancyclic or G is isomorphic to $K_{n/2,n/2}$ in [2]. Pancyclicity is studied by many researchers and so we refer readers to the surveys [16] or [12] for details.

Ore [15] considered a property strengthening hamiltonicity and proved that a graph with $\sigma_2(G) \geq n+1$ is *hamilton-connected*, i.e., for any two vertices in G, there is a hamilton path joining the specified vertices. If the vertices are adjacent, then we can obtain a hamilton cycle from the hamilton path by adding the edge.

Alavi and Williamson [1] introduced *panconnectivity*. A graph is called *panconnected* if for any two vertices and an integer $2 \leq k \leq n-1$, there is a path joining the vertices of length k. Williamson [17] proved a graph with $\delta \geq n/2+1$ is panconnected. As in hamilton-connectivity, panconnected graphs are necessarily pancyclic. A similar result for bipartite graphs, bi-panconnectivity, was given by Du et al. [5].

Enomoto conjectured the following:

Conjecture B ([6]). *If G is a graph with $\delta \geq n/2+1$, then for any two vertices x and y in G, there is a hamilton cycle C of G such that $d_C(x,y) = \lfloor n/2 \rfloor$.*

In this conjecture, the minimum degree condition is sharp because in the graph $K_{(n-3)/2} \vee K_3 \vee K_{(n-3)/2}$, the minimum degree is $(n+1)/2$ and $d_C(x,y) \leq (n-3)/2$ for any x and y in one of $K_{(n-3)/2}$ and any hamilton cycle C.

Motivated by Conjecture B, Kaneko and Yoshimoto [11] showed that if G is a graph with $\delta \geq n/2$ and d an integer such that $0 < d \leq n/4$, then for any vertex subset $A \subset V(G)$ with $|A| \leq n/2d$, there is a hamilton cycle C such that $d_C(x,y) \geq d$ for any x and $y \in A$. Sárkőzy and Selkow [13] generalized this result by applying the Regularity Lemma. Furthermore by using k-linkage, Faudree et al. [7] also gave interesting facts relating to the result.

On the other hand, Faudree and Li gave a natural conjecture generalizing the conjecture by Enomoto.

Conjecture C ([10]). *If G is a graph with $\delta \geq n/2+1$, then for any vertices x and y and any integer $2 \leq k \leq n/2$, there is a hamilton cycle C of G such that $d_C(x,y) = k$.*

This conjecture generalizes also the panconnectivity result by Williamson. Faudree and Li [10] proved that if the order of G is sufficiently large for k, then the statement of Conjecture C holds. Recently Faudree, Lehel and Yoshimoto improved the lower bound of n as follows:

Theorem D ([8]). *If G is a graph with $\delta \geq n/2+1$, then for any vertices x and y and any integer $2 \leq k \leq n/6$, there is a hamilton cycle C of G such that $d_C(x,y)=k$.*

A similar result for bipartite graphs was given by Faudree, Lehel and Yoshimoto [9].

The purpose of this paper is to propose new conjectures implying the conjecture by Enomoto and give partial results for them. A path P with ends x and y is denoted by xPy and for any two vertices u and v of P, the subpath joining u and v in P is denoted by uPv.

Conjecture 1. *If G is a graph with $\delta \geq n/2+1$, then for any three vertices x, y and $z \in V(G)$, there is a hamilton path P joining x and z such that $\lfloor \frac{n}{2} \rfloor \leq ||xPy|| \leq \lceil \frac{n}{2} \rceil$.*

This conjecture implies Conjecture B because if we choose x and z which are adjacent in G, then $P \cup \{xz\}$ is a hamilton cycle satisfying the condition in the conjecture.

Let $u \in V(G)$ and $S \subset V(G)-u$. A path joining u and some vertex in S is called a (u,S)*-path.* A *path factor* of G is a spanning subgraph of G in which all components are paths.

Let $Y = N_G(y)$. If $G-y$ has a path factor consisting of an (x,Y)-path xPy' and a (z,Y)-path $y''Qz$ such that $\lfloor \frac{n}{2} \rfloor - 1 \leq ||P|| \leq \lceil \frac{n}{2} \rceil - 1$, then $xPy'yy''Qz$ is a desired hamilton path in Conjecture 1. Therefore the following conjecture also implies Conjecture B.

Conjecture 2. *If G is a graph with $\delta \geq (n+1)/2$, then for any two vertices x and $z \in V(G)$ and $Y \subset V(G)-\{x,z\}$ with at least $(n-1)/2$ vertices, G has a path factor consisting of an (x,Y)-path P and a (z,Y)-path Q such that $\lfloor \frac{n-1}{2} \rfloor \leq ||P|| \leq \lceil \frac{n-1}{2} \rceil$.*

Our main results are the following:

Theorem 1. *If G is a graph with $\delta \geq (n+1)/2$, then for any two vertices x and $z \in V(G)$ and $Y \subset V(G)-\{x,z\}$ with at least $(n-1)/2$ vertices, there exist disjoint (x,Y)-path P and (z,Y)-path Q such that $\min\{||P||, ||Q||\} \geq n/3-2$.*

Theorem 2. *Let G be a graph with $\delta \geq (n+2)/2$ and x, y and z be any three vertices in G. If there are disjoint paths xPy and yQz such that $s = \min\{||P||, ||Q||\} \geq (n-1)/3-2$, then there is a hamilton path R joining x and z such that*

$$\min\{||xRy||, ||yRz||\} \geq s+1.$$

By Theorem 1 and Theorem 2, we have the following immediately.

Corollary 3. *If G is a graph with $\delta \geq (n+2)/2$, then for any two vertices x and $y \in V(G)$, there is a hamilton cycle C such that $d_C(x,y) \geq (n-4)/3$.*

First we give a proof of Theorem 2 in Section 2, which is easier and the proof of Theorem 1 is given in Section 3.

Notice that in Conjecture 2, it is difficult to improve the minimum degree condition and the lower bound of $|Y|$ at the same time because $K_{(n-2)/2} \vee K_2 \vee K_{(n-2)/2}$ has no desired path factor if we choose the vertices in K_2 as $\{x,z\}$ and one of $K_{(n-2)/2}$ as Y.

Finally, we give some additional notations. For a subgraph H of G, we denote $N_G(x) \cap V(H)$ by $N_H(x)$ and its cardinality by $d_H(x)$. Let $C = v_1v_2 \cdots v_cv_1$ be a cycle with a fixed orientation. The segment $v_iv_{i+1}\cdots v_j$ is written by v_iCv_j where the subscripts are to be taken modulo c. The successor of v_i is denoted by v_i^+ and the predecessor by v_i^-. For a vertex subset A in C, we write $\{u_i^+ : u_i \in A\}$ and $\{u_i^- : u_i \in A\}$ by A^+ and A^-, respectively. For a path with fixed orientation, we define similar notations.

2. Proof of Theorem 2

Let $x, y, z \in V(G)$ and $Y = N(y) - \{x,z\}$. Then

$$\begin{aligned} \delta(G-y) &\geq \frac{n+2}{2} - 1 = \frac{n}{2} = \frac{|G-y|+1}{2} \text{ and} \\ |Y| &\geq \frac{n+2}{2} - 2 = \frac{n-2}{2} = \frac{|G-y|-1}{2}. \end{aligned}$$

Thus by Theorem 1, in $G-y$, there are vertex disjoint (x,Y)-path xPy' and (z,Y)-path zQy'' both of which have length at least $s \geq \frac{n-1}{3} - 2$. Then there is the path $R_0 = xPy'yy''Qz$ in G with

$$|R_0| \geq 2(s+1)+1 \geq 2(\frac{n-1}{3} - 2 + 1) + 1 = \frac{2n-5}{3}$$

and $\min\{d_{R_0}(x,y), d_{R_0}(y,z)\} \geq \min\{||xPy||, ||zQy||\} \geq s+1$.

Let R be a longest path joining x and z such that

$$\min\{d_R(x,y), d_R(z,y)\} \geq s+1. \tag{1}$$

Then

$$|R| \geq |R_0| \geq \frac{2n-5}{3}. \tag{2}$$

Suppose R is not a hamilton path. Since R is longest, no vertex in $G-R$ is adjacent to consecutive vertices on R. Thus $d_R(v) \leq (|R|+1)/2 \leq n/2$ for any $v \in G - R$. Since $d_G(v) \geq \delta \geq n/2 + 1$, there is no isolated vertex in $G - R$. Let $v_1 L v_2$ be a longest path in $G - R$ and $l = |L|$. By (2),

$$2 \leq l \leq n - |R| \leq \frac{n+5}{3}. \tag{3}$$

Let $d_i = d_R(v_i)$ for $i \in \{1, 2\}$. Since $d_{G-R}(v_i) \leq l - 1$,

$$d_i \geq \frac{n+2}{2} - d_{G-R}(v_i) \geq \frac{n}{2} - l + 2. \tag{4}$$

Let $N_R(v_1) \cup N_R(v_2) = \{u_1, u_2, \ldots, u_p\}$ which occur in the order on R. Let $I_i = u_i^+ R u_{i+1}^-$ for $i < p$. Since R is longest, $|I_i| \geq 1$. If $\{v_1, v_2\} \subset N(u_i) \cup N(u_{i+1})$ and $y \notin I_i$, then $|I_i| \geq l$; otherwise we can construct a path satisfying (1) which is longer than R.

Suppose $M = N_R(v_1) \cap N_R(v_2) = \emptyset$. Since every interval I_i contains at least one vertex, by (4), $\sum_{i<p} |I_i| \geq p - 1$, and so

$$\begin{aligned} n - l \geq |R| &\geq \sum_{i<p} |I_i| + |N_R(v_1) \cup N_R(v_2)| \\ &\geq (p-1) + p = 2(d_1 + d_2) - 1 \\ &\geq 4(\frac{n}{2} - l + 2) - 1 = 2n - 4l + 7 \\ \to l &\geq \frac{n+7}{3}. \end{aligned}$$

This contradicts (3).

Suppose $M \neq \emptyset$, and let $m = |M|$.

Case 1. $N_R(v_1) = M$ or $N_R(v_2) = M$.

In this case, $m \geq n/2 - l + 2$ by (4). If $y \in M$, then there are at least $m - 1$ intervals corresponding to vertices in M which contains at least l vertices; otherwise we can construct a path satisfying (1) which is longer than R. If y is in an interval corresponding to a vertex in M, then the interval may contain less than l vertices. Hence there are at least $m - 2$ intervals corresponding to vertices in M which contains at least l vertices. Therefore $\sum_{i<p} |I_i| \geq (m-2)l + 1$. Thus

$$\begin{aligned} n - l \geq |R| &\geq \sum_{i<p} |I_i| + |N_R(v_1) \cup N_R(v_1)| \\ &\geq (m-2)l + 1 + m \\ &\geq (\frac{n}{2} - l + 2 - 2)l + 1 + \frac{n}{2} - l + 2 \end{aligned}$$

$$(5) \qquad \to 0 \quad \geq \quad \frac{(l-1)\,n - 2\,l^2 + 6}{2}.$$

If the equality holds, then

$$l = \frac{\pm\sqrt{n^2 - 8\,n + 48} + n}{4}.$$

Since by (3),

$$\frac{-\sqrt{n^2 - 8\,n + 48} + n}{4} < 2 \leq l \leq \frac{n+5}{3} < \frac{\sqrt{n^2 - 8\,n + 48} + n}{4},$$

the inequality (5) does not hold. This is a contradiction.

Case 2. $N_R(v_i) - M \neq \emptyset$ for each $i \in \{1, 2\}$.

There are $p - 1 = d_1 + d_2 - m - 1$ intervals. Since $N_R(v_i) - M \neq \emptyset$ for each $i \in \{1, 2\}$, there are at least $m + 2 - 1$ intervals $u_i^+ R u_i^-$ such that $\{v_1, v_2\} \subset N(u_i) \cup N(u_{i+1})$. Therefore if $y \in N_R(v_1) \cup N_R(v_2)$, then there are at least $m + 1$ intervals containing at least l vertices. In the case of $y \notin N_R(v_1) \cup N_R(v_2)$, there are at least m such intervals as in Case 1. Thus,

$$\sum_{i<p} |I_i| \geq ml + (d_1 + d_2 - m - 1 - m),$$

and hence by (4)

$$\begin{aligned}
n - l \geq |R| \quad &\geq \quad \sum_{i<p} |I_i| + |N_R(v_1) \cup N_R(v_2)| \\
&\geq \quad ml + (d_1 + d_2 - 2m - 1) + (d_1 + d_2 - m) \\
&\geq \quad ml + 4(\frac{n}{2} - l + 2) - 3m - 1 \\
&\geq \quad ml + 2n - 4l + 7 - 3m \\
\to 0 \quad &\geq \quad ml + n - 3l + 7 - 3m \\
0 \quad &\geq \quad m(l-3) + n - 3l + 7.
\end{aligned}$$

If $l = 2$, then $m \geq n + 1$, a contradiction. If $l \geq 3$, then $l \geq (n + 7)/3$. This contradicts (3). ■

3. Proof of Theorem 1

We will use the following lemma.

Lemma 1. *Let A and B be vertex subsets of a path L. Then there is a subpath in L joining a vertex in A and a vertex in B of length at least $(|A|+|B|)/2-1$.*

Proof. Let $A \cup B = \{u_1, \ldots, u_l\}$ which occur in the order on L. By symmetry, we may assume $u_1 \in A$, and let $s = \min\{i : u_i \in B\}$ and $t = \max\{i : u_i \in B\}$. If $u_1 L u_t$ is not a desired path, i.e., $|u_1 L u_t| < (|A|+|B|)/2$, then $|u_t^+ L u_l| > |A| - (|A|+|B|)/2 = (|A|-|B|)/2$. Thus $|u_s L u_l| > |B| + (|A| - |B|)/2 = (|A|+|B|)/2$. □

Proof of Theorem 1. Let x and z be two distinct vertices in G and $Y \subset V(G) - \{x, z\}$ with at least $\frac{n-1}{2}$ vertices. Without of generality, we may assume Y contains exactly $\lceil \frac{n-1}{2} \rceil$ vertices by ignoring several vertices in Y. We will construct disjoint (x, Y)-path P and (z, Y)-path Q such that $\min\{||P||, ||Q||\} \geq n/3 - 2$.

Since $\delta(G) \geq \frac{n+1}{2}$, G is 3-connected, and so $G' = G - \{x, z\}$ is connected.

Claim 1. *If G' has a cut vertex, then G has desired paths P and Q.*

Proof. Suppose G' has a cut vertex u, and let H_1 and H_2 be two components of $G' - u$. For any vertex $v \in H_i$ for $i \in \{1, 2\}$,

$$|H_i| - 1 \geq d_{H_i}(v) \geq d(v) - |\{x, z, u\}| \geq \frac{n+1}{2} - 3 = \frac{n-5}{2},$$

and so $|H_i| \geq \frac{n-3}{2}$. Since $|H_1| + |H_2| \leq n - 3$, we have $|H_1| = |H_2| = \frac{n-3}{2}$. Thus H_i is isomorphic to $K_{\frac{n-3}{2}}$ and every vertex in H_i is adjacent to all of x, z and u.

Suppose there are $y_1 \in H_i \cap Y$ and $y_2 \in H_j \cap Y$ for $\{i, j\} = \{1, 2\}$. By symmetry, we may assume $y_i \in H_i$. Let $w_i \in H_i - \{y_i\}$ and P_i be a hamilton path of H_i joining w_i and y_i for $i \in \{1, 2\}$. Then $P = x w_1 P_1 y_1$ and $Q = z w_2 P_2 y_2$ are desired paths because $||P|| = (n-3)/2 + 1 - 1 = (n-3)/2$ and also $||Q|| = (n-3)/2$.

Suppose $H_1 \cap Y = \emptyset$ or $H_2 \cap Y = \emptyset$. By symmetry, we may assume $H_1 \cap Y = \emptyset$, and then $Y \subset H_2 \cup \{u\}$. Since $|Y| \geq \frac{n-1}{2}$ and $|H_2| = \frac{n-3}{2}$, we have $Y = V(H_2) \cup \{u\}$. Thus for any hamilton path $w_i P_i w_i'$ of H_i for $i \in \{1, 2\}$, the paths $P = x w_1 P_1 w_1' u$ and $Q = y w_2 P_2 w_2'$ are desired paths as in the previous case. □

Thus we suppose $G' = G - \{x, z\}$ is 2-connected. Let $C = v_1 v_2 \cdots v_c v_1$ be a longest cycle of G'. By Theorem A,

$$n - 2 \geq c = |C| \geq \min\{2(\delta(G) - 2), n - 2\} \geq n - 3. \tag{6}$$

Claim 2. *If $N_C^+(x) \cap N_C(z) = N_C^-(x) \cap N_C(z) = \emptyset$, then there are desired paths P and Q.*

Proof. Suppose that $N_C^{\pm}(x) \cap N_C(z) = \emptyset$.

When $c = n-2$, we have $d_C(x), d_C(z) \geq \frac{n-1}{2}$, then

$$\frac{n-1}{2} \leq d_C(z) \leq |C| - d_C(x) \leq \frac{n-3}{2},$$

a controdiction.

When $c = n-3$, we have $d_C(x), d_C(z) \geq \frac{n-3}{2}$, then

$$\frac{n-3}{2} \leq d_C(z) \leq |C| - d_C(x) \leq \frac{n-3}{2},$$

and we obtain $d_C(z) = d_C(x) = \frac{n-3}{2}$. We claim that the distance of every pair of consecutive neighbors of x along C is exactly 2. Suppose not. If v_i, v_i^+ are consecutive neighbors of x along C, then there exist a pair of consecutive neighbors of x along C such that their distance along C is more than 3, for otherwise we have $c \leq 2(d_C(x) - 1) + 1 = n - 4$. If v_i, v_j are consecutive neighbors of x along C with $v_j = v_i^{+k} (k \geq 3)$, then $v_j^- \neq v_i^+$ and $v_j^- \notin N_C^+(x)$, so $|N_C^{\pm}(x)| \geq |N_C^+(x)| + 1 \geq \frac{n-1}{2}$. Thus we have

$$\frac{n-3}{2} = d_C(z) = |N_C(z)| \leq |C| - |N_C^{\pm}(x)| \leq \frac{n-5}{2},$$

a controdiction.

By the same reason, we obtain that the distance of every pair of consecutive neighbors of z along C is also 2. Without loss of generality, let $N_C(x) = \{v_1, v_3, v_5, \cdots, v_{n-4}\}$. If $N_C(z) = \{v_2, v_4, v_6, \cdots, v_{n-3}\}$, then it contracts to $N_C^{\pm}(x) \cap N_C(z) = \emptyset$, so $N_C(z) = N_C(x) = \{v_1, v_3, v_5, \cdots, v_{n-4}\}$.

Since $|Y \cap C| \geq \frac{n-3}{2}$, there are v_s and $v_t \in Y$ such that $1 \leq t - s \leq 2$. Then for two vertices v_i and v_{i+2} in $v_{t+\lceil \frac{n}{3} \rceil} C v_{s-\lceil \frac{n}{3} \rceil} \cap N_C(x)$, $P = v_t C v_i x$ and $Q = z v_{i+2} C v_s$ are desired paths. □

By Claim 2, we may assume that $(N_C^+(x) \cap N_C(z)) \cup (N_C^-(x) \cap N_C(z)) \neq \emptyset$, say $N_C^+(x) \cap N_C(z) \neq \emptyset$. Without loss of generality, we may assume that $v_1 \in N_C(x)$ and $v_c \in N_C(z)$. Let

$$m_1 = d_C(x),\ m_2 = d_C(z),\ k = |Y \cap C| \text{ and } d = \lceil \frac{n}{3} \rceil - 2.$$

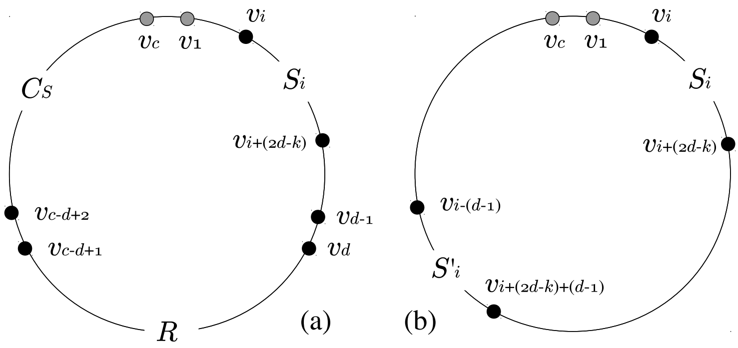

Figure 1.

Notice that

$$(7)\quad \begin{cases} m_1, m_2 \geq \dfrac{n-1}{2} \text{ and } k = \lceil \dfrac{n-1}{2} \rceil & \text{if } c = n-2. \\ m_1, m_2 \geq \dfrac{n-3}{2} \text{ and } \lceil \dfrac{n-1}{2} \rceil \geq k \geq \dfrac{n-3}{2} & \text{if } c = n-3. \end{cases}$$

If $R = v_d C v_{c-d+1}$ contains two vertices v_i and v_j $(i < j)$ in Y, then $P = xv_1 C v_i$ and $Q = v_j C v_c z$ are desired paths because

$$||P|| \geq d + 1 - 1 \geq \lceil \frac{n}{3} \rceil - 2 \text{ and also } ||Q|| \geq \lceil \frac{n}{3} \rceil - 2.$$

See Figure 1a.

Thus we suppose $R = v_d C v_{c-d+1}$ contains at most one vertex in Y. Let

$$C_S = C - R = v_{c-d+2} C v_{d-1}.$$

Then $|C_S| = 2(d-1)$ $(\approx 2n/3)$ and $|C_S \cap Y| \geq k - 1$. We define intervals of length $2d - k$ $(\approx n/6)$ in C_S as follows:

let $S_i = v_i C v_{i+(2d-k)}$ for $v_i \in v_{c-d+2} C v_{d-1-(2d-k)} = v_{c-d+2} C v_{k-d-1}$.

Then $C_S = \bigcup \{S_i : v_i \in v_{c-d+2} C_{k-d-1}\}$. Since $|C_S \cap Y| \geq k - 1$ and

$$|C_S| - |S_i| = 2d - 2 - (2d - k + 1) = k - 3,$$

each interval S_i contains at least two vertices in Y. For each S_i, let

$$S'_i = v_{i+(2d-k)+(d-1)} C v_{i-(d-1)} \ (\subset C - S_i).$$

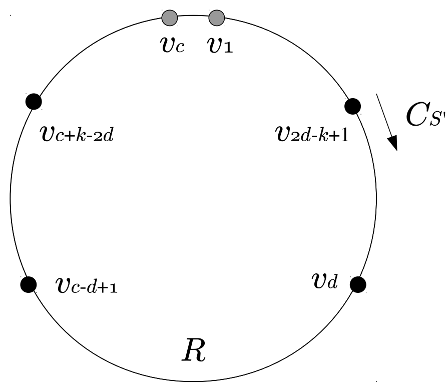

Figure 2.

See Figure 1b. Then

$$|S_i'| = c - |S_i| - 2(d-2) = c - 4d + k + 3 \ (\approx n/6). \tag{8}$$

If there is an S_i' which contains distinct vertices v_s and v_t adjacent to x and z, respectively, then since $S_i \cap Y$ contains at least two vertices, by using the vertices of Y, u_s and u_t, we can construct desired paths P and Q as in the case of $|R \cap Y| \geq 2$.

Thus we suppose that there is no S_i' containing distinct vertices adjacent to x and z, respectively. Let $C_{S'} = \bigcup\{S_i' : v_i \in v_{c-d+2}Cv_{k-d-1}\}$. Then for any $v_s \in N_{C_{S'}}(x)$ and $v_t \in N_{C_{S'}}(z)$,

$$v_s = v_t \text{ or } d_C(v_s, v_t) \geq |S_i'| = c - 4d + k + 3 \ (\approx n/6). \tag{9}$$

Since

$$S'_{c-d+2} = v_{c-d+2+(2d-k)+(d-1)}Cv_{c-d+2-(d-1)} = v_{2d-k+1}Cv_{c-2d+3} \text{ and}$$
$$S'_{k-d-1} = v_{k-d-1+(2d-k)+(d-1)}Cv_{k-d-1-(d-1)} = v_{2d-2}Cv_{c+k-2d},$$

we have

$$\begin{aligned} &C_{S'} = v_{2d-k+1}Cv_{c+k-2d} \text{ and} \\ &|C_{S'}| = c + k - 2d - (2d-k) = c + 2k - 4d \ (\approx 2n/3). \end{aligned} \tag{10}$$

See Figure 2. Thus

$$|N_{C_{S'}}(x)| \geq |N_C(x)| - (|C| - |C_{S'}|) = m_1 - 4d + 2k \ (\approx n/6) \text{ and}$$

(11) $|N_{C_{S'}}(z)| \geq m_2 - 4d + 2k.$

Let

$$\begin{aligned} a &= \min\{i : v_i \in N_{C_{S'}}(x)\}, \quad b = \max\{i : v_i \in N_{C_{S'}}(x)\}, \\ X &= v_a C v_b (\subset C_{S'}), \\ a' &= \min\{i : v_i \in N_{C_{S'}}(z)\}, \quad b' = \max\{i : v_i \in N_{C_{S'}}(z)\}, \\ Z &= v_{a'} C v_{b'} (\subset C_{S'}). \end{aligned}$$

Then by (11),

$$\begin{aligned} &|X| \geq |N_{C_{S'}}(x)| \geq m_1 - 4d + 2k \ (\approx n/6) \text{ and} \\ &|Z| \geq |N_{C_{S'}}(z)| \geq m_2 - 4d + 2k. \end{aligned} \tag{12}$$

Later we will take a path P or Q by using X and Z.

Claim 3. $X \cap Z = \emptyset$.

Proof. First we show $N_{C_{S'}}(x) \cap N_{C_{S'}}(z) = \emptyset$. Suppose $l = |N_{C_{S'}}(x) \cap N_{C_{S'}}(z)| \geq 1$, and let $U = C_{S'} - N_{C_{S'}}(x) \cup N_{C_{S'}}(z)$.

Suppose $l < \min\{|N_{C_{S'}}(x)|, |N_{C_{S'}}(z)|\}$. Then both of $N_{C_{S'}}(x) - N_{C_{S'}}(z)$ and $N_{C_{S'}}(z) - N_{C_{S'}}(x)$ are not empty. Since, by (9), U has at least $l+1$ components containing at least $|S_i| - 1$ vertices, $|U| \geq (l+1)(|S_i'| - 1)$. Therefore by (10), (11), (9) and (8),

$$\begin{aligned} |C_{S'}| &= c + 2k - 4d \\ &= |N_{C_{S'}}(x)| + |N_{C_{S'}}(z)| - l + |U| \\ &\geq (m_1 - 4d + 2k) + (m_2 - 4d + 2k) - l + (l+1)(c - 4d + k + 2) \\ &= m_1 + m_2 - 8d + 4k + l(c - 4d + k + 1) + c - 4d + k + 2 \\ &\geq m_1 + m_2 - 8d + 4k + (c - 4d + k + 1) + c - 4d + k + 2 \\ \to 0 &\geq c - 12d + 4k + m_1 + m_2 + 3 \\ &\geq n - 3 - 12(\lceil \frac{n}{3} \rceil - 2) + 4 \times \frac{n-3}{2} + 2 \times \frac{n-3}{2} + 3 \\ &\geq n - 3 - 12(\frac{n+2}{3} - 2) + 4 \times \frac{n-3}{2} + 2 \times \frac{n-3}{2} + 3 > 0, \end{aligned}$$

a contradiction.

Suppose $l = \min\{|N_{C_{S'}}(x)|, |N_{C_{S'}}(z)|\}$. By (11),

$$l \geq m_i - 4d + 2k \geq \frac{n-3}{2} - 4(\lceil \frac{n}{3} \rceil - 2) + 2\frac{n-3}{2}$$

$$\begin{aligned} &\geq \frac{n-3}{2} - 4(\frac{n+2}{3} - 2) + 2\frac{n-3}{2} \\ &\geq \frac{n+5}{6}. \end{aligned}$$

Since, by (9), U has at least $l-1$ components containing at least $|S_i|-1$ vertices, $|U| \geq (l-1)(|S'_i|-1)$. Thus, by (10), (11), (9) and (8),

$$\begin{aligned} |C_{S'}| &= c + 2k - 4d \\ &= |N_{C_{S'}}(x)| + |N_{C_{S'}}(z)| - l + |U| \\ &\geq (m_1 - 4d + 2k) + (m_2 - 4d + 2k) - l + (l-1)(c - 4d + k + 2) \\ &= m_1 + m_2 - 8d + 4k + l(c - 4d + k + 1) - (c - 4d + k + 2) \\ \to 0 &\geq l(c - 4d + k + 1) + m_1 + m_2 + k - 2c - 2 \\ &\geq \frac{n+5}{6}(n - 3 - 4(\lceil\frac{n}{3}\rceil - 2) + \frac{n-3}{2} + 1) + 2 \times \frac{n-3}{2} \\ &\quad + \frac{n-3}{2} - 2(n-2) - 2 \\ &\geq \frac{n+5}{6}(n - 3 - 4(\frac{n+2}{3} - 2) + \frac{n-3}{2} + 1) + 2 \times \frac{n-3}{2} \\ &\quad + \frac{n-3}{2} - 2(n-2) - 2 \\ &\geq \frac{n^2 - 2n - 35}{36} > 0 \text{ if } n > 7, \end{aligned}$$

a contradiction. Thus $N_{C_{S'}}(x) \cap N_{C_{S'}}(z) = \emptyset$.

If $X \cap Z \neq \emptyset$, then $|U| \geq 2(|S'_i|-1)$, and so again by (10), (11), (9) and (8),

$$\begin{aligned} |C_{S'}| &= c + 2k - 4d \\ &= |N_{C_{S'}}(x)| + |N_{C_{S'}}(z)| + |U| \\ &\geq (m_1 - 4d + 2k) + (m_2 - 4d + 2k) + 2(c - 4d + k + 2) \\ \to 0 &\geq m_1 + m_2 - 12d + 4k + c + 4 \\ &\geq 2 \times \frac{n-3}{2} - 12(\lceil\frac{n}{3}\rceil - 2) + 4 \times \frac{n-3}{2} + n - 3 + 4 \\ &\geq 2 \times \frac{n-3}{2} - 12(\frac{n+2}{3} - 2) + 4 \times \frac{n-3}{2} + n - 3 + 4 > 0, \end{aligned}$$

a contradiction. □

By symmetry, we may assume $a < a'$, i.e., by (10),

$$2d - k + 1 \leq a < b < a' < b' \leq c + k - 2d.$$

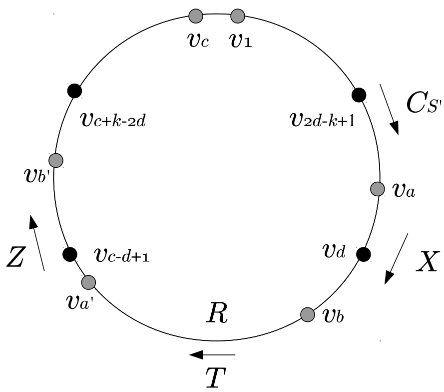

Figure 3.

In the next claim, we show that the ends of $R = v_d C v_{c-d+1}$ are contained in X and Z, respectively. See Figure 3.

Let $T = v_{b+1} C v_{a'-1}$ and then by (9),

$$|T| \geq |S_i'| - 1 = c - 4d + k + 2. \tag{13}$$

Claim 4. *$a < d < b < a' < c - d + 1 < b'$.*

Proof. If $d \leq a$, then by (12) and (13),

$$\begin{aligned} c + k - 2d \geq b' &\geq a - 1 + |X| + |T| + |Z| \\ &\geq d - 1 + (m_1 - 4d + 2k) + (c - 4d + k + 2) \\ &\quad + (m_2 - 4d + 2k) \\ &\geq m_1 + m_2 + 5k - 11d + c + 1 \\ \rightarrow 0 &\geq m_1 + m_2 + 4k - 9d + 1 \\ &\geq 2\frac{n-3}{2} + 4\frac{n-3}{2} - 9(\frac{n+2}{3} - 2) + 1 > 0, \end{aligned}$$

a contradiction. Thus $d > a$. Since $|X| = b - (a-1) \geq |N_{C_{S'}}(x)| \geq m_1 - 4d + 2k$ and $a \geq 2d - k + 1$,

$$\begin{aligned} b &\geq (a-1) + m_1 - 4d + 2k \geq m_1 - 4d + 2k + (2d - k + 1 - 1) \\ &= m_1 + k - 2d \geq \frac{n-3}{2} + \frac{n-3}{2} - 2(\lceil \frac{n}{3} \rceil - 2) \\ &\geq \frac{n-3}{2} + \frac{n-3}{2} - 2(\frac{n+2}{3} - 2) = \frac{n-1}{3} > d. \end{aligned}$$

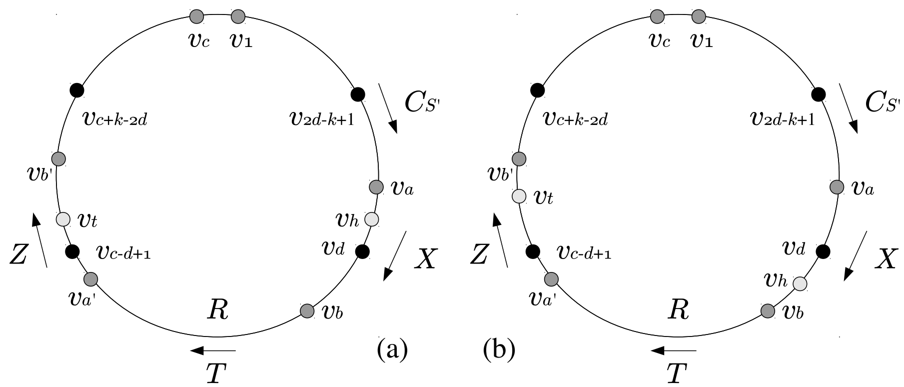

Figure 4.

Thus $a < d < b$. By symmetry, we have $a' < c - (d-1) < b'$. □

Let $v_h \in N_X(x)$ and $v_{h'} \in N_Z(z)$ be vertices which are closest to v_d and v_{c-d+1}, respectively. Possibly $v_h = v_d$ and $v_{h'} = v_{c-d+1}$. By symmetry, we may assume

$$|h - d| \leq |h' - (c - d + 1)|. \tag{14}$$

Since $R = v_d C v_{c-d+1}$ contains at most one vertex of Y, there are at least $k-1$ vertices of Y in $C - R = v_{c-d+2} C v_{d-1}$.

If $v_h \in v_a C v_d$, then let v_t be the vertex in $Y \cap (C-R) = Y \cap v_{c-d+2} C v_{d-1}$ which is closest to v_{c-d+1}. See Figure 4a. Then $P = x v_h C v_t$ is a desired (x, Y)-path because

$$\begin{aligned} ||P|| &\geq |\{x\}| + |v_d C v_{c-d+2}| - 1 \geq c - 2d + 3 \\ &\geq n - 3 - 2 \times (\frac{n+2}{3} - 2) + 3 = \frac{n+8}{3} > d. \end{aligned}$$

If $v_h \in v_{d+1} C v_b$, then let v_t be the vertex in $Y \cap v_{h+d-1} C v_c$ which is closest to v_{h+d-2}. See Figure 4b. Then $P = x v_h C v_t$ is a desired (x, Y)-path because

$$\begin{aligned} ||P|| &\geq |\{x\}| + |v_h C v_{h+d-1}| - 1 \\ &\geq 1 + (h + d - 1) - (h - 1) - 1 = d. \end{aligned}$$

Next we will construct a (z, Y)-path Q by using $C - P$. Since $R = v_d C v_{c-d+1}$ contains at most one vertex in Y,

$$|v_{c-d+2} C v_{d-1} - Y| = (d-1) + (d-2) + 1 - (k-1) \leq 2d - k - 1. \tag{15}$$

We divide our argument into two cases.

Case 1. $v_h \in v_a C v_d$.

Recall that v_t is the vertex in $Y \cap v_{c-d+2} C v_{d-1}$ which is closest to v_{c-d+1} and $P = x v_h C v_t$. See Figure 4a.

Claim 5. $P - x = v_h C v_t \subset C_{S'}$.

Proof. Since $v_h \in N_X(x) \subset C_{S'}$, it is enough to show $v_t \in C_{S'}$. By (15), we have

$$\begin{aligned} c-d+2 \leq t &\leq (c-d+2) + |v_{c-d+2} C v_{d-1} - Y| \\ &\leq (c-d+2) + (2d-k-1) = c+d-k+1. \end{aligned}$$

Since v_{c+k-2d} is an end of $C_{S'}$ and

$$\begin{aligned} (c+k-2d) - (c+d-k+1) &\geq 2k - 3d - 1 \\ &\geq 2 \times \frac{n-3}{2} - 3(\frac{n+2}{3} - 2) - 1 \geq 0, \end{aligned}$$

we have

$$t \leq c+d-k+1 \leq c+k-2d,$$

and so $v_t \in C_{S'}$. □

We will construct a (z, Y)-path Q by using vertices in $N_{C-P}(z)$ and $Y \cap (C-P)$ and Lemma 1.

Claim 6. *1.* $|N_{C-P}(z)| \geq m_1 + m_2 - 4h + k$.
2. $|Y \cap (C-P)| \geq k + h - d - 2$.

Proof. If $v_h = v_d$, by (10), (12) and (13)

$$\begin{aligned} |N_{P-x}(z)| &\leq |N_{C_{S'}}(z)| \leq |Z| \leq |C_{S'}| - |X| - |T| \\ &\leq (c+2k-4d) - (m_1 - 4d + 2k) - (c - 4d + k + 2) \\ &= 4d - m_1 - k - 2, \end{aligned}$$

and so

$$\begin{aligned} |N_{C-P}(z)| &\geq |N_C(z)| - |N_{P-x}(z)| \\ &\geq m_2 - (4d - m_1 - k - 2) = m_2 + m_1 - 4d + k + 2. \end{aligned}$$

Since $|Y \cap P| = |Y \cap R| + |\{v_t\}| \leq 1 + 1 = 2$,

$$|Y \cap (C-P)| \geq k - 2.$$

Suppose $v_h \in v_a C v_{d-1}$. By the definition of v_h, x is adjacent to no vertex in $v_{h+1} C v_{d+(d-h)-1}$. Since $v_{h+1} C v_{d+(d-h)-1} \subset X$, by (12),

$$\begin{aligned} |X| &\geq |N_{C_{S'}}(x)| + |v_{h+1} C v_{d+(d-h)-1}| \\ &\geq (m_1 - 4d + 2k) + (2d - 2h - 1) \\ &= m_1 - 2d - 2h + 2k - 1. \end{aligned} \tag{16}$$

Similarly, by (14), z is adjacent to no vertex in $v_{(c-d+1)-(d-h-1)} \times C v_{(c-d+1)+(d-h-1)}$,

$$\begin{aligned} |Z| &\geq |N_{C_{S'}}(z)| + |v_{(c-d+1)-(d-h-1)} C v_{(c-d+1)+(d-h-1)}| \\ &= |N_{C_{S'}}(z)| + |v_{c-2d+h+2} C v_{c-h}|. \end{aligned}$$

Thus by (10), (16) and (13),

$$\begin{aligned} |N_{P-x}(z)| &\leq |N_{C_{S'}}(z)| \leq |Z| - |v_{c-2d+h+2} C v_{c-h}| \\ &\leq |C_{S'}| - |X| - |T| - ((c-h) - (c - 2d + h + 1)) \\ &\leq (c + 2k - 4d) - (m_1 - 2d - 2h + 2k - 1) - (c - 4d + k + 2) \\ &\quad -(2d - 2h - 1) \\ &= 4h - k - m_1. \end{aligned}$$

Thus, we have $|N_{C-P}(z)| = |N_C(z)| - |N_{P-x}| \geq m_2 + m_1 - 4h + k$.

Since

$$\begin{aligned} |Y \cap P| &= |Y \cap v_h C v_{d-1}| + |Y \cap R| + |\{v_t\}| \\ &\leq (d - 1 - (h-1)) + 1 + 1 = d - h + 2, \end{aligned}$$

we have

$$|Y \cap (C - P)| \geq k - (d - h + 2) = k + h - d - 2. \qquad \square$$

By Lemma 1, there is a subpath Q_0 in $C - P$ joining $N_{C-P}(z)$ and $Y \cap (C - P)$ of length at least $(|N_{C-P}(z)| + |Y \cap (C-P)|)/2 - 1$. Let Q be the path obtained from $Q_0 \cup \{z\}$ by adding the edge joining z and the end of Q_0 in $N_{C-P}(z)$. Then Q is a desired (z, Y)-path because

$$\begin{aligned} ||Q|| &\geq \frac{|N_{C-P}(z)| + |Y \cap (C-P)|}{2} - 1 + 1 \\ &\geq \frac{1}{2}((k + h - d - 2) + (m_2 + m_1 - 4h + k)) \\ &\geq \frac{1}{2}(m_2 + m_1 - 3h - d + 2k - 2) \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{2}(m_2+m_1-4d+2k-2)\\
&\geq \frac{1}{2}(2\times\frac{n-3}{2}-4(\lceil\frac{n}{3}\rceil-2)+2\times\frac{n-3}{2}-2)\\
&\geq \frac{1}{2}(2\times\frac{n-3}{2}-4(\frac{n+2}{3}-2)+2\times\frac{n-3}{2}-2)=\frac{n}{3}-\frac{4}{3}\geq d.
\end{aligned}
$$

Case 2. $v_h \in v_{d+1}Cv_b$.

Recall that v_t is the vertex in $Y \cap v_{h+d-1}Cv_c$ which is closest to v_{h+d-2} and $P = xv_hCv_t$. See Figure 4b. In Case 2, $P - x = v_hCv_t$ may not be in $C_{S'}$.

Claim 7. *1. If $h + (d-1) < c - d + 2$, then $P - x \subset C_{S'}$.*
2. If $h + (d-1) \geq c - d + 2$, then $|P - x - C_{S'}| \leq h - \frac{n+8}{3}$.

Proof. Since $v_h \in C_{S'}$, it is enough to show $t \leq c + k - 2d$.

1. Since

$$t \leq (h+d-1) + |v_{h+d-1}Cv_{d-1} - Y|$$

and by (15),

$$
\begin{aligned}
|v_{h+d-1}Cv_{d-1} - Y| &= |v_{h+d-1}Cv_{c-d+1} - Y| + |v_{c-d+2}Cv_{d-1} - Y|\\
&\leq ((c-d+1)-(h+d-1)+1)+(2d-k-1)\\
&= c-h-k+2,
\end{aligned}
$$

we have $t \leq (h+d-1)+(c-k-h+2) = c-k+d+1$. Therefore

$$
\begin{aligned}
t-(c+k-2d) &\leq (c-k+d+1)-(c+k-2d)\\
&= 3d-2k+1 \leq 3(\frac{n+2}{3}-2)-2\times\frac{n-3}{2}+1=0.
\end{aligned}
$$

2. Notice that

$$h \geq c-d+2-(d-1) = c-2d+3 \geq n-3-2(\frac{n+2}{3}-2)+3=\frac{n+8}{3}.$$

If $t \leq c+k-2d$, then $P - x \subset C_{S'}$ and so we are done. Hence we suppose $t > c+k-2d$. Since by (15),

$$|v_{h+d-1}Cv_{d-1} - Y| \leq |v_{c-d+2}Cv_{d-1} - Y| \leq 2d-k-1,$$

we have $t \leq (h+d-1)+(2d-k-1) = 3d+h-k-2$. Thus

$$|P - x - C_{S'}| = t-(c+k-2d)$$

$$
\begin{aligned}
&\leq (3d + h - k - 2) - (c + k - 2d) \\
&= 5d + h - c - 2k - 2 \\
&\leq 5(\frac{n+2}{3} - 2) + h - (n-3) - 2 \times \frac{n-3}{2} - 2 \\
&\leq h - \frac{n+8}{3}.
\end{aligned}
$$

□

As in Case 1, we will construct a (z, Y)-path Q by using vertices in $N_{C-P}(z)$ and $Y \cap (C - P)$ and Lemma 1.

Claim 8. *1. If $h + (d-1) < c - d + 2$, then*

$$|N_{C-P}(z)| \geq m_1 + m_2 - 8d + 4h + k \text{ and } |Y \cap (C-P)| \geq k - 2.$$

2. If $h + (d-1) \geq c - d + 2$, then

$$
\begin{aligned}
&|N_{C-P}(z)| \geq m_1 + m_2 - 8d + 3h + k + \frac{n+8}{3} \text{ and} \\
&|Y \cap (C-P)| \geq k - h - 2d + c + 1.
\end{aligned}
$$

Proof. Since $v_h \in v_{d+1}Cv_b$, by (12) and (14),

$$
\begin{aligned}
|X| &\geq |N_{C_{S'}}(x)| + |c_{d-(h-d-1)}Cv_{h-1}| \\
&\geq (m_1 - 4d + 2k) + (2h - 2d - 1) \\
&= m_1 - 6d + 2k + 2h - 1 \text{ and} \qquad (17)\\
|Z| &\geq |N_{C_{S'}}(z)| + |v_{(c-d+1)-(h-d-1)}Cv_{(c-d+1)+(h-d-1)}| \\
&\geq |N_{C_{S'}}(z)| + 2h - 2d - 1. \qquad (18)
\end{aligned}
$$

1. Since $P - x \subset C_{S'}$, by (18), (10), (17) and (13), we have

$$
\begin{aligned}
|N_{P-x}(z)| &\leq |N_{C_{S'}}(z)| \leq |Z| - (2h - 2d - 1) \\
&\leq |C_{S'}| - |X| - |T| - (2h - 2d - 1) \\
&\leq (c + 2k - 4d) - (m_1 - 6d + 2k + 2h - 1) - (c - 4d + k + 2) \\
&\quad -(2h - 2d - 1) \\
&= 8d - 4h - m_1 - k.
\end{aligned}
$$

Therefore $|N_{C-P}(z)| = |N_C(z)| - |N_{P-x}(z)| \geq m_1 + m_2 - 8d + 4h + k$.

Since

$$
\begin{aligned}
|P \cap Y| &= |v_hCv_t \cap Y| = |v_hCv_{h+d-2} \cap Y| + |v_{h+d-1}Cv_t \cap Y| \\
&\leq |v_dCv_{c-d+1} \cap Y| + |v_{h+d-1}Cv_t \cap Y| \leq 1 + 1 = 2,
\end{aligned}
$$

we have $|Y \cap (C-P)| \geq k - 2$.

2. Since $|P-x-C_{S'}|\le h-(n+8)/3$, by (18), (10), (17) and (13),

$$\begin{aligned}
|N_{P-x}(z)| &\le |N_{C_{S'}}(z)|+|(P-x)-C_{S'}| \\
&\le |Z|-(2h-2d-1)+(h-\frac{n+8}{3}) \\
&\le |C_{S'}|-|X|-|T|-(2h-2d-1)+(h-\frac{n+8}{3}) \\
&\le (c+2k-4d)-(m_1-6d+2k+2h-1)-(c-4d+k+2) \\
&\quad -(2h-2d-1)+(h-\frac{n+8}{3}) \\
&= 8d-3h-m_1-k-\frac{n+8}{3}.
\end{aligned}$$

Thus

$$|N_{C-P}(z)|=|N_C(z)|-|N_{P-x}(z)|\ge m_1+m_2-8d+3h+k+\frac{n+8}{3}.$$

If $h+d-1=c-d+2$, then

$$\begin{aligned}
|P\cap Y| &= |v_hCv_t\cap Y| \\
&= |v_hCv_{c-d+1}\cap Y|+|v_{h+d-1}Cv_t\cap Y|\le 2.
\end{aligned}$$

Since $h+d-1=c-d+2$, we have $|Y\cap(C-P)|\ge k-2=k-h-2d+c+1$.

If $h+d-1>c-d+2$, then

$$\begin{aligned}
|P\cap Y| &= |v_hCv_t\cap Y| \\
&= |v_hCv_{c-d+1}\cap Y|+|v_{c-d+2}Cv_{h+d-2}\cap Y|+|v_{h+d-1}Cv_t\cap Y| \\
&\le 1+(h+2d-c-3)+1=h+2d-c-1.
\end{aligned}$$

Thus $|Y\cap(C-P)|\ge k-h-2d+c+1$. □

By Lemma 1, there is a subpath Q_0 in $C-P$ joining a vertex in $N_{C-P}(z)$ and a vertex in $Y\cap(C-P)$ of length at least $(|N_{C-P}(z)|+|Y\cap(C-P)|)/2-1$. Let Q be the path obtained from $Q_0\cup\{z\}$ by adding the edge joining z and the end of Q_0 in $N_{C-P}(z)$. Then Q is a desired (z,Y)-path. In fact, if $h+(d-1)<c-d+2$, as $d<h$,

$$\begin{aligned}
||Q|| &\ge \frac{|N_{C-P}(z)|+|Y\cap(C-P)|}{2}-1+1 \\
&\ge \frac{1}{2}((m_1+m_2-8d+4h+k)+(k-2))
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}(m_1 + m_2 - 8d + 4h + 2k - 2) \\
&> \frac{1}{2}(m_1 + m_2 - 4d + 2k - 2) \\
&\geq \frac{1}{2}(2 \times \frac{n-3}{2} - 4(\frac{n+2}{3} - 2) + 2 \times \frac{n-3}{2} - 2) \\
&= \frac{n-4}{3} > d.
\end{aligned}
$$

In the case of $h + (d-1) \geq c - d + 2$,

$$
\begin{aligned}
||Q|| &\geq \frac{|N_{C-P}(z)| + |Y \cap (C-P)|}{2} - 1 + 1 \\
&\geq \frac{1}{2}((m_1 + m_2 - 8d + 3h + k + \frac{n}{3} + \frac{8}{3}) + (k - h - 2d + c + 1)) \\
&= \frac{1}{2}(m_1 + m_2 - 10d + 2h + 2k + c + \frac{n}{3} + \frac{11}{3}) \\
&\geq \frac{1}{2}(m_1 + m_2 - 14d + 2k + 3c + \frac{n}{3} + \frac{29}{3}) \\
&\geq \frac{1}{2}(2 \times \frac{n-3}{2} - 14(\frac{n+2}{3} - 2) + 2 \times \frac{n-3}{2} + 3(n-3) + \frac{n}{3} + \frac{29}{3}) \\
&= \frac{n+20}{3} > d.
\end{aligned}
$$

Now we complete the proof. ■

Acknowledgments

The authors would like to thank Professor Hao Li for his comments. The second author's work was supported by JSPS KAKENHI Grant Number 26400190.

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Received 5 February 2015