Open questions for operators related to rectangular catalan combinatorics[*]

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We formulate many open questions, most ot them new, regarding Schur positivity, Schur inclusion, e-positivity, and e-inclusion of interesting symmetric functions arising from operators in the elliptic Hall algebra, and give supporting evidence for why one should expect such behavior. This ties in with many recent advances in the study of Rectangular Catalan Combinatorics and other subjects pertaining to Algebraic Geometry, Representation Theory, and the Homology of Torus Knots.

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Introduction

The aim of this text is to present, in a concise manner, a set of open questions relating to operators on symmetric functions that are relevant to rectangular

Catalan combinatorics. In some form or another special cases of these questions have partly been considered, but we thought it good to have them all stated together with the new ones that are presented here, so that a clearer picture could emerge. There has been significant recent advances on some of them, such as the still to be thoroughly reviewed proof [8] of the "Shuffle Conjecture", but by and large most questions remain wide open. The questions considered are elegantly linked to the combinatorics of rectangular Dyck paths, and associated parking functions (see [1, 17]). In particular, this is made explicit when one specializes one of the underlying parameters to be equal to 1. In recent years, there has also been a flurry of new developments concerning the link between elliptic Hall algebras and rectangular Catalan combinatorics (see [13]), and this gives more importance to the questions that we state, especially since these developments make apparent deep ties with problems of Algebraic Geometry [15, 19], Representation Theory [12], and Homology of Torus Knots [9, 10]. We would like to stress that, on top of giving proofs for some interesting specializations, the identities and properties considered here have been thoroughly checked by extensive direct computer algebra calculations. Hence they are stated with a rather good degree of confidence.

1. Elliptic Hall algebra

In a fashion somewhat similar to how creation operators are used in quantum mechanics, the main actors of our story are operators on symmetric functions that we eventually apply to the simplest symmetric function 1, aiming at constructing interesting symmetric functions. Although we will not go into much detail here, we recall that these operators belong to a realization of the "positive part" $\mathcal{E}$ of the "elliptic Hall algebra" (see below for more details) as a subalgebra of $\mathrm{End}(\Lambda)$, where

$$\Lambda = \bigoplus_{d \geq 0} \Lambda_d,$$

is the degree-graded algebra of symmetric functions (polynomials) in a denumerable set of variables $\boldsymbol{x} = x_1, x_2, x_3, \ldots$ over the field $\mathbb{Q}(q, t)$. It is shown in [7, 22, 24, 23] that the positive part $\mathcal{E}$ of the elliptic Hall algebra[1] be realized as a $(\mathbb{N} \times \mathbb{N})$-graded algebra of operators on Λ

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[1]The full algebra is $\mathbb{Z}^2$-graded, but we only need the positive components for our purpose.

$$\mathcal{E} = \bigoplus_{(m,n)\in\mathbb{N}^2} \mathcal{E}_{m,n},$$

with the operators in the homogeneous component $\mathcal{E}_{m,n}$ sending Λ_d to Λ_{d+n}. Back in 1999, special cases of these operators were introduced in [5, see Thm 4.4], where relevant properties were also stressed out.

Generating operators

The elliptic Hall algebra operators that we consider here are generated by two families of "well-known" operators. The first of these corresponds simply to multiplication by symmetric functions:

$$(-)\cdot f : \Lambda_d \longrightarrow \Lambda_{d+n}, \qquad \text{(that is } g \mapsto g\cdot f, \text{ for any } f\in\Lambda_n),$$

with $(-)\cdot f$ considered to belong to the $(0,n)$ component of $\mathcal{E}$; while the second is the family $\{\mathbf{D}_k\}_{k\in\mathbb{Z}}$ of some interesting operators that occur (see [5]) in the study of Macdonald polynomials, with $\mathbf{D}_k$ considered to belong to $\mathcal{E}_{1,k}$. Let us recall that these operators $\mathbf{D}_k$, send degree d symmetric function to degree $d+k$

$$\mathbf{D}_k : \Lambda_d \longrightarrow \Lambda_{d+k}.$$

They are jointly defined by the generating function equality

$$\sum_{k=-\infty}^{\infty} \mathbf{D}_k(g(\boldsymbol{x}))\, z^k := g[\boldsymbol{x} + M/z] \sum_{n\geq 0} e_n(\boldsymbol{x})\,(-z)^n,$$

here written using plethystic notation (see [5] for more on this), for any $g(\boldsymbol{x}) \in \Lambda_d$, and writing $M = M(q,t)$ for $(1-t)(1-q)$. It may be shown that $\mathbf{D}_0$ is a **Macdonald eigenoperator**. This is to say that it affords the (combinatorial) Macdonald polynomials as joint eigenfunctions. It may also be worth recalling that, for all k, we have

$$\mathbf{D}_{k+1} = \frac{1}{M}[\mathbf{D}_k, p_1],$$

with $[-,-]$ standing for the usual Lie bracket of operators, and e_1 correspond to multiplication by the degree 1 elementary symmetric function. In other words, all of the $\mathbf{D}_k$ (for $k > 0$) are obtained as order k Lie-derivatives, with respect to the operator of multiplication by p_1/M. Maybe even better for calculation purposes, we have

$$\mathbf{D}_{k+j} = \frac{1}{(1-t^j)(1-q^j)}[\mathbf{D}_k, p_j],$$

for all k and j. Indeed, considering the above operator generating series $\mathcal{D}(z) = \sum_{k=-\infty}^{\infty} \mathbf{D}_k \, z^k$, one may check that

$$z^j[\mathcal{D}(z), p_j(\boldsymbol{x})] = (1-t^j)(1-q^j)\,\mathcal{D}(z),$$

simply by calculating that[2]

$$
\begin{aligned}
z^j[\mathcal{D}(z), p_j]\,g(\boldsymbol{x}) &= \sum_{k=-\infty}^{\infty} [\mathbf{D}_k, p_j](g(\boldsymbol{x}))\,z^k \\
&= z^j\Big(g[\boldsymbol{x} + M/z]\,p_j[\boldsymbol{x} + M/z] - g[\boldsymbol{x} + M/z]\,p_j(\boldsymbol{x})\Big) \\
&\quad \times \sum_{n\geq 0} e_n(\boldsymbol{x})\,(-z)^n \\
&= z^j\Big(g[\boldsymbol{x} + M/z]\,(1-t^j)(1-q^j)/z^j\Big) \sum_{n\geq 0} e_n(\boldsymbol{x})\,(-z)^n \\
&= (1-t^j)(1-q^j)\,\mathcal{D}(z)\,g(\boldsymbol{x}).
\end{aligned}
$$

In particular, we get $(1-t^j)(1-q^j)D_j = [D_0, p_j]$, reducing the calculation of D_j to that of D_0, modulo a single bracket operation.

The special symmetric functions π_d

It is established in [7] that, for each pair of coprime integers (a, b), there are ring monomorphisms

$$\Theta_{a,b} : \Lambda \longrightarrow \mathcal{E},$$

explicitly described below, such that $\Theta_{a,b}(\Lambda_d)) \subseteq \mathcal{E}_{(ad,bd)}$. In particular, this says that one has commutation of operators belonging to the image of $\Theta_{a,b}$, for any given coprime pair (a, b). We will often consider $(m, n) = (ad, bd)$, with (a, b) coprime and thus $d = \gcd(m, n)$, and exploit the fact that operators in $\mathcal{E}_{(m,n)}$ commute with operators in $\mathcal{E}_{(m',n')}$ when $\gcd(m, n) = \gcd(m', n')$.

The easiest way to give an explicit definition for the above monomorphisms (see [13]), is to describe (see next subsection for this) the $\Theta_{a,b}$-image

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[2]Recall that, in plethystic notation, one has $p_j[\boldsymbol{x} + M/z] = p_j(\boldsymbol{x}) + (1-t^j)(1-q^j)/z^j$.

of a "new" family of algebraic generators for Λ. These are simply the following twisted version of the classical power-sums:

$$\pi_d = \pi_d(\boldsymbol{x}; q, t) := \sum_{j+k=d-1} (-qt)^{-j} s_{(j\,|\,k)}(\boldsymbol{x}),$$

where $s_{(j\,|\,k)}(\boldsymbol{x})$ stands[3] for the hook indexed Schur symmetric functions, with one part of size $j + 1$ and k parts of 1. For instance,

$$\pi_1(\boldsymbol{x}) = s_1(\boldsymbol{x}) = e_1(\boldsymbol{x}),$$

$$\pi_2(\boldsymbol{x}) = s_{11}(\boldsymbol{x}) - \frac{1}{qt}\, s_2(\boldsymbol{x}),$$

$$\pi_3(\boldsymbol{x}) = s_{111}(\boldsymbol{x}) - \frac{1}{qt}\, s_{21}(\boldsymbol{x}) + \frac{1}{q^2 t^2}\, s_3(\boldsymbol{x}),$$

$$\pi_4(\boldsymbol{x}) = s_{1111}(\boldsymbol{x}) - \frac{1}{qt}\, s_{211}(\boldsymbol{x}) + \frac{1}{q^2 t^2}\, s_{31}(\boldsymbol{x}) - \frac{1}{q^3 t^3}\, s_4(\boldsymbol{x}),$$

$$\pi_5(\boldsymbol{x}) = s_{11111}(\boldsymbol{x}) - \frac{1}{qt}\, s_{2111}(\boldsymbol{x}) + \frac{1}{q^2 t^2}\, s_{311}(\boldsymbol{x}) - \frac{1}{q^3 t^3}\, s_{41}(\boldsymbol{x}) + \frac{1}{q^4 t^4}\, s_5(\boldsymbol{x}).$$

When the parameters q and t are specialized in such a way that $qt = 1$, the symmetric function $\pi_d(\boldsymbol{x})$ are simply the power sum symmetric functions $(-1)^{d-1} p_d(\boldsymbol{x})$, since we have the well-known identity

$$(-1)^{d-1} p_d(\boldsymbol{x}) = \sum_{j+k=d-1} \sum_{j+k=d-1} (-1)^j s_{(j\,|\,k)}(\boldsymbol{x}).$$

It follows that $\{\pi_d((\boldsymbol{x})\}_d$ is an independent algebraic generator set for Λ. Equivalently, the set of symmetric functions

$$\pi_\mu(\boldsymbol{x}) := \pi_{\mu_1}(\boldsymbol{x}) \pi_{\mu_2}(\boldsymbol{x}) \cdots \pi_{\mu_k}(\boldsymbol{x}),$$

with $\mu = \mu_1 \mu_2 \cdots \mu_k$ running over the set of integer partitions of d, constitutes a linear basis of Λ. It is usual to write $\mu \vdash n$ when μ is a partition of n, and $\ell(\mu) := k$ is the length of the partition. With these definitions at hand, we can now describe the construction of our fundamental operators.

Definition of the basic operators

For coprime $a, b \geq 1$, the basic operators $\Theta_{a,b}(\pi_d)$ are obtained (see [7, 13]) as iterated bracketings of the operators $\mathbf{D}_k$ and multiplication by π_j. These

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[3]This is known as Frobenius notation for hooks.

are respectively considered as being of degree $(1, k)$ and $(0, j)$. The operator Q_{0j} is multiplication by the symmetric function $\pi_j(\boldsymbol{x})$, and $Q_{k1} := (-1)^k D_k$. For $m, n \geq 1$, we then recursively define Q_{mn} by the Lie bracket formula

$$Q_{mn} := \frac{1}{M} [Q_{uv}, Q_{kl}],$$

where we choose (k, l) such that $(m, n) = (k, l) + (u, v)$, with (k, l) and (u, v) lying in $\mathbb{N}^2$, $l - (kn/m)$ is minimal, and such that

$$\det \begin{pmatrix} u & v \\ k & l \end{pmatrix} = d,$$

with d standing for the greatest common divisor of m and n. Moreover, if $(m, n) = (ad, bd)$, we ask that (k, l) be chosen to be the same as it would for (a, b). For example, we get the Lie bracket expressions

$$Q_{43} = \frac{1}{M^6} [[e_1, D_0], [[e_1, D_0], [[e_1, D_0], D_0]]],$$

or

$$Q_{63} = \frac{1}{M^8} [[e_1, D_0], [[[e_1, D_0], D_0], [[[e_1, D_0], D_0], D_0]]].$$

For sure, the monomials that occur in the expansion of Q_{mn} involve m copies of D_0, and n copies of e_1. With these operators at hand, we may now define the monomorphisms $\Theta_{a,b}$ by setting

$$\Theta_{a,b}(\pi_d) := Q_{ad,bd},$$

hence, since the $\Theta_{a,b}$ are ring-homomorphisms, we have

$$\tag{1} \Theta_{a,b}(\pi_\mu) = \Theta_{a,b}(\pi_{\mu_1}) \cdots \Theta_{a,b}(\pi_{\mu_k}).$$

Observe that we need not worry about the order in which the operators $\Theta_{a,b}(\pi_{\mu_i})$ should be applied, since they commute. Clearly $\Theta_{0,1} = \mathrm{Id}_\Lambda$.

Seeds for operator families

For each degree d symmetric function g_d, which we call a **seed**, and any coprime pair (a, b), the construction above allows us to define operators $\Theta_{a,b}(g_d) \in \mathcal{E}_{ad,bd}$. Indeed, assume that we have the expansion

$$\tag{2} g_d(\boldsymbol{x}) = \sum_{\mu \vdash d} c_\mu \, \pi_\mu(\boldsymbol{x}),$$

in the $\{\pi_\mu(\boldsymbol{x})\}_{\mu \vdash d}$ basis. Since the $\Theta_{a,b}$ are ring-homomorphisms, we may set

$$\Theta_{a,b}(g_d) := \sum_{\mu \vdash d} c_\mu \, \Theta_{a,b}(\pi_\mu),$$

which can be evaluated using (1). In this manner g_d becomes a seed for the family of operators $\Theta_{a,b}(g_d)$ (indexed by coprime pairs (a,b)). One of the striking implications of the properties of $\mathcal{E}$, see [3, 4], is that, for all (a,b) coprime, all $d \in \mathbb{N}$, and all g_d, one has the operator identity

$$(3) \qquad\qquad \nabla \, \Theta_{a,b}(g_d) \, \nabla^{-1} = \Theta_{a+b,b}(g_d).$$

Here ∇ stands for the much-discussed Macdonald eigenoperator, which is such that $\nabla(e_n)$ gives the Frobenius transform of the bigraded character of the diagonal coinvariant space of the symmetric group $\mathbb{S}_n$ (see [2] for more on this). Recall that the Frobenius transform is the linear map that sends irreducible characters to Schur functions, hence the resulting symmetric function expands with positive integer (or positive integer coefficient polynomials in the graded version) coefficients in the Schur basis. In terms of the Hall scalar product $\langle -, - \rangle$ on symmetric function, for which the Schur symmetric functions are orthonormal, this means that we have $\langle M(\boldsymbol{x}; q, t), s_\mu(\boldsymbol{x}) \rangle \in \mathbb{N}[q, t]$, for all $\mu \vdash n$, whenever $M(\boldsymbol{x}; q, t)$ corresponds to the (graded) Frobenius transform of some (bi)graded $\mathbb{S}_n$-module

$$M = \bigoplus_{j,j} M_{i,j}.$$

Recall that the graded version of Frobenius transform sends the degree (i,j) homogeneous component $M_{i,j}$ to $q^i t^j M_{i,j}(\boldsymbol{x})$, where $M_{i,j}(\boldsymbol{x})$ stands for the Frobenius transform of the character of $M_{i,j}$. Because of ties with representation theory, we are interested in seeds g_d such that all the associated operators $\Theta_{a,b}(g_d)$ send the constant symmetric function 1 to Schur-positive symmetric functions. In formula, this is to say that

$$\langle \Theta_{a,b}(g_d)(1), s_\mu(\boldsymbol{x}) \rangle \in \mathbb{N}[q, t], \qquad \text{for all } \mu \vdash n.$$

When this is so, we write

$$0 \preceq_s g_d^{(a,b)}(\boldsymbol{x}; q, t),$$

and say that $g_d^{(a,b)}(\boldsymbol{x}; q, t)$ is **Schur-positive**. denoting by $g_d^{(a,b)}(\boldsymbol{x}; q, t)$ the symmetric function $\Theta_{a,b}(g_d)(1)$, and we will say that g_d is a **good seed**.

More generally we consider the partial order relation

(4) $g(\boldsymbol{x}) \preceq_s g'(\boldsymbol{x})$,   if and only if $0 \preceq_s g'(\boldsymbol{x}) - g(\boldsymbol{x})$,

and say that $g(\boldsymbol{x})$ is **Schur-included** in $g'(\boldsymbol{x})$.

Observe that for any seed g_d, and any coprime (a, b), we have

$$
\begin{aligned}
g_d^{(a+b,b)}(\boldsymbol{x}; q, t) &= \Theta_{a+b,b}(g_d)(1) \\
&= \nabla \Theta_{a,b}(g_d) \nabla^{-1}(1) \\
&= \nabla \Theta_{a,b}(g_d)(1) \\
(5) \qquad\qquad &= \nabla \left(g_d^{(a,b)}(\boldsymbol{x}; q, t) \right),
\end{aligned}
$$

since $\nabla(1) = 1$. In particular,

(6) $g_d^{(r,1)}(\boldsymbol{x}; q, t) = \nabla^r(g_d)$,

will be Schur-positive for good seeds.

2. Interesting seeds for family of operators

To get good seeds for a family of operators we need to suitably normalize well known symmetric functions. To make our formulas more elegant, we introduce the following notations, assuming all through that μ is a partition of d, and setting

$$
\iota(\mu) := \sum_{i=1}^{\ell(\mu)} \chi(\mu(i) - i),
$$

where $\chi(k)$ is equal to 1 if k is positive, and 0 if k is negative. We then define:

1. The renormalized Schur functions

   (7) $\widehat{s}_\mu(\boldsymbol{x}) := (-qt)^{-\iota(\mu)} s_\mu(\boldsymbol{x})$,

   with the particular case of hook shapes giving

   $$
   \widehat{s}_{(k \,|\, j)}(\boldsymbol{x}) := (-qt)^{-j} s_{(j \,|\, k)}(\boldsymbol{x}),
   $$

   and the normalized complete homogeneous symmetric functions

   $$
   \widehat{h}_d(\boldsymbol{x}) := (-qt)^{-(d-1)} h_d(\boldsymbol{x}).
   $$

2. The normalized monomial symmetric functions

$$(8) \qquad \widehat{m}_\mu(\boldsymbol{x}) := (-1)^{d-\ell(\mu)}\, m_\mu(\boldsymbol{x}),$$

including the special case $\widehat{m}_{(d)}(\boldsymbol{x}) = \widehat{p}_d(\boldsymbol{x}) := (-1)^{d-1} p_d(\boldsymbol{x})$.

3. The normalized forgotten symmetric functions

$$\widehat{f}_\mu(\boldsymbol{x}) := (-qt)^{-(\ell(\mu)-1)}\, f_\mu(\boldsymbol{x}).$$

Manifestly, all these symmetric functions are equal for $d = 1$, and per-force the associate operators will coincide. Clearly, since $\Theta_{a,b}$ is a ring-homomorphism, the associated operators are linked by the same relations as those between their seeds. In particular, we immediately get the following.

Lemma 1. *For all μ partition of d, and (a, b) coprime, we have*

$$\pi_d^{(a,b)}(\boldsymbol{x}; q, t) = \sum_{j+k=d-1} \widehat{s}_{(j\,|\,k)}^{(a,b)}(\boldsymbol{x}; q, t),$$

$$\widehat{p}_d^{(a,b)}(\boldsymbol{x}; q, t) = \sum_{j+k=d-1} (qt)^j\, \widehat{s}_{(j\,|\,k)}^{(a,b)}(\boldsymbol{x}; q, t).$$

The compositional (ad, bd)-shuffle conjecture (theorem)

Recall from [3] the conjectured combinatorial formula for the effect on 1 of the operators having as seed the symmetric function $C_\alpha(\boldsymbol{x}; q, t) := \boldsymbol{C}_\alpha(1)$. Here, for any composition $\alpha = a_1 a_2 \ldots a_\ell$ of d, one defines the operator $\boldsymbol{C}_\alpha := \boldsymbol{C}_{a_1} \boldsymbol{C}_{a_2} \cdots \boldsymbol{C}_{a_\ell}$ as the composition of the individual integer indexed operators $\boldsymbol{C}_a$ specified by the formula

$$\boldsymbol{C}_a f[\boldsymbol{x}] := (-t)^{1-a} f[\boldsymbol{x} - (t-1)/(tz)] \sum_{m \geq 0} z^m h_m[\boldsymbol{x}] \Big|_{z^a},$$

where $(-)\big|_{z^a}$ means that we take the coefficient of z^a in the series considered. With these notations, the compositional (ad, bd)-shuffle conjecture (of [3]) states that

$$(9) \qquad C_\alpha^{(a,b)}(\boldsymbol{x}; q, t) = \sum_\gamma q^{\mathrm{area}(\gamma)} \sum_\pi t^{\mathrm{dinv}(\pi)} s_{\mathbf{co}(\pi)}(\boldsymbol{x}),$$

where the first sum runs over all (ad, bd)-Dyck paths that return to the diagonal at positions specified by the composition α, and the second is over

parking functions whose underlying path is γ (necessary concepts and notations are defined in the appendix). It is known (see [14]) that for any given (m, n)-Dyck path γ,

$$(10) \qquad \gamma_{m,n}(\boldsymbol{x}; t) := \sum_{\pi} t^{\mathrm{dinv}(\pi)} s_{\mathbf{co}(\pi)}(\boldsymbol{x}),$$

is a LLT-polynomial[4], which is Schur-positive. Hence it would follow from Conjecture (9) that we have the Schur-positivity

$$(11) \qquad 0 \preceq_s C_\alpha^{(a,b)}(\boldsymbol{x}; q, t).$$

In other words, the conjecture states that $C_\alpha(\boldsymbol{x}; q, t)$ is a good seed.

It is also known (*Loc. cit.*) that the particular LLT polynomial that corresponds to (10) specializes at $t = 1$ to the elementary symmetric function $e_{\rho(\gamma)}(\boldsymbol{x})$. Hence proving (9) would also show that, for all α and all coprime (a, b) we have the "e-positivity":

$$0 \leq_e C_\alpha^{(a,b)}(\boldsymbol{x}; q, 1).$$

This is to say that we have an expansion in the elementary symmetric function basis with coefficients in $\mathbb{N}[q, t]$ (for more on this, see Section 4). A proof of (9) has been announced by Carlson and Mellit (see [8]), and their paper is currently under review.

First questions

As underlined previously, see (6), the image under ∇ of good seeds must be Schur-positive. It is thus natural to consider all situations for which this is believed to be the case. The oldest such conjecture (formulated in [5]) suggests the following.

Question. 1. *Can we prove the Schur positivity*

$$(12) \qquad 0 \preceq_s \widehat{s}_\mu^{(a,b)}(\boldsymbol{x}; q, t),$$

for all partition μ of d, and all coprime $a, b \geq 1$? Can we explain this in terms of bigraded subrepresentations[5] of the $\mathbb{S}_n$-module of generalized diagonal harmonics?

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[4]The indices underline that the calculation of $\mathrm{dinv}(\pi)$ and $\mathbf{co}(\pi)$ are dependent on $m = ad$ and $n = bd$.

[5]For a clearer statement concerning this, see Section 3.

For example, we have

$$
\begin{aligned}
\widehat{s}_3^{(1,2)}(\boldsymbol{x};q,t) \ =\ & (q+t)\,s_{222}(\boldsymbol{x}) + s_{321}(\boldsymbol{x}) + (q+t)\,s_{3111}(\boldsymbol{x}) \\
& + \left(q^2 + qt + t^2 + q + t\right) s_{2211}(\boldsymbol{x}) \\
& + (q+t)\left(q^2 + t^2 + q + t\right) s_{21111}(\boldsymbol{x}) \\
& + (q^4 + q^3 t + q^2 t^2 + q t^3 + t^4 + q^2 t + q t^2) s_{111111}(\boldsymbol{x}),
\end{aligned}
$$

$$
\begin{aligned}
\widehat{s}_{21}^{(1,2)}(\boldsymbol{x};q,t) \ =\ & \left(q^2 + qt + t^2\right) s_{222}(\boldsymbol{x}) \\
& + (q+t)\left(q^2 + t^2 + q + t\right) s_{2211}(\boldsymbol{x}) \\
& + (q+t)\,s_{321}(\boldsymbol{x}) + \left(q^2 + qt + t^2\right) s_{3111}(\boldsymbol{x}) \\
& + (q^4 + q^3 t + q^2 t^2 + q t^3 + t^4 + q^3 \\
& \qquad + 2\,q^2 t + 2\,q t^2 + t^3) s_{21111}(\boldsymbol{x}) \\
& + \left(q^2 + qt + t^2\right)\left(q^3 + t^3 + qt\right) s_{111111}(\boldsymbol{x}).
\end{aligned}
$$

Answering Question-1 in the positive would settle many previous conjectures. The special case $\mu = 1^d$ (which coincides for both this open question and the one below) corresponds to the known Schur positivity of $\nabla(e_n)$; and for $(a,1)$, it corresponds the original Shuffle Conjecture [16, Conjecture 3.1]. For $\mu = (d)$, it corresponds to a special case of [18, Conjecture 3.3]. For general μ, and $b = 1$, it corresponds to [5, Conjecture I]. Indeed, this last assertion follows from (3) and (6).

Clearly, if a seed $g_d(\boldsymbol{x})$ expands positively in the basis $\widehat{s}_\mu(\boldsymbol{x};q,t)$, then the associated $g_d^{(a,b)}(\boldsymbol{x};q,t)$ will perforce be Schur-positive if Question-1 is answered positively. Thus we would have

$$
0 \preceq_s e_d^{(a,b)}(\boldsymbol{x};q,t), \qquad \text{and} \qquad 0 \preceq_s \widehat{h}_d^{(a,b)}(\boldsymbol{x};q,t),
$$

as well as

$$
0 \preceq_s \pi_d^{(a,b)}(\boldsymbol{x};q,t), \qquad \text{and} \qquad 0 \preceq_s \widehat{p}_k^{(a,b)}(\boldsymbol{x};q,t),
$$

by Lemma 1.

Another conjecture of [5] asserts the Schur-positivity of ∇ applied to the seeds (8). Together with related experiments, this leads us to our next question.

Question. 2. *Can we prove that the following symmetric functions are Schur-positive*

$$
\text{(13)} \qquad 0 \preceq_s \widehat{m}_\mu^{(a,b)}(\boldsymbol{x};q,t), \qquad \text{and} \qquad 0 \preceq_s \widehat{f}_\mu^{(a,b)}(\boldsymbol{x};q,t),
$$

for all partition μ of d, and all coprime $a, b \geq 1$?

Preferably, this would be explained by introducing adequate bigraded $\mathbb{S}_n$-modules whose bigraded Forbenius characteristic would correspond to these Schur-positive expressions. For example, we have

$$\widehat{m}_{21}^{(1,1)}(\boldsymbol{x}; q, t) = 2\, s_3(\boldsymbol{x}) + (q^2 t + q t^2 + 2\, q^2 + 2\, q t + 2\, t^2 + 2\, q + 2\, t)\, s_{21}(\boldsymbol{x})$$
$$+ (q^3 t + q^2 t^2 + q t^3 + 2\, q^3 + 2\, q^2 t$$
$$+ 2\, q t^2 + 2\, t^3 + 2\, q t)\, s_{111}(\boldsymbol{x}),$$

$$\widehat{f}_{21}^{(1,1)}(\boldsymbol{x}; q, t) = (2\, q t + q + t)\, s_{21}(\boldsymbol{x}) + \left(2\, q^2 t + 2\, q t^2 + q^2 + q t + t^2\right) s_{111}(\boldsymbol{x}).$$

Once again Question-2 relates to previous conjectures. For instance, the case $b = 1$ corresponds to Conjecture IV of [5], which asserts the Schur-positivity of $\nabla^a(\widehat{m}_\mu)$. For both Inequalities (12) and (13), we have checked by explicit computer algebra calculations that we do indeed have Schur-positivity for all possible cases of $\mu \vdash d$ with $1 \leq ad, bd \leq 12$.

Lemma 2. *When ν is a hook, the positivity in* (12) *implies* $0 \preceq_s \widehat{m}_\nu^{(a,b)}(\boldsymbol{x}; q, t)$.

Proof. We need only verify that, for μ a hook, the symmetric function $\widehat{m}_\nu(\boldsymbol{x})$ expands as a linear combination of the symmetric functions $\widehat{s}_\mu(\boldsymbol{x})$, with coefficients in $\mathbb{N}[q, t]$. This simply follows from the fact that, for ν a hook, the sign of $\langle \widehat{m}_\nu(\boldsymbol{x}), s_\mu(\boldsymbol{x}) \rangle$ is precisely $(-1)^{\iota(\mu)}$. $\qquad\square$

3. Schur inclusions

The following considerations (greatly) extend the second observation of [5, Conjecture III]. We now consider Schur-positive differences of operators. From the point of view of representation theory, this corresponds to inclusions of graded $\mathbb{S}_n$-modules. For our current purpose, it is convenient to denote by $g_{m,n}(\boldsymbol{x}; q, t)$ the symmetric function $g_d^{(a,b)}(\boldsymbol{x}; q, t)$, when $(m, n) = (ad, bd)$ and $d = \gcd(m, n)$. As before, we write $g_{m,n}(\boldsymbol{x}; q, t) \preceq_s g_{m,n}(\boldsymbol{x}; q, t)$, if and only if the difference $g_{m,n}(\boldsymbol{x}; q, t) - g_{m,n}(\boldsymbol{x}; q, t)$ is Schur-positive. Our first observation[6] is that

$$(14) \qquad\qquad q^\alpha\, e_{m-1,n}(\boldsymbol{x}; q, t) \ \preceq_s \ e_{m,n}(\boldsymbol{x}; q, t),$$

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[6]Experimentally supported by calculating all cases with $m, n \leq 9$.

where $\alpha = \alpha(m, n)$ is the number of cells between the corresponding staircase paths (see (42) for the definition of the (m, n)-staircase path). At $t = 1$, we may explain combinatorially that

$$(15) \qquad e_{m,n}(\boldsymbol{x}; q, 1) - q^\alpha \, e_{m-1,n}(\boldsymbol{x}; q, 1) \in \mathbb{N}[q][e_1, e_2, \ldots],$$

since the difference between the right-hand side and left-hand side corresponds to a weighted enumeration of the (m, n)-Dyck paths that cannot be obtained from $(m - 1, n)$-Dyck paths by the simple addition of a final horizontal step. On the other hand, the Schur-positivity of (14) is surprising, since it suggests that there is some "dinv" weight-correcting injection between $(m - 1, n)$-Dyck paths and (m, n)-Dyck paths. Such a correction seems far from obvious.

To state our next observed property, we need to introduce the following linear operator. For a partition μ, let us denote by $\overline{\mu}$ the partition obtained by removing the first column of μ. Then, we set $\overline{s_\mu(\boldsymbol{x})} := s_{\overline{\mu}}(\boldsymbol{x})$, and extend linearly to all symmetric functions. We this notation at hand, we have observed that, similarly to (15), we have

$$(16) \qquad q^{\alpha'} \, \overline{e_{m,n-1}(\boldsymbol{x}; q, t)} \ \preceq_s \ \overline{e_{m,n}(\boldsymbol{x}; q, t)}.$$

In this case, much as before, $\alpha' = \alpha'(m, n)$ is the number of integer points between the $(m, n - 1)$-staircase path and the minimal (m, n)-staircase. For example, we have

$$
\begin{aligned}
\overline{e_{4,6}(\boldsymbol{x}; q, t)} - q^2 \, \overline{e_{4,5}(\boldsymbol{x}; q, t)} = {} & \\
& (qt^7 + t^8 + q^2 t^5 + qt^6 + q^4 t^2 + q^3 t^3 + 2\, q^2 t^4 + qt^5)\, s_0(\boldsymbol{x}) \\
& + t\, (q + t)\, (t^5 + qt^3 + t^4 + q^3 + q^2 t + 2\, qt^2 + t^3 + qt) s_1(\boldsymbol{x}) \\
& + t(qt^3 + t^4 + q^3 + q^2 t + 2\, qt^2 + t^3 + q^2 + 2\, qt + t^2) s_2(\boldsymbol{x}) \\
& + (qt^5 + t^6 + q^2 t^3 + 2\, qt^4 + t^5 + q^4 + 2\, q^3 t + 4\, q^2 t^2 \\
& \qquad + 4\, qt^3 + 2\, t^4 + q^2 t + qt^2)\, s_{11}(\boldsymbol{x}) \\
& + t\, (q + t)\, s_3(\boldsymbol{x}) + (q + t)\, (t^3 + q^2 + qt + 2\, t^2 + q + t)\, s_{21}(\boldsymbol{x}) \\
& + (q^2 + qt + t^2)\, (q^3 + t^3 + qt + q + t)\, s_{111}(\boldsymbol{x}) \\
& + (q + t)\, s_{31}(\boldsymbol{x}) + (q^2 + qt + t^2)\, s_{22}(\boldsymbol{x})
\end{aligned}
$$

Among other interesting inequalities, we have

$$(17) \qquad q\, \widehat{s}^{(a,b)}_{(j+1 \mid k-1)}(\boldsymbol{x}; q, t) \ \preceq_s \ \widehat{s}^{(a,b)}_{(j \mid k)}(\boldsymbol{x}; q, t),$$

for two "consecutive" hooks[7]. For instance, the inequalities

$$q^3\,\widehat{s}_4^{(a,b)}(\boldsymbol{x};q,t) \;\preceq_s\; q^2\,\widehat{s}_{31}^{(a,b)}(\boldsymbol{x};q,t)$$
$$\preceq_s\; q\,\widehat{s}_{211}^{(a,b)}(\boldsymbol{x};q,t) \;\preceq_s\; \widehat{s}_{1111}^{(a,b)}(\boldsymbol{x};q,t),$$

correspond to the respective Schur-positive differences

$$\widehat{s}_{31}^{(a,b)}(\boldsymbol{x};q,t) - q\,\widehat{s}_4^{(a,b)}(\boldsymbol{x};q,t)$$
$$= t^2 s_{22}(\boldsymbol{x}) + t s_{31}(\boldsymbol{x}) + t\left(t^2 + q + t\right) s_{211}(\boldsymbol{x}) + t^2\left(t^2 + q\right) s_{1111}(\boldsymbol{x}),$$

$$\widehat{s}_{211}^{(a,b)}(\boldsymbol{x};q,t) - q\,\widehat{s}_{31}^{(a,b)}(\boldsymbol{x};q,t)$$
$$= t^2 s_{31}(\boldsymbol{x}) + t\left(t^2 + q\right) s_{22}(\boldsymbol{x}) + t^2\left(t^2 + q + t\right) s_{211}(\boldsymbol{x})$$
$$+ t^3\left(t^2 + q\right) s_{1111}(\boldsymbol{x}),$$

$$\widehat{s}_{1111}^{(a,b)}(\boldsymbol{x};q,t) - q\,\widehat{s}_{211}^{(a,b)}(\boldsymbol{x};q,t)$$
$$= s_4(\boldsymbol{x}) + \left(t^4 + qt^2 + q^2 + qt + t^2\right) s_{22}(\boldsymbol{x})$$
$$+ \left(t^3 + q^2 + qt + t^2 + q + t\right) s_{31}(\boldsymbol{x})$$
$$+ \left(t^5 + qt^3 + t^4 + q^3 + 2\,q^2 t + 2\,qt^2 + t^3 + qt\right) s_{211}(\boldsymbol{x})$$
$$+ t\left(t^5 + qt^3 + q^3 + q^2 t + qt^2\right) s_{1111}(\boldsymbol{x}).$$

The compositional (ad, bd)-shuffle conjecture implies inequality (17). Indeed, we have the identity (shown in [5])

$$\sum_{\alpha \vdash k} \boldsymbol{C}_j \boldsymbol{C}_\alpha(1) = \widehat{s}_{(j\,|\,k)}(\boldsymbol{x}) + \frac{1}{q}\widehat{s}_{(j+1\,|\,k-1)}(\boldsymbol{x}).$$

This also shows that recent settling of the compositional (ad, bd)-shuffle conjecture answers in the affirmative the first part of Question-1 for any hook shapes (see [18, Proposition 5.3]).

For all (m, n), we have also observed (calculating all cases for $m, n \leq 8$) that the following inequality seems to hold

(18) $q^\beta\,e_{m-1,n}(\boldsymbol{x};q,t) \;\preceq_s\; \widehat{h}_{m,n}(\boldsymbol{x};q,t),$

with $\beta = \alpha(m, n) - d + 1$, for $d = \gcd(m, n)$. In other words, this is the number of points that lie between the diagonal avoiding (m, n)-staircase

---

[7]Notice that j is the ι-function value of the hook $(j\,|\,k)$.

path, and the $(m-1, n)$-staircase path. Once again, there seems to be a transpose version of this

$$(19) \qquad q^{\beta'} \overline{e_{m,n-1}(\boldsymbol{x}; q, t)} \preceq_s \overline{\widehat{h}_{m,n}(\boldsymbol{x}; q, t)},$$

with β' defined suitably. Together with (17), inequality (18) refines the inequality in (14). Hence we are led to ask the following:

Question. 3. *Can we prove that, for all coprime $a, b \geq 1$, all j and k, and all $m, n \geq 1$, the Schur inclusions* (17) *and* (18) *hold?*

As well as

Question. 4. *Can we prove that, for all pair $m, n \geq 1$, the Schur inclusions* (16) *and* (19) *hold?*

Preferably, these "facts" would be explained in terms of inclusion of bigraded representations. Observe that, up to applying a sequence of such inclusions, we may include any of the relevant expressions as sub-expressions of $\nabla^a(e_n) = e_{an,n}(\boldsymbol{x}; q, t)$ which is conjectured to give the bigraded Frobenius characteristic of the $\mathbb{S}_n$-module $\mathcal{C}_n^{(a)}$ of the generalized diagonal coinvariant $\mathbb{S}_n$-module[8]. Hence, Schur-positivity of the above differences would imply that we have bigraded-monomorphism between associated $\mathbb{S}_n$ modules, all of which included in $\mathcal{C}_n^{(a)}$, for a large enough. Experiments suggest that these modules are ideals, generated by correctly chosen lowest degree components.

Transpose sub-symmetry

Following a somewhat different track, we have another kind of inclusion involving a matrix like "transposition". This seems to be a very general phenomenon that we have checked for all positive seeds considered here, as well as in the cases that correspond to the compositional (ad, bd)-shuffle conjecture (see [3]). The most general question may be coined as follows:

Question. 5. *Can we prove the Schur-inclusion*

$$(20) \qquad \overline{g_d^{(b,a)}(\boldsymbol{x}; q, t)} \preceq_s \overline{g_d^{(a,b)}(\boldsymbol{x}; q, t)},$$

for all coprime $b \geq a \geq 1$, and any good seed g_d?

---

[8]Recall that the case $a = 1$ has been shown to hold in [15].

We underline that the functions $g_d^{(b,a)}(\boldsymbol{x}; q, t)$ and $g_d^{(a,b)}(\boldsymbol{x}; q, t)$ are of different degrees; equal to ad in the first case, and bd in the second. Hence they can only be compared after applying the $\overline{(-)}$ operator, which results in a symmetric function having components of various degrees. For example, we have

$$
\begin{aligned}
\overline{e_1^{(5,3)}(\boldsymbol{x}; q, t)} &= (q+t)\, s_2(\boldsymbol{x}) + (q+t)\left(q^2 + t^2 + q + t\right) s_1(\boldsymbol{x}) \\
&\quad + \left(q^4 + q^3 t + q^2 t^2 + q t^3 + t^4 + q^2 t + q t^2\right) s_0(\boldsymbol{x}),
\end{aligned}
$$

$$
\overline{e_1^{(3,5)}(\boldsymbol{x}; q, t)} = \overline{e_1^{(5,3)}(\boldsymbol{x}; q, t)} + s_{21}(\boldsymbol{x}) + \left(q^2 + qt + t^2 + q + t\right) s_{11}(\boldsymbol{x}).
$$

As alluded to above, statement (20) has been checked by explicit computer algebra calculations for all cases involving either $\widehat{s}_\mu^{(a,b)}(\boldsymbol{x}; q, t)$, $\widehat{m}_\mu^{(a,b)}(\boldsymbol{x}; q, t)$, or $\widehat{f}_\mu^{(a,b)}(\boldsymbol{x}; q, t)$, for all partitions μ, all compositions α, and all coprime pairs (a, b) for which the overall degree of the resulting function is at most 12. Hence it holds for all situations that can be expressed as positive linear combinations of these.

4. e-positivity and specializations at $t = 1$, and $t = 1 + r$

Our next considerations concern an interesting feature of the specialization of the operators at $t = 1$. Indeed, the resulting operators appear to be much simpler operators than their general counterpart. Indeed, one observes experimentally[9] that

$$
(21) \qquad \Theta_{a,b}(g_d)(g(\boldsymbol{x}))\Big|_{t=1} = g_d^{(a,b)}(\boldsymbol{x}; q, 1) \cdot g(\boldsymbol{x}).
$$

This states that the effect of the operator $\Theta_{a,b}(g_d)\big|_{t=1}$ on any symmetric function $g(\boldsymbol{x})$ corresponds to multiplication of $g(\boldsymbol{x})$ by the fixed symmetric function[10] $g_d^{(a,b)}(\boldsymbol{x}; q) := g_d^{(a,b)}(\boldsymbol{x}; q, 1)$. In other words, at $t = 1$, the monomorphism $\Theta_{a,b}$ may be considered as graded-algebra homomorphism

$$
\Theta_{a,b}\Big|_{t=1} : \bigoplus_{d \geq 0} \Lambda_d \longrightarrow \bigoplus_{d \geq 0} \Lambda_{bd}.
$$

sending g_d to (multiplication by) $g_d^{(a,b)}(\boldsymbol{x}; q)$. Notice here the "multiplicative" shift in grading, $d \mapsto bd$. Implicit in statement (21) is the "multiplicativity"

---

[9]This will be supported by actual results in the sequel.

[10]Henceforth we make our notation more compact by dropping the "1".

$$(22) \qquad (g_d \, g'_k)^{(a,b)}(\boldsymbol{x}; q) = g_d^{(a,b)}(\boldsymbol{x}; q) \, g'^{(a,b)}_k(\boldsymbol{x}; q).$$

Thus all of this would be implied by a positive answer to the following:

Question. 6. *Can we prove that the operator* $\Theta_{a,b}(g_d)\big|_{t=1}$ *operates by multiplication by* $g_d^{(a,b)}(\boldsymbol{x}; q, 1)$*, for all seed* g_d *and all coprime* $a, b \geq 1$*?*

Observe that it is clearly sufficient to answer this question for any given family of algebraic generators of Λ, say $\{\pi_d\}_{d \in \mathbb{N}}$ or $\{e_d\}_{d \in \mathbb{N}}$. Recall also from [5] that $\widetilde{\nabla}$, the linear operator obtained from ∇ by specializing t to 1, is multiplicative. Hence, we get the following.

Proposition 1. *If* $\Theta_{a,b}(g_d)\big|_{t=1}$ *operates by multiplication by* $g_d^{(a,b)}(\boldsymbol{x}; q)$*, then* $\Theta_{a+b,b}(g_d)\big|_{t=1}$ *also operates by multiplication by* $g_d^{(a+b,b)}(\boldsymbol{x}; q)$*.*

Proof. Using (3) and (5) specialized at $t = 1$, we calculate that, for any symmetric function $g(\boldsymbol{x})$,

$$
\begin{aligned}
\Theta_{a+b,b}(g_d)\big|_{t=1}(g(\boldsymbol{x})) &= \widetilde{\nabla} \Theta_{a,b}(g_d)\big|_{t=1} \widetilde{\nabla}^{-1}(g(\boldsymbol{x})) \\
&= \widetilde{\nabla} \left[g_d^{(a,b)}(\boldsymbol{x}; q) \cdot \widetilde{\nabla}^{-1}(g(\boldsymbol{x})) \right] \\
&= \widetilde{\nabla}(g_d^{(a,b)}(\boldsymbol{x}; q)) \cdot \widetilde{\nabla}(\widetilde{\nabla}^{-1}(g(\boldsymbol{x}))) \\
&= g_d^{(a+b,b)}(\boldsymbol{x}; q) \cdot g(\boldsymbol{x}),
\end{aligned}
$$

which shows the required property. $\qquad \qquad \square$

Observe also that, to answer Question-6 positively in all instances, we need only show that $\Theta_{a,b}(e_d)\big|_{t=1}$ operates by multiplication. To this end, let us recall the following conjectured constant term formula of Negut (see [20]),

$$
\begin{aligned}
\Theta_{a,b}(e_d)(g(\boldsymbol{x})) &= \mathrm{CT}\Big(\frac{g[\boldsymbol{x} + M \sum_{i=1}^m z_i^{-1}]}{\boldsymbol{z}_{m,n}} \prod_{i=1}^{m-1} \frac{z_i \, \Omega'[\boldsymbol{x}; z_i]}{z_i - q t z_{i+1}} \times \\
(23) \qquad & \prod_{1 \leq i < j \leq m} \frac{(z_i - z_j)(z_i - q t z_j)}{(z_i - q z_j)(z_i - t z_j)} \Big)
\end{aligned}
$$

for the calculation of the operators $\Theta_{a,b}(e_d)$, where the constant term is calculated with respect to the variables $\boldsymbol{z} = z_1, \ldots, z_m$, and

$$
\boldsymbol{z}_{m,n} := \prod_{i=1}^m z_i^{\lfloor i\, n/m \rfloor - \lfloor (i-1)\, n/m \rfloor}.
$$

We use here the notation $\Omega'[\boldsymbol{x}; z] := \sum_{n \geq 0} e_n(\boldsymbol{x}) z^n$ for the **dual Cauchy kernel**[11].

Specializing at $t = 1$ this constant term formula, one finds the following further support for the "fact" that our operators have this multiplicative property at $t = 1$.

Proposition 2. *Let (m, n) be equal to (ad, bd), with $d = \gcd(m, n)$, then Negut's conjecture (23) implies that*

$$(24) \qquad \Theta_{a,b}(e_d)\big|_{t=1}(g(\boldsymbol{x})) = \mathrm{CT}\left(\frac{1}{\boldsymbol{z}_{m,n}} \prod_{i=1}^{m-1} \frac{z_i}{z_i - q z_{i+1}} \Omega'[\boldsymbol{x}; z_i]\right) \cdot g(\boldsymbol{x}),$$

It is noteworthy that a combinatorial argument, discussed in [3], shows that the constant term involved in the right-hand side of (24) corresponds to the enumeration of (m, n)-Dyck paths by area and risers, that is

$$
\begin{aligned}
e_{m,n}(\boldsymbol{x}; q) &= \mathrm{CT}\left(\frac{1}{\boldsymbol{z}_{m,n}} \prod_{i=1}^{m-1} \frac{z_i}{z_i - q z_{i+1}} \Omega'[\boldsymbol{x}\, z_i]\right) \\
(25) &= \sum_{\gamma} q^{\mathrm{area}(\gamma)} e_{\rho(\gamma)}(\boldsymbol{x}),
\end{aligned}
$$

with the sum running over the set of (ad, bd)-Dyck paths. One easily gets a similar constant term formula for the enumeration of (m, n)-Dyck paths with no return to the diagonal, except at both ends. To this end, one simply replaces $\boldsymbol{z}_{m,n}$ by $\boldsymbol{z}_{m,n}/(z_1 z_2 \cdots z_m)$, and it corresponds (conjecturally) to the specialization at $t = 1$ of a constant term formula for $\widehat{h}_d^{(a,b)}(\boldsymbol{x}; q, t)$.

Another interesting feature of this specialization at $t = 1$ is made apparent for special seeds. Indeed, for these special cases, the symmetric function $g_d^{(a,b)}(\boldsymbol{x}; q)$ appears to expand with coefficients in $\mathbb{N}[q]$ in the basis of elementary symmetric functions e_μ, for μ partitions of bd. As previously mentioned, we then say that $g_d^{(a,b)}$ is e-**positive**, and write $0 \leq_e g_d^{(a,b)}$. This is clearly stronger than Schur-positivity, since it is classical that each e_μ is itself Schur-positive. In fact, an even stronger version of e-positivity seems to be at play here, as stated by the following, which has been checked explicitly for all $j + k = d - 1$, and all a, b such that $1 \leq ad, bd \leq 8$.

Question. 7. *Can we prove the following e-positivity and e-inclusion*

---

[11]The Cauchy kernel $\Omega[\boldsymbol{x}; z]$, obtained by replacing $e_n(\boldsymbol{x})$ replaced by $h_n(\boldsymbol{x})$, is naturally related to the standard scalar product of symmetric functions.

$$0 \leq_e \widehat{h}_d^{(a,b)}(\boldsymbol{x};q,1+r), \quad \text{and} \quad q\,\widehat{s}_{(j+1\,|\,k-1)}^{(a,b)}(\boldsymbol{x};q,1+r) \leq_e \widehat{s}_{(j\,|\,k)}^{(a,b)}(\boldsymbol{x};q,1+r),$$

for all $j + k = d - 1$, and all coprime $a, b \geq 1$?

Exploiting the transitivity of the order, this would imply that

$$\widehat{s}_{(j\,|\,k)}^{(a,b)}(\boldsymbol{x};q,1+r)$$

itself is e-positive, since $h_d(\boldsymbol{x}) = s_{(d-1|0)}(\boldsymbol{x})$. This also implies (setting $r = 0$) that

$$0 \leq_e e_d^{(a,b)}(\boldsymbol{x};q), \quad \text{and} \quad 0 \leq_e \pi_d^{(a,b)}(\boldsymbol{x};q),$$

in view of the definition of $\pi_d(\boldsymbol{x};q,t)$. For example, for the seed $e_d(\boldsymbol{x})$, some explicit values are

$$e_2^{(1,3)}(\boldsymbol{x};q) = q^3\,e_6(\boldsymbol{x}) + q^2 e_{51}(\boldsymbol{x}) + q\,e_{42}(\boldsymbol{x}) + e_{33}(\boldsymbol{x}),$$

$$\begin{aligned}
e_2^{(1,2)}(\boldsymbol{x};q) = {}& q^6\,e_6(\boldsymbol{x}) + q^4\,(q+1)\,e_{51}(\boldsymbol{x}) + q^2\,(q^2+2)\,e_{42}(\boldsymbol{x}) \\
&+ q^3 e_{411}(\boldsymbol{x}) + q^3 e_{33}(\boldsymbol{x}) + q\,(q+2)\,e_{321}(\boldsymbol{x}) + e_{222}(\boldsymbol{x}),
\end{aligned}$$

$$\begin{aligned}
e_2^{(2,3)}(\boldsymbol{x};q) = {}& q^8\,e_6(\boldsymbol{x}) + q^5\,(q^2+q+1)\,e_{51}(\boldsymbol{x}) + q^4(q^2+2)\,e_{42}(\boldsymbol{x}) \\
&+ q^3(q^2+q+1)\,e_{411}(\boldsymbol{x}) + q^2\,(q^3 + 2\,q^2 + 2\,q + 1)\,e_{33}(\boldsymbol{x}) \\
&+ q\,(q^3 + 3\,q^2 + q + 2)\,e_{321}(\boldsymbol{x}) + q^2 e_{3111}(\boldsymbol{x}) \\
&+ q^2 e_{222}(\boldsymbol{x}) + (q+1)\,e_{2211}(\boldsymbol{x})^2.
\end{aligned}$$

Now, as discussed in [3], the e-positive symmetric functions $g_d^{(a,b)}(\boldsymbol{x};q)$ considered often appear to expand as a weighted sum, over combinatorial objects, of powers of q multiplied by some elementary symmetric function, giving a combinatorial explanation why they are e-positive. The relevant combinatorial objects are discussed in Appendix.

It is interesting to underline the following fact, which reduces the proof of e-positivity to the cases where $a \leq b$.

Proposition 3. If $g_d^{(a,b)}(\boldsymbol{x};q)$ is e-positive, then so is $g_d^{(a+b,b)}(\boldsymbol{x};q)$.

Proof. Recall from [5] that on top of $\widetilde{\nabla}$ being is multiplicative, we have that $\widetilde{\nabla}(e_k)$ is e-positive. Hence, $\widetilde{\nabla}(e_\lambda) = \prod_{k \in \lambda} \widetilde{\nabla}(e_k)$ is e-positive for all λ, and we get the announced property since we get $g_d^{(a+b,b)}(\boldsymbol{x};q)$ by applying $\widetilde{\nabla}$ to the e-positive expression $g_d^{(a,b)}(\boldsymbol{x};q)$. $\square$

Many other instances of e-positivity seem to occur, but they still have to be explained combinatorially. A tantalizing fact along these lines, discussed in [15, see Prop 2.3.4], is that the expression

$$\langle p_1(\boldsymbol{x})^n, e_{n,n}(\boldsymbol{x}; 1, 1+r)\rangle$$

enumerates connected graphs, r-weighted by the number of edges. Extensive experiments, including all cases of degree ≤ 8, lead to the following.

Question. 8. *Can we prove that, for any partition μ of d, and any coprime $a, b \geq 1$, we have the e-positivity*

$$(26) \qquad\qquad 0 \leq_e \widehat{m}_\mu^{(a,b)}(\boldsymbol{x}; q, 1+r),$$

Furthermore, can we explain this e-positivity in terms of a combinatorial enumeration in the style of (25)?

For example, we have

$$\widehat{m}_{21}^{(1,1)}(\boldsymbol{x}; q, 1+r)$$
$$= 2\, e_1(\boldsymbol{x})^3 + \Big(q^2 r + qr^2 + 3\,q^2 + 4\,q\,r + 2\,r^2 + 5\,q + 6\,r\Big) e_1(\boldsymbol{x}) e_2(\boldsymbol{x})$$
$$+ \Big(q^3 r + q^2 r^2 + qr^3 + 3\,q^3 + 3\,q^2 r + 4\,qr^2 + 2\,r^3 + 5\,q\,r + 4\,r^2\Big) e_3(\boldsymbol{x}).$$

Positive answers to these and Question-6 would imply many relations between Schur-positive expressions. For instance, one obtains Bizley-like formulas in the form[12]

$$(27) \qquad \sum_{d \geq 0} e_d^{(a,b)}(\boldsymbol{x}; q)\, z^d \;=\; \exp\Big(\sum_{j \geq 1} \widehat{p}_j^{(a,b)}(\boldsymbol{x}; q)\, z^j / j\Big),$$

$$(28) \qquad \sum_{d \geq 1} \widehat{h}_d^{(a,b)}(\boldsymbol{x}; q)\, z^d \;=\; q - q\, \exp\Big(\sum_{j \geq 1} -\widehat{p}_j^{(a,b)}(\boldsymbol{x}; q)\, (z/q)^j / j\Big).$$

Other interesting observations concern the compositional (ad, bd)-shuffle conjecture of [3], specialized at $t = 1$. Indeed, as discussed in [18], the evaluation at 1 of the operator $\boldsymbol{C}_\alpha$ specializes, at $t = 1$, to the product of the $\widehat{h}_k(\boldsymbol{x})$ with k running over parts of α, where α is a composition of d. Hence, modulo our above observations and if (9) holds, we should have

---

[12]The differences in signs from the analogous generating function formulas for e_d and h_d correspond to the use of $\widehat{p}_d$ and $\widehat{h}_d$ which contain signs.

$$(29) \qquad C_\alpha^{(a,b)}(\boldsymbol{x};q) = \sum_\gamma q^{\mathrm{area}(\gamma)} e_{\rho(\gamma)}(\boldsymbol{x}),$$

where γ runs over the set of (m,n)-Dyck paths that return to the diagonal at the points

$$(a\,\alpha_i, b\,\alpha_i), \qquad \text{for} \quad \alpha_i = k_1 + \ldots + k_i,$$

with i varying between 0 and ℓ. Thus, some cases that are common to (26) and (29) are consequences of the (9), in particular this is so for $e_{m,n}(\boldsymbol{x};q)$.

Using the combinatorial interpretation (29), we may readily see that the specialization at $t = 1$ of (14) and (18) hold. In fact, the relevant differences are in fact Schur-positive, since we have inclusion between the sets of paths enumerated by each expression. Hence we get the following.

Proposition 4. *For all m and n, we have the e-inclusions*

$$(30) \qquad q^\alpha\, e_{m-1,n}(\boldsymbol{x};q) \quad \leq_e \quad e_{m,n}(\boldsymbol{x};q), \qquad \text{and}$$

$$(31) \qquad q^\beta\, e_{m-1,n}(\boldsymbol{x};q) \quad \leq_e \quad \widehat{h}_{m,n}(\boldsymbol{x};q).$$

This raises the question of whether we have e-positivity when setting $t = 1 + r$, namely

Question. 9. *Can we prove the e-inclusions*

$$(32) \qquad q^\alpha\, e_{m-1,n}(\boldsymbol{x};q,1+r) \quad \leq_e \quad e_{m,n}(\boldsymbol{x};q,1+r), \qquad \text{and}$$

$$(33) \qquad q^\beta\, e_{m-1,n}(\boldsymbol{x};q,1+r) \quad \leq_e \quad \widehat{h}_{m,n}(\boldsymbol{x};q,1+r).$$

for all $m,n \geq 1$, and explain this combinatorially?

These statements have been explicitly checked to hold for all $m,n \leq 8$. We might get a possible explanation for these properties if we could show that the relevant symmetric functions expand positively in terms of the LLT-polynomials $\gamma_{m,n}(\boldsymbol{x};t)$ (see (10)) associated to certain paths γ (for some (m,n)). Recall that this is essentially the content of the (ad, bd)-compositional shuffle conjecture, which covers part of the story here. As we have mentioned earlier, it has been shown that $\gamma(\boldsymbol{x};t)$ is Schur-positive, and that $\gamma_{m,n}(\boldsymbol{x};1)$ is an elementary symmetric function.

Moreover, for $m \geq n$, we have experimentally observed that these LLT-polynomials expand positively in the elementary function basis when specialized at $t = 1 + r$, so that

$$(34) \qquad 0 \leq_e \gamma_{m,n}(\boldsymbol{x};1+r).$$

Which is to say that coefficients of the e-expansion of the right-hand side are positive integer polynomials in the variable r. This (apparently) new property has been explicitly checked for all (m, n)-Dyck paths, with $n \leq 6$ and all values[13] of m. Thus, having positive expansions of some $g_d^{(a,b)}(\boldsymbol{x}; q, t)$ in terms of LLT-polynomials satisfying (34) would give a joint explanation for both: the Schur positivity of $g_d^{(a,b)}(\boldsymbol{x}; q, t)$, and the e-positivity of $g_d^{(a,b)}(\boldsymbol{x}; q, 1 + r)$.

5. Specialization at $q = t = 1$

We simplify our notation in this section, writing $g_d^{(a,b)}(\boldsymbol{x})$ instead of $g_d^{(a,b)}(\boldsymbol{x}; 1, 1)$, and follow the logic of our previous conventions so that

$$\pi_\mu^{(a,b)}(\boldsymbol{x}) := \pi_{\mu_1}^{(a,b)}(\boldsymbol{x})\pi_{\mu_2}^{(a,b)}(\boldsymbol{x}) \cdots \pi_{\mu_\ell}^{(a,b)}(\boldsymbol{x}).$$

Once again we assume that $(m, n) = (ad, bd)$, with (a, b) a coprime pair. Then, an argument similar to that of [6] (using (27) and (25)) shows that

$$(35) \qquad \pi_d^{(a,b)}(\boldsymbol{x}) \;=\; \widehat{p}_d^{(a,b)}(\boldsymbol{x})$$

$$(36) \qquad\qquad\qquad = \; \frac{d}{m}\, e_n[m\,\boldsymbol{x}] = \frac{1}{a}e_{db}[da\,\boldsymbol{x}],$$

so that, using the multiplicativity property (22),

$$(37) \qquad \pi_\mu^{(a,b)}(\boldsymbol{x}) = \prod_{k \in \mu} \frac{1}{a} e_{kb}[ka\,\boldsymbol{x}].$$

From this it follows, using the expansion of g_d in the π_μ bases (2), that we have a generalized Bizley-like formula

$$g_d^{(a,b)}(\boldsymbol{x}) \;=\; \sum_{\mu \vdash d} c_\mu\, \pi_\mu^{(a,b)}(\boldsymbol{x})$$

$$(38) \qquad\qquad = \; \sum_{\mu \vdash d} c_\mu \prod_{k \in \mu} \frac{1}{a} e_{kb}[ka\,\boldsymbol{x}],$$

if we have the expansion

---

[13]Up to a global power of t, there are but a finite number of LLT-polynomials that may occur for a given value of n.

$$g_d(\boldsymbol{x}) = \sum_{\mu \vdash d} c_\mu \, \pi_\mu(\boldsymbol{x}).$$

For example,

$$
\begin{aligned}
e_3^{(a,b)}(\boldsymbol{x}) &= \frac{1}{3}\pi_3^{(a,b)}(\boldsymbol{x}) + \frac{1}{2}\pi_2^{(a,b)}(\boldsymbol{x})\,\pi_1^{(a,b)}(\boldsymbol{x}) + \frac{1}{6}\pi_1^{(a,b)}(\boldsymbol{x})^3, \\
&= \frac{1}{3\,a}\,e_{3\,b}[3\,a\,\boldsymbol{x}] + \frac{1}{2\,a^2}\,e_{2\,b}[2\,a\,\boldsymbol{x}]\,e_b[a\,\boldsymbol{x}] + \frac{1}{6\,a^3}\,(e_b[a\,\boldsymbol{x}])^3\,, \\
\widehat{s}_{21}^{(a,b)}(\boldsymbol{x}) &= \frac{1}{3}\pi_3^{(a,b)}(\boldsymbol{x}) - \frac{1}{3}\pi_1^{(a,b)}(\boldsymbol{x})^3, \\
&= \frac{1}{3\,a}\,e_{3\,b}[3\,a\,\boldsymbol{x}] - \frac{1}{3\,a^3}\,(e_b[a\,\boldsymbol{x}])^3\,, \\
h_3^{(a,b)}(\boldsymbol{x}) &= \frac{1}{3}\pi_3^{(a,b)}(\boldsymbol{x}) - \frac{1}{2}\pi_2^{(a,b)}(\boldsymbol{x})\pi_1^{(a,b)}(\boldsymbol{x}) + \frac{1}{6}\pi_1^{(a,b)}(\boldsymbol{x})^3, \\
&= \frac{1}{3\,a}\,e_{3\,b}[3\,a\,\boldsymbol{x}] - \frac{1}{2\,a^2}\,e_{2\,b}[2\,a\,\boldsymbol{x}]\,e_b[a\,\boldsymbol{x}] + \frac{1}{6\,a^3}\,(e_b[a\,\boldsymbol{x}])^3\,.
\end{aligned}
$$

Let us now consider the linear transformations on symmetric functions

$$\delta(g(\boldsymbol{x})) := \langle p_1(\boldsymbol{x})^n, g(\boldsymbol{x})\rangle, \qquad \text{and} \qquad \varepsilon(g(\boldsymbol{x})) := \langle e_n(\boldsymbol{x}), g(\boldsymbol{x})\rangle,$$

for which we clearly have

$$
\begin{aligned}
\delta(g_{d_1}(\boldsymbol{x}))g_{d_2}(\boldsymbol{x}))\cdots g_{d_\ell}(\boldsymbol{x})) &= \binom{n}{d_1, d_2, \ldots, d_\ell} \prod_{i=1}^{\ell} \delta(g_{d_i}(\boldsymbol{x})), \\
\varepsilon(g_{d_1}(\boldsymbol{x}))g_{d_2}(\boldsymbol{x}))\cdots g_{d_\ell}(\boldsymbol{x})) &= \prod_{i=1}^{\ell} \varepsilon(g_{d_i}(\boldsymbol{x})),
\end{aligned}
$$

where $n = d_1 + d_2 + \ldots + d_\ell$. Also recall that

$$\delta(g_d^{(a,b)}(\boldsymbol{x})) = \dim(M_{g_d}^{(a,b)}), \qquad \text{and} \qquad \varepsilon(g_d^{(a,b)}(\boldsymbol{x})) = \dim\left(M_{g_d}^{(a,b)}\right)^{\pm},$$

whenever $g_d^{(a,b)}(\boldsymbol{x})$ may be interpreted as the Frobenius characteristic of some $\mathbb{S}_n$-module $M_{g_d}^{(a,b)}$, with $(M_{g_d}^{(a,b)})^{\pm}$ standing for the alternating isotypic component of this $\mathbb{S}_n$-module. Since

$$\delta(\pi_d^{(a,b)}(\boldsymbol{x})) = d\,m^{n-1} = d^{bd}a^{bd-1},$$

and

$$\varepsilon(\pi_d^{(a,b)}(\boldsymbol{x})) = \frac{d}{m+n}\binom{n+m}{n} = \frac{1}{a+b}\binom{(a+b)d}{bd},$$

for any partition μ of d, with $(m, n) = (ad, bd)$ and $d = \gcd(m, n)$ as before, we have

$$\delta(\pi_\mu^{(a,b)}(\boldsymbol{x})) = \binom{n}{a\mu} a^{n-\ell(\mu)} \prod_{k \in \mu} k^{bk},$$

$$\varepsilon(\pi_\mu^{(a,b)}(\boldsymbol{x})) = \frac{1}{(a+b)^{\ell(\mu)}} \prod_{k \in \mu} \binom{(a+b)k}{bk},$$

where we use[14] the partition multinomial notation

$$\binom{n}{a\,\mu} := \frac{n!}{(a\,\mu_1)! \cdots (a\,\mu_\ell)!}.$$

Thus, for $M_{g_d}^{(a,b)}$ the be the required $\mathbb{S}_n$-module would have to have the dimension formulas

$$(39) \qquad \dim(M_{g_d}^{(a,b)}) = \sum_{\mu \vdash d} c_\mu \binom{n}{a\,\mu} a^{n-\ell(\mu)} \prod_{k \in \mu} k^{kb-1}, \qquad \text{and}$$

$$(40) \qquad \dim(M_{g_d}^{(a,b)})^\pm = \sum_{\mu \vdash d} \frac{c_\mu}{(a+b)^{\ell(\mu)}} \prod_{k \in \mu} \binom{(a+b)k}{bk},$$

with the coefficients c_μ coming from the expansion (2). Observe that, in view of the dual Cauchy formula, the right-hand side of (36) affords a positive integer coefficient expansion in the e-basis given by the formula

$$\frac{d}{m} e_n[m\,\boldsymbol{x}] = \sum_{\lambda \vdash n} e_\lambda(\boldsymbol{x}) \frac{d}{m} h_\lambda[m]$$

$$= \sum_{\lambda \vdash n} e_\lambda(\boldsymbol{x}) \frac{d}{m} \prod_{k \in \lambda} \binom{m+k-1}{k},$$

with $d = \gcd(m, n)$ as before. Recalling that $\langle e_n(\boldsymbol{x}), e_\lambda(\boldsymbol{x}) \rangle = 1$ for all partition λ of n, it follows that the sum of the coefficients of (38), when expanded in the e-basis, must be equal to the number of copies of the alternating representations in M_{g_d}. In other words, it is the dimension of $(M_{g_d}^{(a,b)})^\pm$, as given by (40).

---

[14]Observe that $a\mu$ is a partition of n, with parts $a\mu_i$.

Other specializations

Other specializations of q and t have been considered in the "classical" context of $e_{n,n}(\boldsymbol{x}; q, t)$. The case $t = 1/q$ gives rise to many interesting formulas, with recent advances discussed in [11]. Also, in [21], the authors set $t = -1$ and $q = 1$ and get interesting combinatorial considerations. Similar specializations of $g_d^{(a,b)}(\boldsymbol{x}; q, t)$, for good seeds, seem to give rise to many interesting combinatorial questions.

Appendix: combinatorics of (m, n)-Dyck paths

Recall that a (m, n)-**Dyck path** can be presented as a south-east lattice path, going from $(0, n)$ to $(m, 0)$, which stays above the (m, n)-**diagonal**. This is the line segment joining $(0, n)$ to $(m, 0)$. See Figure 1 for an example.
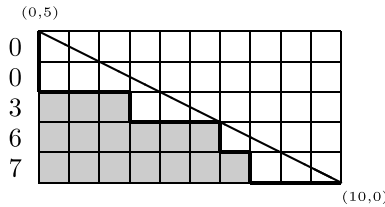

Figure 1: The $(10, 5)$-Dyck path encoded as 00367.

We may encode such paths as (weakly) increasing integer sequences (words)

$$(41) \qquad \gamma = a_1 a_2 \cdots a_n, \qquad \text{with} \quad 0 \le a_k \le (k - 1)\, m/n.$$

Each a_k gives the distance between the y-axis of the (unique) south step that starts at level $n + 1 - k$. If we have equality $a_k = (k - 1)\, m/n$, then k must be equal to $j\, b + 1$, for some $1 \le j < d := \gcd(m, n)$. When this is the case, we say that we have a **return** to the diagonal at position j. The resulting set of returns may be encoded as a composition of d, using a classical correspondence with subsets of $\{1, \ldots, d - 1\}$ and compositions α of d. Recall that this bijective correspondence associates to a composition $\alpha = (c_1, \ldots, c_k)$ the set of partial sums $S(\alpha) = \{s_1, s_2, \ldots, s_k\}$, where

$$s_i = c_1 + c_2 + \cdots + c_i, \qquad \text{with} \qquad 1 \le i < k.$$

The (m, n)-Dyck that stays "closest" to the diagonal is called the (m, n)-**staircase path**

$$(42) \qquad \delta_{m,n} := d_1 d_2 \cdots d_n, \qquad \text{with} \qquad d_k := \lfloor (k-1)\, m/n \rfloor.$$

For example, we have

$$\begin{aligned}
\delta_{1,4} &= 0000, \quad \delta_{2,4} = 0011, \quad \delta_{3,4} = 0012, \quad \delta_{4,4} = 0123, \\
\delta_{5,4} &= 0123, \quad \delta_{6,4} = 0134, \quad \delta_{7,4} = 0135, \quad \delta_{8,4} = 0246, \\
\delta_{9,4} &= 0246, \quad \delta_{10,4} = 0257, \quad \delta_{11,4} = 0258, \quad \delta_{12,4} = 0369.
\end{aligned}$$

It is easy to check that $\delta_{kn,n} = \delta_{kn+1,n}$. We denote by $\mathcal{D}_{m,n}$, the set of (m,n)-Dyck paths, and by $C_{m,n}$ its cardinality. For example, we have

$$\begin{aligned}
\mathcal{D}_{5,4} \;=\; &\{0000, 0001, 0002, 0003, 0011, 0012, 0013, \\
&\;\; 0022, 0023, 0111, 0112, 0113, 0122, 0123\}.
\end{aligned}$$

It follows from the observation that $\delta_{kn,n} = \delta_{kn+1,n}$, that we have the set equality

$$(43) \qquad\qquad\qquad \mathcal{D}_{kn,n} = \mathcal{D}_{kn+1,n}.$$

When m and m are coprime, the enumeration of (m,n)-Dyck path is given by the "well" known formula

$$C_{m,n} = \frac{1}{m+n}\binom{m+n}{n}.$$

For the more general situation, when m and n have greatest common divisor $d \geq 1$, the formula was obtained by Bizley [6] in 1954. His argument may be given a more general understanding, using a symmetric function encoding of the multiplicities of parts in (m,n)-Dyck paths. To this end, we consider the **riser composition** $\rho(\gamma)$ of a path γ, which is simply the sequence of multiplicities of the entries of γ. We may then count (m,n)-paths with weight $e_{\rho(\gamma)}(\boldsymbol{x}) := e_{r_1}(\boldsymbol{x}) e_{r_2}(\boldsymbol{x}) \cdots e_{r_k}(\boldsymbol{x})$, if $\rho(\gamma) = r_1 r_2 \cdots r_k$.

Let $(m,n) = (ad, bd)$, with a and b coprime. It may be shown that (see [3])

$$\pi_d^{(a,b)}(\boldsymbol{x}; 1, 1) = \frac{d}{m}\, e_n[m, \boldsymbol{x}],$$

in which one considers m as a constant[15] for the pletystic evaluation of the right-hand side. Then, a symmetric function version of Bizley's formula may

---

[15]This means that $p_k[m\boldsymbol{x}] = m\, p_k(\boldsymbol{x})$.

be written as

$$(44) \qquad \sum_{\mu \vdash d} \pi_\mu^{(a,b)}(\boldsymbol{x}; 1, 1)/z_\mu = \sum_{\gamma \in \mathcal{D}_{ad,bd}} e_{\rho(\gamma)}(\boldsymbol{x}).$$

Recall that, for a partition μ of d having c_i parts of size i, the integers z_μ are defined as

$$z_\mu := \prod_k k^{c_k} c_k!$$

Expressed in generating function terms, formula 44 takes the form

$$(45) \qquad \sum_{d=0}^{\infty} \sum_{\gamma \in \mathcal{D}_{ad,bd}} e_{\rho(\gamma)}(\boldsymbol{x}) \, x^d = \exp\left(\sum_{k \geq 1} \frac{1}{a} e_{bk}[ak\,\boldsymbol{x}] \frac{x^k}{k}\right).$$

For example, we have

$$\sum_{\gamma \in \mathcal{D}_{2a,2b}} e_{\rho(\gamma)}(\boldsymbol{x}) = \frac{1}{2}\left(\frac{1}{a}e_b[a\,\boldsymbol{x}]\right)^2 + \frac{1}{2}\left(\frac{1}{a}e_{2b}[2a\,\boldsymbol{x}]\right),$$

$$\sum_{\gamma \in \mathcal{D}_{3a,3b}} e_{\rho(\gamma)}(\boldsymbol{x}) = \frac{1}{6}\left(\frac{1}{a}e_b[a\,\boldsymbol{x}]\right)^3 + \frac{1}{2}\left(\frac{1}{a}e_b[a\,\boldsymbol{x}]\right)\left(\frac{1}{a}e_{2b}[2a\,\boldsymbol{x}]\right)$$

$$+ \frac{1}{3}\left(\frac{1}{a}e_{3b}[3a\,\boldsymbol{x}]\right).$$

One obtains Bizley's formula as the coefficient of $e_n(\boldsymbol{x})$ in the resulting elementary symmetric function expansion. Bizley also obtained a formula for the number of **primitive** (ad, bd)-Dyck paths. These are the paths that remain strictly above the diagonal (except at both ends). The symmetric function enumerator for these is

$$(46) \qquad \widehat{h}_d^{(a,b)}(\boldsymbol{x}; 1, 1) = \frac{1}{a} h_{bk}[ak\,\boldsymbol{x}].$$

Using this expression as a basic block, we may easily enumerate (m, n)-Dyck paths with specified return positions to the diagonal.

Area of (m, n)-Dyck paths

The **area** of a (m, n)-Dyck path α is the number of **cells**[16] lying entirely between the path α and the (m, n)-staircase:

---

[16]These are the 1×1 squares in the $\mathbb{N} \times \mathbb{N}$-grid, and they are labeled by their southwest corner.

$$(47) \qquad \mathrm{area}_{m,n}(\alpha) := \sum_{i=k}^{n} d_k - a_k,$$

where the $\delta_{m,n} = d_1 \cdots d_n$ is the (m,n)-staircase. In particular, $\delta_{m,n}$ is the unique (m,n)-Dyck path having area zero.
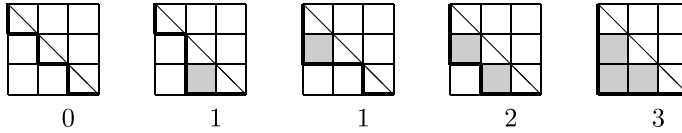

Figure 2: The areas of $(3,3)$-Dyck paths.

Parking functions

A (m,n)-parking function is simply a permutation of the entries of a (m,n)-Dyck path encoded as in (41). As observed by Garsia, it may be represented as a labeling of the south steps of the path. To this end, a step is labeled i if the corresponding entry appears in the ith-position in a parking function π. If this step starts at (x,y), we write $\pi(x,y) = i$. In other words, i appears in the cell having coordinates (x,y). This is illustrated in Figure 3, for the parking functions such that $\pi(0,5) = 2$, $\pi(0,4) = 4$, $\pi(3,3) = 3$, $\pi(6,2) = 1$, and $\pi(7,1) = 5$.
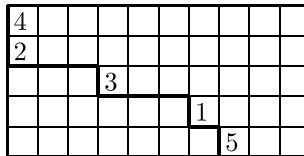

Figure 3: The $(10,5)$-parking function 60307.

As illustrated in Figure 4, the (m,n)-**rank** of a cell (x,y) is defined as being equal to $\mathrm{rank}(x,y) := n\,m - y\,m - x\,n$. The **descent set** $\mathrm{des}(\pi)$ of a parking function π is the set of i $(< n)$ for which $i+1$ sits in a cell of lower or equal rank to that of the cell in which i appears, hence

$$\mathrm{des}(\pi) := \{i \mid \pi(x,y) = i,\ \pi(u,v) = i+1,\ \mathrm{rank}(x,y) \geq \mathrm{rank}(u,v)\}.$$

We write $\mathbf{co}(\pi)$ for the composition of n that encodes this subset of $\{1,\ldots,$ $n-1\}$. In the next section, we will need to consider composition indexed

$$
\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & -5 & -10 & -15 & -20 & -25 & -30 & -35 & \cdots \\
7 & 2 & -3 & -8 & -13 & -18 & -23 & -28 & \cdots \\
14 & 9 & 4 & -1 & -6 & -11 & -16 & -21 & \cdots \\
21 & 16 & 11 & 6 & 1 & -4 & -9 & -14 & \cdots \\
28 & 23 & 18 & 13 & 8 & 3 & -2 & -7 & \cdots \\
35 & 30 & 25 & 20 & 15 & 10 & 5 & 0 & \cdots \\
\end{array}
$$

Figure 4: Examples of (m, n)-ranks (with $m = 7$ and $n = 5$).

Schur functions. These are obtained by extending to compositions the classical Jacobi-Trudi formula. More explicitly, for a composition $\alpha = (c_1, \cdots, c_k)$, one sets

$$
s_\alpha(\boldsymbol{x}) := \det(h_{c_i - i + j}(\boldsymbol{x}))_{1 \leq i, j \leq k}.
$$

It may easily be seen that this evaluates either to 0, or to a single Schur function up to a sign.

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