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The number of permutations with the same peak set for signed permutations

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A signed permutation $\pi = \pi_1\pi_2\dots\pi_n$ in the hyperoctahedral group B_n is a word such that each $\pi_i \in \{\overline{n}, \dots, \overline{1}, 1, \dots, n\}$ where $\overline{i} = -i$, and $\{|\pi_1|, |\pi_2|, \dots, |\pi_n|\} = \{1, 2, \dots, n\}$. An index i is a peak of π if $\pi_{i-1} < \pi_i > \pi_{i+1}$ and $P_B(\pi)$ denotes the set of all peaks of π. Given any set S, we define $P_B(S, n)$ to be the set of signed permutations $\pi \in B_n$ with $P_B(\pi) = S$. In this paper we show that $|P_B(S, n)| = p(n)2^{2n-|S|-1}$ where $p(n)$ is a polynomial that takes integral values when evaluated at integers. In addition, we have partially extended these results to the more complicated case where we add $\pi_0 = 0$ at the beginning of the permutations, which gives rise to the possibility of a peak at position 1, for both the symmetric and the hyperoctahedral groups. In both cases we establish recursive formulae to compute the number of permutations (signed permutations in the case of B_n) with a given peak set.

Keywords and phrases: Peak sets, permutations, signed permutations.

1. Introduction

A permutation $\pi = \pi_1\pi_2\dots\pi_n$ in the symmetric group S_n has a peak at index i if $\pi_{i-1} < \pi_i > \pi_{i+1}$. The peak set of π is defined to be $P(\pi) = \{i \mid i \text{ is a peak of } \pi\}$, then we define

$$P(S, n) = \{\pi \in S_n \mid P(\pi) = S\}$$

to be the set of all permutations with the same peak set S. For example, the permutation $\pi = 2\ 6\ 5\ 1\ 4\ 3$ has peaks at position 2 and 5, hence $P(\pi, 6) = \{2, 5\}$.

Peaks occur frequently in the literature. For instance, Stembridge [12] gave a peak analog of Stanley's theory of poset partitions. Billey and Haiman

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[7] also introduced a version of the fundamental quasisymmetric function that uses peaks instead of descents. Additional interest in the study of peaks arose when Nyman [10] showed that summing permutations according to their peak sets leads to a non-unital subalgebra of the group algebra of the symmetric group.

In a recent paper [5] Billey, Burdzy and Sagan considered the cardinalities of the sets $P(S,n)$. They discovered that $|P(S,n)| = p(n)2^{n-|S|-1}$ for some polynomial $p(n)$ depending on S; they also computed special cases of the polynomial $p(n)$. One motivation for studying peaks of permutations lies in probability theory; in a recent paper Billey, Burdzy, Pal and Sagan [6] studied distributions on graphs that are related to random permutations with certain peak sets. Besides the applicability to probability theory, the problem of enumerating permutations and signed permutations with respect to a given statistic is an interesting problem on its own, for example the enumeration of permutations related to peak sets has also been studied in [9, 13, 14].

It is natural that when a result related to the symmetric group (the Coxeter group of type A) is obtained one wishes to generalize it to other Coxeter groups. In this paper we generalize the results in [5] to the group of signed permutations, the hyperoctahedral group B_n (the Coxeter group of type B). A signed permutation $\pi = \pi_1\pi_2\ldots\pi_n$ is a word such that each $\pi_i \in \{\overline{n},\ldots,\overline{1},1,\ldots,n\}$ where $\overline{i} = -i$, and $\{|\pi_1|,|\pi_2|,\ldots,|\pi_n|\} = \{1,2,\ldots,n\}$. A peak of a signed permutation is defined in exactly the same way as for regular permutations. We will denote by $P_B(\pi)$ the set of peaks of a signed permutation π and define

$$P_B(S,n) = \{\pi \in B_n \mid P_B(\pi) = S\}.$$

We show that $|P_B(S,n)| = 2^{2n-|S|-1}p_B(n)$, where $p_B(n)$ is the same polynomial as for the symmetric group, this generalizes Theorem 3 in [5]. We also consider special cases of the polynomials $p_B(n)$.

Peaks for signed permutations are also of interest in the construction of algebraic structures. In [3] Bergeron and Hohlweg have described peak analogues of the peak algebras for the hyperoctahedral group and Petersen [11] considered peak algebras of the hyperoctahedral group when the signed permutations are grouped by number of peaks.

The second part of our paper considers the enumeration of peak classes when we put a zero at the beginning of the permutations for both the symmetric and hyperoctahedral groups. That is, we consider permutations of the form $\pi_0\pi_1\cdots\pi_n$ where $\pi_0 = 0$. These permutations arose in the study of unital peak algebras of the symmetric group [1, 2]. In the case of the symmetric group, adding a zero at the beginning of every permutation has the effect of having the identity as the unique permutation with no peaks. We denote by $\widehat{P}(S,n)$ the set of permutations with a zero added in S_n with peak set S and $\widehat{P}_B(S,n)$ the corresponding set for the hyperoctahedral group. We generalize results obtained in [5] to $\widehat{P}(S,n)$ and $\widehat{P}_B(S,n)$. In particular, we give a method for computing $|\widehat{P}(S,n)|$ and $|\widehat{P}_B(S,n)|$ and compute these numbers for special sets S.

We now give a more detailed description of the contents of this paper. In Section 2, we prove that $|P_B(S,n)| = p_B(n)2^{2n-|S|-1}$ where $p_B(n)$ is an integral polynomial in terms of n, and equal to the polynomial $p(n)$ in S_n found in [5]. We also show that the values for $|P_B(S,n)|$ are symmetric for a fixed n as we vary the set S. In Section 3, we provide a method to compute $|\widehat{P}_B(S,n)|$ for any S. We find that if $S=\emptyset$, $|\widehat{P}_B(S,n)|$ can be written in terms of the Stirling numbers of the second kind. Another result in this section gives us the parity of $|\widehat{P}_B(S,n)|$. Additionally, we find that $|P_B(S,n)|$ can be written as the sum of $|\widehat{P}_B(S,n)|$ and $|\widehat{P}_B(S\cup\{1\},n)|$ which gives us the results of the previous section when $2\in S$. Finally, we calculate $|\widehat{P}_B(S,n)|$ for various specific sets S.

In Section 4, we focus on the symmetric group, we provide a method to compute $|\widehat{P}(S,n)|$ for any S. We also find that $|P(S,n)|$ can be written as the sum of $|\widehat{P}(S,n)|$ and $|\widehat{P}(S\cup\{1\},n)|$ which gives us the results in [5] when $2\in S$. Another result in this section gives us the parity of $|\widehat{P}(S,n)|$. Finally, we calculate $|\widehat{P}(S,n)|$ for various special cases of the set S.

2. Signed permutations in B_n

Let B_n be the hyperoctahedral group, i.e., the group of signed permutations, and let $\pi=\pi_1\pi_2\dots\pi_n$ be a permutation in B_n. Recall that we define a position $i\in\{2,\dots,n-1\}$ as a peak if $\pi_{i-1}<\pi_i>\pi_{i+1}$, and the set $P_B(\pi)$ as the set of all peaks of π.

Define a set $S=\{i_1<\cdots<i_s\}$ to be *n-admissible* if $|P_B(S,n)|\neq 0$. Note that we assume that the elements in S are listed in increasing order. Notice that S cannot contain two consecutive integers and S is a subset of $\{2,\dots,n-1\}$. The minimum possible value of n for which S is n-admissible is i_s+1, and in that case S is n-admissible for all $n\geq i_s+1$. If we make a statement about an admissible set S, we mean that S is n-admissible for some n and the statement holds for every n such that S is n-admissible. It is well-known that the number of n-admissible sets is the n-th Fibonacci number. We include a proof for completeness.

Proposition 2.1. *Let A_n be the set of n-admissible peak sets S. Then the size of A_n is given by the n-th Fibonacci number.*

Proof. Note that $A_1 = A_2 = \{\emptyset\}$, thus the result holds for $n = 1$ and $n = 2$. For $n \geq 3$ write A_n as a union of disjoint sets A_α and A_β where A_α is the set of n-admissible sets that do not contain $n-1$, and A_β is the set of n-admissible sets that do contain $n-1$. Since A_{n-1} contains all $(n-1)$-admissible peak sets S which cannot contain the element $n-1$, it must be equal to A_α. Also, adding $n-1$ to all the peak sets in A_{n-2} gives us A_β. Therefore we get $|A_n| = |A_{n-1}| + |A_{n-2}|$. □

If we fix n and the cardinality of the set S, then there exists a set T of the same cardinality as S such that $|P_B(S,n)| = |P_B(T,n)|$. We make this symmetry property more explicit in the following proposition.

Proposition 2.2. *Let $S = \{i_1, i_2, \ldots, i_k\}$ and $T = \{n+1-i_k, \ldots, n+1-i_1\}$. Then $|P_B(S,n)| = |P_B(T,n)|$.*

Proof. Let $f : B_n \to B_n$ be the function defined by the rule $f(\pi) = \pi_n \cdots \pi_2\pi_1$ for $\pi = \pi_1\pi_2 \cdots \pi_n \in B_n$. Note that f is an involution (i.e., $f(f(\pi)) = \pi$). Let $\rho = \rho_1\rho_2 \ldots \rho_n \in P_B(S,n)$. If j is a peak of ρ then $n+1-j$ is a peak of $f(\rho)$, hence the peak set of $f(\rho)$ is

$$\{n+1-i_k, \ldots, n+1-i_2, n+1-i_1\}\,.$$

Thus $f(\rho) \in P_B(T,n)$. Similarly, if $f(\pi) \in P_B(T,n)$ then $f(f(\pi)) \in P_B(S,n)$. Therefore $|P_B(S,n)| = |P_B(T,n)|$. □

Remark. Note that since $S_n \subseteq B_n$, this result holds in S_n as well.

We now show that the results in [5] for S_n extend to B_n.

Theorem 2.3. *Let $S = \{i_1, i_2, \ldots, i_s\}$ be admissible. Then*

$$|P_B(S,n)| = p_B(n)2^{2n-s-1},$$

where $p_B(n) = p_B(S,n)$ is a polynomial depending on S such that $p_B(n)$ is an integer for all integral n. In addition, the degree of $p_B(n) = i_s - 1$ (when $S = \emptyset$, the degree of $p_B(n) = 0$). Moreover, $p_B(n)$ is the polynomial $p(n)$ appearing in [5, Theorem 3].

The first proof of this theorem appeared in a pre-print version of this paper, [8, Theorem 2.3]. The proof was similar to that of Theorem 3 in [5]. Later Billey, Farbach and Talmage published a much shorter proof in [4], which we present here for completeness.

Proof. We partition B_n into 2^n disjoint subsets described by the signage of the permutations. For example

$$B_3 = S_{+++} \cup S_{-++} \cup S_{+-+} \cup S_{++-} \cup S_{+--} \cup S_{-+-} \cup S_{+--} \cup S_{---},$$

where S_{++-} is the set of permutations of $\{1, 2, -3\}$ (and similarly for all other sets in the partition). Each set of the partition is a copy of S_n. It follows that $|P_B(S, n)| = 2^n |P(S, n)|$. The result now follows from Theorem 3 in [5]. □

In [5], the polynomials $p(n)$ have been computed for several special cases of S. Hence using Theorem 2.3 we obtain the following corollary.

Corollary 2.4. *If S is admissible then*

$$p_B(S, n) = \begin{cases} \binom{n-1}{m-1} - 1, & \text{if } S = \{m\} \\ (m-3)\binom{n-2}{m-1} + (m-2)\binom{n-2}{m-2} - \binom{n-2}{1}, & \text{if } S = \{2, m\}. \end{cases}$$

If $S = \{2, m, m+2\}$ then

$$p_B(S, n) = m(m-3)\binom{n}{m+1} - 2(m-3)\binom{n-2}{m-1} - 2(m-2)\binom{n-2}{m-2} + 2\binom{n-2}{1}.$$

Proof. The result follows from similar corollaries for $p(n)$ in [5]. □

The following special case of Theorem 2.3 will be used in subsequent sections.

Corollary 2.5. *Let $S = \emptyset$, then $|P_B(S, n)| = 2^{2n-1}$.*

3. Permutations in B_n with $\pi_0 = 0$

Recall that a peak is defined such that the permutation $\pi \in B_n$ has a peak at position i if $\pi_{i-1} < \pi_i > \pi_{i+1}$. Therefore if we introduce the assumption that $\pi_0 = 0$ for all $\pi \in B_n$, then a permutation π can have a peak at position 1 if $0 < \pi_1 > \pi_2$, that is if π_1 is positive and π has a descent at 1 (i.e. $\pi_1 > \pi_2$). We define $\widehat{P}_B(S, n)$ to be the set of all permutations in B_n with peak set S with the assumption that $\pi_0 = 0$. The number of n-admissible sets is also given by the Fibonacci sequence.

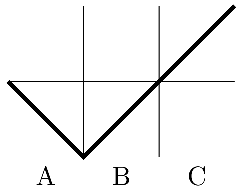

Figure 1: General shape for $\pi \in \widehat{P}_B(\emptyset, n)$.

Proposition 3.1. *Let A_n be the set of n-admissible peak sets S. Then the size of A_n is given by the $(n+1)$th Fibonacci number.*

Proof. The proof is exactly the same as for Proposition 2.1 with initial values $A_1 = \{\emptyset\}$ and $A_2 = \{\emptyset, \{1\}\}$. □

The next two propositions, together with Theorem 3.4, allow us to compute the cardinality of any peak set recursively.

Proposition 3.2. *Let $S = \emptyset$. Then*

$$|\widehat{P}_B(S, n)| = \frac{3^n + 1}{2}.$$

Proof. Let $\pi \in \widehat{P}_B(\emptyset, n)$. A general shape for π is given by Figure 1, where the section labeled A is negative and decreasing, the section labeled B is negative and increasing, and the section labeled C is positive and increasing. According to these sections, we can partition π into sections $\pi = \pi_0 \pi_A \pi_B \pi_C$. In general, up to two of these sections can be empty. We also assume that π_B contains the entire section of the permutation that is negative and ascending, including the minimum of π. For example, if $\pi = 0\overline{4}\,\overline{5}\,\overline{6}\,\overline{2}\,\overline{1}\ 3$, then $\pi_A = \overline{4}\,\overline{5}$, $\pi_B = \overline{6}\,\overline{2}\,\overline{1}$ and $\pi_C = 3$.

Now, define a function f from B_n to the set of partitions of $[n+1]$ into at most 3 blocks. Let $\pi \in B_n$. If $\pi = \pi_0 \pi_A \pi_B \pi_C$, then we let A, B, and C be the subsets of $[n]$ that correspond to the absolute values of the sections π_A, π_B, and π_C, respectively. Then f maps π to the partition of $[n+1]$ into at most 3 blocks, given by $\{A, B, C \cup \{n+1\}\}$, where if a section is empty it is not represented in the partition. Then $f(\pi)$ is in the set of partitions of $[n+1]$ into at most 3 blocks.

Next, we define the inverse of f from the set of partitions of $[n+1]$ into at most 3 blocks to B_n. Let P be such a partition. We write P as a set of three blocks, where we allow some of the blocks to be empty. I.e., if $P = \{P_1, P_2\}$, we write $P = \{P_1, P_2, \emptyset\}$. If P_j is the block containing $n+1$, then we let $C = P_j - \{n+1\}$. If P_i is the block containing the maximum value of the remaining two blocks, then we let $B = P_i$, and we let A be the remaining block. Hence, P maps to the signed permutation $\pi = \pi_0\pi_A\pi_B\pi_C$, such that π_A is given by negating the elements of A and ordering them so they are decreasing, π_B is given by negating the elements of B and ordering them so they are increasing, and π_C is given by ordering the elements of C so they are increasing.

It is known that the size of the set of partitions of $[n+1]$ into at most 3 blocks is given by the first three Stirling numbers of the second kind, $S(n+1,1)+S(n+1,2)+S(n+1,3) = \frac{3^n+1}{2}$. Therefore, the size of $\widehat{P}_B(\emptyset, n)$ is $(3^n+1)/2$. □

We should note $|\widehat{P}(\emptyset, n)|$ is given by sequence A007051 in the OEIS.

Proposition 3.3. *Let $S = \{1\}$ be an admissible set, then*

$$|\widehat{P}_B(S,n)| = 2^{2n-1} - \frac{3^n+1}{2}.$$

Proof. First note that $P_B(\emptyset, n)$ is the disjoint union of $\widehat{P}_B(\emptyset, n)$ and $\widehat{P}_B(\{1\}, n)$. Thus $|P_B(\emptyset, n)| = |\widehat{P}_B(\emptyset, n)| + |\widehat{P}_B(\{1\}, n)|$. Solving for $|\widehat{P}_B(\{1\}, n)|$ we get

$$|\widehat{P}_B(\{1\}, n)| = |P_B(\emptyset, n)| - |\widehat{P}_B(\emptyset, n)|.$$

Applying Corollary 2.5 and Proposition 3.2 we obtain the desired result. □

The next theorem shows a recursive formula for computing the values of $|\widehat{P}_B(S,n)|$. We then proceed to compute special cases for various n-admissible peak sets S.

Theorem 3.4. *Let $S = \{i_1, i_2, \ldots, i_s\}$ be a non-empty admissible set such that $i_1 + i_2 + \cdots + i_s \geq 2$ then,*

$$\left|\widehat{P}_B(S,n)\right| = \binom{n}{i_s-1}\left|\widehat{P}_B(S_1, i_s-1)\right| 2^{2(n-i_s)-1} - \left|\widehat{P}_B(S_1,n)\right| - \left|\widehat{P}_B(S_2,n)\right|.$$

where $S_1 = S - \{i_s\}$ and $S_2 = S_1 \cup \{i_s - 1\}$.

Proof. Let $k = i_s - 1$, thus $k \geq 1$. For any $n \geq i_s$, let Π be the set of all signed permutations $\pi = \pi_0\pi_1\pi_2\ldots\pi_n$ such that $\widehat{P}_B(\pi_0\pi_1\pi_2\ldots\pi_k) = S_1 = S - \{i_s\}$ and $P_B(\pi_{i_s}\ldots\pi_n) = \emptyset$. We can partition Π into blocks by the peak set of π. In addition to the peaks given by $S_1 = S - \{i_s\}$, there could be a peak at π_k, a peak at π_{i_s}, or no peak at both π_k and π_{i_s}. Note that these are all the possibilities, and that the three are disjoint. Thus, if we let $S_2 = S_1 \cup \{i_s - 1\}$, then

$$|\Pi| = |\widehat{P}_B(S_2, n)| + |\widehat{P}_B(S, n)| + |\widehat{P}_B(S_1, n)|. \tag{1}$$

First, we find $|\Pi|$. Recall that for $\pi \in \Pi$, we have $\widehat{P}_B(\pi_0\pi_1 \dots \pi_k) = S_1$ and $P_B(\pi_{i_s} \dots \pi_n) = \emptyset$. Therefore to construct any π, first we choose $k = i_s - 1$ elements to be in the first section. For signed permutations, if an integer $m \in [n]$ is in the permutation, then $-m$ cannot be, and vice versa. Therefore we choose k elements from a set of n elements. Then we create a signed permutation (with $\pi_0 = 0$) from these k elements, arranged in a way such that their peak set is S_1. We have denoted the number of ways to do so by $|\widehat{P}_B(S_1, k)|$. Finally we arrange the last $n - k$ items such that their peak set is $\emptyset$. The number of ways to do this is $|P_B(\emptyset, n - k)|$ (notice that we no longer have the first entry being zero, as we are considering the last $n - k$ entries and $k \geq 1$). Therefore

$$|\Pi| = \binom{n}{k} |\widehat{P}_B(S_1, k)||P_B(\emptyset, n - k)|.$$

Using this expression for $|\Pi|$ and substituting in equation (1), and using Corollary 2.5 we obtain the desired result. □

3.1. Parity of $\widehat{P}_B(S, n)$

In the previous section we showed that $P_B(S, n)$ was always a multiple of a power of 2, and hence always even. This is no longer the case for $\widehat{P}_B(S, n)$ as we show in the next theorem.

Theorem 3.5. *Let $S = \{i_1, i_2, \dots, i_s\}$. Then $|\widehat{P}_B(S, n)|$ is even if S contains some even number or if n is odd, and is odd otherwise.*

Proof. We first consider the case when $S = \emptyset$. Clearly, S contains no even elements. Note that $|\widehat{P}_B(\emptyset, n)| = (3^n + 1)/2$ is even if n is odd and odd if n is even, thus our claim holds.

For the general case we induct on $i_1 + i_2 + \cdots + i_s$ and will make use of Theorem 3.4. Our base case is $i_1 + i_2 \cdots + i_s = 1$, thus $S = \{1\}$. We note S contains no even elements. Note that $|\widehat{P}_B(\{1\}, n)| = 2^{2n-1} - (3^n + 1)/2$ is even if n is odd and odd if n is even, thus our claim holds.

Recall Theorem 3.4, which states that if $S_1 = S - \{i_s\}$ and $S_2 = S_1 \cup \{i_s - 1\}$, then

$$|\widehat{P}_B(S, n)| = \binom{n}{i_s - 1} |\widehat{P}_B(S_1, i_s - 1)| 2^{2(n - i_s) + 1} - |\widehat{P}_B(S_1, n)| - |\widehat{P}_B(S_2, n)|.$$

Note that the first term will always be even, since it is multiplied by 2 with some positive exponent. Therefore $|\widehat{P}_B(S,n)|$ is even if and only if $|\widehat{P}_B(S_1,n)|+|\widehat{P}_B(S_2,n)|$ is even. If n is odd, then by our inductive assumption, $|\widehat{P}_B(S_1,n)|$ and $|\widehat{P}_B(S_2,n)|$ are both even, so their sum is even.

Now, consider the case where n is even. If S has at least one even element, let i_j be the first even element in S. Either $i_j \in S_1$ or $i_j = i_s$. In the first case, our inductive hypothesis implies that $|\widehat{P}_B(S_1,n)|$ and $|\widehat{P}_B(S_2,n)|$ are both even, then their sum is even. In the second case, S_1 has no even elements, thus by our inductive hypothesis, $|\widehat{P}_B(S_1,n)|$ is odd. Note that if i_s is even, then $i_s - 1$ is odd and S_2 has no even elements. Therefore $|\widehat{P}_B(S_2,n)|$ is also odd, thus their sum is even.

Now consider the case where S contains no even elements and n is still even. Since S_1 contains no even elements, by our inductive hypothesis $|\widehat{P}_B(S_1,n)|$ is odd. But since i_s is odd, $i_s - 1$ must be even. Therefore by our inductive hypothesis $|\widehat{P}_B(S_2,n)|$ is even, hence their sum is odd. □

3.2. Relationship between $|P_B(S,n)|$ and $|\widehat{P}_B(S,n)|$

The following relation between $|P_B(S,n)|$ and $|\widehat{P}_B(S,n)|$ allows us to extrapolate some results from Section 2.

Proposition 3.6. *If S is admissible, then*

$$|P_B(S,n)| = |\widehat{P}_B(S,n)| + |\widehat{P}_B(S \cup \{1\},n)|.$$

Proof. For any $\pi \in P_B(S,n)$, either π has a descent at position 1 (i.e. $\pi_1 < \pi_2$), or it does not. Therefore we can write $P_B(S,n)$ as a union of disjoint sets $P_B(S,n) = P_\alpha(S,n) \cup P_\beta(S,n)$ where $\pi \in P_\alpha(S,n)$ has a descent at position 1 and $\pi \in P_\beta(S,n)$ does not. Note that $\pi \in P_\alpha(S,n)$ correspond to an element in $\widehat{P}_B(S\cup\{1\},n)$ by adding a zero at the beginning of π. Hence, $|P_\alpha(S,n)| = |\widehat{P}_B(S\cup\{1\},n)|$. Similarly, any $\pi \in P_\beta(S,n)$ corresponds to an element in $\widehat{P}_B(S,n)$ and thus $|P_\beta(S,n)| = |\widehat{P}_B(S,n)|$. Therefore $|P_B(S,n)| = |\widehat{P}_B(S,n)| + |\widehat{P}_B(S\cup\{1\},n)|$. □

Proposition 3.6 implies the following corollary.

Corollary 3.7. *If S is admissible and $2 \in S$, then*

$$|P_B(S,n)| = |\widehat{P}_B(S,n)|.$$

Proof. If $2 \in S$, then $1 \notin S$. Thus, $|\widehat{P}_B(S\cup\{1\},n)| = 0$, implying that $|P_B(S,n)| = |\widehat{P}_B(S,n)|$ using Proposition 3.6. □

Below, we compute $|\widehat{P}_B(S,n)|$ for different sets S.

Proposition 3.8. *Let $S=\{m\}$ be admissible, then*

$$|\widehat{P}_B(\{m\},n)| = 4^{n-m-1}\sum_{i=1}^{m}\binom{n}{m-i}(3^{m-i}+1)4^i(-1)^{i+1} - (m \bmod 2)\left[\frac{3^n+1}{2}\right].$$

Proof. We will induct on m. First let $m=1$ then using Proposition 3.2 and Proposition 3.6,

$$\begin{aligned}|\widehat{P}_B(\{1\},n)| &= |P_B(\emptyset,n)| - |\widehat{P}_B(\emptyset,n)| \\ &= 2^{2n-1} - \left[\frac{3^n+1}{2}\right] \\ &= 4^{n-2}(2)(4) - \left[\frac{3^n+1}{2}\right] \\ &= 4^{n-1-1}\sum_{i=1}^{1}\binom{n}{1-i}(3^{1-i}+1)4^i(-1)^{i+1} - (1 \bmod 2)\left[\frac{3^n+1}{2}\right].\end{aligned}$$

We assume our claim is true for m and we consider $m+1$. Apply Theorem 3.4 for the peak set $S=\{m+1\}$ to obtain the following

$$|\widehat{P}_B(\{m+1\},n)| = \binom{n}{m}|\widehat{P}_B(\emptyset,m)||P_B(\emptyset,n-m)| - |\widehat{P}_B(\emptyset,n)| - |\widehat{P}_B(\{m\},n)|.$$

Apply Corollary 2.5, Proposition 3.2 and the inductive hypothesis, then

$$\begin{aligned}|\widehat{P}_B(\{m+1\},n)| &= \binom{n}{m}\left(\frac{3^m+1}{2}\right)2^{2(n-m)-1} - \left(\frac{3^n+1}{2}\right) \\ &\quad - 4^{n-m-1}\sum_{i=1}^{m}\left[\binom{n}{m-i}(3^{m-i}+1)4^i(-1)^{i+1}\right] + (m \bmod 2)\left(\frac{3^n+1}{2}\right) \\ &= -4^{n-m-1}\left(\binom{n}{m-0}(3^{m-0}+1)(4)^0(-1)^{0+1}\right. \\ &\quad \left.+\sum_{i=1}^{m}\binom{n}{m-i}(3^{m-i}+1)4^i(-1)^{i+1}\right) - (m+1 \mod 2)\left(\frac{3^n+1}{2}\right) \\ &= -4^{n-m-1}\sum_{i=0}^{m}\binom{n}{m-i}(3^{m-i}+1)4^i(-1)^{i+1} - (m+1 \bmod 2)\left(\frac{3^n+1}{2}\right)\end{aligned}$$

$$= -4^{n-(m+1)-1}\sum_{i=1}^{m+1}\binom{n}{(m+1)-i}(3^{(m+1)-i}+1)4^i(-1)^{i+1}$$
$$-(m+1 \mod 2)\left(\frac{3^n+1}{2}\right). \qquad \square$$

Corollary 3.9. *Let $S=\{1,m\}$ be admissible, then*

$$|\widehat{P}_B(S,n)|$$
$$=2^{2n-2}\left[\binom{n-1}{m-1}-1\right]-4^{n-m-1}\sum_{i=1}^{m}\binom{n}{m-i}(3^{m-i}+1)(4)^i(-1)^{i+1}$$
$$+(m \bmod 2)\left[\frac{3^n+1}{2}\right].$$

Proof. Apply Proposition 3.6 to obtain the following

$$|P_B(\{m\},n)|=|\widehat{P}_B(\{m\},n)|+|\widehat{P}_B(\{m\}\cup\{1\},n)|.$$

We then use Theorem 2.3 together with Corollary 2.4 for $|P_B(\{m\},n)|$ and Proposition 3.8 for $|\widehat{P}_B(\{m\},n)|$. The rest follows. $\square$

The following result is a general result for a two-element peak set S.

Proposition 3.10. *Let $S=\{m,m+z\}$ be admissible, then $|\widehat{P}_B(S,n)|$ equals*

$$\sum_{i=0}^{z-2}(-1)^i 2^{2(n-m-z+i+1)-1}\binom{n}{m+z-1-i}$$
$$\times\left[4^{z-i-2}\sum_{j=1}^{m}\left((3^{m-j}+1)4^j(-1)^{j+1}\binom{m+z-i-1}{m-j}\right)\right.$$
$$\left.-(m \mod 2)\left(\frac{3^{m+z-i-1}+1}{2}\right)\right]$$
$$-(z-1 \mod 2)\left[4^{n-m-1}\sum_{i=1}^{m}\left((3^{m-i}+1)4^i(-1)^{i+1}\binom{n}{m-i}\right)\right.$$
$$\left.-(m \mod 2)\left(\frac{3^n+1}{2}\right)\right].$$

Proof. Let $S=\{m,m+z\}$ be admissible, let $S_1=\{m\}$ and $S_2=\{m,m+z-1\}$. Apply Theorem 3.4 to obtain the following recursive formula,

$$|\widehat{P}_B(S,n)| \quad = \quad \binom{n}{m+z-1}|\widehat{P}_B(S_1,m+z-1)||P_B(\emptyset,n-(m+z)+1)|$$

$$-|\widehat{P}_B(S_1,n)| - |\widehat{P}_B(S_2,n)|.$$

Then apply the recursion $|\widehat{P}_B(S_2,n)|$ until $m+z-1$ approaches $m+1$, then if $a = m+z-1$ we arrive at the following formula for $|\widehat{P}_B(S,n)|$

$$\sum_{i=0}^{z-2}\left[(-1)^i\binom{n}{a-i}|\widehat{P}_B(\{m\}, m+z-1-i)||P_B(\emptyset, n-(m+z)+1-i)|\right]$$
$$-(z-1 \mod 2)|\widehat{P}_B(\{m\},n)|.$$

Using Proposition 3.8 and Corollary 2.5 we obtain the result. □

We have the following special case when S is a three element set.

Proposition 3.11. *Let $S = \{1, m, m+2\}$ be admissible, then*

$$\begin{aligned}
&|\widehat{P}_B(S,n)| \\
&= 4^{n-m-1}\sum_{i=1}^{m}(3^{m-i}+1)(4)^i(-1)^{i+1}\left[\binom{n}{m-i} - \frac{1}{2}\binom{n}{m+1}\binom{m+1}{m-i}\right] \\
&\quad + 4^{n-1}\left[\frac{m-1}{2}\binom{n}{m+1} + 1 - \binom{n-1}{m-1}\right] \\
&\quad + (m \mod 2)\left[\binom{n}{m+1}\left(\frac{3^{m+1}+1}{2}\right)2^{2(n-m-1)-1} - \frac{3^n+1}{2}\right].
\end{aligned}$$

Proof. Let $S_1 = \{1, m\}$ and let $S_2 = \{1, m, m+1\}$ and apply Theorem 3.4. Note that $|\widehat{P}_B(S_2,n)| = 0$, then

$$|\widehat{P}_B(S,n)| = \binom{n}{m+1}|\widehat{P}_B(S_1, m+1)||P_B(\emptyset, n-(m+2)+1)| - |\widehat{P}_B(S_1,n)|.$$

The result follows from Corollary 3.9 and Corollary 2.5. □

4. Permutations in S_n with $\pi_0 = 0$

Let S_n be the set of all permutations $\pi = \pi_1\pi_2\ldots\pi_n$ of $[n]$. Recall that we define the set $P(\pi)$ as the set of all peaks of π. Now, we introduce the condition $\pi_0 = 0$, which will allow our peak set to contain $i = 1$. Define $\widehat{P}(\pi)$ as the set of all peaks of π with $\pi_0 = 0$, and $\widehat{P}(S,n)$ as the set of all permutations of S_n having $\pi_0 = 0$ and peak set S.

We need the following result in order to prove future cases.

Lemma 4.1. *(I) Let $S = \emptyset$ then $|\widehat{P}(S,n)| = 1$.*
(II) Let $S = \{1\}$ then $|\widehat{P}(S,n)| = 2^{n-1} - 1$.

Proof. To prove part (I), let $\pi = \pi_0\pi_1 \ldots \pi_n$ be a permutation in $\widehat{P}(\emptyset, n)$. Since $\pi_1 > \pi_0 = 0$ then π must be always increasing, otherwise we would have a peak. Clearly, the identity i.e., $\pi = 012\ldots n$ is the only permutation satisfying this condition.

To prove part (II), let $\pi \in P(\emptyset, n)$. Then either $\pi \in \widehat{P}(\{1\}, n)$ or by Lemma 4.1 (I), π is the identity in S_n, therefore $|\widehat{P}(\{1\}, n)| = |P(\emptyset, n)| - 1$. By [5, Proposition 2] we have $|\widehat{P}(\{1\}, n)| = 2^{n-1} - 1$. □

We first give the recursive method that will allow us to compute formulas for the special peak sets $S = \{\{m\}, \{1, m\}, \{1, m, m+2\}, \{1, m, n-1\}\}$. This recursive formula is based on Theorem 3.4.

Theorem 4.2. *Let $S = \{i_1, i_2, \ldots, i_s\}$ be a non-empty admissible set such that $i_1 + i_2 + \cdots + i_n \geq 2$, then*

$$|\widehat{P}(S, n)| = \binom{n}{i_s - 1} |\widehat{P}(S_1, i_s - 1)| 2^{n-i_s} - |\widehat{P}(S_1, n)| - |\widehat{P}(S_2, n)|,$$

where $S_1 = S - \{i_s\}$ and $S_2 = S_1 \cup \{i_s - 1\}$.

Proof. For any $n \geq i_s$ we let Π be the set of all permutations $\pi = \pi_0\pi_1\pi_2\ldots\pi_n$ such that $\widehat{P}(\pi_0\pi_1\pi_2\ldots\pi_{i_s-1}) = S_1 = S - \{i_s\}$ and $P(\pi_{i_s}\ldots\pi_n) = \emptyset$. By a similar argument to that in the proof of Theorem 3.4 we get that

$$|\Pi| = |\widehat{P}(S_2, n)| + |\widehat{P}(S, n)| + |\widehat{P}(S_1, n)|,$$

and

$$|\Pi| = \binom{n}{i_s - 1} |\widehat{P}(S_1, i_s - 1)| |P(\emptyset, n - (i_s - 1))|.$$

Equaling both equations and solving for $|\widehat{P}(S, n)|$ we get the desired result. We have used the fact that by Proposition 2 in [5] we get $|P(\emptyset, n-(i_s-1))| = 2^{n-i_s}$. □

Now we make use of the recursive formula in Theorem 4.2 and the result in Lemma 4.1 (I) to obtain a recursive formula for the case when $S = \{m\}$. This will lead us to have a closed formula for this case.

Proposition 4.3. *Let $S = \{m\}$ be admissible. Then*

$$|\widehat{P}(S, n)| = \sum_{i=1}^{m} 2^{n-i} \binom{n}{i-1} (-1)^{m-i} - (m \mod 2).$$

Proof. We induct on m. Our base case is $m = 1$. Then by Lemma 4.1 (II), our claim is true. Now, by our inductive assumption,

$$\widehat{P}(\{m-1\},n) = \sum_{i=1}^{m-1} 2^{n-i}\binom{n}{i-1}(-1)^{m-1-i} - (m-1 \mod 2).$$

Using this value for $\widehat{P}(\{m-1\},n)$ in the recursive formula given in Theorem 4.2, we find

$$\begin{aligned}
&|\widehat{P}(S,n)| \\
&= 2^{n-m}\binom{n}{m-1} - 1 - \left(\sum_{i=1}^{m-1} 2^{n-i}\binom{n}{i-1}(-1)^{m-1-i} - (m-1 \mod 2)\right) \\
&= 2^{n-m}\binom{n}{m-1}(-1)^0 + \sum_{i=1}^{m-1} 2^{n-i}\binom{n}{i-1}(-1)^{m-i} - (m \mod 2) \\
&= \sum_{i=1}^{m} 2^{n-i}\binom{n}{i-1}(-1)^{m-i} - (m \mod 2). \qquad \square
\end{aligned}$$

From Proposition 4.3 we notice that we can factor a power of two out of the summation, in this way we obtain a new formula for the case $S = \{m\}$.

Proposition 4.4. *Let $S = \{m\}$ be admissible, then*

$$|\widehat{P}(\{m\},n)| = \frac{p_{m-1}(n)2^{n-m}}{(m-1)!} - (m \mod 2) \tag{2}$$

where $p_{m-1}(n) = p(S,n)$ is a polynomial depending on S such that $p(n)$ is an integer for all integral n. Also, $deg(p_{m-1}(n)) = m-1$.

Proof. We will prove this by induction on m. The case where $m = 1$ is true since we already found that $|\widehat{P}(\{1\},n)| = 2^{n-1} - 1$ where $p(n) = 1$ and is of degree $1 - 1 = 0$. We will first use the recurrence relation in Theorem 4.2 with our inductive assumption to get

$$\begin{aligned}
|\widehat{P}(\{m+1\},n)| &= \binom{n}{m}2^{n-(m+1)} - 1 - \left(\frac{p_m(n)2^{n-m}}{(m-1)!} - (m \mod 2)\right) \\
&= \frac{2^{n-(m+1)}}{m!}\left(\frac{n!}{(n-m)!} - 2mp_m(n)\right) - (m+1 \mod 2).
\end{aligned}$$

We now have to prove that

$$\frac{n!}{(n-m)!} - 2mp_m(n)$$

is a polynomial in terms of n with degree m. Because m is fixed, we can see that this expression is a polynomial in terms of n and we can also clearly see that $n!/(n-m)!$ has degree m and because of the inductive hypothesis $2m \cdot p_m(n)$ has degree $m-1$ which means that the whole expression has degree m which completes the induction. □

Corollary 4.5. *Additionally,*

$$p_m(n) = m! \sum_{i=0}^{m} 2^i (-1)^i \binom{n}{m-i}. \tag{3}$$

Proof. From Proposition 4.4 we have a recursive formula for $p_m(n)$,

$$p_m(n) = \frac{n!}{(n-m)!} - 2mp_{m-1}(n).$$

We will prove this by induction on m. The case where $m = 0$ is obviously true since using the formula we get that $p_0(n) = 1$ which agrees with $|\widehat{P}(\{1\}, n)| = 2^{n-1} \cdot 1 - 1$. We will assume that the proposition is true for m and we will it prove it for the $m+1$ case. Using the recursive formula and the inductive hypothesis we get

$$\begin{aligned}
&p_{m+1}(n) \\
&\quad = \frac{n!}{(n-(m+1))!} - 2(m+1) \cdot p_m(n) \\
&\quad = \frac{n!}{(n-(m+1))!} - 2(m+1) \left(\frac{n!}{(n-m)!} + m! \sum_{i=1}^{m} 2^i (-1)^i \binom{n}{m-i} \right) \\
&\quad = \frac{n!}{(n-(m+1))!} - 2(m+1) \left(m! \binom{n}{m} + m! \sum_{i=1}^{m} 2^i (-1)^i \binom{n}{m-i} \right) \\
&\quad = \frac{n!}{(n-(m+1))!} + (m+1)! \sum_{i=1}^{m+1} 2^i (-1)^i \binom{n}{m+1-i} \\
&\quad = (m+1)! \sum_{i=0}^{m+1} 2^i (-1)^i \binom{n}{m+1-i}.
\end{aligned}$$

which is what we wanted thus completing the induction. □

In the following proposition we compute $|\widehat{P}(\{m\}, n)|$ using a different approach. The new formula we obtain will help us to compute other special cases such when $S = \{1, n-1\}$ in a simpler way.

Proposition 4.6. *Let $S = \{n-m\}$ be admissible. Then*

$$|\widehat{P}(S,n)| = \sum_{i=0}^{m-1} 2^i \binom{n-(m-i)}{i+1}.$$

Proof. Let $\pi = \pi_0\pi_1\pi_2\ldots\pi_n$ be a permutation in S_n. We will prove this proposition by induction on m. We will first prove the base case, when $m = 1$. Letting $m = 1$ means that we will have a peak only in the $(n-1)$-th position. Note that $\pi_{n-1} = n$ because otherwise there would either be no peaks (if $\pi_n = n$) or more than one peaks (if $\pi_i = n$ for some $i \in \{1, 2 \ldots, n-2\}$). We know that the numbers before the $(n-1)$-th position must be in increasing order, thus the permutation is completely determined by the element in the n-th position. There are $\binom{n-1}{1} = n-1$ ways to choose the last element.

Now assume the proposition is true for $m \geq 1$, we will prove that it is true for $m+1$. This means that we have a peak at the $n-(m+1)$-th position, then using reasoning similar to the one used in the inductive hypothesis, we know that n is either in position $n-(m+1)$ or in the last position. If n is in the position of the peak, the number of permutations that satisfy this condition is equal to the number of ways to choose the last $m+1$ numbers in the permutation times the number of ways to arrange these $m+1$ numbers so that they do not form a peak. This number is equal $2^m\binom{n-1}{m+1}$.

If n is in the n-th position of the permutation, then we can reduce the computation to the m-th case of the induction. Thus,

$$|\widehat{P}(S,n)| = 2^m\binom{n-1}{m+1} + \sum_{i=0}^{m-1} 2^i \binom{n-(m-i)}{i+1} = \sum_{i=0}^{m} 2^i \binom{n-(m+1-i)}{i+1}.$$

□

Note that doing a change of variable in the previous result will lead us to obtain better results for the case $S = \{m\}$.

Remark 4.7. *Let $S = \{m\}$ be admissible. Notice from the Proposition 4.6 that we can write $|\widehat{P}(S,n)|$, as*

$$|\widehat{P}(S,n)| = \sum_{i=0}^{n-(m+1)} 2^i \binom{m+i}{i+1}.$$

Proposition 4.8. *Let $S = \{1, m\}$ be admissible, then*

$$|\widehat{P}(\{1,m\},n)| = \sum_{i=1}^{m-2} \binom{n}{m-i} (2^{m-i-1}-1)(2^{n-(m-i+1)})(-1)^{i+1}$$

$$- (m \mod 2)(2^{n-1} - 1).$$

Proof. Let $S = \{1, m\}$ be admissible and let $S_1 = \{1\}$ and $S_2 = \{1, m-1\}$. Recall Theorem 4.2 provides the following recursive fomula

$$\begin{aligned} |\widehat{P}(S,n)| &= \binom{n}{m-1}|\widehat{P}(\{1\}, m-1)||P(\emptyset, n-m+1)| \\ &\quad - |\widehat{P}(\{1\}, n)| - |\widehat{P}(\{1, m-1\}, n)| \\ &= \binom{n}{m-1}(2^{m-2} - 1)(2^{n-m}) - (2^{n-1} - 1) - |\widehat{P}(\{1, m-1\}, n)|. \end{aligned}$$

To obtain the terms $(2^{m-2} - 1)$ and $(2^{n-1} - 1)$ apply Lemma 4.1 (II) and the term $(2^{n-m+1-1})$ follows from Proposition 2 in [5]. The result follows by induction. ☐

Proposition 4.9. *Let $S = \{1, m, m+2\}$ be admissible, then $|\widehat{P}(S,n)|$ equals*

$$\sum_{i=1}^{m-2}(2^{m-i-1} - 1)(-1)^{i+1}(2^{n-m+i-1})\left[\binom{n}{m+1}\binom{m+1}{m-i}\frac{1}{2} - \binom{n}{m-i}\right]$$
$$+ (m \mod 2)\left(2^{n-1} - 1 - \binom{n}{m+1}2^{n-m-2}(2^m - 1)\right).$$

Proof. We apply the same method as in Proposition 4.8. For this we let $S_1 = \{1, m\}$ and $S_2 = \{1, m, m+1\}$. Note that $|\widehat{P}(\{1, m, m+1\}, n)| = 0$ since we can not have consecutive peaks. Then we construct Π based on the number of ways to arrange the permutations in S_1 and S_2 and use Theorem 4.2 to obtain the recursive formula.

$$\begin{aligned} |\widehat{P}(\{1,m, m+2\}, n)| &= |\Pi| - |\widehat{P}(\{1, m\}, n)| \\ &= \binom{n}{m+1}|\widehat{P}(\{1, m\}, m+1)||P(\emptyset, n-(m+2)+1)| - |\widehat{P}(\{1, m\}, n)| \end{aligned}$$

Now for the terms $|\widehat{P}(\{1, m\}, m+1)|$ and $|\widehat{P}(\{1, m\}, n)|$ apply Proposition 4.8, and for $|P(\emptyset, n-(m+2)+1)|$ apply Proposition 2 in [5]. The result follows. ☐

Proposition 4.10. *Let $S = \{1, m, n-1\}$ be admissible, then we have the following recursive formula for $|\widehat{P}(S,n)|$*

$$\sum_{i=1}^{m-2}(2^{m-i-1} - 1)(2^{n-m+i-1})(-1)^{i+1}\left(\frac{1}{2}\binom{n}{n-2}\binom{n-2}{m-i} - \binom{n}{m-i}\right)$$

$$-(m \mod 2)\left((2^{n-2}-2)\binom{n}{2}-2^{n-1}+1\right)-|\widehat{P}(\{1,m,n-2\},n)|.$$

Proof. Let $S_1=\{1,m\}$ and $S_2=\{1,m,n-2\}$. Apply Theorem 4.2 to obtain the recursive formula.

$$\begin{aligned}|\widehat{P}(\{1,m,n-1\},n)| &= 2\binom{n}{2}|\widehat{P}(\{1,m\},n-2)||P(\emptyset,n-(n-1)+1)| \\ &\quad -|\widehat{P}(\{1,m\},n)|-|\widehat{P}(\{1,m,n-2\},n)|.\end{aligned}$$

Now for $|\widehat{P}(\{1,m\},n-2)|$ and $|\widehat{P}(\{1,m\},n)|$ apply Proposition 4.8. The result follows. ☐

4.1. Relationship between $|P(S,n)|$ and $|\widehat{P}(S,n)|$

In this section we use the relationship between $|P(S,n)|$ and $|\widehat{P}(S,n)|$ to find new formulas for special cases. We begin by giving this relationship.

Proposition 4.11. *If S is admissible then*

$$|P(S,n)|=|\widehat{P}(S,n)|+|\widehat{P}(S\cup\{1\},n)|.$$

We omit the proof of this proposition since it is identical to the proof of Proposition 3.6.

Corollary 4.12. *Let $S=\{1,m\}$ be an admissible set, then*

$$|\widehat{P}(\{1,m\},n)|=\left(\binom{n-1}{m-1}-1\right)2^{n-2}-|\widehat{P}(\{m\},n)|.$$

Proof. We apply Proposition 4.11 to the case $S=\{m\}$, using the fact that $|P(\{m\},n)|=\left(\binom{n-1}{m-1}-1\right)2^{n-2}$ from Theorem 6 in [5]. ☐

Corollary 4.13. *Let $S=\{1,n-1\}$ be an admissible set then,*

$$|\widehat{P}(S,n)|=2^{n-2}(n-2)-(n-1).$$

Proof. Note that $S=\{1,n-1\}$ is a special case of $S=\{1,m\}$ with $m=n-1$. Now apply Corollary 4.12 and Remark 4.7. Then we have,

$$\begin{aligned}|\widehat{P}(\{1,n-1\},n)| &= \left(\binom{n-1}{(n-1)-1}-1\right)2^{n-2}-\sum_{i=0}^{n-((n-1)+1)}2^i\binom{(n-1)+i}{i+1} \\ &= 2^{n-2}(n-2)-(n-1).\end{aligned}$$ ☐

The following is a consequence of Proposition 4.11.

Corollary 4.14. *Let $S = \{i_1, i_2, \ldots, i_s\}$ where $i_1 = 2$. Then*

$$|\widehat{P}(S,n)| = p(n)2^{n-s-1}$$

where $p(n) = p(S,n)$ is an polynomial depending on S with degree $i_s - 1$, such that $p(n)$ is an integer for all integral n.

Furthermore, if we let $m = max(S)$, $S_1 = S - \{m\}$, and $S_2 = S_1 \cup \{m-1\}$, then

$$p(S,n) = p(S_1, m-1)\binom{n}{m-1} - 2p(S_1,n) - p(S_2,n).$$

Proof. We apply Proposition 4.11, and note that $|\widehat{P}(S \cup \{1\}, n)| = 0$ if $2 \in S$. Thus,

$$|\widehat{P}(S,n)| = |P(S,n)|.$$

Hence, we can apply Theorem 3 in [5] for $|P(S,n)|$ to this special case of $|\widehat{P}(S,n)|$. ☐

4.2. Parity of $|\widehat{P}(S,n)|$

Notice from the previous results how the number of permutations varies according to the parity of some integer related to the peaks. This lead us to establish a relation between the parity of the numbers of permutations with a given peak set and the parity of the numbers in the peak set.

Theorem 4.15. *Let $S = \{i_1, i_2, \ldots, i_s\}$ be admissible. Then $|\widehat{P}(S,n)|$ is even if and only if there exists $i_j \in S$ such that i_j is even.*

Proof. We induct on $i_1 + i_2 + \cdots + i_s$. Our base case is $i_1 + i_2 \cdots + i_s = 0$, where, $S = \emptyset$. Clearly, S contains no even elements. By Lemma 4.1 (I), $|P(\emptyset, n)| = 1$ for all n, thus our claim holds.

Now, by Theorem 4.2 if we let $S_1 = S - \{i_s\}$ and $S_2 = S_1 \cup \{i_s - 1\}$, then

$$|\widehat{P}(S,n)| = \binom{n}{i_s - 1} 2^{n-i_s} |\widehat{P}(S_1, i_s - 1)| - |\widehat{P}(S_1,n)| - |\widehat{P}(S_2,n)|.$$

S admissible implies $i_s < n$. Since $n - i_s > 0$, then $\binom{n}{i_s-1} 2^{n-i_s} |\widehat{P}(S_1, i_s - 1)|$ is even in all cases.

Assume that $|\widehat{P}(S,n)|$ is even. Then either $|\widehat{P}(S_1,n)|$ and $|\widehat{P}(S_2,n)|$ are both even, or they are both odd. If $|\widehat{P}(S_1,n)|$ is even, then, by the inductive hypothesis, S_1 contains some even element. Since $S_1 \subset S$, S then contains some even element. If $|\widehat{P}(S_2,n)|$ is odd, then, by the inductive hypothesis, all its elements are odd, including $i_s - 1$. Therefore i_s is even, hence S contains some even element.

Assume that S contains some even element. Then either S_1 contains some even element and i_s is even, or S_1 contains some even element and i_s is odd, or S_1 does not contain any even elements and i_s is even. In the first case and the second case, by inductive hypothesis $|\widehat{P}(S_1,n)|$ and $|\widehat{P}(S_2,n)|$ are both even, then $|\widehat{P}(S,n)|$ is even. In the third case, $i_s - 1$ is odd, thus by the inductive hypothesis $|\widehat{P}(S_1,n)|$ and $|\widehat{P}(S_2,n)|$ are both odd, hence $|\widehat{P}(S,n)|$ is even. □

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