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Hook length formulas for partially colored labeled forests

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Motivated by the study of the invariant theory of some finite groups, we introduce and study the notion of partially colored labeled forest. A flag-major index is defined on these forests and we study the distribution of this statistic on all partially colored labeled forests and on linear extensions of a fixed partially colored labeled forest. The main results that we obtain are formulas for such distributions which have a very simple factorization form and generalize and unify several known results present in the literature.

Keywords and phrases: Complex reflection groups, flag-major index, invariant algebras.

Introduction

In the early 1900s Percy MacMahon [14] introduced and studied the *greater index* of a permutation of a totally ordered set, and probably his most known result on it is that it is equidistributed with the inversion number. Apparently because of the military degree of MacMahon the greater index was later renamed and is now widely known as *major index*. In the last thirty years, this index has been generalized in two directions which are of interest in this work.

In 1989 Björner and Wachs [6] generalized the major index defining a new statistic on labeled forests (i.e. partially ordered sets whose Hasse diagram is a rooted forest) in a very natural way. They presented in particular two q-hook length formulas: one for the distribution of the major index over permutations which correspond to linear extensions of a labeled forest, and the other for the distribution of the new statistic over all labelings of a fixed forest. These results have recently been extended to "signed-labeled" forests by Chen, Gao and Guo [10].

In the early 2000s, Adin and Roichman [2] generalized the major index to the case of colored permutation groups $G(r,n)$, which are wreath products of the form $\mathbb{Z}_r \wr \mathcal{S}_n$, where $\mathbb{Z}_r$ is the cyclic group of order r. They called this new statistic the *flag-major index* because of a specific algebraic property that it satisfies and showed, in particular, that it is equidistributed with the length function for the classical Weyl group of type B (i.e. in the case $r = 2$). In 2004 Biagioli and the second author [4] defined an analogous statistic for the Weyl groups of type D and in 2007 Bagno and Biagioli [3] extended the definition of the flag-major index for complex reflection groups $G(r,p,n)$, which are normal subgroups of $G(r,n)$ of index p. Finally, in 2011 the second author [8] introduced a new family of groups $G(r,p,q,n)$, which are concrete examples of a more general class of groups called projective reflection groups, and that can be described as quotients of $G(r,p,n)$ modulo the cyclic scalar subgroup C_q. He extended the notion of flag-major index to these groups and showed how the combinatorics, and in particular the flag-major index, of a group $G(r,p,q,n)$ can be used to describe certain aspects of the representation theory of the "dual" group $G(r,q,p,n)$, providing a unified description of many of the main results appearing in [2, 1, 4, 5].

In this work we give new definitions of labelings of a forest, which generalize the standard type in [6] and the signed type in [10] (strictly speaking the specialization of our definitions and results to the signed type does not coincide with those given in [10] but one can easily show that they are equivalent in the most relevant cases). A first natural generalization is to consider labels which are colored integers. We generalize the major index defined in [6] introducing the *flag-major index of a colored labeled forest*. This allows us to generalize in a natural way the two hook-length formulas recalled above. As particular cases of them, we recover some known results for the distribution of the flag-major index on projective reflection groups $G^* = G(r,n)/C_p$ [8] and on sets of cosets representatives for some special subgroups of G^* [9]. Motivated by the study of invariant and coinvariant algebras of some groups related to the projective reflection groups $G(r,p,q,n)$ in §2 and the above mentioned study of the distribution of the flag-major index on sets of cosets representatives, we have been naturally led to consider a more general class of labelings that we call partially colored labelings (and also equivalence classes of such labelings) in order to provide a general statement (Theorem 4.4), which is our main result and includes as special cases all the above mentioned results.

This paper is structured as follows. In §1 we collect some notations and preliminaries for the necessary background. In §2 we study invariant and coinvariant algebras of some finite groups in order to have deeper motivations to introduce and study the notion of partially colored labeling in §3. In §4 we make a further generalization considering orbits of partially colored labelings under the action of a specific cyclic group. We define also the flag-major index for these labelings and we present an analogue of the q-hook length formula over all linear extensions of a colored labeled forests. Finally, in §5 we give a generalized version of the second q-hook length formula of Björner and Wachs.

1. Notation and preliminaries

1.1. Some notations

We let $\mathbb{Z}$ be the set of integer numbers and $\mathbb{N}$ be the set of non-negative integers. For $a, b \in \mathbb{Z}$, $a \leq b$, we let $[a, b] := \{a, a+1, \ldots, b\}$. For $n \in \mathbb{N}$, $n \neq 0$, we let also $[n] := [1, n]$. If q is an indeterminate, we let

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$$

be the *q-analogue* of n, and $[n]_q! := [1]_q[2]_q \cdots [n]_q$. We let

$$\mathscr{P}_n := \{f = (f_1, f_2, \ldots, f_n) \in \mathbb{N}^n : \ f_1 \geq f_2 \geq \cdots \geq f_n\}$$

be the set of *partitions* of *length* at most n, and $|f| := f_1 + f_2 + \cdots + f_n$ be the *size* of f.

Let $\mathcal{S}_n$ be the symmetric group on n letters. A permutation $\sigma \in \mathcal{S}_n$ will be denoted by $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n]$, where $\sigma_i = \sigma(i)$ for $i \in [n]$. We denote the *descent set* of σ by

$$\mathrm{Des}(\sigma) := \{i \in [n-1] : \ \sigma_i > \sigma_{i+1}\},$$

and the *major index* of σ by

$$\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{Des}(\sigma)} i.$$

If $r \in \mathbb{N}$, we let $\mathbb{Z}_r := \mathbb{Z}/r\mathbb{Z}$. If $p|r$ and $a \in \mathbb{Z}_r$, when no confusion arises, we will usually still denote by a the projection of a on $\mathbb{Z}_p$, including the case $r = 0$, i.e. the case when a is a genuine integer. The following convention will be very useful in this paper: if $r, r' \in \mathbb{N}$, $p|r, r'$, $a \in \mathbb{Z}_r$ and $b \in \mathbb{Z}_{r'}$ we write "$a = b \in \mathbb{Z}_p$" to mean that the projections of a and b in $\mathbb{Z}_p$ coincide. Moreover, if $a \in \mathbb{Z}_r$ we let $\mathrm{res}_p(a)$ be the smallest non-negative representative of a in $\mathbb{Z}_p$, and for $a, b \in \mathbb{Z}_r$ we write $a \prec_p b$ if $\mathrm{res}_p(a) < \mathrm{res}_p(b)$.

A *r-colored integer* is a pair (i, a), denoted also i^a, where $i \in \mathbb{N} \setminus \{0\}$ and $a \in \mathbb{Z}_r$. We define its *absolute value* to be $|i^a| := i$, and its *color* to be $c(i^a) := a$.

Finally, we denote by ζ_r the primitive r-th root of the unity $e^{2\pi i/r}$.

1.2. Complex reflection groups and $G(r, p, n)$

Let V be a complex vector space of finite dimension. An element $r \in GL(V)$ is called a *pseudo-reflection* if it has finite order and its fixed point space is of codimension 1. A finite subgroup $W \subseteq GL(V)$ is a (finite) *complex reflection group* if it is generated by pseudo-reflections.

In this paper we deal with the infinite family of complex reflection groups $G(r, p, n)$, where r, p, n are positive integers with $p \mid r$, that we are going to describe.

When $r = p = 1$, the group $G(1, 1, n)$ is the symmetric group $\mathcal{S}_n$, the group of the $n \times n$ permutation matrices.

When $p = 1$, the group $G(r, n) := G(r, 1, n)$ is the *wreath product* $\mathbb{Z}_r \wr \mathcal{S}_n$, also called *generalized symmetric group*, or *group of colored permutations*. $G(r, n)$ consists of all $n \times n$ matrices satisfying the following conditions:

- the entries are either 0 or r-th roots of unity;
- there is exactly one non-zero entry in every row and every column.

If p divides r, then $G(r, p, n)$ is the subgroup of $G(r, n)$ given by the matrices such that:

- the product of the non-zero entries is a r/p-th root of unity.

For our exposition it is more convenient to consider wreath products not as groups of complex matrices but as groups of colored permutations. So we recall the following alternative notation.

Notation 1. If $g \in G(r, n)$, we write $g = [\sigma_1^{c_1}, \sigma_2^{c_2}, \ldots, \sigma_n^{c_n}]$ if the non-zero entry in the i-th row of g is $\zeta_r^{c_i}$ and appears in the σ_i-th column.

In this notation we reinterpret $G(r, n)$ as the group of permutations g of the set of r-colored integers i^a, where $i \in [n]$ and $a \in \mathbb{Z}_r$, such that if $g(i^0) = j^b$ then $g(i^a) = j^{a+b}$. In fact the element $g = [\sigma_1^{c_1}, \sigma_2^{c_2}, \ldots, \sigma_n^{c_n}]$ represents the unique such permutation such that $g_i := g(i^0) = \sigma_i^{c_i}$. If $g = [\sigma_1^{c_1}, \sigma_2^{c_2}, \ldots, \sigma_n^{c_n}] \in G(r, n)$, we let $|g| := \sigma \in \mathcal{S}_n$ and we denote by

$$\mathrm{col}(g) := \sum_{i=1}^{n} c_i \in \mathbb{Z}_r$$

the *color weight* of g. In this notation we have that

$$G(r, p, n) := \{g \in G(r, n) : \mathrm{col}(g) = 0 \in \mathbb{Z}_p\}.$$

Note that $G(r, p, n)$ is a normal subgroup of $G(r, n)$ of index p, as it is the kernel of the surjective map $G(r, n) \to \mathbb{Z}_p$ given by $g \mapsto \mathrm{col}(g)$.

Example 1. If $r = 2$, the group $G(2, n)$ is the Weyl group B_n. In this case one usually reinterprets 2-colored integers as signed integers, i.e. for any positive integer i, we write i instead of i^0 and $-i$ instead of i^1, so that the group B_n is called the group of *signed* permutations. For example, the element $\beta = [2^0,\ 4^1,\ 3^0,\ 5^0,\ 1^0] \in G(2, 5)$ becomes $\beta = [2,\ -4,\ 3,\ 5,\ 1]$.

Example 2. If $r = p = 2$, the group $G(2, 2, n)$ is the Weyl group D_n, also known as the group of *even-signed* permutations. In fact D_n is the subgroup of B_n consisting of signed permutations with an even number of minus signs, or equivalently of 2-colored permutations in which the color 1 appears an even number of times:

$$D_n := \{g \in B_n : \operatorname{neg}(g) = 0 \in \mathbb{Z}_2\} = \{g \in B_n : \operatorname{col}(g) = 0 \in \mathbb{Z}_2\},$$

where $\operatorname{neg}(g) = |\{i \in [n] : g(i) < 0\}|$. For example, $\gamma = [2,\ -4,\ 3,\ -5,\ 1] = [2^0,\ 4^1,\ 3^0,\ 5^1,\ 1^0] \in G(2, 2, 5)$.

1.3. Projective reflection groups and $G(r, p, q, n)$

Let V be a complex vector space of finite dimension n and $S^q(V)$ the q-th symmetric power of V. Let C_q be the cyclic scalar subgroup of $GL(V)$ of order q generated by $\zeta_q I$. Finally, let G be a finite subgroup of $GL(S^q(V))$. Then, according to [8], we say that the pair (G, q) is a (finite) *projective reflection group* if there exists a finite complex reflection group $W \subset GL(V)$ such that $C_q \subseteq W$ and $G = W/C_q$.

The infinite family of groups $G(r, p, n)$ gives rise to the following family of projective reflection groups.

Definition 1. Let r, p, q, n be positive integers such that $p \mid r$, $q \mid r$ and $pq \mid rn$. Then we let

$$G(r, p, q, n) := \frac{G(r, p, n)}{C_q},$$

where C_q is the cyclic group generated by $\zeta_q I$.

In [7, 8, 9] it has been shown that the combinatorics and the representation theory of the two projective reflection groups $G(r, p, q, n)$ and $G(r, q, p, n)$ are intimately related. This is why it has been natural to let $G(r, p, q, n)^* = G(r, q, p, n)$ and call this group the *dual* group of $G(r, p, q, n)$.

Following our notation, for an element $g \in G(r, p, q, n)$ we also write $g = [\sigma_1^{c_1}, \sigma_2^{c_2}, \ldots, \sigma_n^{c_n}]$ to mean that g can be represented by $[\sigma_1^{c_1}, \sigma_2^{c_2}, \ldots, \sigma_n^{c_n}]$ in $G(r, p, n)$.

Example 3. We have $D_n^* = G(2,1,2,n) = B_n/\pm id$, where $id := id_{B_n}$ is the identity element of B_n. For example, $g = [2,\ -4,\ 3,\ 5,\ 1] \in G(2,1,2,5)$ can be represented by $g_1 = [2,\ -4,\ 3,\ 5,\ 1]$ or $g_2 = [-2,\ 4,\ -3,\ -5,\ -1]$ in $G(2,5)$.

1.4. Flag-major index on $G(r,p,q,n)$

Let $g = [\sigma_1^{c_1}, \sigma_2^{c_2}, \dots, \sigma_n^{c_n}] \in G(r,p,q,n)$. According to [8], we let

$$\mathrm{HDes}(g) := \{i \in [n-1] :\ c_i = c_{i+1} \text{ and } \sigma_i > \sigma_{i+1}\}$$

be the *homogeneous descent set* of g (note that, while the colors c_i's depend on the chosen representative, the condition $c_i = c_{i+1}$ is independent of such choice),

$$d_i(g) := |\{j \in [i, n-1] :\ j \in \mathrm{HDes}(g)\}|$$

for all $i \in [n]$, and

$$k_i(g) := \begin{cases} \mathrm{res}_{r/q}(c_n) & \text{if } i = n, \\ k_{i+1}(g) + \mathrm{res}_r(c_i - c_{i+1}) & \text{if } i \in [n-1]. \end{cases}$$

Note that the sequence $d(g) := (d_1(g), d_2(g), \dots, d_n(g))$ is a partition, and observe that $k(g) := (k_1(g), k_2(g), \dots, k_n(g))$ can be characterized as the smallest element in $\mathscr{P}_n$ (with respect to the entrywise order) such that

$$g = [\sigma_1^{k_1(g)}, \sigma_2^{k_2(g)}, \dots, \sigma_n^{k_n(g)}].$$

We also let

$$\lambda_i(g) := r d_i(g) + k_i(g)$$

for all $i \in [n]$, and similarly we note that $\lambda(g) := (\lambda_1(g), \lambda_2(g), \dots, \lambda_n(g))$ is a partition such that

$$g = [\sigma_1^{\lambda_1(g)}, \sigma_2^{\lambda_2(g)}, \dots, \sigma_n^{\lambda_n(g)}]. \tag{1.1}$$

Finally, we define the *flag-major index* of an element $g \in G(r,p,q,n)$ as

$$\mathrm{fmaj}(g) := |\lambda(g)|.$$

Note that these definitions do not depend on the choice of the representative of g in $G(r,p,n)$.

Example 4. Let $g = [2^2, 7^3, 6^3, 4^5, 8^1, 1^7, 5^3, 3^2] \in G(6,2,3,8)$. Then $\mathrm{HDes}(g) = \{2,5\}$, $d(g) = (2,2,1,1,1,0,0,0)$, $k(g) = (18,13,13,9,5,5,1,0)$, $\lambda(g) = (30,25,19,15,11,5,1,0)$ and $\mathrm{fmaj}(g) = 106$.

All the interest around the fmaj statistic probably originated from the following result.

Proposition 1.1. ([2], Theorem 4.1) *Let t be an indeterminate. We have*

$$\sum_{g\in G(r,n)} t^{\mathrm{fmaj}(g)} = [d_1]_t[d_2]_t\cdots[d_n]_t,$$

where d_i's are the fundamental degrees of $G(r,n)$ (see §1.5).

This result has been extended to all groups $G(r,p,q,n)$ in [8] and in particular we have the following fact which is of interest in this paper.

Proposition 1.2. *Let $G = G(r,p,n)$ and $G^* = G(r,n)/C_p$. Then*

$$\sum_{g\in G^*} t^{\mathrm{fmaj}(g)} = [d_1]_t[d_2]_t\cdots[d_n]_t,$$

where d_i's are the fundamental degrees of G.

In the rest of this section we let $G = G(r,p,n)$ and $G^* = G(r,n)/C_p$. Inspired by work of Garsia [12] the second author also studied in [9] the distribution of the flag-major index on sets of cosets representatives for some special subgroups of G^*, defined as follows. For $1 \le k < n$, let

$$\mathscr{C}_k := \{[\sigma_1^0, \sigma_2^0, \ldots, \sigma_k^0, g_{k+1}, \ldots, g_n] \in G^* : \sigma_1 < \sigma_2 < \cdots < \sigma_k\}. \tag{1.2}$$

We note that the subgroup of G^* given by

$$\{g \in G^* : g = [g_1, g_2, \ldots, g_k, (k+1)^0, \ldots, n^0]\}$$

is isomorphic to $G(r,k)$ for all $k < n$. We may observe that $\mathscr{C}_k$ contains exactly p representatives for each (right) coset of $G(r,k)$ in G^*. Then we have the following distribution which can be seen as a generalization of Proposition 1.2.

Theorem 1.3. ([9], Theorem 5.5) *Let $\mathscr{C}_k$ be defined as in (1.2). Then*

$$\sum_{g\in\mathscr{C}_k} t^{\mathrm{fmaj}(g^{-1})} = [p]_{t^{kr/p}}[(k+1)r]_t\,[(k+2)r]_t\cdots[(n-1)r]_t\,[nr/p]_t.$$

The following is an immediate consequence.

Corollary 1.4. ([9], Corollary 5.6) *If $p=1$, then $\mathscr{C}_k$ is a complete system of coset representatives for the subgroup $G(r,k)$ and*

$$\sum_{g\in\mathscr{C}_k} t^{\mathrm{fmaj}(g^{-1})} = [(k+1)r]_t\,[(k+2)r]_t \cdots [nr]_t.$$

We recall now some further technical results we will use in the present work.

Lemma 1.5. ([9], Lemma 5.1) *There exists a bijection*

$$G^*\times\mathscr{P}_n\times[0,p-1]\to\mathbb{N}^n, \qquad (g,\lambda,h)\mapsto f=(f_1,f_2,\ldots,f_n),$$

where $f_i=\lambda_{|g^{-1}(i)|}(g)+r\lambda_{|g^{-1}(i)|}+h\frac{r}{p}$ for all $i\in[n]$. In this case we say that f is g-compatible.

Lemma 1.6. ([9], Lemma 5.2) *If $g\in G^*$ we let S_g be the set of g-compatible vectors in $\mathbb{N}^n$. Then*

$$\sum_{f\in S_g} x_1^{f_1}\cdots x_n^{f_n} = \frac{x_{|g_1|}^{\lambda_1(g)}\cdots x_{|g_n|}^{\lambda_n(g)}}{(1-x_{|g_1|}^r)\cdots(1-x_{|g_1|}^r\cdots x_{|g_{n-1}|}^r)(1-x_{|g_1|}^{r/p}\cdots x_{|g_n|}^{r/p})}.$$

Lemma 1.7. ([9], Lemma 5.3) *If $g\in G^*$ then there exists $h\in[0,p-1]$ such that $\lambda_i(g)+\lambda_{|g_i|}(g^{-1})=h\frac{r}{p}\in Z_r$, for all $i\in[n]$.*

1.5. Invariants and descent basis

Let V be a complex vector space of finite dimension n and W a finite complex reflection group. Then W is characterized by the structure of its invariant ring, in the following sense.

Let $S[V^*]$ be the symmetric algebra on V^*, which can be seen as the algebra of polynomial functions on V. Any finite subgroup W of $GL(V)$ acts naturally on $S[V^*]$. Denote by $S[V^*]^W$ the invariant ring of W. Then Chevalley [11] and Shephard-Todd [15] proved that W is a complex reflection group if and only if $S[V^*]^W$ is generated by (1 and by) n algebraically independent homogeneous elements, called *basic invariants*. Although these polynomials are not uniquely determined, their degrees $d_1,\ldots,d_n$ are basic numerical invariants of W, and they are called *fundamental degrees* of W. Denote by $I(W)$ the ideal of $S[V^*]$ generated by the homogeneous elements of strictly positive degree in $S[V^*]^W$. Then we recall that the *coinvariant algebra* of W is defined by

$$R(W) := \frac{S[V^*]}{I(W)}.$$

Since $I(W)$ is W-invariant, the group W acts naturally on $R(W)$. We recall that $R(W)$ is isomorphic to the left regular representation of W and in particular that its dimension as a $\mathbb{C}$-vector space is $|W|$.

In [8] the second author generalized this result to the case of projective reflection groups. Let $S_q[V^*]$ be the q-th Veronese subalgebra of $S[V^*]$, i.e. the algebra of polynomial functions on V generated by 1 and the homogeneous polynomial functions of degree q. For the reader's convenience we recall some results in the invariant theory of projective reflection groups which are proved in [8].

Let G be any finite subgroup of graded automorphisms of $S_q[V^*]$. Then (G,q) is a projective reflection group if and only if the invariant algebra $S_q[V^*]^G$ is generated by (1 and by) n algebraically independent homogeneous elements.

We denote by $I(G)$ the ideal of $S_q[V^*]$ generated by homogeneous elements of positive degree in $S_q[V^*]^G$. Then the *coinvariant algebra* of G is defined by

$$R(G) := \frac{S_q[V^*]}{I(G)}.$$

Let W be the complex reflection group such that $G = W/C_q$. We recall that

$$S_q[V^*]^G = S[V^*]^W. \tag{1.3}$$

It follows that $R(G)$ is the subalgebra of $R(W)$ generated by the homogeneous elements of degree multiple of q. Moreover, we recall that $R(G)$ is isomorphic to the left regular representation as a G-module and in particular that its dimension as a $\mathbb{C}$-vector space is $|G|$.

If we let $X := (x_1, \ldots, x_n)$ be a basis of V^*, then $S[V^*]$ and $S_q[V^*]$ can be identified respectively with the polynomial algebra $\mathbb{C}[X]$ and its subalgebra $S_q[X]$ generated by 1 and the monomials of degree q.

Observe that $G(r,n)$ acts on $\mathbb{C}[X]$ as follows:

$$[\sigma_1^{c_1}, \sigma_2^{c_2}, \ldots, \sigma_n^{c_n}] \cdot P(X) = P\left(\zeta_r^{c_{\sigma_1}} x_{\sigma_1}, \zeta_r^{c_{\sigma_2}} x_{\sigma_2}, \ldots, \zeta_r^{c_{\sigma_n}} x_{\sigma_n}\right).$$

A set of basic invariants under this action is given by

$$e_i(x_1^r, \dots, x_n^r), \quad i \in [n], \tag{1.4}$$

where the e_i's are the elementary symmetric functions. It follows that the fundamental degrees of $G(r,n)$ are

$$r,\ 2r,\ \dots,\ nr.$$

Moreover, $\dim R\big(G(r,n)\big) = |G(r,n)| = n!r^n$.

Now, consider the restriction to $W = G(r,p,n)$ of the action of $G(r,n)$ on $\mathbb{C}[X]$. Then a set of basic invariants is given by

$$\begin{cases} e_i(x_1^r, \dots, x_n^r) & \text{if } i \in [n-1] \\ x_1^{r/p} \cdots x_n^{r/p} & \text{if } i = n, \end{cases} \tag{1.5}$$

and the fundamental degrees of W are

$$r,\ 2r,\ \dots,\ (n-1)r,\ nr/p.$$

Moreover, $\dim R(W) = |W| = n!r^n/p$.

Finally, consider the action of $G = G(r,p,q,n)$ on $S_q[X]$. From (1.3) we recall that a set of basic invariants is given by (1.5). Moreover,

$$\dim R(G) = |G| = \frac{n!r^n}{pq} = |G^*|.$$

The following result shows that a basis of the coinvariant algebra of G is naturally described by its dual group G^*.

Theorem 1.8. ([8], Theorem 5.3) *Let $G = G(r,p,q,n)$. Then the set $\{a_g : g \in G^*\}$, where*

$$a_g(X) := \prod_{i=1}^{n} x_{|g_i|}^{\lambda_i(g)}$$

is a monomial of degree $\mathrm{fmaj}(g)$*, represents a basis for $R(G)$.*

1.6. Labeled forests and q-hook length formulas

According to [6] we consider a finite poset F in which every element is covered by at most one element, or equivalently such that its Hasse diagram is a rooted forest with roots on top. For this reason we call also F a *forest* and we let $V(F)$ and $E(F)$ be the vertex set and the edge set of the Hasse diagram of F, respectively, and $\prec$ the order relation in F. We can also denote an *edge* in $E(F)$ by an ordered pair (x, y) of elements of F such that x is covered by y. Let

$$h_x := |\{a \in F : a \preceq x\}|$$

be the *hook length* of the element x, for each $x \in F$, and

$$h_{(x,y)} := h_x$$

the hook length of the edge (x, y), for each $(x, y) \in E(F)$. We recall that a *linear extension* of F is an indexing x of the vertices of $V(F) = \{x_1, \ldots, x_n\}$ such that $x_i \prec x_j$ only if $i < j$, and we denote by $\mathscr{L}(F)$ the set of linear extensions of F. Let

$$\mathscr{W}(F) := \{w : V(F) \to [n] \text{ s.t. } w \text{ is a bijection}\}$$

be the *set of labelings* of F.

For $w \in \mathscr{W}(F)$ we denote the *descent set* of w by

$$\mathrm{Des}(w) := \{(x, y) \in E(F) : w(x) > w(y)\},$$

the *major index* of w by

$$\mathrm{maj}(w) = \sum_{e \in \mathrm{Des}(w)} h_e,$$

and the *set of linear extensions* of w by

$$\mathscr{L}(w) = \{\sigma \in \mathcal{S}_n : \text{ if } x \prec y \text{ then } \sigma^{-1}(w(x)) < \sigma^{-1}(w(y))\}.$$

Equivalently, if $x_1, x_2, \ldots, x_n$ is a linear extension of F, then the element $[w(x_1), w(x_2), \ldots, w(x_n)] \in \mathcal{S}_n$ is a linear extension of w and $\mathscr{L}(w)$ is the set of all such permutations.

Example 5. Let w be the labeling in Figure 1 (where the letters x, y, z denote the vertices of interest in this example and the numbers $1, 2, 3, 4, 5$ the labels). Then for example we have $[3, 2, 5, 4, 1] \in \mathscr{L}(w)$. Moreover, $\mathrm{Des}(w) = \{(x, y),\ (y, z)\}$ and $\mathrm{maj}(w) = h_x + h_y = 1 + 3 = 4$.

The distributions of the major index on linear extensions of a fixed labeling and on all labelings of a fixed forest have very nice factorization formulas which have been obtained by Björner and Wachs.

Theorem 1.9. ([6], Theorem 1.2) *Let F be a finite forest with n elements and w a labeling of F. Then*

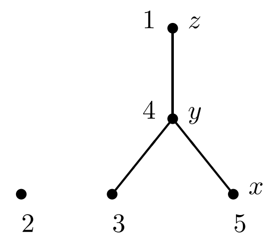

Figure 1: Example of labeling.

$$\sum_{\sigma \in \mathscr{L}(w)} q^{\mathrm{maj}(\sigma)} = q^{\mathrm{maj}(w)} \frac{[n]_q!}{\prod_{x \in F} [h_x]_q}.$$

Theorem 1.10. ([6], Theorem 1.3) *Let F be a finite forest with n elements and $\mathscr{W}(F)$ the set of all labelings of F. Then*

$$\sum_{w \in \mathscr{W}(F)} q^{\mathrm{maj}(w)} = \frac{n!}{\prod_{x \in F} h_x} \prod_{x \in F} [h_x]_q.$$

2. Invariants and coinvariants

In this section we study the structure (such as generators, bases and Hilbert series/polynomials) of the algebra of invariant polynomials and of the algebra of coinvariant polynomials for some finite groups which are strictly related to the groups $G(r,p,q,n)$. This algebraic setting will serve as inspiration for the appropriate definitions and as a motivation for the generalization and unification of all the main results collected in §1.

Let $N = (n_1, \ldots, n_k) \in \mathbb{N}^k$ and $r \in \mathbb{N}$, $r > 0$. Consider the direct product

$$G(r,N) := G(r,n_1) \times \cdots \times G(r,n_k)$$

of k groups of r-colored permutations. Let p be a positive divisor of r. By extending the definitions given in §1, we consider the following two groups obtained from $G(r,N)$: its subgroup

$$\begin{aligned} G &:= G(r,p,N) \\ &:= \left\{ (g_1, \ldots, g_k) \in G(r,N) : \sum \mathrm{col}(g_i) = 0 \in \mathbb{Z}_p \right\}, \end{aligned}$$

and its quotient

$$H := \frac{G(r,N)}{C_p},$$

where C_p is the cyclic subgroup of $G(r,N)$ of order p generated by

$$([1^{r/p}, 2^{r/p}, \ldots, n_1^{r/p}], \ldots, [1^{r/p}, 2^{r/p}, \ldots, {n_k}^{r/p}]).$$

For notational convenience we let $X_i := (x_{i,1}, \ldots, x_{i,n_i})$, where $x_{i,j}$'s are variables, $\mathbb{C}[X_i] = \mathbb{C}[x_{i,1}, \ldots, x_{i,n_i}]$, and

$$\mathbb{C}[\mathcal{X}] = \mathbb{C}[x_{i,j} : i \in [k],\, j \in [n_i]]$$

be a polynomial ring in $n_1 + \cdots + n_k$ variables. We also let $e_j^{(r)}(X_i) := e_j(x_{i,1}^r, \ldots, x_{i,n_i}^r)$, where e_j is the j-th elementary symmetric function. The group H is a projective reflection group, since it is the quotient of a reflection group modulo a cyclic scalar subgroup of order p, while the group G is not in general. If we denote by

$$S_p[\mathcal{X}] \subset \mathbb{C}[\mathcal{X}]$$

the subalgebra spanned by the homogeneous elements of total degree divisible by p, we have that H acts on $S_p[\mathcal{X}]$ and its invariants coincide with the invariants of $G(r,N)$, i.e.

$$S_p[\mathcal{X}]^H = \mathbb{C}[e_j^{(r)}(X_i),\, i \in [k],\, j \in [n_i]].$$

Observe that we already knew from §1.5 that the invariant algebra of H is generated as a $\mathbb{C}$-algebra by $n_1 + \cdots + n_k$ algebraically independent homogeneous polynomials (together with 1).

Denote by $I(H)$ the ideal of $S_p[\mathcal{X}]$ generated by the homogeneous invariants of (strictly) positive degree and let

$$R(H) = \frac{S_p[\mathcal{X}]}{I(H)}$$

be the coinvariant algebra of H. We define the *flag-major index* of an element $\gamma = (g_1, \ldots, g_k) \in G(r,N)$ as

$$\mathrm{fmaj}(\gamma) := \sum_i \mathrm{fmaj}(g_i).$$

We also let in this case

$$a_\gamma(\mathcal{X}) := a_{g_1}(X_1)\cdots a_{g_k}(X_k),$$

and we note that $\deg a_\gamma = \operatorname{fmaj}(\gamma)$.

Proposition 2.1. *The set $\{a_\gamma : \gamma \in G\}$ represents a basis for the coinvariant algebra $R(H)$.*

Proof. We observe that, since the invariants of H coincide with the invariants of $G(r, N)$ by Eq. (1.3), we have that $R(H)$ is the subalgebra of

$$R\big(G(r,N)\big) = \frac{\mathbb{C}[\mathcal{X}]}{I\big(G(r,N)\big)}$$

spanned by homogeneous elements of degree divisible by p. As $R(G(r, N)) \cong R(G(r, n_1)) \otimes \cdots \otimes R(G(r, n_k))$ we have that the set $\{a_\gamma : \gamma \in G(r, N)\}$ is a basis of $R(G(r, N))$, by Theorem 1.8. Therefore, a basis for $R(H)$ is given by

$$\{a_\gamma : \deg a_\gamma = 0 \in \mathbb{Z}_p\} = \{a_\gamma : \operatorname{fmaj}(\gamma) = 0 \in \mathbb{Z}_p\} = \{a_\gamma : \gamma \in G\},$$

as $\operatorname{fmaj}(g) = \operatorname{col}(g) \in \mathbb{Z}_r$ for all $g \in G(r, n)$. □

Therefore we have that a basis of $R(H)$ can be naturally described by elements in G and in particular $\dim R(H) = |G| = |H|$. The next target is to show a sort of inverse of this result: we will show that a basis of $R(G)$ can be described using elements in H, although this result is not as neat as in the case of projective reflection groups.

To study the G-invariant polynomials we need the following technical result.

Lemma 2.2. *Let G be a finite group and V a complex vector space of finite dimension n. Consider a representation of G on V and suppose that such representation is monomial, i.e. there exists a basis $B = (b_1, b_2, \ldots, b_n)$ of V such that, for every $g \in G$ and $i \in [n]$, $g(b_i)$ is a scalar multiple of some basis element b_j. Let $v = a_1 b_1 + \ldots + a_n b_n$ be an invariant element of V and suppose that there exists a subgroup H of G and $l \in [n]$ such that*

$$\sum_{h \in H} h(b_l) = 0.$$

Then $a_l = 0$.

Proof. Say that two basis elements b_i and b_j are in the same G-orbit if $g(b_i)$ is a scalar multiple of b_j for some $g \in G$. Consider the G-orbit $\mathcal{O}$ of the basis B containing b_l and consider the projection $v_{\mathcal{O}} = \sum_{b_i \in \mathcal{O}} a_i b_i$ of v. The element $v_{\mathcal{O}}$ is still invariant and therefore, by restricting the representation of G to the vector subspace spanned by the elements in $\mathcal{O}$, we can assume that the action of G on B is transitive.

Let S be a set of representatives of (left) cosets of H in G, i.e. $G = S \cdot H = \biguplus_{s\in S} sH$, where $\biguplus$ denotes the disjoint union. Then

$$\sum_{g\in G} g(b_l) = \sum_{s\in S}\sum_{h\in H} sh(b_l) = 0.$$

Since the representation is monomial and G is transitive, for any $j \in [n]$ there exists an element $\widetilde{g} \in G$ such that $b_j = c\,\widetilde{g}(b_l)$ for a suitable $c \in \mathbb{C}$. So

$$\sum_{g\in G} g(b_j) = \sum_{g\in G} g(c\,\widetilde{g}(b_l)) = \sum_{g\in G} c\, g\widetilde{g}(b_l) = c \sum_{g'\in G} g'(b_l) = 0.$$

Then, since v is invariant,

$$v = \frac{1}{|G|}\sum_{g\in G} g(v) = \frac{1}{|G|}\sum_{g\in G}\sum_{i} a_i g(b_i) = \frac{1}{|G|}\sum_{i} a_i \sum_{g\in G} g(b_i) = 0,$$

completing the proof. $\square$

The following result provides a precise description of the G-invariant polynomials.

Proposition 2.3. *Let $d = r/p$. The invariant ring of $G(r,p,N)$ is generated as a $\mathbb{C}$-algebra by (1 and by) the homogeneous polynomials $e_j^{(r)}(X_i)$, $i \in [k]$, $j \in [n_i]$, and*

$$e_N^{(d)}(\mathcal{X}) := e_{n_1}^{(d)}(X_1)\cdots e_{n_k}^{(d)}(X_k) = \prod_{i,j} x_{i,j}^d.$$

Proof. It is a simple verification that all polynomials $e_j^{(r)}(X_i)$ and $e_N^{(d)}(\mathcal{X})$ are G-invariant, so that we only have to show that if P is a G-invariant polynomial then P can be expressed as a polynomial in the $e_j^{(r)}(X_i)$ and $e_N^{(d)}(\mathcal{X})$. We can clearly assume that P is homogeneous (since otherwise we can consider its homogeneous components) and proceed by induction on $\deg P$, the case $\deg P = 0$ being trivial. If $e_N^{(d)}(\mathcal{X})$ divides P, the result easily follows by induction. If $e_N^{(d)}(\mathcal{X})$ does not divide P, then if we expand P as a linear combination of monomials

$$P = \sum_M c_M M,$$

there exists a monomial $M_0 = \prod x_{i,j}^{d_{i,j}}$ with a pair (i_0, j_0) such that $0 \leq d_{i_0,j_0} < d$. Suppose $c_{M_0} \neq 0$. We first show that $d_{i_0,j_0} = 0$. In fact, consider the element $\gamma = (\gamma_1, \ldots, \gamma_k) \in G$ given by

$$\gamma_i(j) = \begin{cases} j^p & \text{if } (i,j) = (i_0, j_0) \\ j & \text{otherwise,} \end{cases}$$

for all $i \in [k]$ and $j \in [n_i]$. If $d_{i_0,j_0} \neq 0$ we have $\gamma(M_0) = \zeta_r^{pd_{i_0,j_0}} M_0 \neq M_0$ and therefore, letting $H = \langle \gamma \rangle$ we have

$$\sum_{h \in H} h(M_0) = \sum_{s=1}^{r/p} \zeta_r^{psd_{i_0,j_0}} M_0 = 0.$$

Hence, by Lemma 2.2, we have $c_{M_0} = 0$ which contradicts our assumption and therefore $d_{i_0,j_0} = 0$.

Now suppose that there exists a pair (i_1, j_1) such that $r \nmid d_{i_1,j_1}$. We first claim that there exists a such that $(ap-1)d_{i_1,j_1} \neq 0 \in \mathbb{Z}_r$. In fact, if $d \nmid d_{i_1,j_1}$ and $(p-1)d_{i_1,j_1} = 0 \in \mathbb{Z}_r$ then $(2p-1)d_{i_1,j_1} = pd_{i_1,j_1} \neq 0 \in \mathbb{Z}_r$. If $d \mid d_{i_1,j_1}$ then $(p-1)d_{i_1,j_1} = -d_{i_1,j_1} \neq 0 \in \mathbb{Z}_r$ by the choice of (i_1, j_1).

Choose such element a and consider the element $\gamma := (\gamma_1, \ldots, \gamma_k) \in G$ given by

$$\gamma_i(j) = \begin{cases} j^1 & \text{if } (i,j) = (i_0, j_0) \\ j^{ap-1} & \text{if } (i,j) = (i_1, j_1) \\ j & \text{otherwise,} \end{cases}$$

for all $i \in [k]$ and $j \in [n_i]$.

If we let $s := (ap-1)d_{i_1,j_1} \neq 0 \in \mathbb{Z}_r$ we have that $\gamma(M_0) = \zeta_r^s M_0$. Moreover, the subgroup $\langle \gamma \rangle$ has order r as $\gamma_{i_0}(j_0) = j_0^1$ and we have

$$\sum_{m=0}^{r-1} \gamma^m(M_0) = \Big(\sum_{m=0}^{r-1} \zeta_r^{ms} \Big) M_0 = 0,$$

since $r \nmid s$. Then, by Lemma 2.2, we have $c_{M_0} = 0$.

Now we observe that, since the action of G preserves the set of exponents of any monomial M, we can assume that every monomial M with $c_M \neq 0$ has an exponent $< d$. So we can repeat the above argument to every monomial M with $c_M \neq 0$ and conclude that all variables appear in P with exponents divisible by r. In particular, if $\gamma = (\gamma_1, \dots, \gamma_k) \in G(r,N)$ then $\gamma(P) = (|\gamma_1|, \dots, |\gamma_k|)(P) = P$ as, clearly, $(|\gamma_1|, \dots, |\gamma_k|) \in G$. It follows that P is also $G(r,N)$-invariant and in particular it is a polynomial in the $e_j^{(r)}(X_i)$, $i \in [k]$, $j \in [n_i]$. $\square$

Now we note that, since

$$\mathrm{Inv}\big(G(r,N)\big) \subset \mathrm{Inv}(G),$$

then $R(G)$ is a quotient of $R(G(r,N))$. More precisely, by Proposition 2.3, we have

$$R(G) = \frac{R\big(G(r,N)\big)}{(e_N^{(d)}(\mathcal{X}))},$$

where $(e_N^{(d)}(\mathcal{X}))$ is the ideal generated by $e_N^{(d)}(\mathcal{X})$ in $R(G(r,N))$.

For $\gamma \in G(r,N)$ we let $\gamma_i \in G(r,n_i)$ be its i-th coordinate, so that $\gamma = (\gamma_1, \dots, \gamma_k)$.

Proposition 2.4. *Let $d = r/p$. We have:*

- *The set*

$$\{a_\gamma(\mathcal{X}) : \gamma \in G(r,N) \textit{ is such that } c((\gamma_i)_n) \succeq_r d \textit{ for all } i \in [k]\}$$

  is a basis for the ideal $(e_N^{(d)}(\mathcal{X}))$ in $R(G(r,N))$,
- *The set*

$$\{a_\gamma(\mathcal{X}) : \gamma \in G(r,N) \textit{ is such that } c((\gamma_i)_n) \prec_r d \textit{ for some } i \in [k]\}$$

  is a basis for $R(G)$.

Proof. We make the following claim.

Let $P \in \mathbb{C}[X]$ be a polynomial such that $x_1^d \cdots x_n^d$ divides P. Then P admits the following expansion in $R(G(r,n))$:

$$P = \sum_{g \in \Omega} \eta_g a_g,$$

where $\eta_g \in \mathbb{C}$ and $\Omega := \{g \in G(r,n) : c(g_n) \succeq_r d\}$.

Since P is divisible by $x_1^d \cdots x_n^d$ we can write $P = Qx_1^d \cdots x_n^d$. We can expand the polynomial Q in $R(G(r,n))$ with respect to the basis $\{a_g : g \in G(r,n)\}$ and obtain the following expression for P:

$$P = \sum_{g \in G(r,n)} \eta_g a_g x_1^d \cdots x_n^d \in R(G(r,n)).$$

The result will follow if we can show that, for all $g \in G(r,n)$, either $a_g x_1^d \cdots x_n^d = 0 \in R(G(r,n))$ or $a_g x_1^d \cdots x_n^d = a_{g'}$, for some $g' \in \Omega$. In fact, let $g = [\sigma_1^{c_1}, \ldots, \sigma_n^{c_n}]$. Then if $c_n \succeq_r r - d$ all the exponents in the monomial a_g are at least $r - d$ and so $x_1^r \cdots x_n^r \mid a_g x_1^d \cdots x_n^d$, therefore $a_g x_1^d \cdots x_n^d = 0 \in R(G(r,n))$. Otherwise, if $c_n \prec_r r - d$ we consider the element $g' \in G(r,n)$ given by

$$g' = [\sigma_1^{c_1+d}, \ldots, \sigma_n^{c_n+d}].$$

It is clear that $g' \in \Omega$ and a simple verification shows that $\lambda_i(g') = \lambda_i(g) + d$ for all $i \in [n]$ and therefore $a_{g'} = a_g x_1^d \cdots x_n^d$. Now the first part of the statement is a straightforward consequence of the claim and the second part of the statement is an immediate consequence of the first, together with the fact that $R(G(r,p,N)) = R(G(r,N))/(e_N^{(d)}(\mathcal{X}))$ and Theorem 1.8. $\square$

We can finally describe the desired basis of $R(G)$ in terms of the elements of the group H, a result which can be seen as the "dual" version of Proposition 2.1. For $\delta \in H$, let Π_δ be the set of lifts $\gamma = (\gamma_1, \ldots, \gamma_n)$ of δ in $G(r,N)$ such that $c((\gamma_i)_{n_i}) \prec_r d$ for some $i \in [k]$.

Corollary 2.5. *It follows from Proposition 2.4 that the set*

$$\{a_\gamma : \delta \in H, \gamma \in \Pi_\delta\}$$

is a basis for $R(G)$.

So we have that the group H can still be taken as an indexing set for a basis of the coinvariant algebra $R(G)$ though the elements $\delta \in H$ must be considered with multiplicity given by $|\Pi_\delta|$. And in some sense we may say that the cardinalities of the sets Π_δ measure a defect for G from being a projective reflection group.

The description of the coinvariant algebras for the groups G and H that we have obtained serves as a motivation and inspiring example in the development of a theory of "partially colored labeled forests" which is the main subject of this paper. As an application we will also be able to describe the Hilbert polynomial of the coinvariant algebras of G and H explicitly.

3. Linear extensions of partially colored forest labelings

Let F be a finite forest with n vertices (see §1.6). The following is a natural generalization of a labeling of F.

Definition 2. A r*-colored labeling* of F is a pair $w = (\sigma, c)$ where σ is a bijection $\sigma : V(F) \to [n]$ and c is a map $c : V(F) \to \mathbb{Z}_r$. We denote by $\mathscr{W}_r(F)$ the set of all r-colored labelings of F.

A r-colored labeling $w = (\sigma, c)$ of F can be thought as the assignement of the *colored label* $w_x := \sigma_x^{c_x}$ to each vertex $x \in V(F)$. As customary, we also identify a colored integer i^0 with the integer i for each $i \in [n]$, and vice versa. Then, for $w \in \mathscr{W}_r(F)$ we define the *set of linear extensions* of w as

$$\begin{aligned}\mathscr{L}(w) := \big\{ g \in G(r,n) :\ & c(g^{-1}(w_x)) = 0 \text{ for all } x \in V(F), \text{ and} \\ & \text{if } x, y \in V(F) \text{ are such that } x \prec y, \text{ then } g^{-1}(w_x) < g^{-1}(w_y) \big\}.\end{aligned}$$

Note that this definition generalizes the one in §1.6, since the element $[w(x_1), w(x_2), \ldots, w(x_n)] \in G(r,n)$ is a linear extension of w if $x_1, x_2, \ldots, x_n$ is a linear extension of F and then $\mathscr{L}(w)$, as defined above, is the set of all such colored permutations.

If $x \in V(F)$ and x is not a root, we let $p(x)$ be the unique element that covers x in the forest. For each $x \in F$ we let

$$z_x(w) := \begin{cases} \operatorname{res}_r(c_x) & \text{if } x \text{ is a root of } F, \\ \operatorname{res}_r\left(c_x - c_{p(x)}\right) & \text{otherwise} \end{cases}$$

and we define the *homogeneous descent set* of w as

$$\operatorname{HDes}(w) := \{(x,y) \in E(F) : c_x = c_y \text{ and } \sigma_x > \sigma_y\}.$$

Finally we define the *flag-major index* of w as

$$\operatorname{fmaj}(w) := \sum_{e \in E(F)} r\chi_e(w) h_e + \sum_{v \in V(F)} z_v(w) h_v,$$

where

$$\chi_e(w) := \begin{cases} 1 & \text{if } e \in \operatorname{HDes}(w), \\ 0 & \text{otherwise.} \end{cases}$$

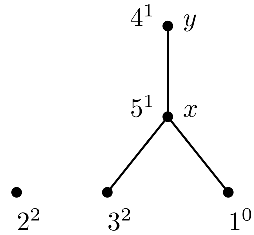

Figure 2: Example of 3-colored labeling.

Example 6. Let w be the 3-colored labeling in Figure 2. Then the element $[1^0, 2^2, 3^2, 5^1, 4^1] \in G(3,5)$ is a linear extension of w. We also have $\mathrm{HDes}(w) = \{(x,y)\}$ and so $\mathrm{fmaj}(w) = 3 \cdot 3 + (1 \cdot 4 + 1 \cdot 1 + 2 \cdot 1 + 2 \cdot 1) = 18$.

Remark 3.1. Note that if $r = 1$ then a 1-colored labeling $w \in \mathscr{W}_1(F)$ can be thought as a standard labeling in $\mathscr{W}(F)$. Then we have $\mathrm{HDes}(w) = \mathrm{Des}(w)$ and $\mathrm{fmaj}(w) = \mathrm{maj}(w)$. Moreover, if F is a *linear tree* (i.e. a totally ordered set $\{x_1, x_2, \ldots, x_n\}$ in which $x_i \prec x_{i+1}$ for $i \in [n-1]$) we note that a r-colored labeling w of F can be thought as the unique linear extension $g \in G(r,n)$ of w. And in this case we have $\mathrm{fmaj}(w) = \mathrm{fmaj}(g)$.

Now we can give a generalized version of Theorem 1.9, which we can recover from the following result when $r = 1$:

Theorem 3.2. *Let F be a finite forest with n elements and w a r-colored labeling of F. Then*

$$\sum_{g \in \mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = q^{\mathrm{fmaj}(w)} \frac{[d_1]_q [d_2]_q \cdots [d_n]_q}{\prod\limits_{x \in F} [h_x r]_q},$$

where $d_i = ri$, $i = 1, \ldots, n$ are the fundamental degrees of $G(r,n)$.

As we are planning to further generalize this result, we postpone its proof to a more general case (see Proof of Theorem 3.3).

Starting from Theorem 3.2 and partly inspired by Corollary 1.4, we can introduce a new notion of labeling that will allow us to generalize and unify these results.

Definition 3. We define the set $\mathscr{P}_r(F)$ of *r-partial labelings* of F as the set of triples $w = (\sigma, \iota, j)$ such that:

- σ is a bijection $\sigma : V(F) \to [n]$;

- ι is a map $\iota : V(F) \to \mathbb{N}$ such that $\iota_x := \iota(x)$ is a divisor of r for all $x \in V(F)$ and $x \prec y$ implies $\iota_x \mid \iota_y$;
- j is a map $j : V(F) \to \bigcup_{m>0} \mathbb{Z}_m$ such that $j_x := j(x) \in \mathbb{Z}_{\iota_x}$ for all $x \in V(F)$.

It is clear that a colored labeling can be interpreted as a partial labeling with $\iota_x = r$ for all $x \in V(F)$. A partial labeling $w = (\sigma, \iota, j)$ assigns to each vertex $x \in V(F)$ the *partial label* $w_x := \sigma_x^{\iota_x, j_x}$: the color of a label is not uniquely determined modulo r, but only modulo a divisor of r. In this sense such partial label can sometimes be interpreted as the set of r/ι_x distinct r-colored integers:

$$\left\{\sigma_x^{j_x}, \sigma_x^{j_x+\iota_x}, \sigma_x^{j_x+2\iota_x}, \dots\right\}.$$

For $w = (\sigma, \iota, j) \in \mathscr{P}_r(F)$ we define the *set of linear extensions* of w as

$$\begin{aligned}\mathscr{L}(w) := \{g \in G(r,n) :\ & c(g^{-1}(\sigma_x)) = -j_x \in \mathbb{Z}_{\iota_x} \text{ for all } x \in V(F), \text{ and} \\ & \text{if } x, y \in V(F) \text{ are such that } x \prec y \text{ then } |g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|\}\end{aligned}$$

and for each $x \in V(F)$ we let

$$z_x(w) := \begin{cases} \operatorname{res}_{\iota_x}(j_x) & \text{if } x \text{ is a root of } F, \\ \operatorname{res}_{\iota_x}\left(j_x - j_{p(x)}\right) & \text{otherwise.} \end{cases}$$

Finally we let

$$\operatorname{HDes}(w) := \{(x,y) \in E(F) :\ j_x = j_y \in \mathbb{Z}_{\iota_x} \text{ and } \sigma_x > \sigma_y\}$$

be the *homogeneous descent set* of w and we define the *flag-major index* of w as

$$\operatorname{fmaj}(w) := \sum_{e \in E(F)} \iota_e \chi_e(w) h_e + \sum_{v \in V(F)} z_v(w) h_v,$$

where

$$\iota_{(x,y)} := \iota_x \text{ for each } (x,y) \in E(F) \qquad \text{and} \qquad \chi_e(w) := \begin{cases} 1 & \text{if } e \in \operatorname{HDes}(w), \\ 0 & \text{otherwise.} \end{cases}$$

Example 7. Let w be the 6-partial labeling in Figure 3. Then the element $[3^a, 4^b, 2^0, 1^4, 5^2] \in \mathscr{L}(w)$ for all $a \in \{1, 4\}$ and $b \in \{1, 3, 5\}$. Moreover, $\operatorname{HDes}(w) = \{(x, y)\}$ and $\operatorname{fmaj}(w) = 3 \cdot 1 + (2 \cdot 4 + 2 \cdot 3 + 1 \cdot 1) = 18$.

The following result is the natural generalization of Theorem 3.2 to this context.

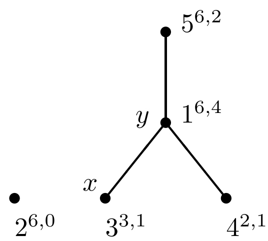

Figure 3: Example of 6-partial labeling.

Theorem 3.3. *Let F be a finite forest with n elements and w a r-partial labeling of F. Then*

$$\sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = q^{\mathrm{fmaj}(w)} \frac{[d_1]_q[d_2]_q\cdots[d_n]_q}{\prod\limits_{x\in F}[h_x\iota_x]_q},$$

where $d_i = ri$, $i = 1,\ldots,n$ are the fundamental degrees of $G(r,n)$.

Before proving Theorem 3.3 we need some further preliminary results. Let $w = (\sigma,\iota,j)$ be a r-partial labeling of F and let

$$\begin{aligned}\mathscr{A}_w = \big\{f\in\mathbb{N}^n :\ & f_{\sigma_x} = j_x \in \mathbb{Z}_{\iota_x} \text{ for all } x\in V(F),\ f_{\sigma_x}\geq f_{\sigma_y} \text{ for each}\\ & (x,y)\in E(F), \text{ and } f_{\sigma_x} = f_{\sigma_y} \text{ only if } j_x = j_y\in\mathbb{Z}_{\iota_x} \text{ and } \sigma_x<\sigma_y\big\}.\end{aligned}$$

The next result gives an alternative description of the set $\mathscr{A}_w$.

Proposition 3.4. *Let w be a r-partial labeling of F and $f\in\mathbb{N}^n$. Then $f\in\mathscr{A}_w$ if and only if f is g-compatible for some $g\in\mathscr{L}(w)$.*

Proof. Let $g\in G(r,n)$ be such that f is g-compatible, i.e. there exists $\lambda\in\mathscr{P}_n$ such that

$$f_i = \lambda_{|g^{-1}(i)|}(g) + r\lambda_{|g^{-1}(i)|} \tag{3.1}$$

for all $i\in[n]$. We make two claims.

i) If $x\in V(F)$, then $c(g^{-1}(\sigma_x)) = -j_x\in\mathbb{Z}_{\iota_x}$ if and only if $f_{\sigma_x} = j_x\in\mathbb{Z}_{\iota_x}$.

In fact we have

$$\begin{aligned} f_{\sigma_x} = j_x\in\mathbb{Z}_{\iota_x} &\Leftrightarrow \lambda_{|g^{-1}(\sigma_x)|}(g) = j_x\in\mathbb{Z}_{\iota_x} && \text{(by the g-compatibility)},\\ &\Leftrightarrow \lambda_{\sigma_x}(g^{-1}) = -j_x\in\mathbb{Z}_{\iota_x} && \text{(by Lemma 1.7)},\end{aligned}$$

$$\Leftrightarrow c(g^{-1}(\sigma_x)) = -j_x \in \mathbb{Z}_{\iota_x} \qquad \text{(by Eq. (1.1))}.$$

ii) If $(x,y) \in E(F)$, then $|g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|$ if and only if $f_{\sigma_x} \geq f_{\sigma_y}$ and equality $f_{\sigma_x} = f_{\sigma_y}$ holds only if $\sigma_x < \sigma_y$.

Let us prove *ii)*.

$\Leftarrow$) If $f_{\sigma_x} > f_{\sigma_y}$ then $|g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|$ by Eq. (3.1) since $\lambda(g)$ and λ are both partitions. If $f_{\sigma_x} = f_{\sigma_y}$ with $\sigma_x < \sigma_y$ then $\lambda_{|g^{-1}(\sigma_x)|}(g) = \lambda_{|g^{-1}(\sigma_y)|}(g)$. Now we make an easy observation that follows from the definition of the partition $\lambda(g)$: if h, k are such that $\lambda_h(g) = \lambda_k(g)$ then $|g(h)| < |g(k)|$ if and only if $h < k$. Applying this observation to $h = |g^{-1}(\sigma_x)|$ and $k = |g^{-1}(\sigma_y)|$ we conclude that $|g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|$.

$\Rightarrow$) If $|g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|$ then $f_{\sigma_x} \geq f_{\sigma_y}$ since $\lambda(g)$ and λ are both partitions, by Eq. (3.1). Moreover, if $f_{\sigma_x} = f_{\sigma_y}$ then necessarily $\lambda_{|g^{-1}(\sigma_x)|}(g) = \lambda_{|g^{-1}(\sigma_y)|}(g)$ and by the same observation above it follows that $\sigma_x < \sigma_y$.

By *i)* and *ii)* to complete the proof we only have to show that if f is g-compatible, $g \in \mathscr{L}(w)$ and $f_{\sigma_x} = f_{\sigma_y}$ then $j_x = j_y \in \mathbb{Z}_{\iota_x}$. But this follows easily since by *i)* we have $f_{\sigma_x} = j_x \in \mathbb{Z}_{\iota_x}$ and $f_{\sigma_y} = j_y \in \mathbb{Z}_{\iota_y}$. $\square$

For $x \in F$ we let $\mathscr{F}_x = \{a \in F : a \succeq x\}$ be the filter at x, which is a chain, and $\mathscr{E}_x = \{(y,z) \in E(F) : y \in \mathscr{F}_x\}$ be the set of edges of $\mathscr{F}_x$.

If $w = (\sigma, \iota, j)$ is a fixed r-partial labeling of F and $m : V(F) \to \mathbb{N}$, we let $f[m] \in \mathbb{N}^n$ be given by

$$f[m]_{\sigma_x} := \sum_{y \in \mathscr{F}_x} (z_y(w) + \iota_y m_y) + \sum_{e \in \mathscr{E}_x} \iota_e \chi_e(w)$$

and

$$\mathscr{B}_w = \left\{ f[m] : m \in \mathbb{N}^{V(F)} \right\}.$$

Proposition 3.5. *For all $w \in \mathscr{P}_r(F)$ we have $\mathscr{A}_w = \mathscr{B}_w$.*

Proof. We first show that $\mathscr{B}_w \subseteq \mathscr{A}_w$, so let $f \in \mathscr{B}_w$. We show by reverse induction on $\prec$ that for all $x \in V(F)$ we have $f_{\sigma_x} = j_x \in \mathbb{Z}_{\iota_x}$, $f_{\sigma_x} \geq f_{\sigma_y}$ for each $(x,y) \in E(F)$, and $f_{\sigma_x} = f_{\sigma_y}$ only if $j_x = j_y \in \mathbb{Z}_{\iota_x}$ and $\sigma_x < \sigma_y$. If x is a root we have $f_{\sigma_x} = z_x(w) \in \mathbb{Z}_{\iota_x}$ and the result is clear. Otherwise, by definition, $f_{\sigma_x} = f_{\sigma_y} + \big(z_x(w) + \iota_x m_x + \iota_x \chi_{(x,y)}(w)\big)$, where $y = p(x)$. Then our claim follows immediately from the definition of $z_x(w)$. It follows that $f \in \mathscr{A}_w$.

Now we show that $\mathscr{A}_w \subseteq \mathscr{B}_w$ so let $f \in \mathscr{A}_w$. If u is a root then $f_{\sigma_u} = j_u \in \mathbb{Z}_{i_u}$, so there exists $m_u \in \mathbb{N}$ such that $f_{\sigma_u} = \mathrm{res}_{\iota_u}(j_u) + \iota_u m_u = z_u + \iota_u m_u$. Let x be an element covered by u. Then there exists $m_x \in \mathbb{N}$ such that

$f_{\sigma_x} = f_{\sigma_u} + \mathrm{res}_{\iota_x}(j_x - j_u) + \iota_x \chi_{(x,u)} + \iota_x m_x = f_{\sigma_u} + z_x + \iota_x \chi_{(x,u)} + \iota_x m_x$. We note that $f_{\sigma_x} = j_x \in \mathbb{Z}_x$. We obtain the result extending this argument to every $x \in F$. □

Now we are ready to complete the proof of the main result of this section:

Proof of Theorem 3.3. We follow a general idea that goes back at least to Garsia and Gessel [13] and we compute the formal power series $\sum_{f \in \mathscr{A}} q^{|f|}$ in two different ways. In the first computation we use Lemma 1.6 (for $p = 1$) and Proposition 3.4 and we have

$$\sum_{f \in \mathscr{A}} q^{|f|} = \sum_{g \in \mathscr{L}(w)} \frac{q^{\lambda_1(g)} \cdots q^{\lambda_n(g)}}{(1-q^r)\cdots(1-q^{nr})} = \frac{\sum_{g \in \mathscr{L}(w)} q^{\mathrm{fmaj}(g)}}{(1-q^r)(1-q^{2r})\cdots(1-q^{nr})}.$$

In the second computation we use Proposition 3.5: using the same notations, for all $m \in \mathbb{N}^{V(F)}$ we have

$$\begin{aligned}|f[m]| = \sum_{x \in F} f[m]_{\sigma_x} = \sum_{v \in V(F)} (z_v + \iota_v m_v) h_v + \sum_{e \in E(F)} \iota_e \chi_e h_e = \\ = \mathrm{fmaj}(w) + \sum_{x \in F} \iota_x m_x h_x\end{aligned}$$

and then

$$\sum_{f[m] \in \mathscr{B}_w} q^{|f[m]|} = \sum_{m \in \mathbb{N}^{V(F)}} q^{\mathrm{fmaj}(w) + \sum_x \iota_x m_x h_x} = q^{\mathrm{fmaj}(w)} \frac{1}{\prod\limits_{x \in V(F)} (1 - q^{\iota_x h_x})}.$$

Therefore

$$\sum_{g \in \mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = q^{\mathrm{fmaj}(w)} \frac{(1-q^r)(1-q^{2r})\cdots(1-q^{nr})}{\prod\limits_{x \in F}(1-q^{\iota_x h_x})}.$$ □

To complete this section we show that some of the results appearing in §1 can be seen as particular cases of Theorem 3.3.

Remark 3.6. Consider the poset $V_n = \{x_1, x_2, \ldots, x_n\}$ with no order relation between any two distinct elements. The Hasse diagram of this poset is a forest consisting of n disjoint vertices. Consider now the r-partial labeling $w = (\sigma, \iota, j) \in \mathscr{P}_r(V_n)$ such that $\sigma(x_s) = s$, $\iota_{x_s} = 1$ and $j_{x_s} = 0$ for all $s \in [n]$. Then $\mathrm{fmaj}(w) = 0$ and $\mathscr{L}(w) = G(r, n)$. Therefore in this

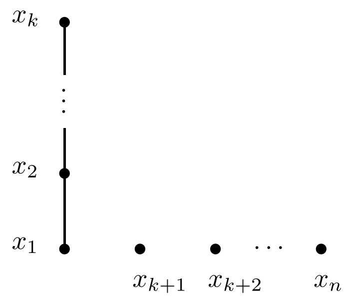

Figure 4: $T_{n,k}$ poset.

case Theorem 3.3 reduces to the distribution of fmaj on the group $G(r,n)$ (Proposition 1.1):

$$\sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = \sum_{g\in G(r,n)} q^{\mathrm{fmaj}(g)} = [r]_q[2r]_q\cdots[nr]_q.$$

Remark 3.7. Let $1 \leq k < n$ and consider the poset $T_{n,k} = \{x_1, x_2, \ldots, x_n\}$ with the ordering given by $x_s \prec x_t$ if and only if $s < t \leq k$. The Hasse diagram of $T_{n,k}$ is a forest consisting of a linear tree of length k and $n-k$ disjoint vertices (see Figure 4). Consider now the r-partial labeling $w = (\sigma, \iota, j) \in \mathscr{P}_r(T_{n,k})$ such that $\sigma(x_s) = s$ for all $s \in [n]$,

$$\iota_{x_s} = \begin{cases} r & \text{if } s \in [k] \\ 1 & \text{otherwise,} \end{cases}$$

and $j_{x_s} = 0$ for all $s \in [n]$. We observe that the hook lengths are $h_{x_s} = s$ for $s \in [k]$ and $h_{x_s} = 1$ otherwise, that $\mathrm{fmaj}(w) = 0$ and $\mathscr{L}(w) = \{g \in G(r,n) : c(g^{-1}(i)) = 0 \text{ if } i \in [k] \text{ and } g^{-1}(1^0) < g^{-1}(2^0) < \cdots < g^{-1}(k^0)\}$. We finally note that if $g \in \mathscr{L}(w)$ then $g^{-1} \in \mathscr{C}_k$, where $\mathscr{C}_k$ is the same set defined in (1.2) when $p = 1$. Then in this case Theorem 3.3 reduces to Corollary 1.4:

$$\sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = \sum_{g\in\mathscr{C}_k} q^{\mathrm{fmaj}(g^{-1})} = [(k+1)r]_q[(k+2)r]_q\cdots[nr]_q.$$

4. (r,p)-partial labelings

Inspired by the theory of projective reflection groups and the study of invariant and coinvariant algebras in §2 we are naturally lead to consider the

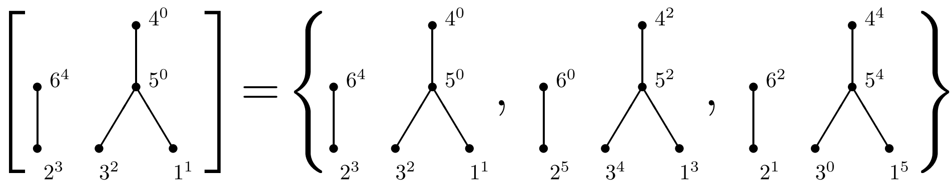

Figure 5: Example of $(6, 3)$-colored labeling.

following generalization of a partial labeling of a forest. So let F be a finite forest with n vertices and $\mathscr{W}_r(F)$ be the set of all colored labelings of F. Let C_p be a cyclic group of order p generated by an element δ and consider the action of C_p on the set $\mathscr{W}_r(F)$ defined by

$$\delta.(\sigma, c) = (\sigma, c'),$$

where $c'_x = c_x + \frac{r}{p}$, for all $x \in V(F)$.

Definition 4. A (r, p)-*colored labeling* of F is an orbit of $\mathscr{W}_r(F)$ under the action of C_p; the set of all (r, p)-colored labelings of F is denoted by

$$\mathscr{W}_{r,p}(F) := \mathscr{W}_r(F)/C_p.$$

See an example in Figure 5.

The action of $C_p = \langle\delta\rangle$ can also be extended to the set $\mathscr{P}_r(F)$ of all partial labelings by

$$\delta.(\sigma, \iota, j) = (\sigma, \iota, j')$$

where $j'_x = j_x + \frac{r}{p}$, for all $x \in V(F)$.

Definition 5. A (r, p)-*partial labeling* of F is an orbit of $\mathscr{P}_r(F)$ under the action of C_p; the set of all (r, p)-partial labelings of F is denoted by

$$\mathscr{P}_{r,p}(F) := \mathscr{P}_r(F)/C_p.$$

See an example of an orbit consisting of three partial labelings in Figure 6.

A C_p-orbit in $\mathscr{W}_r(F)$ always consists of exactly p elements, but it is not always the case for partial labelings. The following lemma is useful to determine the cardinality of these orbits:

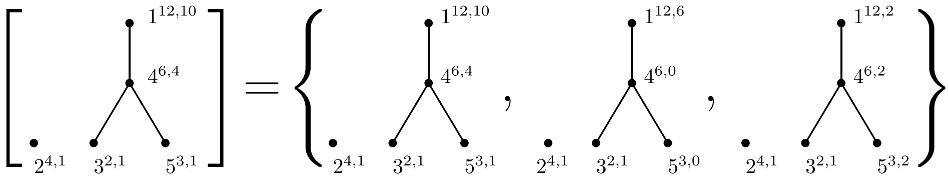

Figure 6: Example of $(24, 3)$-partial labeling.

Lemma 4.1. *Let F be a forest and $v_1, v_2, \ldots, v_l$ its roots. Let $w = (\sigma, \iota, j) \in \mathscr{P}_r(F)$ and consider the action of C_p on $\mathscr{P}_r(F)$ defined as above. Then the orbit of w contains p/d distinct elements, where*

$$d := \gcd\left(\frac{r}{\operatorname{lcm}(\iota_1, \iota_2, \ldots, \iota_l)}, p\right), \tag{4.1}$$

and ι_t denotes ι_{v_t}, for $t \in [l]$.

Proof. We consider first the case in which F is a tree and then the case of a general forest. So let F be a tree and v be its root. In this case we have to show that the orbit of w contains exactly p/d distinct elements, where

$$d = \gcd(r/\iota_v, p).$$

For this it is enough to show that the number of distinct residue classes in $\mathbb{Z}_{\iota_v}$ of the form $j_v + kr/p$, for $k \in [p]$, is p/d. In other words, we have to show that the order of r/p in $\mathbb{Z}_{\iota_v}$ is p/d. And in fact such order is

$$\frac{\iota_v}{\gcd\,(r/p,\ \iota_v)} = \frac{\iota_v\, p}{\gcd(r,\ \iota_v\, p)} = \frac{p}{\gcd(r/\iota_v,\ p)} = \frac{p}{d}.$$

Now let F be any forest with connected components $T_1, T_2, \ldots, T_l$ and roots $v_1, v_2, \ldots, v_l$. It follows from the previous discussion that the orbit of w contains exactly

$$\operatorname{lcm}\left(\frac{p}{d_1}, \frac{p}{d_2}, \ldots, \frac{p}{d_l}\right)$$

elements, where $d_t = \gcd(r/\iota_t,\ p)$ and $\iota_t = \iota_{v_t}$, for $t \in [l]$.

Therefore we have to show that

$$\operatorname{lcm}\left(\frac{p}{d_1}, \frac{p}{d_2}, \ldots, \frac{p}{d_l}\right) = \frac{p}{d},$$

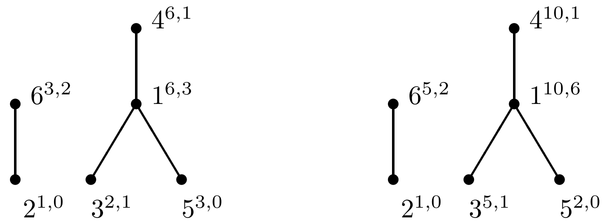

Figure 7: Examples of $(30, 6)$-partial labelings.

where p/d_t is the order of r/p in $\mathbb{Z}_{\iota_t}$.

Let π be any prime number that divides p. Let a and b be positive integers and c a non-negative integer, $c \leq a$, such that $\pi^a \parallel p$, $\pi^b \parallel r$ and $\pi^c \parallel d$, where the symbol $\parallel$ means "exactly divides".

If $c < a$ we have

$$\pi^c \parallel \frac{r}{\operatorname{lcm}(\iota_1, \iota_2, \ldots, \iota_l)},$$

so there exists $t \in [l]$ such that $\pi^{b-c} \mid \iota_t$. Then $\pi^c \parallel d_t$ and $\pi^{a-c} \mid p/d_t$. So

$$\pi^{a-c} \mid \operatorname{lcm}\left(\frac{p}{d_1}, \frac{p}{d_2}, \ldots, \frac{p}{d_l}\right).$$

If $a = c$ we have

$$\pi^a \mid \frac{r}{\operatorname{lcm}(\iota_1, \iota_2, \ldots, \iota_l)}$$

and so there exists $t \in [l]$ such that $\pi^{b-a+1} \nmid \iota_t$. It follows that $\pi^a \mid d_s$ and therefore $\pi \nmid \operatorname{lcm}(p/d_1, \ldots, p/d_l)$.

By repeating the same argument for each prime in the factorization of p, we have

$$\frac{p}{d} \mid \operatorname{lcm}\left(\frac{p}{d_1}, \frac{p}{d_2}, \ldots, \frac{p}{d_l}\right).$$

The result follows, since $d \mid d_t$ for all $t \in [l]$, and so we have

$$\operatorname{lcm}\left(\frac{p}{d_1}, \frac{p}{d_2}, \ldots, \frac{p}{d_l}\right) \mid \frac{p}{d}. \qquad \square$$

Example 8. Let w be the labeling in Figure 7 (left). Note that

$$d = \gcd\left(\frac{30}{\operatorname{lcm}(3, 6)}, 6\right) = \gcd(5, 6) = 1,$$

and in fact one can easily check that the orbit of such partial labeling contains 6 elements, while the orbit of the labeling in Figure 7 (right) contains 2 elements only, as in this case

$$d = \gcd\left(\frac{30}{\mathrm{lcm}(5,10)}, 6\right) = \gcd(3,6) = 3.$$

For $w = (\sigma, \iota, j) \in \mathscr{P}_r(F)$, we denote by $[w]$ the corresponding class in $\mathscr{P}_{r,p}(F)$. We extend the map j to the set of edges of F by

$$j_{(x,y)} := j_x - j_y \in \mathbb{Z}_{\iota_x}$$

and we observe that this map depends on the class $[w]$ of w only. Then, for $[w] \in \mathscr{P}_{r,p}(F)$ we define the *set of linear extensions* of $[w]$ as

$$\begin{aligned}\mathscr{L}([w]) := \big\{ & g \in G^* : \ \forall \widetilde{g} \text{ lift of } g \text{ in } G(r,n),\ \exists \widetilde{w} \text{ lift of } [w] \text{ in } \mathscr{P}_r(F), \\ & \widetilde{w} = (\sigma, \iota, j), \text{ s.t. } c(\widetilde{g}^{-1}(\sigma_x)) = -j_x \in \mathbb{Z}_{\iota_x} \text{ for all } x \in V(F), \text{ and} \\ & \text{if } x, y \in V(F) \text{ are such that } x \prec y, \text{ then } |g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)| \big\}.\end{aligned}$$

Example 9. Let w be the labeling in Figure 7 (right). Then for example the element $g = [5^4, 3^{11}, 1^{26}, 4^1, 2^{19}, 6^{27}] \in G(30,6,6)^*$ is a linear extension of $[w]$.

For $[w] \in \mathscr{P}_{r,p}(F)$ we let

$$\mathrm{HDes}([w]) := \{(x,y) \in E(F) : \ j_{(x,y)} = 0 \in \mathbb{Z}_{\iota_x} \text{ and } \sigma_x > \sigma_y\}$$

be the *homogeneous descent set* of $[w]$ and finally we define the *flag-major index* of $[w]$ as the multiset

$$\mathrm{Fmaj}([w]) := \left\{\!\!\left\{ \sum_{e \in E(F)} \iota_e \chi_e([w]) h_e + \sum_{v \in V(F)} z_v(\widetilde{w}) h_v, \quad \text{for each } \widetilde{w} \text{ lift of } [w] \text{ in } \mathscr{P}_r(F) \right\}\!\!\right\}$$

where

$$\chi_e([w]) := \begin{cases} 1 & \text{if } e \in \mathrm{HDes}([w]), \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.2. Let $w = (\sigma, \iota, j) \in \mathscr{P}_r(F)$ and let d be defined as in (4.1). Then $|\mathrm{Fmaj}([w])| = p/d$.

Example 10. Let w be the labeling in Figure 7 (left). Then the flag-major index of w is the multiset:

$$\begin{aligned}\mathrm{Fmaj}(w) = \{\!\{ &(2\cdot 1 + 3\cdot 1) + (2\cdot \mathrm{res}_3(2+5k) + 4\cdot \mathrm{res}_6(1+5k) + 3\cdot 2),\\ &k = 0,1,\dots,5\}\!\} = \{\!\{19,13,31,31,25,19\}\!\}.\end{aligned}$$

Let w be the labeling in Figure 7 (right). Then the flag-major index of w is the multiset:

$$\begin{aligned}\mathrm{Fmaj}(w) = \{\!\{ &(5\cdot 1 + 2\cdot 1) + (2\cdot \mathrm{res}_5(2+5k) + 4\cdot \mathrm{res}_{10}(1+5k) + 3\cdot 5),\\ &k = 0,1\}\!\} = \{\!\{30,50\}\!\}.\end{aligned}$$

For $[w] \in \mathscr{P}_{r,p}(F)$ with $w = (\sigma, \iota, j)$, let

$$\begin{aligned}\mathscr{A}_{[w]} = \big\{ f \in \mathbb{N}^n : \exists\, \widetilde{w} = (\sigma,\iota,j') \in [w] \text{ s.t. } & f_{\sigma_x} = j'_x \in \mathbb{Z}_{\iota_x}\\ \text{for all } x \in V(F),\ f_{\sigma_x} \geq f_{\sigma_y} \text{ for each } (x,y) \in E(F),\ & \text{and}\\ f_{\sigma_x} = f_{\sigma_y} \text{ only if } j_{(x,y)} = 0 \in \mathbb{Z}_{\iota_x} & \text{ and } \sigma_x < \sigma_y \big\}.\end{aligned}$$

Proposition 4.3. *Let $[w]$ be a (r,p)-partial labeling of F and $f \in \mathbb{N}^n$. Then $f \in \mathscr{A}_{[w]}$ if and only if f is g-compatible for some $g \in \mathscr{L}([w])$.*

Proof. Let $f \in \mathbb{N}^n$ and $g \in G(r,p,n)^*$ be such that f is g-compatible, i.e. there exist $\lambda \in \mathscr{P}_n$ and $h \in [0, p-1]$ such that

$$f_i = \lambda_{|g^{-1}(i)|}(g) + r\lambda_{|g^{-1}(i)|} + h\frac{r}{p}$$

for all $i \in [n]$. We make the following claim: for any $\widetilde{g}$ lift of g in $G(r,n)$ there exists $\widetilde{w} = (\sigma, \iota, j')$ lift of $[w]$ in $\mathscr{P}_r(F)$ such that

$$c(\widetilde{g}^{\,-1}(\sigma_x)) = -j'_x \in \mathbb{Z}_{\iota_x}$$

for all $x \in V(F)$ if and only if there exists $\widehat{w} = (\sigma, \iota, j'')$ lift of $[w]$ in $\mathscr{P}_r(F)$ such that $f_{\sigma_x} = j''_x \in \mathbb{Z}_{\iota_x}$. This is a consequence of the following facts.

- There exists $k \in [0, p-1]$ such that $f_{\sigma_x} = \lambda_{|g^{-1}(\sigma_x)|}(g) + kr/p \in \mathbb{Z}_r$ for all $x \in V(F)$, since f is g-compatible;

- if $\widetilde{g}$ is a lift of g in $G(r,n)$ then there exists $l \in [0,p-1]$ such that

$$\widetilde{g}\left(|g^{-1}(\sigma_x)|\right) = {\sigma_x}^{\lambda_{|g^{-1}(\sigma_x)|}(g)+lr/p}$$

by Eq. (1.1), and therefore

$$c(\widetilde{g}^{-1}(\sigma_x)) = -\lambda_{|g^{-1}(\sigma_x)|}(g) - lr/p \in \mathbb{Z}_r;$$

- there exists $h \in [0,p-1]$ such that $\lambda_{|g^{-1}(\sigma_x)|}(g) = -\lambda_{\sigma_x}(g^{-1}) + hr/p \in \mathbb{Z}_r$ by Lemma 1.7.

The rest of this proof is analogous to the proof of Proposition 3.4 and is therefore omitted. □

Now we are ready to give a generalization of Theorem 3.2 to (r,p)-partial labelings:

Theorem 4.4. *Let F be a finite forest with n elements and $[w]$ a (r,p)-partial labeling of F. Then*

$$\sum_{g\in\mathscr{L}([w])} q^{\mathrm{fmaj}(g)} = \sum_{s\in\mathrm{Fmaj}([w])} q^s \frac{[d_1]_q[d_2]_q\cdots[d_n]_q}{\prod_{x\in F}[h_x\iota_x]_q},$$

where $d_i = ri$ if $i < n$ and $d_n = rn/p$ are the fundamental degrees of $G(r,p,n)$.

Proof. The strategy of this proof is the same as in the proof of Theorem 3.3 and so we present only a sketch of it.

We observe that from the definition of $\mathscr{A}_{[w]}$ and Proposition 3.5 we have that $\mathscr{A}_{[w]}$ is the (disjoint) union of the sets $\mathscr{B}_{\widetilde{w}}$ as $\widetilde{w}$ varies in the orbit $[w]$. Computing the series

$$\sum_{f\in\mathscr{A}_{[w]}} q^{|f|}$$

in two different ways using Proposition 4.3 and using the observation above, the result follows. □

Also in this case some known results described in §1 can be obtained as special cases of Theorem 4.4.

Remark 4.5. Consider the poset $V_n = \{x_1, x_2, \ldots, x_n\}$ with no order relation between any two distinct elements. The Hasse diagram of this poset is a forest consisting of n disjoint vertices. Consider now the (r,p)-partial labeling $[w]$ of V_n such that $w(x_s) = s^{1,0}$ for all $s \in [n]$. Then $\mathrm{Fmaj}([w]) = \{0\}$

and $\mathscr{L}([w]) = G(r,p,n)^*$. Therefore in this case Theorem 4.4 reduces to the distribution of fmaj on the group $G(r,p,n)^*$ (Proposition 1.2).

Remark 4.6. Let $1 \leq k < n$ and consider the poset $T_{n,k} = \{x_1, x_2, \dots, x_n\}$ with the ordering given by $x_s \prec x_t$ if and only if $s < t \leq k$ (see again Figure 4). Consider now the (r,p)-partial labeling $[w]$ of $T_{n,k}$ with $w = (\sigma, \iota, j)$ given by $\sigma(x_s) = s$ for all $s \in [n]$,

$$\iota_{x_s} = \begin{cases} r & \text{if } s \in [k] \\ 1 & \text{otherwise,} \end{cases}$$

and $j_{x_s} = 0$ for all $s \in [n]$. Then $h_{x_s} = s$ for $s \in [k]$ and $h_{x_s} = 1$ otherwise, $\mathrm{Fmaj}([w]) = \{0, kr/p, 2kr/p, \dots, (p-1)kr/p\}$ and $\mathscr{L}([w]) = \{g \in G(r,p,n)^* : \exists\, h \in \{0,1,\dots,p-1\}$ s.t. $c(\widetilde{g}^{-1}(s)) = hr/p$ for each $\widetilde{g}$ lift of g in $G(r,n)$, $s \in [k]$ and $|g^{-1}(1)| < |g^{-1}(2)| < \cdots < |g^{-1}(k)|\}$. We finally note that if $g \in \mathscr{L}([w])$ then $g^{-1} \in \mathscr{C}_k$, where $\mathscr{C}_k$ is the same set defined in (1.2). Then in this case Theorem 4.4 reduces to Theorem 1.3.

5. q-counting colored labelings

Let F be a finite forest with n vertices (see §1.6). In this section we generalize the result in Theorem 1.10 by q-counting the set of all partial labelings of a fixed forest F using the fmaj statistic. We recall from [6] that, for any fixed $\sigma \in \mathcal{S}_n$, there are

$$\frac{n!}{\prod\limits_{x \in F} h_x}$$

labelings w of F such that σ is a linear extension of w, since there is a bijection between the set $\{w \in \mathscr{W}(F) : \sigma \in \mathscr{L}(w)\}$ and the set $\mathscr{L}(F)$ of linear extensions of F (see §1.6). An analogous argument also applies to any element $g \in G(r,n)$.

For $g \in G(r,n)$ we let

$$\mathscr{W}(g) := \{w \in \mathscr{W}_r(F) : g \in \mathscr{L}(w)\}$$

Lemma 5.1. *For $g \in G(r,n)$, $x \in \mathscr{L}(F)$ and $u \in \mathscr{W}_r(F)$, the maps $\phi : \mathscr{W}(g) \to \mathscr{L}(F)$ and $\psi : \mathscr{L}(F) \to \mathscr{L}(u)$, given by*

$$\phi(w)_i = w^{-1}(g_i)$$

and

$$\psi(x)_i = u(x_i)$$

are both bijections. In particular,

$$|\mathscr{W}(g)| = |\mathscr{L}(u)| = \frac{n!}{\prod\limits_{x\in V(F)} h_x}.$$

Proof. All the statements are simple verifications based on the definitions of the involved sets. We prove only one of the corresponding statements, namely that if $w \in \mathscr{W}(g)$ then $\phi(w) \in \mathscr{L}(F)$, and we leave the rest of the proof to the reader. So assume that $\phi(w)_i \prec \phi(w)_j$ and we show that $i < j$. By definition of ϕ we have $w^{-1}(g_i) \prec w^{-1}(g_j)$. Letting $x = w^{-1}(g_i)$ and $y = w^{-1}(g_j)$ we have $x \prec y$. But since $g \in \mathscr{L}(w)$ this implies $g^{-1}(w_x) < g^{-1}(w_y)$, i.e. $i < j$. □

Theorem 5.2. *Let F be a finite forest with n elements and $\mathscr{W}_r(F)$ the set of all r-colored labelings of F. Then*

$$\sum_{w\in\mathscr{W}_r(F)} q^{\mathrm{fmaj}(w)} = \frac{n!}{\prod\limits_{x\in F} h_x} \prod_{x\in F} [h_x r]_q.$$

Remark 5.3. We recall from the Introduction that for $r = 2$ an equivalent result was given in [10, Theorem 2.3] by Chen, Gao and Guo.

Proof. We consider the double sum

$$\sum_{w\in\mathscr{W}_r(F)} \sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)}$$

and we evaluate it in two different ways. In the first computation we use Theorem 3.2 and we have

$$\begin{aligned}\sum_{w\in\mathscr{W}_r(F)} \sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)} &= \sum_{w\in\mathscr{W}_r(F)} q^{\mathrm{fmaj}(w)} \frac{[r]_q[2r]_q\cdots[nr]_q}{\prod\limits_{x\in F}[h_x r]_q}\\ &= \frac{[r]_q[2r]_q\cdots[nr]_q}{\prod\limits_{x\in F}[h_x r]_q} \sum_{w\in\mathscr{W}_r(F)} q^{\mathrm{fmaj}(w)}.\end{aligned}$$

In the second computation we exchange the order of summations and use Lemma 5.1 and Proposition 1.1. We have

$$\begin{aligned}\sum_{w\in\mathscr{W}_r(F)}\sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)} &= \sum_{g\in G(r,n)}\sum_{w\in\mathscr{W}(g)} q^{\mathrm{fmaj}(g)}\\ &= |\mathscr{L}(F)|\sum_{g\in G(r,n)} q^{\mathrm{fmaj}(g)}\\ &= \frac{n!}{\prod_{x\in F} h_x}[r]_q[2r]_q\cdots[nr]_q.\end{aligned}$$

Therefore we conclude that

$$\frac{[r]_q[2r]_q\cdots[nr]_q}{\prod_{x\in F}[h_x r]_q}\sum_{w\in\mathscr{W}_r(F)} q^{\mathrm{fmaj}(w)} = \frac{n!}{\prod_{x\in F} h_x}[r]_q[2r]_q\cdots[nr]_q$$

and we are done. □

If we consider the analogous result for (r,p)-colored labelings, we do not obtain anything new. A more interesting result shows up if we consider (r,p)-colored labelings with a multiplicity motivated by the study of coinvariant algebras in §2; this multiplicity will be determined by the possible coloring of its lifts in $\mathscr{W}_r(F)$. More precisely, we let $\tilde{\mathscr{W}}_{r,p}(F)$ be the set of labelings in $\mathscr{W}_r(F)$ where some root receives a label with color in $\{0,1,\dots,\frac{r}{p}-1\}$.

Theorem 5.4. *Let F be a forest with k connected components $F_1,\dots,F_k$ of cardinality $n_1,\dots,n_k$, respectively, and $R=\{\rho_1,\dots,\rho_k\}$ be the set of roots of F. Then*

$$\sum_{w\in\tilde{\mathscr{W}}_{r,p}(F)} q^{\mathrm{fmaj}(w)} = \frac{n!}{\prod h_x}\prod_{x\notin R}[h_x r]_q\prod_{x\in R}[h_x r/p]_q\sum_{\emptyset\neq I\subseteq[k]}(-1)^{|I|-1}\prod_{i\notin I}[p]_{q^{n_i r/p}}.$$

Proof. We first assume that $k=1$. In this case we have that the set $\tilde{\mathscr{W}}_{r,p}(F)$ is a set of orbit representatives of $\mathscr{W}_{r,p}(F)$ and the other elements in the same orbit are obtained by adding a multiple of $\frac{r}{p}$ to the color of all its labels. From this observation one can deduce that

$$\sum_{w\in\mathscr{W}_r(F)} q^{\mathrm{fmaj}(w)} = \sum_{w\in\tilde{\mathscr{W}}_{r,p}(F)} q^{\mathrm{fmaj}(w)}(1+q^{n\frac{r}{p}}+\cdots+q^{(p-1)n\frac{r}{p}})$$

and so, by Theorem 5.2, we have

$$\sum_{w\in\tilde{\mathscr{W}}_{r,p}(F)} q^{\mathrm{fmaj}(w)} = \frac{1}{[p]_{q^{nr/p}}}\cdot\frac{n!}{\prod_{x\in F} h_x}\prod_{x\in F}[h_x r]_q = \frac{n!}{\prod_{x\in F} h_x}\prod_{x\notin R}[h_x r]_q\cdot[nr/p]_q.$$

In the general case one can split the sum over all labelings in $\tilde{\mathscr{W}}_{r,p}(F)$ according to the set of roots which receive a label colored in $\{0,1,\ldots,\frac{r}{p}-1\}$ and then use a standard inclusion-exclusion argument to show that

$$\begin{aligned}
&\sum_{w\in\tilde{\mathscr{W}}_{r,p}(F)} q^{\mathrm{fmaj}(w)} = \\
&= \binom{n}{n_1,\ldots,n_k} \sum_{\emptyset\neq I\subseteq[k]} (-1)^{|I|-1} \prod_{i\in I} \sum_{w\in\tilde{\mathscr{W}}_{r,p}(F_i)} q^{\mathrm{fmaj}(w)} \prod_{i\notin I} \sum_{w\in\mathscr{W}_r(F_i)} q^{\mathrm{fmaj}(w)} \\
&= \binom{n}{n_1,\ldots,n_k} \sum_{\emptyset\neq I\subseteq[k]} (-1)^{|I|-1} \prod_{i\in I} \frac{n_i!}{\prod\limits_{x\in F_i} h_x} \prod_{x\in F_i\setminus\{\rho_i\}} [h_x r]_q \cdot [n_i r/p]_q \cdot \\
&\qquad\qquad \cdot \prod_{i\notin I} \frac{n_i!}{\prod\limits_{x\in F_i} h_x} \prod_{x\in F_i} [h_x r]_q \\
&= \frac{n!}{\prod h_x} \prod_{x\notin R} [h_x r]_q \prod_{x\in R} [h_x r/p]_q \sum_{\emptyset\neq I\subseteq[k]} (-1)^{|I|-1} \prod_{i\notin I} [p]_{q^{n_i r/p}}. \qquad \square
\end{aligned}$$

Corollary 5.5. *Let $G(r,p,N)$ be the group studied in §2 and $R(G(r,p,N))$ its coinvariant algebra. Then*

$$Hilb_{R(G(r,p,N))}(q) = \prod_{i=1}^{k} \prod_{j=1}^{n_i-1} [jr]_q \prod_{i=1}^{k} [n_i r/p]_q \sum_{\emptyset\neq I\subseteq[k]} (-1)^{|I|-1} \prod_{i\notin I} [p]_{q^{n_i r/p}}.$$

Proof. This follows easily from Theorem 5.4 in the special case where F is the union of k disjoint linear trees of cardinality $n_1,\ldots,n_k$ respectively. $\square$

One can observe that in the special case where $k=1$ Corollary 5.5 reduces to the well-known fact that the Hilbert series of the coinvariant algebra of $G(r,p,n)$ is

$$\prod_{j=1}^{n-1} [jr]_q [rn/p]_q.$$

We conclude our work by showing how one can generalize Theorem 5.2 to the context of partial labelings of a fixed forest F. Let $x_1, x_2, \ldots, x_n$ be a linear extension of F. We fix a map $\iota : V(F) \to \mathbb{N}$ such that $\iota_k := \iota(x_k)$ is a positive divisor of r for $k=1,2,\ldots,n$, and ι_j is a divisor of ι_k if x_j is covered by x_k in the forest F. We let

$$\mathscr{P}_{r,\iota}(F) := \{w \in \mathscr{P}_r(F) : w = (\sigma,\iota,j) \text{ for some } \sigma \text{ and } j\}.$$

Let $\mathscr{W}_\iota(g) := \{w \in \mathscr{P}_{r,\iota}(F) : g \in \mathscr{L}(w)\}$. The next result is analogous to Lemma 5.1 and therefore we omit its proof.

Lemma 5.6. *Let $g \in G(r,n)$. Then there exists a bijection $\phi : \mathscr{W}_\iota(g) \to \mathscr{L}(F)$.*

Theorem 5.7. *Let F be a finite forest with n elements and $\mathscr{P}_{r,\iota}(F)$ the set of all r-partial labelings of F associated to ι. Then*

$$\sum_{w \in \mathscr{P}_{r,\iota}(F)} q^{\mathrm{fmaj}(w)} = \frac{n!}{\prod_{x \in F} h_x} \prod_{x \in F} [h_x \iota_x]_q.$$

Proof. We consider the double sum

$$\sum_{w \in \mathscr{P}_{r,\iota}(F)} \sum_{g \in \mathscr{L}(w)} q^{\mathrm{fmaj}(g)}$$

and we evaluate it in two different ways. In the first computation by Theorem 3.3 we have

$$\sum_{w \in \mathscr{P}_{r,\iota}(F)} \sum_{g \in \mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = \frac{[r]_q[2r]_q \cdots [nr]_q}{\prod_{x \in F} [h_x \iota_x]_q} \sum_{w \in \mathscr{P}_{r,\iota}(F)} q^{\mathrm{fmaj}(w)}.$$

In the second computation we use Lemma 5.6 and Proposition 1.1 and we have

$$\begin{aligned}
\sum_{w \in \mathscr{P}_{r,\iota}(F)} \sum_{g \in \mathscr{L}(w)} q^{\mathrm{fmaj}(g)} &= \sum_{g \in G(r,n)} \sum_{w \in \mathscr{W}_i(g)} q^{\mathrm{fmaj}(g)} \\
&= |\mathscr{W}_i(g)| \sum_{g \in G(r,n)} q^{\mathrm{fmaj}(g)} \\
&= \frac{n!}{\prod_{x \in F} h_x} [r]_q[2r]_q \cdots [nr]_q.
\end{aligned}$$

□

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