Schubert duality for $SL(n,\mathbb{R})$-flag domains[*]

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This paper is concerned with the study of spaces of naturally defined cycles associated to $SL(n,\mathbb{R})$-flag domains. These are compact complex submanifolds in open orbits of real semisimple Lie groups in flag domains of their complexification. It is known that there are optimal Schubert varieties which intersect the cycles transversally in finitely many points and in particular determine them in homology. Here we give a precise description of these Schubert varieties in terms of certain subsets of the Weyl group and compute their total number. Furthermore, we give an explicit description of the points of intersection in terms of flags and their number.

1. Introduction

Here we deal with complex flag manifolds $Z = G/P$, with G a complex semisimple Lie group and P a complex parabolic subgroup, and consider the action of a real form G_0 of G on Z. The real form G_0 is the connected real Lie group associated to the fixed point Lie algebra $\mathfrak{g}_0$ of an antilinear involution $\tau : \mathfrak{g} \to \mathfrak{g}$. It is known that G_0 has only finitely many orbits in Z and therefore it has at least one open orbit. These and many other results relevant at the foundational level are proved in [7]. They are also summarised in the research monograph [4]. Our work here is motivated by recent developments in the theory of cycle spaces of such a flag domain D. These arise as follows. Consider a choice K_0 of a maximal compact subgroup of G_0, i.e. K_0 is given by the fixed point set of a Cartan involution $\theta : G_0 \to G_0$. Then K_0 has a unique orbit in $D \subset Z$, denoted by C_0, which is a complex submanifold of D. Equivalently, if K denotes the complexification of K_0, one could look at C_0 as the unique minimal dimensional closed K-orbit in D. If $\dim C_0 = q$, then C_0 can be regarded as a point in the Barlet

space associated to D, namely $C^q(D)$, [1]. By definition, the objects of this space are formal linear combinations $C = n_1 C_1 + \cdots + n_k C_k$, with positive integral coefficients, where each C_j is an irreducible q-dimensional compact subvariety of D. In this context C_0 is called the *base cycle* associated to K_0. It is known that $C^q(D)$ is smooth at C_0 and thus one can talk about the irreducible component of $C^q(D)$ that contains C_0.

It is a basic method of Barlet and Koziarz, [2], to transform functions on transversal slices to the cycles to functions on the cycle space. In the case at hand these transversal slices can be given using a special type of Schubert varieties which are defined with the help of the Iwasawa decomposition of G_0 (see part II of [4] and the references therein). Recall that this is a global decomposition that exhibits G_0 as a product $K_0 A_0 N_0$, where each of the members of the decomposition are closed subgroups of G_0, K_0 is a maximal compact subgroup and $A_0 N_0$ is a solvable subgroup. The Iwasawa decomposition is used to describe a type of Borel subgroups of G which in a sense are as close to being real as possible. We define an *Iwasawa-Borel* subgroup B_I of G to be a Borel subgroup that contains an Iwasawa-component $A_0 N_0$ and we call the closure of an orbit of such a B_I in Z an *Iwasawa-Schubert variety*. The Iwasawa-Borel subgroup can be equivalently obtained at the level of complex groups as follows. If (G, K) is a symmetric pair, i.e. K is defined by a complex linear involution, and $P = MAN$ where as usual M is the centraliser of A in K, then any such B_I is given by choosing a Borel subgroup in M and adjoining it to AN.

The following result, [4, p.101-104], has provided the motivation for our work.

Theorem 1.1. *If S is an Iwasawa-Schubert variety such that* $\dim S + \dim C_0 = \dim D$ *and* $S \cap C_0 \neq \emptyset$ *then the following hold:*

1. *S intersects C_0 in only finitely many points $z_1, \ldots, z_{d_s}$,*
2. *For each point of intersection the orbits $A_0 N_0 . z_j$ are open in S and closed in D,*
3. *The intersection $S \cap C_0$ is transversal at each intersection point z_j.*

For the next steps in this general area, for example computing in concrete terms the trace transform indicated above, we feel that it is important to understand precisely the combinatorial geometry of this situation. This means in particular to describe precisely which Schubert varieties intersect the base cycle C_0, their points of intersection and the number of these points. In particular such results will describe the base cycle C_0 (or any cycle in the corresponding cycle space) in the homology ring of the flag manifold Z. We

have done this for the classical semisimple Lie group $SL(n, \mathbb{C})$ and its real forms $SL(n, \mathbb{R})$, $SU(p, q)$ and $SL(n, \mathbb{H})$ using methods which would seem sufficiently general to handle all classical semisimple Lie groups.

The present paper deals with the real form $SL(n, \mathbb{R})$, while the results for the other two real forms can be found in the author's thesis, [3]. The description of the Schubert varieties is formulated combinatorially in terms of elements of the Weyl group of G. Interesting combinatorial conditions arise and the tight correspondence between combinatorics and geometry is made explicit. Here, up to orientation we have only one open orbit. In the case of the full flag manifold, the Weyl group elements that parametrize the Schubert varieties of interest can be obtained from a simple game that chooses pairs of consecutive numbers from the ordered set $\{1, \ldots, n\}$. Moreover, their total number is also a well-known number, the double-factorial. Surprisingly, the number of intersection points with the base cycle does not depend on the Schubert variety but only on the dimension sequence and in each case it is $2^{\lfloor n/2 \rfloor}$. The main results are presented in Theorem 2.7 and Theorem 2.9. In the case of the partial flag manifold the results depend on whether the open orbit is measurable or not. In the measurable case the main results are found in Theorem 2.13 and Theorem 2.14, and in the non-measurable case in Proposition 2.17.

2. The case of the real form $SL(n, \mathbb{R})$

2.1. Preliminaries

Let $G = SL(n, \mathbb{C})$ and P be a parabolic subgroup of G corresponding to a dimension sequence $d = (d_1, \ldots, d_s)$ with $d_1 + \cdots + d_s = n$, i.e. P is given by block upper triangular matrices of sizes d_1 up to d_s, respectively. Recall that in this case the flag manifold $Z = G/P$ can be identified with the set of all *partial flags of type d*, namely $\{V : 0 \subset V_1 \subset \cdots \subset V_s = \mathbb{C}^n\}$, where $\dim(V_i/V_{i-1}) = d_i$, $\forall 1 \leq i \leq s$, $\dim V_0 = 0$. Equivalently, Z can be defined with the help of the sequence $\delta = (\delta_1, \ldots, \delta_s)$, with $\delta_i := \sum_{k=1}^{i} d_k = \dim V_i$, for all $1 \leq i \leq s$. If $(e_1, \ldots, e_n)$ is the standard basis in $\mathbb{C}^n$, the flags consisting of subspaces spanned by elements of this basis are called *coordinate flags*. In the particular case when each $d_i = 1$ we have a complete flag and the corresponding full flag variety is identified with the homogeneous space $\hat{Z} = G/B$, with B the Borel subgroup of upper triangular matrices in G. In terms of the dimension sequence d we have that $\dim Z = \sum_{1 \leq i < j \leq s} d_i d_j$. For each d a fibration $\pi : \hat{Z} \to Z$ is defined by sending a complete flag to its corresponding partial flag of type d.

Let us look at $\mathbb{C}^n$ equipped with the standard real structure $\tau : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $\tau(v) = \overline{v}$ and the standard non-degenerate complex bilinear form $b : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$, $b(v, w) = v^t \cdot w$ and view G as the group of complex linear transformations on $\mathbb{C}^n$ of determinant 1. Moreover, let $h : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$ be the standard Hermitian form defined by $h(v, w) = b(\tau(v), w) = \overline{v}^t \cdot w$. It follows that $G_0 := \{A \in G : \tau \circ A = A \circ \tau\} = SL(n, \mathbb{R})$. If θ denotes both the Cartan involution on G_0 and on G defined by $\theta(A) = (A^{-1})^t$, then $K_0 := SO(n, \mathbb{R})$ and its complexification $K := SO(n, \mathbb{C})$ are both obtained as fixed points of the respective θ's. Fix the Iwasawa decomposition $G_0 = SL(n, \mathbb{R}) = K_0 A_0 N_0$, where $A_0 N_0$ are the upper triangular matrices with positive diagonal entries in $SL(n, \mathbb{R})$. Thus, in this special case, the Iwasawa Borel subgroup B_I is just the standard Borel subgroup of upper triangular matrices in $SL(n, \mathbb{C})$.

The following definitions give a geometric description in terms of flags of the open G_0-orbits in Z and the base cycles associated to this open orbits. These results can be found in [5] and [6].

Definition 2.1. A flag $z = (0 \subset V_1 \subset \cdots \subset V_s \subset \mathbb{C}^n)$ in $Z = Z_d$ is said to be τ-**generic** if $dim(V_i \cap \tau(V_j)) = max\{0, \delta_i + \delta_j - n\}$, $\forall 1 \le i, j \le s$. In other words, the dimensions of the intersections $V_i \cap \tau(V_j)$ should be minimal.

Note that in the case of $Z = G/B$ a flag z is τ-generic if and only if

$$\tau(V_j) \oplus V_{n-j} = \mathbb{C}^n, \quad \forall 1 \le j \le \lfloor n/2 \rfloor.$$

Definition 2.2. A flag $z = (0 \subset V_1 \subset \cdots \subset V_s \subset \mathbb{C}^n)$ in $Z = Z_d$ is said to be **isotropic** if either $V_i \subseteq V_j^\perp$ or $V_i^\perp \subseteq V_j$, $\forall 1 \le i, j \le s$. In other words,

$$dim(V_i \cap V_j^\perp) = min\{\delta_i, n - \delta_j\}.$$

Note that in the case of $Z = G/B$ and $m = \lfloor n/2 \rfloor$ a flag z is isotropic if and only if $V_i \subset V_i^\perp$ for all $1 \le i \le m$, $V_m = V_m^\perp$, if n is even and the flags V_{n-i} are determined by $V_{n-i} = V_i^\perp$, for all $1 \le i \le m$.

If $n = 2m + 1$ the unique open G_0-orbit is described by the set of τ-generic flags. If $n = 2m$, denote by $\mathbb{C}_\mathbb{R}^n$ the decomplexification of $\mathbb{C}^n$, namely the vector space $\mathbb{C}^n$ viewed as a real vector space. For two basis of $\mathbb{C}_\mathbb{R}^n$ there exist a unique linear transformation, call it A, which takes one bases into the other. If the determinant of A is positive we say that the two basis have the same orientation and opposite orientation otherwise. Since G_0 preserves orientation in this case we have two open orbits defined by the set of positively oriented τ-generic flags and by the set of negatively

oriented $\tau-$generic flags. One can define a map that reverses orientation and interchanges the two open orbits. It is therefore enough to only consider the open orbit defined by the positively-oriented flags. In each of the open orbits the base cycle C_0 is characterised by the set of isotropic flags.

Finally, recall the definition of Schubert varieties in a general flag manifold $Z = G/P$ and a few interesting properties in order to establish notation. In general, for a fixed Borel subgroup B of G, a B-orbit $\mathcal{O}$ in Z is called a Schubert cell and the closure of such an orbit is called a Schubert variety. A Schubert cell $\mathcal{O}$ in Z is parametrized by an element w of the Weyl group of G and Z is the disjoint union of finitely many such Schubert cells. Furthermore, the integral homology ring of Z, $H_*(Z, \mathbb{Z})$ is a free $\mathbb{Z}$-module generated by the set of Schubert varieties.

If $G = SL(n, \mathbb{C})$ and T is the maximal torus of diagonal matrices in G, then the Weyl group of G with respect to T can be identified with Σ_n, the permutation group on n letters. Moreover, the complete coordinate flags in G/B are in $1-1$ correspondence with elements of Σ_n. Given a complete coordinate flag

$$< e_{i_1} > \subset \cdots \subset < e_{i_1}, e_{i_2}, \ldots, e_{i_k} > \subset \cdots \subset \mathbb{C}^n,$$

one can define a permutation w by $w(k) = i_k$ for all k and viceversa. The complete coordinate flags are also in $1-1$ correspondence with permutation matrices in $GL(n, \mathbb{C})$. Given an element $w \in \Sigma_n$ one obtains a permutation matrix with column i^{th} equal to $e_{w(i)}$ for each i. For this reason we use the symbol w for both an element of Σ_n in one line notation, i.e. w(1)w(2)...w(n), or for the corresponding permutation matrix. It will be clear from the context to which kind of representation we are referring to.

The fixed points of the maximal torus T in G/B are the coordinate flags V_w for $w \in W$ and G/B is the disjoint union of the Schubert cells $\mathcal{O}_w := B.V_w$, where $w \in W$. The dimension of the Schubert cell $\mathcal{O}_w$ is given by the number of inversions in the permutation w, that is the length of w. Furthermore, B is the isotropy subgroup of G at the base flag:

$$< e_1 > \subset < e_1, e_2 > \subset \cdots \subset < e_1, e_2, \ldots, e_n >$$

and the zero-dimensional cell corresponds to the identity permutation $w = 123 \ldots n$. Let $0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n$ be an arbitrary point in $\mathcal{O}_w$. Then there exists $(v_1, \ldots, v_n)$ an ordered basis of $\mathbb{C}^n$ adapted to this flag, i.e. $V_j = < v_1, \ldots, v_j >$ is the span of the first j vectors in the basis, such that

$$v_j := \sum_{i=1}^{w(j)-1} v_{ij}e_j + e_{w(j)}, \text{ with } v_{ij} \in \mathbb{C}, v_{ij} = 0, \forall i \in \{w(1), \dots, w(j-1)\}.$$

This gives another useful way of visualizing Schubert cells via their matrix canonical form, which is a matrix $[v_1 \dots v_n]$ with the jth column v_j having the properties that:

- the last nonzero coordinate of v_j is $v_{w(j)j}$ and it is normalized to be 1,
- the $w(j)^{th}$ coordinate of v_k is set to be zero for all $k > j$, i.e.

$$v_{w(j)k} = 0, \forall k > j,$$

- v_j is well-defined only modulo $\text{span}\{v_1, \dots, v_{j-1}\}$.

2.2. Dimension-related computations

It is important for our discussion to compute the dimension of the base cycle and of the respective dual Schubert varieties in both the case of G/B and of G/P. In the case when B is the standard Borel subgroup in $SL(n, \mathbb{C})$ and $K = SO(n, \mathbb{C})$, the cycle C_0 is a compact complex submanifold of D represented in the form $C_0 = K.z_0 \cong K/(K \cap B_{z_0})$ for a base point $z_0 \in D$, where $K \cap B_{z_0}$ is a Borel subgroup of K. Since C_0 is a complex manifold

$$dim\, C_0 = dim\, T_{z_0} C_0 = dim\, \mathfrak{k}/\mathfrak{k} \cap \mathfrak{b}_{z_0},$$

where $\mathfrak{k}$ is the Lie algebra associated to K and $\mathfrak{b}_{z_0}$ is the Borel subalgebra associated to B_{z_0}. Thus in the case when $n = 2m$, $dim\, C_0 = m^2 - m$ and the Schubert varieties of interest must be of dimension m^2. If $n = 2m + 1$, then $dim\, C_0 = m^2$ and the Schubert varieties of interest are among those of dimension $m^2 + m$.

2.3. Introduction to the combinatorics

The next two sections give a full description of the Schubert varieties of interest that intersect the base cycle C_0, the points of intersection and their number, in the case of an open $SL(n, \mathbb{R})$-orbit D in Z. The first case to be considered is the case of $Z = G/B$, where

$$\mathcal{S}_{C_0} := \{S_w \text{ Schubert variety } : dimS_w + dimC_0 = dimZ \text{ and } S_w \cap C_0 \neq \emptyset\}.$$

In what follows we describe the conditions that the element w of the Weyl group that parametrizes the Schubert variety S_w must satisfy in order for S_w

to be in S_{C_0}. One of the main ingredients for this is the fact that $S_w \cap D \subset \mathcal{O}_w$ and the fact that if $S_w \cap D \neq \emptyset$, then $S_w \cap C_0 \neq \emptyset$. Moreover, no Schubert variety of dimension less than the codimension of the base cycle intersects the base cycle. These are general results that can be found in [4, p.101–104].

Definition 2.3. A permutation $w = k_1 \ldots k_m l_* l_m \ldots l_1$ is said to satisfy the **spacing condition** if $l_i < k_i$, $\forall 1 \leq i \leq m$, where l_* is removed from the representation in the case $n = 2m$.

For example, 265431 satisfies the spacing condition, while 261534 does not satisfy the spacing condition.

Definition 2.4. A permutation $w = k_1 \ldots k_m l_* l_m \ldots l_1$ is said to satisfy the **double box contraction** condition if w is constructed by the **immediate predecessor algorithm**:

    • Start by choosing k_1 and $l_1 := k_1 - 1$ from the ordered set $\{1, \ldots, n\}$. If we have chosen all the numbers up to k_i and l_i, then to go to the step $i + 1$ we make a choice of k_{i+1} and l_{i+1} from the ordered set $\{1, \ldots, n\} - \{k_1, l_1, \ldots, k_i, l_i\}$ in such a way that l_{i+1} sits inside the ordered set at the left of k_{i+1}.

Remark that a permutation that satisfies the double box contraction automatically satisfies the spacing condition as well, but not conversely. For example, 256341 satisfies both the double box contraction and consequently the spacing condition while 265431 does not satisfy the double box contraction even though it satisfies the spacing condition.

The next results are meant to establish a tight correspondence between the combinatorics of the Weyl group elements that parametrize the Schubert varieties in $\mathcal{S}_{C_0}$ and the geometry of flags that describe the intersection points. Namely, we prove that the spacing condition on Weyl group elements corresponds to the $\tau-$generic condition on flags. Similarly, the double box contraction condition on Weyl group elements corresponds to the isotropic condition on flags.

2.4. Main results

The first result of this section describes the Schubert varieties that intersect the base cycle independent of their dimension.

Proposition 2.5. *A Schubert variety S_w corresponding to a permutation*

$$w = k_1 \ldots k_m l_* l_m \ldots l_1,$$

where l_* is removed from the representation in the case $n = 2m$, has non-empty intersection with C_0 if and only if w satisfies the spacing condition.

Proof. We use the fact that if a Schubert variety intersects the open orbit D, then it also intersects the cycle C_0 and prove that S_w contains a τ-generic point if and only if w satisfies the spacing condition.

First assume that $l_i < k_i$ for $i \leq m$, $n = 2m$. Under this assumption we need to prove that

$$\tau(V_i) \oplus V_{2m-i} = \mathbb{C}^n \quad \forall i \leq m,$$

where V is an arbitrary flag in S_w. This is equivalent to showing that the matrix formed from the vectors generating $\tau(V_i)$ and V_{2m-i} has maximal rank. Form the following pairs of vectors $(v_i, \bar{v}_i)$ and the matrices

$$[v_1 \bar{v}_1 \ldots v_i \bar{v}_i v_{i+1} \ldots v_{2m-i}], \quad \forall i \leq m.$$

These are the matrices corresponding to

$$\tau(V_i) = < \bar{v}_1, \ldots, \bar{v}_i >$$

and

$$V_{2m-i} = < v_1, \ldots, v_i, v_{i+1}, \ldots, v_{2m-i} > .$$

We carry out the following set of operations keeping in mind that the rank of a matrix is not changed by row or column operations. The initial matrix is the canonical matrix representation of the Schubert cell $\mathcal{O}_w$. For the step $j = 1$, it follows that $l_1 < k_1$ and the last column corresponding to l_1 is eliminated from the initial matrix. Furthermore, at this step a new column corresponding to $\bar{v}_1$ is introduced between the first and second column of the canonical matrix which now becomes

$$[v_1 \bar{v}_1 v_2 \ldots v_{2m-1}].$$

Next denote by c_h the hth column of this matrices and obtain the following:

On the second column c_2 zeros are created on all rows starting with k_1 and going down to row $l_1 + 1$. This is done by subtracting suitable multiplies of c_2 from the columns in the matrix having a 1 on these rows and putting the result on c_2. For example to create a zero on the spot corresponding to k_1 the second column is subtracted from the first column and the result is left on the second column. On row l_1 a 1 is created by normalising. This 1 is the only 1 on row l_1, because the last column that contained a 1 on row l_1 was removed from the matrix. We now want to create zeros on c_j for $j > 2$ on

row l_1. It is enough to consider those columns which have 1's on rows greater than l_1, because the columns with 1's on rows smaller than l_1 already have zeros below them. Finally, subtract from these columns suitable multiplies of c_2. We thus create a matrix that represents points in the Schubert variety $k_1 l_1 k_2 \ldots k_m l_m \ldots l_2$, which obviously has maximal rank.

Assume by induction that we have created the maximal rank matrix corresponding to points in the Schubert variety $k_1 l_1 k_2 l_2 \ldots k_j l_j k_{j+1} \ldots k_m l_m \ldots l_{j+1}$. To go to the step $j+1$ remove from the matrix the last column corresponding to l_{j+1} and add in between c_{2j+1} and c_{2j+2} the conjugate of c_{2j+1} and reindex the columns.

On column $c_{2(j+1)}$ zeros are created on all rows starting with k_{j+1} and going down to $l_{j+1} + 1$ by subtracting suitable multiplies of $c_{2(j+1)}$ from the columns in the matrix having a 1 on this rows and putting the result on $c_{2(j+1)}$. Next a 1 is created on row $l_{2(j+1)}$ by normalisation. Again this is the only spot on row $l_{2(j+1)}$ with value 1, because the column which had a 1 on this spot was removed from the matrix. By subtracting suitable multiplies of $c_{2(j+1)}$ from columns having a 1 on spots greater than $l_{(j+1)}$, we create zeros on row l_{j+1} at the right of the 1 on column $c_{2(i+1)}$. We thus obtain points in the maximal rank matrix of the Schubert variety indexed by $k_1 l_1 k_2 l_2 \ldots k_j l_j k_{j+1} l_{j+1} \ldots k_m l_m \ldots l_{j+2}$.

At step $j = m$ we obtain points in the maximal rank matrix of the Schubert variety indexed by $k_1 l_1 k_2 l_2 \ldots k_m l_m$. For the odd dimensional case we insert in the middle a column corresponding to l_* and observe that this of course does not change the rank of the matrix.

Conversely, if the spacing condition is not satisfied let i be the smallest such that $k_i < l_i$ and look at the matrix

$$[v_1 \overline{v_1} \ldots v_i \overline{v_i} v_{i+1} \ldots v_{2m-i}].$$

Then use the same reasoning as above to create a 1 on the spot in the matrix corresponding to row l_j and column $2j$ for all $j < i$ using our chose of i. Now there is a 1 on each row in the matrix except on row l_i. Column c_{2i} and c_{2i-1} both have a 1 on position k_i and zeros bellow. Since $l_i > k_i$ this implies that for each $j < i$ there exist a column in the matrix that has 1 on row j. Using these 1's we begin subtracting c_{2i} from suitable multiplies of each such column, starting with c_{2i-1} then going to the one that has 1 on the spot $k_i - 1$, then to the one the has 1 on the spot $k_i - 2$ and so on and at each step the result is left on c_{2i}. This creates zeros on all the column c_{2i} and proves that the matrix does not have maximal rank. $\square$

Remark. For dimension computations, recall how to compute the length $|w|$, of an element w of Σ_n. Start with the number 1 and move it from its position toward the left until it arrives at the beginning and associate to this its distance p_1 which is the number of other numbers it passes. Then move 2 to the left until it is adjacent from the right to 1 and compute p_2 in the analogous way. Continuing on compute p_i for each i and then the length of w is just $\sum p_i$.

Lemma 2.6. *If $w = k_1 \ldots k_m l_* l_m \ldots l_1$ satisfies the spacing condition and $|w| = m^2$ for the even dimensional case, or $|w| = m^2 + m$ for the odd dimensional case, then $l_1 = k_1 - 1$.*

Proof. Suppose that $l_1 < k_1 - 1$. Then there exist $j > 1$ such that $p_j = k_1 - 1$ sits on position j inside w. If p_j sits among the l's, then construct $\tilde{w}$ by making the transposition (j, n) that interchanges p_j and l_1. If p_j sits among the k's then construct $\tilde{w}$ by interchanging k_1 with p_j. Observe that $\tilde{w}$ still satisfies the spacing condition and $|\tilde{w}| \leq m^2 - 1$ since in the first case all elements smaller then $k_1 - 1$ (at least one element, namely l_1) do not need to cross over $k_1 - 1$ anymore. In the second case p_j does not need to cross over k_1 anymore and since $p_j = k_1 - 1$, the elements that need to cross k_1 on position j remain the same as the elements that needed to cross p_j in the initial permutation. But this contradicts the fact that no Schubert variety of dimension less than m^2 intersects the cycle.

For the odd dimensional case just add l_* to the representation and consider $|w| = m^2 + m$. The only case remaining to be considered is that where $l_* = k_1 - 1$. In this case we interchange l_* with l_1 and observe that this still satisfies the spacing condition and it is of dimension strictly smaller then $m^2 + m$. As above this implies a contradiction. □

Theorem 2.7. *A Schubert variety S_w belongs to $\mathcal{S}_{C_0}$ if and only if w satisfies the double box contraction condition. In particular, in this case w satisfies the spacing condition and $|w| = m^2$, for $n = 2m$, and $|w| = m^2 + m$.*

Proof. We prove the theorem using induction on dimension, the case $n = 2$ being clear. Denote by $\tilde{Z}$ the flag variety of complete flags in $\mathbb{C}^{n-2}$, by $\tilde{D}$ the unique (up to orientation) open $\tilde{G}_0 := SL(n-2, \mathbb{R})$-orbit in $\tilde{Z}$ and by $\tilde{C}_0$ the set of isotropic flags in $\tilde{D}$. Furthermore, denote by $\mathcal{S}_{\tilde{C}_0}$ the set of Iwasawa-Schubert varieties $\tilde{S}$ corresponding to the Borel subgroup of upper triangular matrices in $SL(n-2, \mathbb{C})$, having non-empty intersection with $\tilde{C}_0$ and such that $\dim \tilde{S} + \tilde{C}_0 = \dim \tilde{Z}$.

The notation $w = (k, l_*, l)$, respectively $\tilde{w} = (\tilde{k}, \tilde{l}_*, \tilde{l})$ is used to represent the full sequence $w = k_1 \ldots k_m l_* l_m \ldots l_1$, respectively

$$\tilde{w} = \tilde{k}_1 \ldots \tilde{k}_{m-1} \tilde{l}_* \tilde{l}_{m-1} \ldots \tilde{l}_1,$$

for an element of Σ_n, respectively Σ_{n-2}.

By induction we assume that $\tilde{S}_{\tilde{w}} \in \mathcal{S}_{\tilde{C}_0}$ if and only if $\tilde{w}$ satisfies the double box contraction condition, that is if and only if $\tilde{w}$ satisfies the spacing condition and the length of $\tilde{w}$ is equal to $(m-1)^2 + (m-1)$ for $n = 2m+1$ and $(m-1)^2$ for $n = 2m$, respectively. We want to prove that this implies $S_w \in \mathcal{S}_{C_0}$ if and only if w satisfies the double box contraction condition.

From Proposition 2.5 it follows that a Schubert variety S_w has non-empty intersection with C_0 if and only if w satisfies the spacing condition. Then Lemma 2.6 tells us that a necessary condition for $S_w \in \mathcal{S}_{C_0}$ is that $l_1 = k_1 - 1$. By definition this is also a necessary condition for an arbitrary element of Σ_n to satisfy the double box contraction condition.

Let $\mathcal{C}_{C_0}$, respectively $\mathcal{C}_{\tilde{C}_0}$, be the set of Schubert varieties S_w, respectively $\tilde{S}_{\tilde{w}}$, such that $S_w \cap C_0 \neq \emptyset$, respectively $\tilde{S}_{\tilde{w}} \cap \tilde{C}_0 \neq \emptyset$ and $w(n) = w(1) - 1$. We define a map

$$\mathcal{C}_{C_0} \to \mathcal{C}_{\tilde{C}_0}, \; S_w \mapsto \tilde{S}_{\tilde{w}},$$

as follows. Let $z_0 \in C_0 \cap S_w$. Then there exists an ordered basis $(v_1, \ldots, v_n)$ of $\mathbb{C}^n$ such that

$$v_j := \sum_{i=1}^{w(j)-1} v_{ij} e_j + e_{w(j)}, \text{ with } v_{ij} \in \mathbb{C}, v_{ij} = 0, \forall i \in \{w(1), \ldots, w(j-1)\},$$

and z_0 is the complete flag

$$0 \subset < v_1 > \subset < v_1, v_2 > \subset \cdots \subset < v_1, v_2, \ldots, v_n > .$$

Remove $w(1)$ and $w(n)$ from the set $\{1, 2, \ldots, n\}$ to obtain a set Σ_w with $n-2$ elements. Define a bijective map $\phi_w : \Sigma_w \to \{1, 2, \ldots, n-2\}$ by $\phi_w(x) = x$ for $x < w(n)$ and $\phi(x) = x - 2$, for $x > w(1)$. Furthermore let

$$\tilde{w} := \phi(w(2)) \ldots \phi(w(n-1)),$$
$$\tilde{v}_j := \sum_{i=1}^{\tilde{w}(j)-1} v_{ij} e_{\tilde{w}(i)} + e_{\tilde{w}(j)}, \; \forall j = \overline{2, n-1},$$

and $\tilde{z}_0$ the flag in $\mathbb{C}^{n-2}$ given by

$$0 \subset < \tilde{v}_2 > \subset \cdots \subset < \tilde{v}_2, \tilde{v}_3, \ldots, \tilde{v}_{n-2} > .$$

By construction it is clear that $\tilde{z}_0$ is an isotropic flag in $\mathbb{C}^{n-2}$ and $\tilde{z}_0 \in \tilde{S}_{\tilde{w}}$. Therefore $\tilde{z}_0 \in \tilde{C}_0 \cap \tilde{S}_{\tilde{w}}$ and thus $\tilde{S}_{\tilde{w}} \in \mathcal{C}_{\tilde{C}_0}$. Since ϕ_w is a bijection, one can go back and forth between w and $\tilde{w}$, i.e. $w(i+1) = \phi_w^{-1}(\tilde{w}(i))$.

Next we express the length of w, i.e. the dimension of the Schubert variety S_w, as a function of the length of $\tilde{w}$, i.e. of the dimension of $\tilde{S}_{\tilde{w}}$. Let $\tilde{p}_j$ be the distances for the permutation $|(\tilde{k}, \tilde{l}_*, \tilde{l})|$. First consider those elements ε of the full sequence (k, l_*, l) which are smaller than l_1, in particular which are smaller than k_1. In order to move them to their appropriate position one needs the number of steps $\tilde{p}_\varepsilon$ to do the same for their associated point in $(\tilde{k}, \tilde{l}_*, \tilde{l})$ plus 1 for having to pass k_1. Thus in order to compute $|(k, l_*, l)|$ from $|(\tilde{k}, \tilde{l}_*, \tilde{l})|$ we must first add $k_1 - 2$ to the former. Having done the above, we now move l_1 to its place directly to the left of k_1. This requires crossing $2m + 1 - (k_1 - 1)$ larger numbers in the odd dimensional case and $2m - (k_1 - 1)$ numbers in the even dimensional case. So together we have now added $2m$ in the odd dimensional case and $2m - 1$ in the even dimensional case to $|(\tilde{k}, \tilde{l}_*, \tilde{l})|$ and $|(\tilde{k}, \tilde{l})|$, respectively. All other necessary moves are not affected by the transfer to $|(\tilde{k}, \tilde{l}_*, \tilde{l})|$ and back. So for those elements we have $\tilde{p}_\varepsilon = p_\varepsilon$ and it follows that

$$|(k, l_*, l)| = |(\tilde{k}, \tilde{l}_*, \tilde{l})| + 2m,$$

in the odd dimensional case and

$$|(k, l)| = |(\tilde{k}, \tilde{l})| + 2m - 1,$$

in the even dimensional case.

Above it was shown that, in the odd dimensional case, $S_w \in \mathcal{S}_{C_0}$ if and only if $|w| = m^2 + m$ and thus if and only if

$$|\tilde{w}| = m^2 + m - 2m = m^2 - 2m + 1 + m - 1$$
$$= (m-1)^2 + (m-1).$$

In turn this is equivalent to $\tilde{S}_{\tilde{w}} \in \mathcal{S}_{\tilde{C}_0}$, which by the induction assumption is equivalent to $\tilde{w}$ satisfying the double box contraction condition. Finally, this is equivalent to w satisfying the double box contraction condition.

In the even dimensional case $S_w \in \mathcal{S}_{C_0}$ if and only if $|w| = m^2$, which is equivalent to $|\tilde{w}| = m^2 - 2m + 1 = (m-1)^2$. In turn this is equivalent to $\tilde{S}_{\tilde{w}} \in \mathcal{S}_{\tilde{C}_0}$, which by the induction assumption is equivalent to $\tilde{w}$ satisfying the double box contraction condition. Finally, this is equivalent to w satisfying the double box contraction condition. □

Corollary 2.8. *The number of Iwasawa-Schubert varieties which intersect the base cycle is the double factorial*

$$(n-1)!! = \begin{cases} (n-1) \cdot (n-3) \cdot \cdots \cdot 5 \cdot 3 \cdot 1, n \text{ odd}, \\ (n-1) \cdot (n-3) \cdot \cdots \cdot 6 \cdot 4 \cdot 2, n \text{ even}. \end{cases}$$

Proof. From Theorem 2.7 it follows that $w \in \Sigma_n$ parametrizes an element of $\mathcal{S}_{C_0}$ if and only if w satisfies the double box contraction condition. We prove using induction on dimension that there are $(n-1)!!$ permutations in Σ_n which satisfy the double box contraction condition. For $n = 2$ the result is clear.

By induction assume that there are $(n-3)!!$ elements of Σ_{n-2} satisfying the double box contraction condition. Observe that to construct an element $w \in \Sigma_n$ which satisfies the double box contraction condition k_1 can be arbitrarily chosen from $\{2, \ldots, n\}$ and once k_1 is chosen l_1 is fixed. This amounts to $(n-1)$ possibilities for the placement of k_1 and l_1. For each k_1 and l_1 define the bijective map $\phi : \Sigma \to \{1, 2, \ldots, n-2\}$ by $\phi(x) = x$ for $x < l_1$ and $\phi(x) = x - 2$ for $x > k_1$. One can then construct using the immediate predecessor algorithm and the induction hypothesis the $(n-3)!!$ elements $\tilde{w}$ of Σ_{n-2} satisfying the double box contraction condition. By definition of the immediate predecessor algorithm, for fixed k_1 and l_1, each w in Σ_n satisfying the double box contraction condition is obtained as $w = k_1 \phi^{-1}(\tilde{w}(1)) \ldots \phi^{-1}(\tilde{w}(n-2))l_1$, for some $\tilde{w}$ in Σ_{n-2} satisfying the double box contraction condition. Thus in total we obtain $(n-1)!! = (n-1)(n-3)!!$ elements of Σ_n which satisfy the double box contraction condition. $\square$

The next theorem gives a geometric description in terms of flags of the intersection points. The complete flags describing the intersection points are obtained in the following way from the Weyl group element that parametrizes S_w. In the case when $n = 2m + 1$ and $w = k_1 \ldots k_m l_* l_m \ldots l_1$ the points of intersection are given by the following flags

$$(1) \quad \begin{aligned} &< (\pm\sqrt{-1})e_{l_1} + e_{k_1} > \subset \cdots \subset < (\pm\sqrt{-1})e_{l_1} + e_{k_1}, \ldots, \\ &(\pm\sqrt{-1})e_{l_m} + e_{k_m}, e_{l_*} > \subset < (\pm\sqrt{-1})e_{l_1} + e_{k_1}, \ldots, \\ &(\pm\sqrt{-1})e_{l_m} + e_{k_m}, e_{l_*}, e_{l_m} > \subset \cdots \subset \mathbb{C}^n. \end{aligned}$$

In the case when $n = 2m$, the complete flags are given by the same expression with the exception that the span of e_{l_*} is removed from the representation. Of course, one must not forget that the positively oriented flags correspond

to one open orbit and the negatively oriented flags to the other, but since this are symmetric with respect to the map that reverses orientation we have the same number of intersection points with the base cycle independently of the chosen open orbit.

Theorem 2.9. *A Schubert variety S_w in $\mathcal{S}_{C_0}$ intersects the base cycle C_0 in 2^m points in the case when $n = 2m + 1$ and 2^{m-1} points in the case when $n = 2m$. The points are given by* (1).

Proof. Let $w = k_1 \ldots k_m l_* l_m \ldots l_1$ and z_0 be an arbitrary flag in $S_w \cap C_0$. Then there exists and ordered basis $(v_1, \ldots, v_n)$ of $\mathbb{C}^n$ such that

$$v_j := \sum_{i=1}^{w(j)-1} v_{ij} e_j + e_{w(j)}, \text{ with } v_{ij} \in \mathbb{C}, v_{ij} = 0, \forall i \in \{w(1), \ldots, w(j-1)\},$$

and z_0 is the complete flag

$$0 \subset\, < v_1 > \,\subset\, < v_1, v_2 > \,\subset\, \cdots \,\subset\, < v_1, v_2, \ldots, v_n > .$$

Let $V_i := < v_1, \ldots, v_i >$ and $[v_1 \ldots v_i]$ the $n \times i$ matrix corresponding to the column vectors spanning V_i, for all $1 \leq i \leq n$. Then for $i = n$, $[v_1 \ldots v_n]$ is just the matrix canonical form and each matrix $[v_1 \ldots v_i]$ can be obtained from it by removing the last $n - i$ columns.

Since $z_0 \in S_w \cap C_0$ it follows that z_0 must be an isotropic flag, i.e. $V_{n-i}^{\perp} = V_i$ for all $1 \leq i \leq m$. In terms of the basis vectors these m conditions translate to the fact that each matrix $[v_1 \ldots v_{n-i}]$ must have the column vector v_i perpendicular to itself and all other vectors in the matrix for each $1 \leq i \leq m$. We now analyze these conditions step by step to give an explicit description of all the intersection points.

For $i = 1$ we look at the matrix $[v_1 \ldots v_{n-1}]$ and thus discard the last column $v_n = v_{1n} e_1 + \cdots + v_{1,l_1-1} e_{l_1-1} + e_{l_1}$, where $l_1 = k_1 - 1$. If $l_1 = 1$, then $k_1 = 2$ and thus $v_1 = v_{11} e_1 + e_2$. From the isotropic condition $v_1 \cdot v_1 = 0$ it follows that $v_{11} \in \{\pm\sqrt{-1}\}$.

If $l_1 > 1$, observe first that $[v_1 \ldots v_{n-1}]$ contains column vectors which have a 1 on entry p for all $1 \leq p < l_1$. Denote such column vectors with f_p. Using the isotropic relations $v_1 \cdot f_1 = 0, \ldots, v_1 \cdot (f_{l_1} - 1) = 0$, and computing step by step it follows that $v_{11} = 0, \ldots, v_{1,l_1-1} = 0$. Now the only remaining freedom on v_1 is on v_{1,l_1} and using the isotropic relation $v_1 \cdot v_1 = 0$ it follows that $(v_{1,l_1})^2 + 1 = 0$. Therefore

$$v_1 = (\pm\sqrt{-1}) e_{l_1} + e_{k_1}.$$

The condition $v_1 \cdot v_p = 0$, for all $2 \le p \le n - 1$, is equivalent to

$$v_{l_1,p} \cdot (\pm\sqrt{-1}) = 0, \text{ for all } 2 \le p \le n - 1,$$

which is further equivalent to $v_{l_1,p} = 0$, for all $2 \le p \le n-1$. The entries $v_{p,n}$, for $1 \le p \le l_1 - 1$ are all zero by definition of the canonical form $[v_1, \ldots v_n]$.

For $i = 2$ we look at the matrix $[v_1 \ldots v_{n-2}]$ and thus discard v_{n-1} and v_n from $[v_1 \ldots v_n]$. From the immediate predecessor algorithm it follows that either $l_2 = k_2 - 1$ or $l_2 = l_1 - 1$ and $v_{k_1,2} = 0$. Furthermore, even though at this step v_n, the column vector corresponding to l_1, is discarded we have from the step $i = 1$ that $v_{l_1,2} = 0$. Following the same algorithm as before we show that the isotropic condition forces $v_{p,2} = 0$, for all $1 \le p \le l_2 - 1$, i.e. compute coefficients step by step starting with the equation $v_2 \cdot f_1 = 0$ and going further to $v_2 \cdot f_p = 0$, where f_p is the column with entry 1 on row p, for all $1 \le p \le l_2 - 1$, except of rows l_1 and k_1 which entries were already forced to be zero. The only freedom that remains on v_2 is on the spot corresponding to l_2. Here, using $v_2 \cdot v_2 = 0$, it follows that

$$v_2 = (\pm\sqrt{-1})e_{l_2} + e_{k_2}.$$

Assume that we have shown that $v_s = (\pm\sqrt{-1})e_{l_s} + e_{k_s}$, for all $1 \le s \le j - 1$, with $j \le m$, that $v_{l_s,p} = 0$, for all $s + 1 \le p \le n - s$, and that $v_{p,n-s+1} = 0$, for all $1 \le p \le l_s - 1$. To go to step j one usees the fact that l_j belongs to the set

$$\{k_j - 1, l_1 - 1, \ldots, l_{j-1} - 1\}$$

and repeats the procedure. In this case the columns v_s for $n - j + 1 \le s \le n$ are removed from the matrix canonical form. As before zeros are created step by step starting with $v_j \cdot f_1 = 0$ and going up to $v_j \cdot f_p = 0$, where f_p is the column where a 1 sits on row p for all $1 \le p \le l_j - 1$, except of course when $p \in \{k_1, \ldots, k_{j-1}, l_1, \ldots l_{j-1}\}$ in which case even though the columns corresponding to this elements are not in the matrix their spots in column j were already made zero in the previous step.

It thus follows that the points of intersection $< v_1 > \subset \cdots \subset \mathbb{C}^n$ are given by the following flags $< (\pm\sqrt{-1})e_{l_1} + e_{k_1} > \subset \cdots \subset < (\pm\sqrt{-1})e_{l_1} + e_{k_1}, \ldots, (\pm\sqrt{-1})e_{l_m} + e_{k_m}, e_{l_*} > \subset < (\pm\sqrt{-1})e_{l_1} + e_{k_1}, \ldots, (\pm\sqrt{-1})e_{l_m} + e_{k_m}, e_{l_*}, e_{l_m} > \subset \cdots \subset \mathbb{C}^n$. $\square$

Therefore the homology class $[C_0]$ of the base cycle inside the homology ring of Z is given in terms of the Schubert classes of elements in $\mathcal{S}_{C_0}$ by:

$$[C_0] = 2^m \sum_{S \in \mathcal{S}_{C_0}} [S], \text{ if } n = 2m + 1, \text{ and } [C_0] = 2^{m-1} \sum_{S \in \mathcal{S}_{C_0}} [S], \text{ if } n = 2m.$$

2.5. Main results for measurable open orbits in $Z = G/P$

This section treats the case when D is an open orbit in $Z = G/P$ and C_0 is the base cycle in D.

Schubert cells and varieties can also be defined in G/P. Those will be indexed by elements of the coset space $\Sigma_n / \Sigma_{d_1} \times \Sigma_{d_2} \times \cdots \times \Sigma_{d_s}$ and each right coset contains a unique minimal representative, i.e. a permutation w such that $w(1) < \cdots < w(d_1), w(d_1 + 1) < \cdots < w(d_1 + d_2), \ldots, w(d_1 + \ldots d_{s-1} + 1) < \cdots < w(d_1 + \cdots + d_s) = w(n)$. The dimension of the Schubert cell $C_{wP} := BwP/P$ is the length of the minimal representative w and there is a unique lift to a Schubert cell in G/B of the same dimension, namely $\mathcal{O}_w := BwB/B$. When working with $\mathcal{O}_{wP}$ we can thus use the same matrix representation as for $\mathcal{O}_w$ and many times we will refer to $\mathcal{O}_{wP}$ just by $\mathcal{O}_w$.

Recall the notation

$$\mathcal{S}_{C_0} := \{S_w \text{ Schubert variety } : dim S_w + dim C_0 = dim Z \text{ and } S_w \cap C_0 \neq \emptyset\}.$$

The main idea is to lift Schubert varieties $S_w \in \mathcal{S}_{C_0}$ to minimal dimensional Schubert varieties $S_{\hat{w}}$ in $\hat{Z} := G/B$ that intersect the open orbit $\hat{D}$ and consequently the base cycle $\hat{C}_0$.

The first step is to consider the situation when D is a measurable open orbit. There are many equivalent ways of defining measurability in general. In our context however, this depends only on the dimension sequence d that defines the parabolic subgroup P. Namely, an open $SL(n, \mathbb{R})$ orbit D in $Z = G/P$ is called *measurable* if P is defined by a symmetric dimension sequence as follows:

- $d = (d_1, \ldots, d_s, d_s, \ldots, d_1)$ or $e = (d_1, \ldots, d_s, e', d_s, \ldots, d_1)$, for $n = 2m$,
- $e = (d_1, \ldots, d_s, e', d_s, \ldots, d_1)$, for $n = 2m + 1$.

A general definition of measurability can be found in [4].

As discussed in the preliminaries, a Schubert variety S_w in Z is indexed by the minimal representative w of the parametrization coset associated to the dimension sequence defining P. Corresponding to the symmetric dimension sequence d, each such w can be divided into blocks B_j and $\tilde{B}_j$, both having the same number of elements d_j for $1 \leq i \leq s$. Set $d_0 := 0$ and $d_{s+1} := 0$. The block B_j is defined to be the sequence of numbers

$$\{w(k) : \sum_{i=0}^{j-1} d_i + 1 \leq k \leq \sum_{i=0}^{j} d_i\}$$

of the minimal representative w, and the block $\tilde{B}_j$ is defined to be the sequence of numbers

$$\{w(k): \sum_{i=j+1}^{s+1} d_i + \sum_{i=1}^{s} d_i + 1 \leq k \leq \sum_{i=1}^{s} d_i + \sum_{i=j}^{s+1} d_i\}.$$

In the case of the symmetric symbol e the block $B_{e'}$ is defined as the sequence of numbers

$$\{w(k): \sum_{i=1}^{s} d_i + 1 \leq k \leq \sum_{i=1}^{s} d_i + e'\}.$$

For example, $B_1 = (w(1) < w(2) < \cdots < w(d_1))$ and $\tilde{B}_1 = (w(d_1 + \cdots + d_s + d_s + \cdots + d_2 + 1) < \cdots < w(n))$. The pairs $(B_j, \tilde{B}_j)$ are called **symmetric block pairs**. In the case of the symmetric dimension symbol e, one single block $B_{e'}$ of length e' is introduced in the middle of w. In what follows w will always correspond to a symmetric dimension sequence and it will satisfy the conditions of a minimal representative.

Definition 2.10. A permutation w is said to satisfy the **generalized spacing condition**, if for each symmetric block pair $(B_j, \tilde{B}_j)$ the elements of $\tilde{B}_j$ can be arranged in such a way that if the elements of B_j are denoted by $k_1^j \ldots k_{d_j}^j$ and the rearranged elements of $\tilde{B}_j$ by $l_1^j \ldots l_{d_j}^j$, then $l_i^j < k_i^j$ for all $1 \leq i \leq d_j$.

Observe that in the case of the symmetric dimension sequence e we can always rearrange the elements in the single block $B_{e'}$ so that they satisfy the spacing condition inside the block.

Definition 2.11. A permutation $B_1 \ldots B_s B_{e'} \tilde{B}_s \ldots \tilde{B}_1$ is said to satisfy the **generalized double box contraction** condition if w is constructed by the **generalized immediate predecessor algorithm**:

• The first symmetric block pair $(B_1 = (k_i^1), \tilde{B}_1 = (l_i^1))$ is constructed by choosing d_1 pairs (l_i^1, k_i^1) of consecutive numbers from the ordered set $\{1, \ldots, n\}$

• If all symmetric block pairs up to $(B_j = (k_i^j), \tilde{B}_j = (l_i^j))$ have been chosen, then to go to the step $j+1$ choose d_{j+1} pairs (l_i^{j+1}, k_i^{j+1}) in such a way that l_i^{j+1} sits at the immediate left of k_i^{j+1} in the ordered set $\{1, \ldots, n\} - \{\cup_{i=1}^{j} B_i\} - \{\cup_{i=1}^{j} \tilde{B}_i\}$, for all $1 \leq i \leq d_{j+1}$.

Observe that in the case of the symmetric symbol e the elements in the single block $B_{e'}$ can always be rearranged so that they satisfy the double

box contraction condition inside the block. The symbol $B_{e'}$ will always be written in a representation of w and disregarded in the case of the symbol d.

If w satisfies the generalized spacing condition, $\tilde{w}$ denotes the permutation obtained from w by replacing each block $\tilde{B}_j = l_1^j \ldots l_{d_j}^j$ with a choice of rearrangement of its elements $\tilde{l}_1^j \ldots \tilde{l}_{d_j}^j$ required so that w satisfies the generalized spacing condition. Further inside each rearranged block $\tilde{B}_j$ in $\tilde{w}$, $\tilde{l}_1^j \ldots \tilde{l}_{d_j}^j$ is rewritten as $\tilde{l}_{d_j}^j \tilde{l}_{d_j-1}^j \ldots \tilde{l}_1^j$. In the case of $B_{e'}$ being part of the representation of w the following rearrangement is chosen: if e' is even then $B_{e'}$ is rearranged as $l'_{e'/2+1} \, l'_{e'/2+2} \ldots l'_{e'} l'_1 \ldots l'_{e'/2-1} l'_{e'/2}$ and if e' is odd then $B_{e'}$ is rearranged as $l'_{(e'+1)/2+1} \, l'_{(e'+1)/2+2} \ldots l'_{e'} l'_1 \ldots l'_{(e'+1)/2-1} l'_{(e'+1)/2}$. Observe that now $\tilde{w}$ is a permutation that satisfies the spacing condition for the G/B case and it is of course also just another representative of the coset that parametrizes S_w.

Proposition 2.12. *A Schubert variety S_w parametrized by the permutation*

$$w = B_1 B_2 \ldots B_s B_{e'} \tilde{B}_s \ldots \tilde{B}_2 \tilde{B}_1$$

has non-empty intersection with C_0 if and only if w satisfies the generalized spacing condition, i.e. if and only if there exists a lift of S_w to a Schubert variety $S_{\tilde{w}}$ that intersects the base cycle in $\hat{Z} = G/B$.

Proof. If w satisfies the generalized spacing condition, then by the above observation one can find another representative $\tilde{w}$ of the parametrizing coset of S_w, that satisfies the spacing condition and thus a Schubert variety $S_{\tilde{w}}$ that intersects $\hat{C}_0$. Because the projection map π is equivariant it follows that $\pi(S_{\tilde{w}}) = S_w$ intersects C_0.

Conversely, suppose $S_w \cap C_0 \neq \emptyset$ and assume without loss of generality that w is the minimal representative of the right coset class indexing S_w. Then for every point $p \in S_w \cap C_0$ there exists $\hat{p} \in S_{\tilde{w}} \cap \hat{C}_0$ with $\pi(\hat{p}) = p$ and $\pi(S_{\tilde{w}}) = S_w$ for some Schubert variety in G/B indexed by $\tilde{w}$. It follows that $\tilde{w}$ and w determine the same coset in G/P, because π is B-equivariant and from Proposition 2.5. $\tilde{w}$ satisfies the spacing condition. By definition of the minimal representative w is obtained from $\tilde{w}$ by dividing $\tilde{w}$ into blocks $B_1 \ldots B_s B_{e'} \tilde{B}_s \ldots \tilde{B}_1$ and arranging the elements in each such block in increasing order. This shows that w satisfies the generalized spacing condition □

If $w = B_{d_1} \ldots B_{d_s} \tilde{B}_{d_s} \ldots \tilde{B}_1$, with $\tilde{B}_{d_j} = l_1^j \ldots l_{d_j}^j$ for each $1 \leq j \leq s$, then let $\hat{w} := B_{d_1} \ldots B_{d_s} \tilde{C}_{d_s} \ldots \tilde{C}_{d_1}$, where $\tilde{C}_{d_j} = l_{d_j}^j l_{d_j-1}^j \ldots l_2^j l_1^j$ for each $1 \leq j \leq s$. If $B_{e'} = l'_1 \ldots \, l'_{e'}$ is part of the representation of w, then

let $\tilde{B}_{e'}$ be the single middle block in the representation of $\hat{w}$ defined by: $l'_{e'/2+1}l'_{e'/2+2}\ldots l'_{e'}l'_1 \ldots l'_{e'/2-1}l'_{e'/2}$ if e' is even and $l'_{(e'+1)/2+1}l'_{(e'+1)/2+2}\cdots$ $l'_{e'}l'_1 \ldots l'_{(e'+1)/2-1}l'_{(e'+1)/2}$ if e' is odd. Call such a choice of rearrangement for w a **canonical rearrangement**.

Note that if $S_w \in \mathcal{S}_{C_0}$ lifts to $S_{\hat{w}}$, such that $\hat{w}$ satisfies the double box contraction condition, then $\dim S_{\hat{w}} - \dim S_w = (\dim \hat{Z} - \dim \hat{C}_0) - (\dim Z - \dim C_0) = (\dim \hat{Z} - \dim Z) - (\dim \hat{C}_0 - \dim C_0)$. Since π is a G_0 and K_0 equivariant map, if F denotes the fiber of π over a base point $z_0 \in C_0$, then the fiber of $\pi|_{\hat{C}_0} : \hat{C}_0 \to C_0$ over z_0 is just $F \cap \hat{C}_0$ and $\dim S_{\hat{w}} - \dim S_w$ must equal $\dim F - \dim(F \cap \hat{C}_0)$.

As stated in the preliminaries in the case of $\hat{Z} = G/B$ and $m = \lfloor n/2 \rfloor$, the isotropic condition on flags is equivalent to $V_i \subset V_i^\perp$ for all $1 \leq i \leq m$, $V_m = V_m^\perp$, if n is even and the flags V_{n-i} are determined by $V_{n-i} = V_i^\perp$, for all $1 \leq i \leq m$. Thus in the case of the dimension sequence $d = (d_1, \ldots, d_s, d_s, \ldots, d_1)$, $\dim F - \dim(F \cap C_0)$ is equal to $2 \sum_{i=1}^{s} d_i(d_i - 1)/2 - \sum_{i=1}^{s} d_i(d_i - 1)/2$ which is equal to $\sum_{i=1}^{s} d_i(d_i - 1)/2$. In the case when the dimension sequence is given by $e = (d_1, \ldots, d_s, e', d_s, \ldots, d_1)$ and the base point z_0 contains a middle flag of length e, it remains to add to the above number the difference in between the dimension of the total fiber over this flag and the dimension of the isotropic flags in this fiber. This is just a special case of the G/B case for a full flag of length e' and the number is $e'(e' - 1)/2 - (e'/2)^2 + (e'/2)$ which equals to $(e'/2)^2$ when e' is even and $e'(e' - 1)/2 - [(e' - 1)/2]^2$ which equals to $(e' - 1)(e' + 1)/2$, when e' is odd.

Thus if $S_w \in \mathcal{S}_{C_0}$ lifts to $S_{\hat{w}}$ such that $\hat{w}$ satisfies the double box contraction condition, then we have that $\dim S_{\hat{w}} - \dim S_w$ is equal to:

$$(2) \qquad \sum_{i=1}^{s} d_i(d_i - 1)/2, \text{ if } d = (d_1, \ldots, d_s, d_s, \ldots, d_1) \text{ and}$$

$$(3) \qquad \sum_{i=1}^{s} d_i(d_i - 1)/2 + d_e, \text{ if } e = (d_1, \ldots, d_s, e', d_s \ldots, d_1),$$

with

$$d_e = \begin{cases} (e'/2)^2, & e' \text{ even,} \\ (e' - 1)(e' + 1)/4, & e' \text{ odd.} \end{cases}$$

Now if w satisfies the generalized double box contraction condition, then $\hat{w}$ satisfies the double box contraction condition. Furthermore, by construction, it is immediate that if w satisfies the generalized double box contraction condition, then $|w| = |\hat{w}| - \sum_{i=1}^{s} d_i(d_i - 1)/2$. That is because the block $\tilde{B}_{d_j}$

is formed by arranging the block $\tilde{C}_{d_j}$ in increasing order and thus crossing l_1^j over $d_j - 1$ numbers, and more generally l_i^j over $d_j - i$ numbers for all $1 \leq i \leq d_j$. Similarly, if e' is part of the representation of w and e' is even then $|w| = |\hat{w}| - \sum_{i=1}^s d_i(d_i - 1)/2 - (e'/2)^2$ and if e' is odd then $|w| = |\hat{w}| - \sum_{i=1}^s d_i(d_i - 1)/2 - (e' - 1)(e' + 1)/4$. Thus if w satisfies the generalized double box contraction condition, then $\hat{w}$ satisfies the double box contraction condition and $|\hat{w}| - |w|$ is equal to:

$$(4) \qquad \sum_{i=1}^s d_i(d_i - 1)/2, \text{ if } d = (d_1, \ldots, d_s, d_s, \ldots, d_1) \text{ and}$$

$$(5) \qquad \sum_{i=1}^s d_i(d_i - 1)/2 + d_e, \text{ if } e = (d_1, \ldots, d_s, e', d_s \ldots, d_1),$$

with

$$d_e = \begin{cases} (e'/2)^2, & e' \text{ even}, \\ (e' - 1)(e' + 1)/4, & e' \text{ odd}. \end{cases}$$

Theorem 2.13. *A permutation $w = B_1 B_2 \ldots B_s B_{e'} \tilde{B}_s \ldots \tilde{B}_2 \tilde{B}_1$ satisfies the generalized double box contraction condition if and only if w parametrizes an Iwasawa-Schubert variety $S_w \in \mathcal{S}_{C_0}$. The lifting map $f : \mathcal{S}_{C_0} \to \mathcal{S}_{\hat{C}_0}$ defined by $S_w \mapsto S_{\hat{w}}$, with $\hat{w}$ the canonical rearrangement of w, is injective.*

Proof. If w satisfies the generalized double box contraction condition, then by the above observation the canonical rearrangement $\hat{w}$ of w is just another representative of the parametrization coset of S_w. Moreover, $\hat{w}$ satisfies the double box contraction condition and thus parametrizes a Schubert variety $S_{\hat{w}}$ that intersects $\hat{C}_0$. Because the projection map π is equivariant it follows that $\pi(S_{\hat{w}}) = S_w$ intersects C_0. From the remarks before the statement of the theorem it follows that if w satisfies the generalized double box contraction condition, then $|\hat{w}| - |w|$ is given by 4 and 5. This is equivalent to the fact that the difference in dimensions between $S_{\hat{w}}$ and S_w is achieved, i.e. $\dim S_{\hat{w}} - \dim S_w$ is given by 2 and 3, and this happens when one needs to write the elements of each block $\tilde{B}_i$ in strictly decreasing order to form the block $\tilde{C}_j$.

Conversely, suppose that $S_w \in \mathcal{S}_{C_0}$ but w does not satisfy the generalized double box contraction condition. It then follows that there exists a first block pair $(B_j, \tilde{B}_j)$ and a first pair (k_i^j, l_i^j), for some i in between 1 and d_j such that l_{i-1}^j sits at the immediate left of l_i^j and k_i^j sits at the immediate right of k_{i-1}^j in the ordered set $\{1, \ldots, n\} - \{\cup_{s=1}^{j-1} B_s\} - \{\cup_{s=1}^{j-1} \tilde{B}_s\}$. This

means that when w is lifted to $\hat{w}$, the place of l_{i-1}^j and l_i^j remain the same because otherwise w will not satisfy the double box contraction condition. But this implies that the difference between the length of $\hat{w}$ and the length of w is strictly smaller than the difference $\dim S_{\hat{w}} - \dim S_w$ computed in 2 and 3. $\qquad\square$

Theorem 2.14. *A Schubert variety S_w in S_{C_0} intersects the base cycle C_0 in $2^{d_1+\cdots+d_s}$ points in the case where w is given by a symmetric symbol $e = (d_1, \ldots, d_s, e', d_s, \ldots, d_1)$ and $n = 2m+1$, while in the even dimensional case we have 2^{m-1} points in the case $d = (d_1, \ldots, d_s, d_s, \ldots, d_1)$ and $2^{d_1+\cdots+d_s-1}$ in the case $e = (d_1, \ldots, d_s, e', d_s, \ldots, d_1)$.*

Proof. The result follows from the lifting principle, because the intersection points of S_w can be identified in a one-to-one manner with a subset of the intersection points of $S_{\hat{w}}$. Thus the cardinality of this subset can be directly computed. $\qquad\square$

It thus follows that the homology class $[C_0]$ of the base cycle inside the homology ring of Z is given in terms of the Schubert classes of elements in $\mathcal{S}_{C_0}$ by:

$$[C_0] = 2^{m-1} \sum_{S \in \mathcal{S}_{C_0}} [S], \text{ if } n = 2m, \text{ and } Z = Z_d,$$

$$[C_0] = 2^{d_1+\cdots+d_s-1} \sum_{S \in \mathcal{S}_{C_0}} [S], \text{ if } n = 2m, \text{ and } Z = Z_e,$$

$$[C_0] = 2^{d_1+\cdots+d_s} \sum_{S \in \mathcal{S}_{C_0}} [S], \text{ if } n = 2m + 1, \text{ and } Z = Z_e.$$

2.6. Main results for non-measurable open orbits in $Z = G/P$

The last case to be considered is the case of $Z = G/P$ and $D \subset Z$ a non-measurable open orbit, i.e. the dimension sequence $f = (f_1, \ldots, f_u)$ that defines P is not symmetric. Associated with the flag domain D there exists its measurable model, a canonically defined measurable flag domain $\hat{D}$ in $\hat{Z} = G/\hat{P}$ together with the projection map $\pi : \hat{Z} \to Z$. If $\hat{z}_0$ is a base point in $\hat{C}_0$, $z_0 = \pi(\hat{z}_0)$ and we denote by $\hat{F}$ the fiber over z_0, by H_0 and $\hat{H}_0$ the isotropy of G_0 at z_0 and $\hat{z}_0$ respectively, then $\hat{F} \cap \hat{D} = H_0/\hat{H}_0$, is holomorphically isomorphic with $\mathbb{C}^k$, where k is root theoretically computable.

Moreover, if one considers the extension of the complex conjugation τ from $\mathbb{C}^n$ to $SL(n, \mathbb{C})$, defined by $\tau(s)(v) = \tau(s(\tau(v)))$, for all $s \in SL(n, \mathbb{C})$,

$v \in \mathbb{C}^n$, then one obtains an explicit construction of $\hat{P}$ as follows. Consider the Levi decomposition of P as the semidirect product $P = L \ltimes U$, where L denotes its Levi part and U the unipotent radical. If $P^- = L \ltimes U^-$ denotes the opposite parabolic subgroup to P, namely the block lower triangular matrix with blocks of size $f_1, \ldots, f_u$, then $\hat{P} = P \cap \tau(P^-)$. Furthermore, the parabolic subgroup $\tau(P^-)$ has dimension sequence $(f_u, \ldots, f_1)$. These results are proved in complete generality in [8].

Because π is a K_0 equivariant map, the restriction of π to $\hat{C}_0$ maps $\hat{C}_0$ onto C_0 with fiber $\hat{C}_0 \cap (\hat{F} \cap \hat{D})$. In this case this fiber is a compact analytic subset of $\mathbb{C}^k$ and consequently it is finite, i.e. the projection $\pi|_{\hat{C}_0} : \hat{C}_0 \to C_0$ is a finite covering map. Because C_0 is simply-connected it follows that:

Proposition 2.15. *The restriction map*

$$\pi|_{\hat{C}_0} : \hat{C}_0 \to C_0$$

is biholomorphic. In particular, if q and $\hat{q}$ denote the respective codimensions of the cycles, it follows that $\hat{q} = dim(\hat{F}) + q$

If we denote with $\mathcal{S}_{C_0}$ the set of Schubert varieties S_w in Z that intersect the base cycle C_0 and $\dim S_w + \dim C_0 = \dim Z$, where w is a minimal representative of the parametrization coset of S_w and by $\mathcal{S}_{\hat{C}_0}$ the analogous set in $\hat{Z}$, then the above discussion implies the following:

Proposition 2.16. *The map $\Phi : \mathcal{S}_{C_0} \to \pi^{-1}(\mathcal{S}_{C_0}) \subset \mathcal{S}_{\hat{C}_0}$ is bijective.*

If $d = (d_1, \ldots, d_s, d_s, \ldots, d_1)$ or $e = (d_1, \ldots, d_s, e', d_s, \ldots, d_1)$ is a symmetric dimension sequence, then one can construct another dimension sequence out of it, not necessarily symmetric, by the following method. Consider an arbitrary sequence $t = (t_1, \ldots, t_p)$ such that each $t_i \geq 1$ for all $1 \leq i \leq p$, at least one t_i is strictly bigger than 1 and $t_1 + \cdots + t_p = 2s$ or $2s + 1$ depending on wether one considers d or e, respectively. Associated to t the sequence $\delta = (\delta_1, \ldots, \delta_p)$ is defined by $\delta_j := \sum_{i=1}^j t_i$. With the use of δ the new dimension sequence $f_\delta = (f_{\delta_1}, \ldots, f_{\delta_p})$ is defined by $f_{\delta_1} := \sum_{i=1}^{\delta_1} d_i$,

$$f_{\delta_j} := \sum_{i=\delta_{j-1}+1}^{\delta_j} d_i, \text{ for all } 2 \leq j \leq p.$$

Because $\hat{P}$ is obtained as the intersection of two parabolic subgroups P and $\tau(P^-)$, it follows that the dimension sequence f of P is obtained as above, from the dimension sequence of $\hat{P}$, as f_δ for a unique choice of t. For

ease of computation we do not break up anymore the dimension sequence of $\hat{P}$ into its symmetric parts and we simply write it as $d = (d_1, \ldots, d_s)$, where s can be both even or odd. Using the usual method of computing the dimension of Z it then follows that

$$\dim Z = \dim \hat{Z} - \sum_{t_j > 1} \sum_{\delta_{j-1}+1 \leq h < g \leq \delta_j} d_h d_g.$$

For example if P corresponds to the dimension sequence $(2, 4, 3)$, then an easy computation with matrices shows that $\hat{P}$ corresponds to the dimension sequence $(2, 1, 3, 1, 2)$, $t = (1, 2, 2)$ and $\delta = (1, 3, 5)$. Moreover, $\dim Z = \dim \hat{Z} - 1 \cdot 3 - 1 \cdot 2$.

Given the sequence $f = f_\delta$, we are now interested in describing the set $\mathcal{S}_{C_0}$. Let $S_{\hat{w}}$ in $\mathcal{S}_{\hat{C}_0}$ be the unique Schubert variety associated to a given $S_w \in \mathcal{S}_{C_0}$ such that $\pi(S_{\hat{w}}) = S_w$. If $\hat{w}$ is given in block form by $B_1 \ldots B_s$, where here again the notation used does not take into consideration the symmetric structure of $\hat{w}$, then w is given in block form by $C_1 \ldots C_p$ corresponding to the dimension sequence f_δ. The blocks C_j are given by $C_1 = \bigcup_{i=1}^{\delta_1} B_{d_i}$ and

$$C_j = \bigcup_{i=\delta_{j-1}+1}^{\delta_j} B_{d_i}, \text{ for all } 2 \leq j \leq p,$$

arranged in increasing order. Moreover,

$$\begin{aligned} \dim S_w &= \dim Z - \dim C_0 = \dim Z - \dim \hat{C}_0 \\ &= \dim \hat{Z} - \sum_{t_j > 1} \sum_{\delta_{j-1}+1 \leq h < g \leq \delta_j} d_h d_g - \dim \hat{C}_0 \\ &= \dim S_{\hat{w}} - \sum_{t_j > 1} \sum_{\delta_{j-1}+1 \leq h < g \leq \delta_j} d_h d_g. \end{aligned}$$

Finally, understanding what conditions $\hat{w}$ satisfies in order for the above equality to hold amounts to understanding the difference in length that the permutation $\hat{w}$ looses when it is transformed into w. If C_j contains only one B-block from $\hat{w}$, i.e. $t_j = 1$, then this is already ordered in increasing order and it does not contribute to the decrease in dimension. If C_j contains more B-blocks, say

$$C_j = \bigcup_{i=\delta_{j-1}+1}^{\delta_j} B_{d_i},$$

we start with the first block $B_{\delta_{j-1}+1}$ which is already in increasing order and bring the elements from $B_{\delta_{j-1}+2}$ to their right spots inside the first block. As usual to each number we can associate a distance, i.e. the number of elements it needs to cross in order to be brought on the right spot, and we denote by $\alpha_{\delta_{j-1}+2}$ the sum of this distances. The maximum value that $\alpha_{\delta_{j-1}+2}$ can attain is when all the elements in the second block are smaller than each element in the first block, i.e. the last element in the second block is smaller than the first element in the first block. In this case $\alpha_{\delta_{j-1}+2} = d_{\delta_{j-1}+1}d_{\delta_{j-1}+2}$, the product of the number of elements in the first block with the number of elements in the second block. Next we bring the elements in the 3rd block among the already ordered elements from the first and second block. Observe that the maximal value that $\alpha_{\delta_{j-1}+3}$ can attain is $d_{\delta_{j-1}+1}d_{\delta_{j-1}+3} + d_{\delta_{j-1}+2}d_{\delta_{j-1}+3}$ when the last element in the 3rd block is smaller than all elements in the first two blocks. In general we say that the group of blocks used to form C_j is in **strictly decreasing order** if it satisfies the following: the last element of block B_{i+1} is smaller than the first element of block B_i for all $\delta_{j-1} + 1 \leq i \leq \delta_j - 1$. Consequently, if one wants to order this sequence of blocks into increasing order one needs to cross over

$$\sum_{\delta_{j-1}+1 \leq h < g \leq \delta_j} d_h d_g.$$

Thus if all the blocks C_j with $t_j > 1$ among w come from groups of blocks arranged in strictly decreasing order in $\hat{w}$, then

$$|w| = |\hat{w}| - \sum_{t_j > 1} \sum_{\delta_{j-1}+1 \leq h < g \leq \delta_j} d_h d_g.$$

What was just proved is the following:

Proposition 2.17. *A Schubert variety $S_{\hat{w}} \in \mathcal{S}_{\hat{C}_0}$ is mapped under the projection map to a Schubert variety $S_w \in \mathcal{S}_{C_0}$ if and only if all the blocks C_j with $t_j > 1$ among w come from groups of blocks arranged in strictly decreasing order in $\hat{w}$.*

As an example consider the complex projective space $Z = \mathbb{P}_5$. The dimension sequence of the measurable model in this case is given by $d = (1, 4, 1)$ and the Schubert varieties in $\mathcal{S}_{\hat{C}_0}$ are parametrized by the following permutations: $(2)(3456)(1)$, $(3)(1456)(2)$, $(4)(1256)(3)$, $(5)(1236)(4)$, $(6)(1234)(5)$. The only permutation that satisfies the strictly decreasing order among the last two blocks is the permutation $(2)(3456)(1)$ and this gives the only Schubert variety in $\mathcal{S}_{C_0}$ parametrized by $(2)(13456)$. More generally,

for $Z = \mathbb{P}_n$ we have only one element in $\mathcal{S}_{C_0}$ parametrized by $(2)(13\ldots n+1)$. For a more complicated example see [3].

Concluding remark

The cases of the other real forms, $SL(m, \mathbb{H})$ and $SU(p,q)$, of $SL(n, \mathbb{C})$ are handled in detail in the author's thesis [3]. Let us close our discussion here by briefly commenting on these results.

For $n = 2m$, $SL(m, \mathbb{H})$ is defined to be the group of operators in $SL(n, \mathbb{C})$ which commute with the antilinear map $j : \mathbb{C}^n \to \mathbb{C}^n$, $v \mapsto J\bar{v}$, where J is the usual symplectic matrix with $J^2 = -\text{Id}$. For simplicity we only comment here on the case where $Z = G/B$ is the full flag manifold (the other cases are handled by the expected lifting procedures.). In that case a certain parity condition on a permutation $w = (k, l)$ plays the role of the spacing condition used for $G_0 = SL(n, \mathbb{R})$. This condition tightens up to the strict parity condition where it is required that the k_i are odd and $l_i = k_i + 1$. The main result for $Z = G/B$ can be stated as follows: There is only one open orbit D, a Schubert variety (with B an Iwasawa-Borel) has non-empty intersection with C_0 if and only if w satisfies the parity condition and intersects it in isolated points if and only if it satisfies the strict parity condition. In the later case the intersection consists of exactly one point.

The case of the Hermitian groups $SU(p,q)$, $p+q = n$, seems to be vastly more complicated than the others. This is in particular due to the large number of open orbits. For example, if z_0 is a full flag $V_1 \subset V_2 \subset \ldots \subset V_n$ in $Z = G/B$, then $D = G_0.z_0$ is open if and only if the restriction of the mixed signature form $\langle \, , \, \rangle_{p,q}$ to each V_i is non-degenerate. Thus the open orbits are parameterised by pairs (a, b) of integer vectors with (a_i, b_i) being the signature of the restriction of the form to V_i.

While our combinatorial conditions are in a certain sense analogous to those for the other real forms, here we only give algorithms for determining exactly which Iwasawa-Schubert varieties have non-empty intersection with a given flag domain D and which intersect C_0 in only isolated points. Analogous to the $SL(m, H)$-case, in the later case there is exactly one point of intersection. As we show in typical examples and by giving detailed descriptions of $\mathcal{S}_{C_0}$ for the Hermitian symmetric spaces, i.e., $Z = G/P$ with P maximal, the algorithms are quite efficient (see [3]).

Acknowledgements

under his valuable guidance. The current paper is part of my dissertation at Jacobs University Bremen. I would also like to thank Prof. Allen Knutson for pointing out to me [9]. In particular, Wyser's results describe the fundamental classes of base cycles in the equivarient cohomology ring of the full flag manifold. It would be interesting to determine the explicit representation of the Iwasawa-Schubert varieties in terms of that polynomial ring. Finally, I would like to thank the referee for his critical reading of the first version of the paper and for making very useful comments.

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