Journal of Combinatorics
Volume 8, Number 2, 349–370, 2017

Sum-free graphs

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Dedicated to Professor J. A. Bondy

An n-vertex graph is sum-free if the vertices can be labelled with $\{1, 2, \ldots, n\}$ such that no vertex gets a label which is the sum of the labels of two of its neighbours. We prove that non-complete graphs with average degree two or less are sum-free. We also prove that graphs with maximum degree three and at least seven vertices are sum-free.

Keywords and phrases: Graph labelling.

In [4] the authors discuss combination graphs, in which vertices of G are labelled using a bijection $f : V(G) \to \{1, \ldots, |V(G)|\}$ and an injective edge-labelling is induced by assigning the label $\binom{a}{b}$ to the edge $e = uv$ if $\{f(u), f(v)\} = \{a, b\}$, with $a > b$. This labelling problem is somewhat complicated by the fact that it is possible to have $\binom{a}{b} = \binom{c}{d}$, for $a \neq c$.

Here we consider the presumably easier problem of characterizing which graphs have sum-free labellings. For an n-vertex simple graph G, with vertex set $V(G)$, a *sum-free labelling* of G is a bijection $f : V(G) \to I_n = \{1, \ldots, n\}$, such that for every path $P = uvw$, $f(v) \neq f(u) + f(w)$. If G is a simple graph and G has a sum-free labelling, we call G sum-free. Obviously, a combination graph is a sum-free graph, but a graph could be sum-free without being a combination graph. In the special case that $V(G) = I_n$ and the sum-free function f is the identity function on I_n, we say that G is a labelled sum-free graph.

All graphs in this note will be finite and simple. Throughout, unless otherwise stated, we shall assume that G is a finite simple graph with n vertices and m edges. Our notation will generally follow that of [2]. For a graph G, $V(G)$ is the vertex set of G, and $E(G)$ the edge set. For $v \in V(G)$, the degree of v is denoted $d(v)$, the set of neighbours of v is $N(v)$, the closed neighbourhood is $N[v] = N(v) \cup \{v\}$. If G is assumed to be sum-free, we will use f for some sum-free labelling of G.

Observation 1. Following [4], we note that if G is sum-free, then $|E(G)| \leqslant \frac{1}{4}|V(G)|^2$.

Proof. Let $v = f^{-1}(n) \in V(G)$. Then, since G is sum-free, no two neighbours of G can have labels in $\{j, n-j\}$, for $1 \leqslant j \leqslant \frac{n}{2}$. Hence, $d_G(v) \leqslant \left\lfloor \frac{n}{2} \right\rfloor$. Note that f when restricted to $V(G) - \{v\}$ yields a sum-free labelling of $G - v$. Hence, for $w = f^{-1}(n-1) \in V(G-v)$, $d_{G-v}(w) \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor$, and $d_G(v) + d_{G-v}(w) \leqslant \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor = n-1$. Now, if $|E(G - \{v,w\})| \leqslant \frac{1}{4}|V(G-\{v,w\})|^2 = \frac{1}{4}(n-2)^2$, then $|E(G)| \leqslant \frac{1}{4}(n-2)^2 + (n-1) = \frac{1}{4}n^2$. To complete a proof by induction, it is only necessary to note that the desired inequality holds for simple graphs on one or two vertices. $\square$

Observation 2. Any n-vertex sum-free graph G can be embedded as a subgraph in an n-vertex sum-free graph H with $|E(H)| = \left\lfloor \frac{1}{4}n^2 \right\rfloor$. Hence, a list of all of the n-vertex m-edge sum-free graphs, with $m = \left\lfloor \frac{1}{4}n^2 \right\rfloor$, contains all of the n-vertex sum-free graphs as subgraphs.

Proof. Let $v = f^{-1}(n) \in V(G)$, and again note that $G - v$ is sum-free with labelling $g = f - \{(v,n)\}$. By the obvious induction, we may embed $G - v$ in a sum-free graph H_1 with the same labelling g, and with $\left\lfloor \frac{1}{4}(n-1)^2 \right\rfloor$ edges. Now, select $X \subseteq V(H_1)$ such that $N(v) \subseteq X$ and for each i, $1 \leqslant i \leqslant \left\lfloor \frac{n}{2} \right\rfloor$, $|f(X) \cap \{i, n-i\}| = 1$. Let $H = H_1 \cup \{vw : w \in X\}$. Then, H is sum-free with labelling f, and $|E(H)| = \left\lfloor \frac{1}{4}n^2 \right\rfloor$. $\square$

Observation 3. The processes in Observation 1 and Observation 2 specify an algorithm for generating all edge-maximal labelled sum-free graphs on n vertices. For a labelled sum-free graph G on $n-1$ vertices, let $Y(G) = \left\{ X \subseteq V(G) = I_{n-1} : |X| = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } |X \cap \{i, n-i\}| = 1,\ 1 \leqslant i \leqslant \left\lfloor \frac{n}{2} \right\rfloor \right\}$.

For $X \in Y(G)$, let $s(G,X) = G \cup \{n\} \cup \{\{n,x\} : x \in X\}$.

Suppose that $L_{n-1} = \{G_k\}_{k=1}^{q}$ is the set of all labelled n-vertex sum-free graphs. Then, $L_n = \bigcup_{k=1}^{q} \{s(G_k, X) : X \in Y(G_k)\}$.

The definition of $L_0 = K_0$ gets the ball rolling. Every graph in L_n has $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges. $\square$

We note that $|L_n| = 2^q |L_{n-1}|$, where $q = \left\lfloor \frac{n-1}{2} \right\rfloor$, and $|L_n| = 2^t$, where $t = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$. For example, L_4 has four labelled graphs. One of these is a 4-cycle, the other three are isomorphic. In general, one would expect the number of non-isomorphic graphs in L_n to be far smaller than $|L_n|$. Of course, the number of non-isomorphic graphs in L_n is at least $\frac{1}{n!} |L_n| = \frac{2^{\lfloor (n-1)^2/4 \rfloor}}{n!}$, which grows rather quickly.

A simple graph on n vertices and $\left\lfloor \frac{1}{4} n^2 \right\rfloor$ edges is not necessarily sum-free. We will show later that $K_{3,3}$ is not sum-free, yet $K_{3,3}$ has 6 vertices and $\left\lfloor \frac{1}{4} \cdot 6^2 \right\rfloor = 9$ edges.

For a graph G, let $R(G,k) = \{v \in V(G) : d(v) \geqslant k\}$. For a positive integer n, let $r(n,k) = \max\{|R(G,k)| : G \in L_n\}$.

Lemma 4. If f is a sum-free labelling of G, and if $v \in V(G)$ with $d(v) = n - j$, then $f(v) \leqslant 2j$ and $R(n, n-j) \subseteq \left\{ f^{-1}(k) : 1 \leqslant k \leqslant 2j \right\}$. Hence, $r(n, n-j) \leqslant 2j$.

Proof. Suppose that $f(v) \geqslant 2j + 1$. Then, for $1 \leqslant i \leqslant j$,

$$\left| \left\{ f^{-1}(i), f^{-1}(n-i) \right\} \cap N(v) \right| \leqslant 1,$$

and $d(v) \leqslant (n-1) - j$. since this is impossible, $f(v) \leqslant 2j$. $\square$

We noted that $K_{2,2}$ is sum-free. (It is in L_4.) Assign $\{2,3\}$ to one colour class, and $\{1,4\}$ to the other.

Lemma 5. $K_{2,3}$ is not sum-free and $K_{k,k}$ is not sum-free, for $k \geqslant 3$.

Proof. We consider the case of $G \cong K_{k,k}$, $k \geqslant 3$, first. Let (X,Y) be a bipartition of G, and let f be a sum-free labelling of G. Without loss of generality, $f^{-1}(1) \in X$. Now, if $f^{-1}(i) \in X$, for $2 \leqslant i \leqslant 2k-1$, then $f^{-1}(i+1) \notin Y$, and we must have $f^{-1}(i+1) \in X$. But then $X = \left\{ f^{-1}(1), f^{-1}(k+2), f^{-1}(k+3), \ldots, f^{-1}(2k) \right\}$, and

$$Y = \left\{ f^{-1}(2), f^{-1}(3), f^{-1}(4), \ldots, f^{-1}(k+1) \right\}.$$

Since $k \geqslant 3$, $f^{-1}(k+1), f^{-1}(k-1) \in Y = N\left(f^{-1}(2k)\right)$, contradicting the choice of f as a sum-free labelling. Hence, $K_{k,k}$ is not sum-free, for $k \geqslant 3$.

Now, let $G \cong K_{2,3}$, with bipartition (X, Y), where $|X| = 2$. By Lemma 4, $f(X) \subseteq \{1,2,3,4\}$. Thus, $f^{-1}(5) \in Y$, and $f(X) \notin \{\{1,4\}, \{2,3\}\}$. Now, if $f(X) = \{1, j\}$, with $j \in \{2,3\}$, then $j+1 \in F(Y)$, which is impossible. If $f(X) = \{2,4\}$, then $\{1,3\} \in f(Y)$, which is impossible. Finally, if $f(X) = \{3,4\}$, then $\{1,2\} \in f(Y)$, which is impossible. Thus, $K_{2,3}$ is not sum-free. □

Corollary 6. Let $n \geqslant 5$. If $K_{k,n-k} \subseteq G$, for $2 \leqslant k \leqslant n-2$, then G is not sum-free. Since $K_{2,n-2}$ cannot be a subgraph of G if $n \geqslant 5$, $|R(G, n-1)| \leqslant 1$ and any two vertices of degree $n-2$ are adjacent.

Proof. Suppose that $K_{k,n-k} \cong H \subseteq G$, where $n \geqslant 5$ and $2 \leqslant k \leqslant n-2$, and that f is a sum-free labelling of G. Then, f is a sum-free labelling of H. Let H have bipartition (X, Y), with $|X| \leqslant |Y|$. By Lemma 4, $f^{-1}(X) \subseteq \{1, 2, \ldots, 2|X|\}$. Let $M = G\left[f^{-1}(\{1, 2, \ldots, 2|X|\})\right]$, if $|X| \geqslant 3$, and let $M = G\left[f^{-1}(\{1, 2, \ldots, 5\})\right]$, if $|X| = 2$. Then, f restricted to M is a sum-free labelling of M. But, by Lemma 5, M has no sum-free labelling. Hence, G has no sum-free labelling. □

By Corollary 6, there are examples of graphs on n vertices, $n \geqslant 5$, which are not sum-free but have only $2n-4$ edges, far below the $\frac{1}{4}n^2$ bound.

It may be possible to limit the total number of vertices of high degree, as in the next two Lemmas.

Lemma 7. If G is sum-free, $n \geqslant 5$, and G has a vertex of degree $n-1$, then G has at most one vertex of degree $n-2$.

Proof. Suppose that $d(u) = n-1$, $d(v) = n-2$, and $d(w) = n-2$, with $v \neq w$. By Corollary 6, $vw \in E(G)$. By Lemma 4, $f(\{u, v, w\}) \subseteq \{1,2,3,4\}$. Since $G[\{u, v, w\}] \cong K_3$ and G is sum-free,

$$f(\{u, v, w\}) \in \{\{1,2,4\}, \{2,3,4\}\}.$$

Suppose that $f(\{u, v, w\}) = \{2,3,4\}$. Then, $f(u) = 2$. Without loss of generality, $f(v) = 3$ and $f(w) = 4$. Let $z = f^{-1}(1)$ and $y = f^{-1}(5)$. Then, $zv \notin E(G)$. Hence, $yv \in E(G)$, $yu \in E(G)$, and $f(y) = f(u) + f(v)$, contradicting the choice of f as sum-free. Thus, $f(\{u, v, w\}) \neq \{2,3,4\}$.

Now, suppose that $f(\{u, v, w\}) = \{1,2,4\}$ and $f(u) = 1$. Then,

$$\left|N\left(f^{-1}(4)\right) \cap \left\{f^{-1}(3), f^{-1}(5)\right\}\right| \geqslant 1,$$

and both of these are impossible.

Finally, suppose that $f(\{u,v,w\}) = \{1,2,4\}$ and $f(u) = 2$. Without loss of generality, $f(v) = 1$ and $f(w) = 4$. Let $z = f^{-1}(3)$ and $y = f^{-1}(5)$. Since $wz, vz \notin E(G)$, we must have $wy, vy \in E(G)$. But then, $f(y) = f(w) + f(v)$, contradicting the choice of f as sum-free. This exhausts all cases. Hence, G has at most one vertex of degree $n-2$. $\square$

Lemma 8. If G is sum-free, $n \geqslant 5$, and G has no vertex of degree $n-1$, then G has at most three vertices of degree $n-2$.

Proof. By Corollary 6, if G has four vertices of degree $n-2$, these four vertices are pairwise adjacent, and by Lemma 4, they have labels in $\{1,2,3,4\}$. Hence, $\{f^{-1}(1), f^{-1}(2)\} \in N(f^{-1}(3))$, which is impossible. $\square$

Lemma 9. If G is sum-free, $n \geqslant 6$, and G has no vertex of degree $n-1$, then G has at most two vertices of degree $n-2$. Hence, $r(n, n-2) \leqslant 2$ for $n \geqslant 6$.

Proof. Suppose that u, v, w are distinct vertices in G, each with degree $n-2$. By Corollary 6, if $G[\{u,v,w\}] \cong K_3$, and, by Lemma 4, $f(\{u,v,w\}) \subseteq \{1,2,3,4\}$. Since G is sum-free, $f(\{u,v,w\}) \in \{\{1,2,4\}, \{2,3,4\}\}$.

Suppose that $f(\{u,v,w\}) = \{1,2,4\}$. Without loss of generality, $f(u) = 1$, $f(v) = 2$, $f(w) = 4$. Let $z = f^{-1}(3)$ and $y = f^{-1}(5)$. Then, $z \notin N(u) \cap N(v)$, $z \notin N(w)$. But then, $y \in N(w)$, and thus $y \notin N(u)$. Then, $z \in N(u)$ and $z \notin N(v)$. Now, $f^{-1}(6) \in N(v) \cap N(w)$, which is impossible.

Suppose that $f(\{u,v,w\}) = \{2,3,4\}$. Without loss of generality, $f(u) = 2$, $f(v) = 3$, $f(w) = 4$. Let $z = f^{-1}(1)$ and $y = f^{-1}(5)$. Here, $z \notin N(v) \cup N(w)$. If $n \geqslant 7$, then $f^{-1}(7) \in N(v) \cap N(w)$, which is impossible. For $n = 6$, $f^{-1}(5), f^{-1}(6) \in N(v) \cap N(w)$. Hence, $f^{-1}(5), f^{-1}(6) \notin N(u)$, and $d(u) \leqslant n-3$, contradicting our assumptions. $\square$

Lemma 10.

(*a*) If $n \geqslant 2j$, then $r(n+1, n+1-j) \leqslant r(n, n-j)$. Hence, $r(n, n-j) \leqslant r(2j, j)$, for $n \geqslant 2j$.

(*b*) Suppose that $G \in L_N$ has $R(G_0, N-j) = I_k$, where $j, k \leqslant \left\lfloor \frac{N}{2} \right\rfloor$. Then, $r(n, n-j) \geqslant k$, for $n \geqslant N$.

Proof. We only need prove part (*a*) for graphs on L_{n+1}. Let $G \in L_{n+1}$ and let $H = G - \{v_{n+1}\}$, where we use v_{n+1} for the vertex of G with label $n+1$ merely to remind the reader that this is a vertex of G. Note that $R(G, n+1-j) \subseteq R(H, n-j) \cup \{v_{n+1}\}$, but $d_G(v_{n+1}) = \left\lfloor \frac{n+1}{2} \right\rfloor <$

$n+1-j$. Thus, $R(G,n+1-j) \subseteq R(H,n-j)$. Since this is true for any $G \in L_{n+1}$, $r(n+1,n+1-j) \leqslant r(n,n-j)$. Now, a straightforward induction gives $r(2j+k,j+k) \leqslant r(2j,j)$, for $k \geqslant 0$, proving part (a).

Now, suppose that $G \in L_N$ has $R(G_0,N-j) = I_k$, where $j,k \leqslant \left\lfloor \dfrac{N}{2} \right\rfloor$. Then, there is some $X \subseteq Y(G)$ such that $I_k \subseteq X$. Thus, $H = s(G,X) \in L_{N+1}$ has $R(H,N+1-j) = I_k$. Induction then establishes $r(n,n-j) \geqslant |R(G,N-j)| = k$, for $n \geqslant N$, proving part (b). □

We now have simple proofs of Lemma 4, of one of the statements in Corollary 6, and of Lemma 9 using the lists L_n. We state this as Corollary 11.

Corollary 11.

(a) $r(n,n-1) = 1$ for $n \geqslant 4$.
(b) $r(n,n-2) = 2$ for $n \geqslant 6$.
(c) $r(n,n-3) = 4$ for $n \geqslant 8$.
(d) $r(n,n-j) \leqslant 2j$, for all $n \geqslant 0$.

Proof. Since $|R(G,3)| \leqslant 1$ for any of the four graphs $G \in L_4$, $r(n,n-1) \leqslant 1$ for $n \geqslant 4$. There is a graph $G_4 \in L_4$ with adjacency matrix M_4 shown below. $R(G_4,3) = \{1\}$. Thus, by Lemma 10, (a) holds.

Since $|R(G,4)| \leqslant 2$ for any of the 64 graphs $G \in L_6$, $r(n,n-2) \leqslant 2$ for $n \geqslant 6$. There is a graph $G_6 \in L_4$ with adjacency matrix M_6 shown below. $R(G_6,4) = \{1,2\}$. Thus, by Lemma 10, (b) holds.

Let G_{11} denote the labelled sum-free graph with adjacency matrix M_{11}, displayed below. Note that $R(G_{11},8) = \{1,2,3,4\}$. From the possible degree sequences in L_8, we know that $r(8,5) = 4$ and, hence, by Lemma 10(a), $r(n,n-3) \leqslant 4$, for $n \geqslant 8$. But, by Lemma 10(b), $r(n,n-3) \geqslant |R(G_{11},8)| = 4$, for $n \geqslant 11$. Hence, $r(n,n-3) = 4$, for $n \geqslant 8$. This established part (c).

Since $r(n,j) \leqslant 2j$ trivially for $n \leqslant 2j$, we deduce that $r(n,n-j) \leqslant 2j$, for all $n \geqslant 0$. This proves part (d). □

$$M_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \ M_6 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$M_{11} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now for a few positive examples.

Lemma 12. If $n \geqslant 4$, and if G has maximum degree two, then G is sum-free.

Proof. Suppose that G is a minimum counterexample. That is, for any graph H with $\Delta(H) \leqslant 2$ and $4 \leqslant |V(H)| < n$, H is sum-free.

Suppose that G has a component C with k vertices, $k \leqslant 3$, and let $H = G - V(C)$. If $|V(H)| \geqslant 4$, label C with $\{j\}_{j=n-k+1}^{n}$. Note that if C has a path uvw, then $f(u) \leqslant n < 2n-3 \leqslant f(v) + f(w)$. Let g be a sum-free labelling of H, and set $f(x) = g(x)$ for $x \in V(H) \subseteq V(G)$. Then, f is a sum-free labelling of G.

If $|V(H)| = 3$, label H with $\{1, 2, 4\}$ and C with $\{3, 5, 6\} \cap \{j\}_{j=1}^{n}$. If $|V(H)| = 2$, label H with $\{1, 2\}$ and C with $\{3, 4, 5\} \cap \{j\}_{j=1}^{n}$. If $|V(H)| = 1$, label H with $\{1\}$ and C with $\{2, 3, 4\}$.

We may now assume that all components of G have at least four vertices. Let C be a component of G, and let $P = v_1 v_2 \cdots v_k$ be a longest path in C. (This is $C = P$ or $C = P \cup \{v_k v_1\}$.) Consider the function $g(v_i) = n + 1 - i$, $1 \leqslant i \leqslant k$. Note that for $1 < i < k$, $g(v_i) = n+1-i$ and $g(v_{i-1}) + g(v_{i+1}) = 2(n+1-i)$. If $C = P$, then using a sum-free labelling h of $H = G - V(C)$, the labelling f give by $f(x) = g(x)$ for $x \in C$ and $f(x) = h(x)$ for $x \in H$ is a sum-free labelling of G. Hence, we may assume that $C = P \cup \{v_k v_1\}$.

We note that $g(v_k) = n-k+1$ and $g(v_{k-1}) + g(v_1) = (n-k+2) + n > g(v_k)$. Also, $g(v_k) + g(v_2) - g(v_1) = (n-k+1) + (n-1) - n = n-k \neq 0$ if $k < n$. Hence, for $k < n$, G is sum-free.

We have one remaining case and in this case, G is a cycle $v_1 v_2 \cdots v_n v_1$, with $n \geqslant 4$. Let $f(v_i) = i$, for $1 \leqslant i \leqslant n-2$, and let $f(v_{n-1}) = n$ and $f(v_n) = n-1$. Now, as above, for $1 < i < n-2$, $2f(v_i) = f(v_{i-1}) + f(v_{i+1})$, and we need only check the sums at v_{n-2}, v_{n-1}, v_n, and v_1.

$$f(v_{n-3}) + f(v_{n-1}) - f(v_{n-2}) = (n-3) + n - (n-2) = n-1 \neq 0.$$

$f(v_{n-2}) + f(v_n) - f(v_{n-1}) = (n-2) + (n-1) - n = n - 3 \neq 0.$
$f(v_{n-1}) + f(v_1) - f(v_n) = n + 1 - (n-1) = 2 \neq 0.$
$f(v_n) + f(v_2) - f(v_1) = (n-1) + 2 - 1 = n \neq 0.$

Hence, f is a sum-free labelling of G in this case. □

The next result is fairly trivial, but perhaps useful.

Lemma 13. If G is acyclic, then G is sum-free.

Proof. We proceed by induction on $n = |V(G)|$. By reviewing L_n, $n \leqslant 3$, or by Lemma 12, the result is true for any acyclic graph on at most three vertices, establishing our base cases. Since G is acyclic, there is a vertex $v \in V(G)$ such that $d_G(v) \leqslant 1$. Let $H = G - v$ and note that H is acyclic. Let g be a sum-free labelling of H and let $f(x) = g(x)$ for $x \in V(H)$ and $f(v) = n$. Then, f is a sum-free labelling of G. □

In the following Lemma, $K_1 \vee 2K_3$ is the graph with one vertex v of degree six and with $G - v$ the disjoint union of two triangles.

Lemma 14. If $n = |V(G)| \geqslant 7$, $v \in V(G)$, $\Delta(G - v) \leqslant 2$, and G is not $K_1 \vee 2K_3$, then G is sum-free.

Proof. Let $H = G - v$. Let the connected components of $H - v$ be A_1, A_2, $\ldots$, A_s, with A_1 not a triangle, if possible. We will first consider the case in which A_1 is a triangle, since it gives an introduction to the method for the remaining cases.

Suppose that $A_1 \cong K_3$. Then, $A_1 \cong A_2 \cong \cdots \cong A_s \cong K_3$. Note that, $s \geqslant 2$. If $s = 2$, then since G is not $K_1 \vee 2K_3$, without loss of generality, there is a vertex $w \in V(A_1)$ such that $vw \notin E(G)$. Let $f(v) = 1$, $f(w) = 6$, $f(N(w) - \{v\}) = \{3, 5\}$ and $f(V(A_2)) = \{2, 4, 7\}$. Then, f is a sum-free labelling of G.

Suppose that $A_1 \cong K_3$ and $s = 3$. Here, $n = 10$. Select any bijection $f : V(G) \to I_{10}$ such that $f(v) = 1$, $f(V(A_1)) = \{2, 4, 8\}$, $f(V(A_2)) = \{3, 6, 10\}$, and $f(V(A_3)) = \{5, 7, 9\}$. Then, f is a sum-free labelling of G.

Next, we suppose that $A_1 \cong K_3$ and $s = 2k$, $k \geqslant 2$. Here, n is odd. By Lemma 12, $A = A_1 \cup A_2 \cup \cdots \cup A_k$ has a sum-free labelling g. Let $B = A_{k+1} \cup A_{k+2} \cup \cdots \cup A_s$. Let h be any bijection from $V(B)$ to $\{3, 5, \ldots, n\}$ and define $f : V(G) \to I_n$ by $f(v) = 1$, $f(x) = 2g(x)$ for $x \in V(A)$, and $f(x) = h(x)$ for $x \in V(B)$. Then, f is a sum-free labelling of G.

The remaining case with $A_1 \cong K_3$ has $s = 2k+1$, $k \geqslant 2$, and $n = 6k+4 \geqslant 16$. Let $A = A_1 \cup A_2 \cup \cdots \cup A_k$ and let $B = A_{k+2} \cup A_{k+3} \cup \cdots \cup A_s$. Let g be a sum-free labelling of A, let h be any bijection from $V(B)$ to $\{5, 7, \ldots, n-1\}$, and let r be a bijection from $V(A_{k+1})$ to $\{3, n-2, n\}$. Define $f : V(G) \to I_n$

by $f(v) = 1$, $f(x) = 2g(x)$ for $x \in V(A)$, $f(x) = r(x)$ for $x \in V(A_{k+1})$, and $f(x) = h(x)$ for $x \in V(B)$. Then, f is a sum-free labelling of G.

We may now assume that A_1 is not a triangle. As in the constructions above, we shall use $f(v) = 1$, unless otherwise specified.

Let t be the integer such that $|V(A_1 \cup A_2 \cup \cdots \cup A_t)| \leqslant \left\lfloor \frac{n}{2} \right\rfloor$ and

$$|V(A_1 \cup A_2 \cup \cdots \cup A_{t+1})| > \left\lfloor \frac{n}{2} \right\rfloor.$$

Let $A = A_1 \cup A_2 \cup \cdots \cup A_t$. If $|V(A)| = \left\lfloor \frac{n}{2} \right\rfloor$, $B = H - V(A)$. Since A is not a triangle, there is a sum-free labelling g of A. Let h be any bijection from $V(B)$ to $\left\{3, 5, ..., 2\left\lfloor \frac{n-1}{2} \right\rfloor\right\}$. Let $f(v) = 1$, $f(x) = 2g(x)$ for $x \in V(A)$, and $f(x) = h(x)$ for $x \in V(B)$. Then, f is a sum-free labelling of G.

We may now assume that $|V(A_1 \cup A_2 \cup \cdots \cup A_t)| < \left\lfloor \frac{n}{2} \right\rfloor$ and

$$|V(A_1 \cup A_2 \cup \cdots \cup A_{t+1})| > \left\lfloor \frac{n}{2} \right\rfloor.$$

Let $A = A_1 \cup A_2 \cup \cdots \cup A_t$, $M = A_{t+1}$, and let $B = A_{t+2} \cup A_{t+3} \cup \cdots \cup A_s$.

We note that M is either a path or a cycle. We intend to use a mixture of even and odd labels on M. There are $k = \left\lfloor \frac{n}{2} \right\rfloor - |V(A)|$ even labels and $\ell = \left\lfloor \frac{n-1}{2} \right\rfloor - |V(B)|$ odd labels available for M. We note that $k > 1$ and $\ell > 1$, since $|V(A)| < \left\lfloor \frac{n}{2} \right\rfloor < |V(A \cup A_{t+1})|$. Let $V(M) = \{m_j\}_{j=1}^{q}$ and either $E(M) = \{m_j m_{j+1}\}_{j=1}^{q-1}$ or $E(M) = \{m_j m_{j+1}\}_{j=1}^{q-1} \cup \{m_q m_1\}$. For convenience, let $n_1 = 2\left\lfloor \frac{n}{2} \right\rfloor \geqslant 6$ denote the largest possible even label and $n_2 = 2\left\lfloor \frac{n-1}{2} \right\rfloor + 1$ denote the largest possible odd label.

Let $f(m_\ell) = 3$ and, if $\ell \geqslant 2$, let $f(m_j) = 2j + 3$ for $1 \leqslant j < \ell$. Let $f(m_{k+\ell}) = n_1$ and, if $k \geqslant 2$, let $f(m_{\ell+j}) = n_1 - 2j$ for $1 \leqslant j < k$. Since A is not a triangle, we pick a sum-free labelling g of A and let $f(x) = 2g(x)$ for $x \in V(A)$. We pick an arbitrarily bijection h from $V(B)$ to $\{2\ell + 3, 2\ell + 5, \ldots, n_2\}$ and let $f(x) = h(x)$ for $x \in V(B)$. Also, we have $f(v) = 1$. The function f defined this way is a sum-free labelling of G if $|(n_1 - 2) - 3| \neq 1$ and $n_1 \neq 6$. Thus, if $n \geqslant 8$, the function f is a sum-free labelling of G. The only remaining case is $n = 7$.

If H is contained in a 6-cycle C assign the labels $(2, 4, 6, 3, 7, 5)$ cyclically along C. We may therefore assume that H has a cycle of length at most 5.

If H has a 5-cycle C, assign the labels $(2, 4, 6, 3, 7)$ cyclically along C, and assign the label 5 to the vertex in $V(H) - V(C)$.

If H has a 4-cycle C, assign the labels $(2, 4, 7, 5)$ cyclically along C, and assign $\{3, 6\}$ to $V(H) - V(C)$.

Finally, if H has a 3-cycle C, then we may assume that $D = H - V(C)$ is not a 3-cycle, since we have already considered this case. Assign the labels $(2, 4, 7)$ cyclically along C, and assign $\{5, 3, 6\}$ to $V(H) - V(C)$ so that the vertex of degree two in D, if any, receives the label 3. □

We next investigate some of the properties of a smallest 3-regular non-sum-free graph G.

Lemma 15. If $|E(G)| \leqslant |V(G)|$ and $|V(G)| \geqslant 4$, then G is sum-free.

Proof. We consider a smallest counterexample. Let G be a simple graph which is not sum-free but which has $|E(G)| \leqslant |V(G)|$ and $|V(G)| \geqslant 4$, and, subject to this, suppose that $n = |V(G)|$ is as small as possible.

By Lemma 12, we may assume that G has a vertex of degree at least three. Suppose that G has a vertex v of degree one, and let $H = G - v$. Note that $|E(H)| = |E(G)| - 1 \leqslant |V(G)| - 1 = |V(H)|$.

If $|V(H)| \geqslant 4$, then H has a sum-free labelling g, and we extend this labelling to a sum-free labelling f of G which agrees with g on H and has $f(v) = n$. On the other hand, if $|V(H)| = 3$, then let w be the neighbour of v in G, let $f(V(H) - w) = \{2, 4\}$ and set $f(w) = 1$ and $f(v) = 3$, yielding a sum-free labelling of G. Hence, we may assume that G has no vertex of degree one.

Suppose that M is a connected component of G, with $2 \leqslant |V(M)| \leqslant 1 + \left\lfloor \frac{n}{2} \right\rfloor$, and let $H = G - V(M)$. Note that every vertex of M has degree at least two and, thus, $|E(M)| \geqslant |V(M)|$ and $|E(H)| \leqslant |V(H)|$. If $|V(H)| \leqslant 3$, then either H is sum-free or H is a 3-cycle. In the case that H is a 3-cycle, $|E(M)| = |E(G)| - 3 \leqslant |V(G)| - 3 = |V(M)|$, but M has no vertex of degree one, and thus G is regular of degree two and is sum-free. Hence, we may assume that H is sum-free for $|V(H)| \leqslant 3$. By our choice of G, H is sum-free if $|V(H)| \geqslant 4$. Thus, for any value of $|V(H)|$, we may assume that H is sum-free. Let g be a sum-free labelling of H, and extend g to any bijection $f : V(G) \to I_n$, that agrees with g on H. Since f restricted to M has no values less than $n + 1 - |V(M)| = n - \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil$, f is a sum-free labelling of G.

Note that G cannot have two connected components H_1 and H_2 with $|V(H_2)| \geqslant |V(H_1)| \geqslant 2$, since then H_1 is a connected component with $|V(H_1)| \leqslant \left\lfloor \frac{n}{2} \right\rfloor$. Hence, we may now assume that G has one non-trivial connected component M. Since every vertex of M has degree at least two, and since M has at least one vertex of degree at least three, $|E(M)| > |V(M)|$. All of the other components of G are isolated vertices. Furthermore, there must be at least $|E(M)| - |V(M)| > 0$ isolated vertices in G, and $|V(M)| \geqslant 2 + \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n+4}{2} \right\rfloor$.

Suppose that G has a vertex v with $d(v) = 2$. Let w be a vertex of G with $d(w) = 0$, let $N(v) = \{x, y\}$, and let $H = G - \{v, w\}$. Then, $H \cong K_3$ or H has a sum-free labelling f. If $H \cong K_3$, then we assign $\{2, 4\}$ to $\{x, y\}$, $\{1, 5\}$ to the vertices of degree two in G and 3 to the vertex of degree zero. Hence, we may assume that H is sum-free, with labelling f. If $f(x) + f(y) = n$, extend f to $V(G)$ by assigning $f(v) = n - 1$ and $f(w) = n$. Otherwise, assign $f(v) = n$ and $f(w) = n - 1$. In either case, f is a sum-free labelling of G. Hence, we may assume that G has no vertices of degree two.

We can bound the number of vertices of degree three, if there are enough vertices of degree zero. Suppose that u and v are non-adjacent vertices of degree three in G, and that W is a set of four vertices of degree zero. Let $N(u) = \{u_1, u_2, u_2\}$ and $N(v) = \{v_1, v_2, v_3\}$. We allow the possibility that $|N(u) \cap N(v)| > 0$. Let $H = G - \{u, v\} - W$. Again, $H \cong K_3$ or H has a sum-free labelling f. If $H \cong K_3$, then $|V(M)| = 5 = \frac{n+1}{2}$, contradicting our previous bound that $|V(M)| \geqslant \left\lfloor \frac{n+4}{2} \right\rfloor$. Thus, H has a sum-free labelling f. For a vertex $z \in V(G) - V(H)$, let $B(z) = \{f(x) + f(y) : x, y \in N(z), x \neq y\}$ and let $Q = I_n - I_{n-6}$. We extend f to $V(G)$ by selecting $f(u) \in Q - B(u)$, $f(v) \in Q - B(v) - \{f(u)\}$, and assigning $Q - \{f(u), f(v)\}$ to W. This is a sum-free labelling of G. Hence, we may assume that either G has at most four vertices of degree three (and the vertices of degree three are pairwise adjacent) or G has at most three vertices of degree zero. Note that if G has four vertices of degree three, then $M \cong K_4$, $4 = |V(M)| \geqslant \left\lfloor \frac{n+4}{2} \right\rfloor$, $n \leqslant 5 < |E(M)|$. Hence, we may assume that G has at most three vertices of degree three.

Let k_0 denote the number of vertices of degree zero in G, k_3 denote the number of vertices of degree three in G, and k_4 denote the number of vertices of degree four or more in G. Then, $n = k_0 + k_3 + k_4$. Since $n \geqslant |E(G)| \geqslant 2k_4 + \frac{3}{2}k_3$, $k_0 + k_3 + k_4 \geqslant 2k_4 + \frac{3}{2}k_3$, and $k_0 \geqslant k_4 + \frac{1}{2}k_3$. From $k_3 + k_4 = |V(M)| \geqslant \left\lfloor \frac{n+4}{2} \right\rfloor \geqslant \frac{n+3}{2}$, we have $2k_3 + 2k_4 \geqslant k_0 + k_3 + k_4 + 3$, and $k_3 + k_4 \geqslant k_0 + 3$. Combining $k_0 \geqslant k_4 + \frac{1}{2}k_3$ and $k_3 + k_4 \geqslant k_0 + 3$, we obtain $k_3 + k_4 \geqslant k_0 + 3 \geqslant k_4 + \frac{1}{2}k_3 + 3$ and $\frac{1}{2}k_3 \geqslant 3$. Hence $k_3 \geqslant 6$, contradicting our previous bound, $k_3 \leqslant 4$. Therefore, there is no graph G that remains, proving the Lemma. $\square$

Sometimes, we prefer a different set of labels. For a set of k integers X, $X \subseteq I_n$, we say an n-vertex graph G is *X-skew* if there is a bijection $f : V(G) \to I_{n+k} - X$, such that for any path uvw in G, $f(v) \neq f(u) + f(w)$. The function f will be called an *X-skew labelling.* If $X = I_k$, we refer to G as *k-skew* and f as a *k-skew* labelling. We establish the following lemma to demonstrates a few small constructions similar to those we will use in the study of graphs with maximum degree three, and which prove useful for Conjecture 29.

Lemma 16.

(a) For any n-vertex graph G, any bijection $f : V(G) \to I_{2n-1} - I_{n-1}$ is an $(n-1)$-skew labelling and any bijection $g : V(G) \to I_{2n-2} - I_{n-2}$ is an $(n-2)$-skew labelling.

(b) If G is a 4-vertex graph, $v \in V(G)$, and $j \in I_7 - \{1, 2, 4\}$, then G has a $\{1, 2, 4\}$-skew labelling f such that $f(v) = j$.

(c) If G is a 5-vertex graph with maximum degree at most three, $v \in V(G)$, and $j \in I_7 - I_2$, then G has a 2-skew labelling f such that $f(v) = j$.

Proof. Statement (a) is obvious, since there are no triples (a, b, c) in $I_{2n-1} - I_{n-1}$ with a,b,c distinct and $a + b = c$.

Statement (b) is also obvious, since any bijection $f : V(G) \to I_7 - \{1, 2, 4\}$ is a $\{1, 2, 4\}$-skew labelling. Select one such that $f(v) = j$.

Now suppose that If G is a 5-vertex graph with maximum degree at most three, $v \in V(G)$, and $j \in I_7 - I_2$. If $j \notin \{3, 7\}$, then let $w \in N(v)$ and let $z \in V(G) - N[v]$. Let $f : V(G) \to I_7 - I_2$ be a bijection such that $f(v) = j$, $f(w) = 3$, and $f(z) = 7$. Then, f is a 2-skew labelling of G with the required property. If $j \in \{3, 7\}$, then let $w \in N(v)$ and let $z \in V(G) - N[v]$. Let $g : V(G) \to I_7 - I_2$ be a bijection such that $g(v) = j$, and $f(z) = 10 - j$. Then, f is a 2-skew labelling of G with the required property, completing the proof of (c). $\square$

We now consider 3-regular graphs, and more generally graphs with maximum degree three. Let $\mathcal{F}$ denote the set of finite graphs each of which has maximum degree three, is not sum-free and has at least seven vertices. We would like to show that $|\mathcal{F}| = 0$. Suppose that $|\mathcal{F}| \neq 0$, and let G_* denote a graph in $\mathcal{F}$ with the least possible number of vertices. By Lemma 12, we know that G_* has maximum degree three. As usual, $n = |V(G_*)|$ in the following lemmas.

Lemma 17. K_3 is not a connected component of G_*.

Proof. Suppose that $K_3 \cong M \subseteq G_*$ and let $H = G_* - V(M)$. Note that H has a vertex of degree three. If $|V(H)| \geqslant 7$, then H has a sum-free labelling f and for any bijection $g : V(M) \to \{n, n-1, n-2\}$, $f \cup g$ is a sum-free labelling of G_*. If $|V(H)| = 4$, let $f : V(M) \to \{3, 5, 6\}$ and $g : V(H) \to \{1, 2, 4, 7\}$ be bijections. Then, $f \cup g$ is a sum-free labelling of G_*. Hence, we know that $|V(H)| \in \{5, 6\}$.

Suppose that $|V(H)| = 5$. Let $v \in V(H)$ such that $d(v) = 3$ and let $w \in V(H) - N[v]$. Let $f : V(M) \to \{1, 2, 8\}$ and $g : V(H) \to \{3, 4, 5, 6, 7\}$ be bijections, such that $g(v) = 7$ and $g(w) = 3$. Then, $f \cup g$ is a sum-free labelling of G_*.

Finally, suppose that $|V(H)| = 6$. Let $v \in V(H)$ such that $d(v) = 3$, let $N(v) = \{x_1, x_2, x_3\}$ and let $V(H) - N[v] = \{w_1, w_2\}$. If $x_i w_j \notin E(H)$, then let $f : V(M) \to \{1, 2, 4\}$ and $g : V(H) \to \{3, 5, 6, 7, 8, 9\}$ be bijections, such that $g(v) = 9$, $g(x_i) = 8$, and $g(w_j) = 3$. Then, $f \cup g$ is a sum-free labelling of G_*. Hence, we may assume that $x_i w_j \in E(H)$, for $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$, and thus $H \cong K_{3,3}$ and $G_* \cong K_3 \cup K_{3,3}$. Let $f : V(M) \to \{4, 5, 6\}$, $g : \{v, w_1, w_2\} \to \{1, 2, 3\}$, and $h : \{x_1, x_2, x_3\} \to \{7, 8, 9\}$ be bijections. Then, $f \cup g \cup h$ is a sum-free labelling of G. This eliminates all possibilities and hence G_* cannot have K_3 as a connected component. $\square$

Lemma 18. K_4 is not a connected component of G_*. Hence, K_4 is not a subgraph of G_*.

Proof. Suppose that $K_4 \cong M \subseteq G_*$ and let $H = G_* - V(M)$. If $|V(H)| \geqslant 7$, then H has a sum-free labelling f and for any bijection $g : V(M) \to \{n, n-1, n-2, n-3\}$, $f \cup g$ is a sum-free labelling of G_*. If $|V(H)| = 3$, let $f : V(H) \to \{3, 5, 6\}$ and $g : V(M) \to \{1, 2, 4, 7\}$ be bijections. Then, $f \cup g$ is a sum-free labelling of G_*. If $|V(H)| = 4$, let $f : V(M) \to \{3, 5, 6, 7\}$ and $g : V(H) \to \{1, 2, 4, 8\}$ be bijections. Then, $f \cup g$ is a sum-free labelling of G_*. Hence, we know that $|V(H)| \in \{5, 6\}$.

If $|V(H)| = 5$, then H has a vertex v with $d(v) \leqslant 2$. Let $\{w, z\} \subseteq V(H) - N[v]$, with $w \neq z$. Let $f : V(M) \to \{1, 2, 4, 7\}$ and $g : V(H) \to \{3, 5, 6, 8, 9\}$ be bijections, with $g(w) = 9$ and $g(z) = 8$. Then, $f \cup g$ is a sum-free labelling of G_*.

Finally, suppose that $|V(H)| = 6$. Let $v \in V(H)$ and let $\{w, z\} \subseteq V(H) - N[v]$, with $w \neq z$.. Let $f : V(M) \to \{1, 2, 4, 7\}$ and $g : V(H) \to \{3, 5, 6, 8, 9, 10\}$ be bijections, with $g(w) = 9$ and $g(z) = 8$. Then, $f \cup g$

is a sum-free labelling of G_*. This eliminates all possibilities and hence G_* cannot have K_3 as a subgraph. $\square$

Lemma 19. Neither $K_{3,3}$ nor $K_{2,3}$ is a connected component of G_*.

Proof. Suppose that $M \subseteq G_*$, and that $K_{2,3} \cong M$ or $K_{3,3} \cong M$. Let (X, Y) be a bipartition of M with $|X| = 3$, and let $H = G_* - V(M)$. If $|V(M)| = 5$, let $Q = \{n, n-1\}$ and $R = \{n, n-1, n-2, n-3, n-4\}$. If $|V(M)| = 6$, let $Q = \{n, n-1, n-2\}$ and $R = \{n, n-1, n-2, n-3, n-4, n-5\}$.

If $|V(H)| \geqslant 7$, then H has a sum-free labelling f and for any bijection $g : V(M) \to R$, $f \cup g$ is a sum-free labelling of G_*. If $|V(H)| = 1$, then $|V(M)| = 6$. Let $f : X \to \{2, 4, 6\}$, $g : Y \to \{3, 5, 7\}$ and $h : V(H) \to \{1\}$ be bijections. Then, $f \cup g \cup h$ is a sum-free labelling of G_*. Hence, we may assume that $2 \leqslant |V(H)| \leqslant 6$.

If $|V(H)| = k$, $2 \leqslant k \leqslant 5$, let $f : V(H) \to I_{3+k} - I_3$, $g : X \to \{1, 2, 3\}$, and $h : Y \to Q$ be bijections. Then, $f \cup g \cup h$ is a sum-free labelling of G_*.

In the remaining case, $|V(H)| = 6$. Let $v \in V(H)$ and let $w \in V(H) - N[v]$. Let $f : V(H) \to I_9 - I_3$ be a bijection such that $f(v) = 4$ and $f(w) = 9$. Let $g : X \to \{1, 2, 3\}$, and $h : Y \to \{n, n-1\}$ be bijections. Then, $f \cup g \cup h$ is a sum-free labelling of G_*. Hence G_* cannot have $K_{2,3}$ or $K_{3,3}$ as a connected component. $\square$

Lemma 20. G_* is connected and 3-colourable.

Proof. We note that G_* has no component isomorphic to any of K_3, K_4, $K_{2,3}$, $K_{3,3}$, and that G_* has at least one vertex of degree three. That G_* is 3-colourable follows directly from Brooks' theorem [3].

Now, suppose that $G_* = M \cup K$, where $|V(M) \cap V(K)| = 0$, and $|V(M)| \neq 0$ but, subject to this, with $m = |V(M)|$ as small as possible. Hence, $|V(M)| \leqslant |V(K)|$, and M is connected. If $|V(K)| \geqslant 7$, or if K has no vertex of degree three, then K has a sum-free labelling f. Let $g : V(M) \to I_n - I_{n-m}$ be a bijection. Then, $f \cup g$ is a sum-free labelling of G_*. Thus, we may assume that K has a vertex of degree three and that $4 \leqslant |V(K)| \leqslant 6$.

If $|V(K)| = 4$, then $|V(M)| \leqslant 4$. If $|V(M)| \geqslant 2$, use $I_n - \{3, 4, 5, 6\}$ on $V(M)$ and $\{3, 4, 5, 6\}$ on $V(K)$. If $|V(M)| = 1$, use $\{1\}$ on $V(M)$ and $\{2, 3, 4, 5\}$ on $V(K)$, assigning labels 2 and 5 to a pair of non-adjacent vertices.

If $|V(K)| = 5$, then $|V(M)| \leqslant 5$. If $|V(M)| = 5$, let v be a vertex of degree two or less in M and let $w, z \in V(M) - N[v]$, with $w \neq z$. Let $f : V(M) \to \{1, 2, 3, 9, 10\}$ such that $f(v) = 1$, $f(w) = 3$ and $f(z) = 10$. Let $g : V(K) \to \{4, 5, 6, 7, 8\}$ be a bijection. Then, $f \cup g$ is a sum-free labelling of G_*.

If $|V(K)| = 5$ and $|V(M)| = 4$, let v and w be non-adjacent vertices of M. Let $f : V(M) \to \{1, 2, 3, 9\}$ such that $f(v) = 1$, and $f(w) = 3$. Let $g : V(K) \to \{4, 5, 6, 7, 8\}$ be a bijection. Then, $f \cup g$ is a sum-free labelling of G_*.

If $|V(K)| = 5$ and $|V(M)| = 3$, let v and w be non-adjacent vertices of M. Let $f : V(M) \to \{1, 2, 3\}$ such that $f(v) = 1$, and $f(w) = 3$. Let $g : V(K) \to \{4, 5, 6, 7, 8\}$ be a bijection. Then, $f \cup g$ is a sum-free labelling of G_*.

If $|V(K)| = 5$ and $|V(M)| = 2$, let v and w be non-adjacent vertices of K. Let $f : V(M) \to \{1, 2\}$ and let $g : V(K) \to \{3, 4, 5, 6, 7\}$ be a bijection such that $g(v) = 3$ and $g(w) = 7$. Then, $f \cup g$ is a sum-free labelling of G_*.

If $|V(K)| = 5$ and $|V(M)| = 1$, let $v \in V(K)$, with $d(v) \leqslant 2$, and let $w, z \in V(K) - N[v]$, with $w \neq z$. Let $f : V(M) \to \{1\}$ and let $g : V(K) \to \{2, 3, 4, 5, 6\}$ be a bijection such that $g(v) = 2$, $g(w) = 5$ and $g(z) = 6$. Then, $f \cup g$ is a sum-free labelling of G_*.

In the remaining cases, $|V(K)| = 6$.

If $|V(K)| = 6$ and $|V(M)| = 1$, let $V(M) = \{q\}$, and let $v \in V(K)$, with $d(v)$ as small as possible. If $d(v) = 3$, then K is the prism $K_2 \times K_3$, with two 3-cycles joined by a matching. Label one 3-cycle $1, 2, 5$ and the other $3, 4, 6$, so that the vertices labelled 1 and 3 are neighbours, and the vertices 5 and 6 are neighbours. Then, label $V(M)$ with $\{7\}$. In the remaining cases, $d(v) \leqslant 2$. Let $\{x, y, z\} \subseteq V(H) - N[v]$, with $|\{x, y, z\}| = 3$. Suppose that there is a vertex w such that $w \notin \{v, x, y, z\}$ and w is not adjacent to x. Let $g : V(K) \to \{2, 3, 4, 5, 6, 7\}$ be a bijection such that $g(v) = 2$, $g(w) = 3$, $g(x) = 7$, $g(y) = 5$, $g(z) = 6$. Let $f : V(M) \to \{1\}$. Then, $f \cup g$ is a sum-free labelling of G_*. Hence, we may assume that x is adjacent to every vertex of $Y = V(H) - \{v, x, y, z\} = \{r, s\}$ and, similarly, we may assume that each of y and z is adjacent to every vertex of Y. At this point, $d(r) = d(s) = 3$ and, thus, $d(v) = 0$. We also note that $K[\{x, y, z\}]$ has at most one edge, since each of these vertices has two edges to $\{r, s\}$. We may therefore assume that $xz, yz \notin E(K)$. Then, $f = \{(q, 5), (s, 2), (r, 3), (x, 1), (y, 6), (z, 4), (v, 7)\}$ is a sum-free labelling of G_*.

If $|V(K)| = 6$ and $|V(M)| = 2$, let $v \in V(K)$, and let $w, z \in V(K) - N[v]$, with $w \neq z$. Let $f : V(M) \to \{1, 2\}$ and let $g : V(K) \to \{3, 4, 5, 6, 7, 8\}$ be a bijection such that $g(v) = 3$, $g(w) = 7$ and $g(z) = 8$. Then, $f \cup g$ is a sum-free labelling of G_*.

If $|V(K)| = 6$ and $|V(M)| = 3$, let $v \in V(K)$, and let $w \in V(K) - N[v]$. Since M is not K_3, there is a sum-free labelling f of M, and let $g : V(K) \to \{4, 5, 6, 8, 9, 10\}$ be a bijection such that $g(v) = 4$, and $g(w) = 9$. Then, $f \cup g$ is a sum-free labelling of G_*.

If $|V(K)| = 6$ and $|V(M)| = 4$, let $v \in V(K)$, and let $w, z \in V(K) - N[v]$, with $w \neq z$. Let $f : V(M) \to \{1, 2, 4, 7\}$ and let $g : V(K) \to \{3, 5, 6, 8, 9, 10\}$ be a bijection such that $g(v) = 3$, $g(w) = 8$ and $g(z) = 9$. Then, $f \cup g$ is a sum-free labelling of G_*.

If $|V(K)| = 6$ and $|V(M)| \in \{5, 6\}$, let $Q = \{4, 5, 8, 9, 10, 11\}$ and $R = I_n - Q \subseteq \{1, 2, 3, 6, 7, 12\}$. In K, we select non-adjacent vertices v and w, and let $g : V(K) \to Q$ be a bijection with $g(v) = 4$ and $g(w) = 9$. In M select a vertex v' of minimum degree. Then there are at least two distinct vertices w' and z' which are not adjacent to v'. Let $f : V(M) \to R$ be a bijection with $f(v') = 1$, $f(w') = 3$, and $f(z') = 7$. Then, $f \cup g$ is a sum-free labelling of G_*. This concludes the proof of the lemma. □

Lemma 21. Suppose that G is 3-regular, not K_4 or $K_{3,3}$, and suppose that $|V(G)| \leqslant 8$. Then, G is sum-free.

Proof. There are two non-isomorphic connected 3-regular graphs on 6 vertices and five non-isomorphic connected 3-regular graphs on eight vertices. (See, for example, [1].)

The two non-isomorphic connected 3-regular graphs on 6 vertices are $K_{3,3}$ and the prism $K_2 \times K_3$. We need only show that the prism has a sum-free labelling and this is easily shown by displaying the adjacency matrix M_6 of a labelled sum-free prism.

$$M_6 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Similarly, adjacency matrices M_{8a}, M_{8b}, M_{8c}, M_{8d}, M_{8e}, for the five non-isomorphic connected 3-regular graphs on eight vertices are shown below, completing the proof of this lemma. Each of these matrices corresponds to a labelled sum-free version of a graph.

$$M_{8a} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad M_{8b} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix},$$

$$M_{8c} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \ M_{8d} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix},$$

$$M_{8e} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$ □

Lemma 22. G_* has no vertex of degree one.

Proof. Suppose that v is a vertex of degree one, let $H = G_* - v$, and let q be the neighbour of v in G_*. If $|V(H)| \geqslant 7$, then H has a sum-free labelling f and, then, $f \cup \{(v, n)\}$ is a sum-free labelling of G_*. Hence, we may assume that $|V(H)| = 6$. Note that H is connected and $d(q) \leqslant 2$. Let x, y, z be distinct vertices which are not adjacent to q, and let r and s be the remaining two vertices of H. Since H is connected, we may assume that $qr \in E(H)$. But then, without loss of generality, $rx \notin E(H)$. Now, $f = \{(v,1), (q,2), (r,3), (s,4), (y,5), (z,6), (x,7)\}$ is a sum-free labelling of G_*. Thus G_* can have no vertex of degree one. □

Lemma 23. G_* cannot have two adjacent vertices of degree two.

Proof. Suppose that $v, w \in V(G_*)$, $vw \in E(G_*)$, $d(v) = d(w) = 2$, let $u, x \in V(G_* - \{v, w\})$ with $uv, wx \in E(G_*)$, and let $H = G_* - \{v, w\}$. If H is connected, let $M = H$ and if H is not connected, we note that $u \neq x$, and we let $M = H \cup \{ux\}$. In either case, M is a simple graph. If H has a sum-free labelling f, without loss of generality, we may assume that $f(u) \neq 1$. Now, $f \cup \{(v, n), (w, n-1)\}$ is a sum-free labelling of G_*. If $|V(H)| \geqslant 7$, then M has a sum-free labelling f, and this provides a sum-free labelling of H. In the remaining cases, $|V(H)| \in \{5, 6\}$.

Suppose that $|V(H)| = 5$. Let $z \in V(K) - N[u] - \{x\}$ and let $f : V(K) \to \{3,4,5,6,7\}$ be a bijection such that $f(u) = 3$, $f(x) = 4$ and $f(z) = 7$. Then, $f \cup \{(v,1), (w,2)\}$ is a sum-free labelling of G_*.

In the last case, $|V(H)| = 6$. Let $w, z \in V(K) - N[u] - \{x\}$ and let $f : V(K) \to \{3, 4, 5, 6, 7, 8\}$ be a bijection such that $f(u) = 3$, $f(x) = 4$, $f(w) = 8$, and $f(z) = 7$. Then, $f \cup \{(v, 1), (w, 2)\}$ is a sum-free labelling of G_*. Thus, G_* cannot have two adjacent vertices of degree two. □

Lemma 24. G_* cannot have two triangles which share an edge.

Proof. By Lemma 18, G_* has no K_4. Let M be a subgraph of G_* with $V(M) = \{w, x, y, z\}$ and with $E(M) = \{wy, wz, xy, xz, yz\}$. Then, $wx \notin V(G_*)$. Let $H = G_* - V(M)$. If H has a sum-free labelling g, then $g \cup \{(w, n-3), (x, n-2), (y, n-1), (z, n)\}$ is a sum-free labelling of G_*. We need only consider the cases in which H does not have a sum-free labelling. Note that H is not 3-regular. We may assume that w has a neighbour w' in H. If x also has a neighbour in H, we label this neighbour x'. (Possibly, $x' = w'$)

If $|V(G_*)| \geqslant 11$, then $|V(H)| \geqslant 7$, H has a sum-free labelling.

If $|V(G_*)| = 10$, then $|V(H)| = 6$. There is no harm in assuming x' exists. (If it doesn't, pick some arbitrary vertex of H and call it x'.) Suppose that $x' \neq w'$. Let $f : V(H) \to \{4, 5, 6, 7, 8, 9\}$ be any bijection with $f(w') = 4$ and $f(x') = 6$, and let $g : V(M) \to \{1, 2, 3, 10\}$ with $g(z) = 10$, $g(y) = 1$, $g(w) = 2$, $g(x) = 3$. Then, $f \cup g$ is a sum-free labelling of G_*. The same argument holds if $x' = w'$, with the exception that now $f(w') = f(x') = 4$.

If $|V(G_*)| = 9$, then $|V(H)| = 5$. As above, there is no harm in assuming x' exists. If $x' \neq w'$, let $f : V(H) \to \{4, 5, 6, 7, 8\}$ be any bijection with $f(w') = 4$ and $f(x') = 5$, and let $g : V(M) \to \{1, 2, 3, 9\}$ by a bijection with $g(z) = 9$ and $g(y) = 1$. Then, $f \cup g$ is a sum-free labelling of G_*. The same argument holds if $x' = w'$, with the exception that now $f(w') = f(x') = 4$.

If $|V(G_*)| = 8$, then $|V(H)| = 4$. By Lemma 15, we may assume that $|E(H)| \geqslant 5$, and by Lemma 18, $|E(H)| \leqslant 5$. Thus G_* consists of two copies of $K_4 - e$, linked by one or two independent edges. There is no harm in assuming both edges appear. Let y' and z' denote the two vertices in G_* which do not yet have names. Then,

$$\left\{(w, 1), (x, 5), (y, 4), (z, 8), \left(x', 6\right), \left(w', 7\right), \left(y'2\right), \left(z', 3\right)\right\}$$

is a sum-free labelling of G_*.

If $|V(G_*)| = 7$, then $|V(H)| = 3$. Since H has no sum-free labelling, $H \cong K_3$. Let $V(H) = \{w', u, v\}$. Then,

$$\left\{(u, 1), (v, 2), \left(w', 4\right), (w, 5), (x, 3), (y, 7), (z, 6)\right\}$$

is a sum-free labelling of G_*. □

Lemma 25. G_* has no triangles.

Proof. Let T be a triangle in G_*, $V(T) = \{x, y, z\}$ and $E(T) = \{xy, xz, xy\}$. Let $M = G_* - V(T)$. By Lemma 24, no vertex of M is adjacent to more than one vertex of T. If a vertex t of T has a neighbour in M, we call that neighbour t'. In the arguments that follow, if some vertex t of T does not have a neighbour in M, then t' could be considered to be any vertex of M that is not already used for one of the other vertices of T. (Thus, we can avoid breaking into cases depending on the number of neighbours of T.)

Suppose that $|V(G_*)| \geqslant 10$. Let g be any sum-free labelling of $G_* - \{x, y, z\}$. Without loss of generality, $g(x') \notin \{1, 2\}$, and $g(y') \neq 1$. Then, $g \cup \{(x, n), (y, n-1), (z, n-2)\}$ is a sum-free labelling of G_*.

In the remaining three cases, let $f(x) = 1$, $f(y) = 2$, $f(z) = 3$, $f(x') = 4$, $f(y') = 5$, $f(z') = 6$.

If $|V(G_*)| = 9$, then $|V(M)| = 6$. There are three vertices in M which have not yet been labelled. At most two of these are in $N(x') \cap N(y')$. Let $q \in V(M) - \{x', y', z'\} - N(x') \cap N(y')$. Let $f(q) = 9$, and assign the labels 7, 8 to the remaining two vertices of G_*, one to each vertex. Then, f is a sum-free labelling of G_*.

If $|V(G_*)| = 8$, then $|V(M)| = 5$. Assign the labels 7, 8 to the remaining two vertices of G_*, one to each vertex. Then, f is a sum-free labelling of G_*.

If $|V(G_*)| = 7$, then $|V(M)| = 4$. Assign the label 7 to the remaining vertex of G_*. Then, f is a sum-free labelling of G_*. $\square$

Lemma 26. G_* has no 4-cycles.

Proof. By Lemma 25, any 4-cycle of G_* is induced subgraph. Let $C = x_1x_2x_3x_4x_1$ be a 4-cycle in G_*,, and let $M - G_* - V(C)$. As above, let x'_j denote the neighbour of x_j in M, if such exists. Note that each x'_j is adjacent to at most vertex in C. If x_j has no neighbour in M, and $|V(G_*)| \geqslant 7$, we select x'_j as in the previous lemma.

Suppose that $|V(M)| \geqslant 7$. Then, M has a sum-free labelling f. At most one vertex of $N = \{x'_1, x'_2, x'_3, x'_4\}$ receives label 1 and at most one receives label 2. If 1 appears as label on a vertex on N, then, by rotating the labels on the cycle, we may assume that $f(x'_3) = 1$. If $f(x'_2) = 2$, replace C, by $x_1x_4x_3x_2x_1$ and relabel, so that now $f(x'_4) = 2$, and if $f\left(x'_j\right) = 1$, then $j = 3$. Now, extend f by setting $f(x_1) = n$, $f(x_2) = n-1$, $f(x_3) = n-3$, and $f(x_4) = n-2$. Now, f is a sum-free labelling of G_*.

Now, suppose that $4 \leqslant |V(M)| \leqslant 6$. Let $f(x_1) = 1$, $f(x_2) = 3$, $f(x_3) = 4$, $f(x_4) = 2$, $f(x'_1) = 5$, $f(x'_2) = 6$, $f(x'_3) = 7$, $f(x'_4) = 8$, and assign the remaining labels from $I_n - I_8$ to the remaining vertices, one to each vertex. Then, f is a sum-free labelling of G_*.

The remaining case has $|V(M)| = 3$. Relabel C so that x_4 has no neighbour in M, and choose $N = \{x'_1, x'_2, x'_3\}$ as these have been previously chosen. The labelling for $|V(M)| = 4$, with the exception of labelling the now nonexistent x'_4 is a sum-free labelling of G_*. □

Lemma 27. G_* has no cycles.

Proof. Assume that G_* has a cycle, and let $C = x_1x_2 \cdots x_kx_1$ be a shortest cycle. We may therefore assume that C has no chords, and that $k \geqslant 5$. Let $M = G_* - V(C)$. We use x'_j as above: $x'_j \in V(M) \cap N(x_j)$, if possible; otherwise, we do not define x'_j. (There may not be enough room in M to allow for distinct x'_j for each x_j. When an undefined x'_j appears in a set, treat it as not being part of the set.)

Suppose that $|V(M)| \geqslant 7$. Then, M has a sum-free labelling f. At most one vertex of $N = \{x'_1, x'_2, \cdots, x'_k\}$ receives label 1 and at most one receives label $k-1$. We relabel the vertices of C, if necessary, so that $f(x'_1) \neq k-1$ and $f\left(x'_j\right) \neq 1$ for $1 \leqslant j \leqslant k-1$. Now, assign $f(x_j) = n+1-j$, for $1 \leqslant j \leqslant k$. Then, f is a sum-free labelling of G_*.

Now, suppose that $|V(M)| \leqslant 6$. Let f be a sum-free labelling of C. If x'_j exists, and if $f(x_j) + k \leqslant n$, then set $f\left(x'_j\right) = f(x_j) + k$. Arbitrarily assign the remaining labels to the remaining vertices of M, one to each vertex. Note that if $x'_jw \in E(M)$, then $f(x_j) + f(w) \neq f\left(x'_j\right)$, since $f(w) \neq k$. If $w_1, w_2, w_3 \in V(M)$, then $f(w_1) + f(w_2) \geqslant 2k+2 = k+(k+2) > k + |V(M)| = n \geqslant f(w_3)$. Hence, f is a sum-free labelling of G_*. □

Theorem 28. G_* cannot exist. Thus, every graph with maximum degree at most three and with at least seven vertices is sum-free.

Proof. By Lemma 27, G_* has no cycles. By Lemma 20, G_* is connected. Hence, G_* is a tree (on at least seven vertices). By Lemma 13, G_* is sum-free. □

Further Directions. Proving 3-regular graphs are almost all sum-free seemed a natural goal. But, there may be a stronger result. From Lemma 4, we know that there are some graphs with n vertices and $2n-4$ edges but which are not sum-free. One might hope that these are the extremal non-sum-free graphs.

Conjecture 29. If $|E(G)| \leqslant 2|V(G)| - 5$, then G is sum-free.

Lemmas 15 and Theorem 28 are weaker forms of this.

Computational Results. Degree sequences of L_n, $1 \leqslant n \leqslant 9$. Here, for example, $3^2 4^5 6^1$ means 2 vertices of degree 3, 5 vertices of degree 4 and 1 vertex of degree 6. Note that $r(8,5) = 4$ and not 3 as we might have hoped.

$L_1 : 0^1$

$L_2 : 1^2$

$L_3 : 1^2 2^1$

$L_4 : 1^1 2^2 3^1,\ 2^4$

$L_5 : 1^1 2^2 3^1 4^1,\ 1^1 2^1 3^3,\ 2^4 4^1,\ 2^3 3^2$

$L_6 : 1^1 3^4 5^1,\ 1^1 3^3 4^2,\ 2^3 3^1 4^1 5^1,\ 2^2 3^3 5^1,\ 2^2 3^2 4^2,\ 2^1 3^4 4^1,\ 3^6$

L_7 : $1^2 3^3 4^2 6^1$, $1^1 3^3 4^1 5^2$, $1^1 3^2 4^3 5^1$, $1^1 3^1 4^5$, $2^2 3^3 5^1 6^1$, $2^2 3^2 4^2 6^1$, $2^2 3^2 4^1 5^2$, $2^2 3^1 4^3 5^1$, $2^1 3^4 4^1 6^1$, $2^1 3^4 5^2$, $2^1 3^3 4^2 5^1$, $2^1 3^2 4^4$, $3^6 6^1$, $3^5 4^1 5^1$, $3^4 4^3$,

L_8 : $1^1 3^1 4^4 5^1 7^1$, $1^1 3^1 4^4 6^2$, $1^1 3^1 4^3 5^2 6^1$, $1^1 3^1 4^2 5^4$, $1^1 4^6 7^1$, $1^1 4^5 5^1 6^1$, $1^1 4^4 5^3$, $2^2 4^4 5^1 7^1$, $2^2 4^4 6^2$, $2^2 4^3 5^2 6^1$, $2^2 4^2 5^4$, $2^1 3^3 4^2 6^1 7^1$, $2^1 3^3 4^1 5^2 7^1$, $2^1 3^3 4^1 5^1 6^2$, $2^1 3^2 4^3 5^1 7^1$, $2^1 3^2 4^3 6^2$, $2^1 3^2 4^2 5^2 6^1$, $2^1 3^2 4^1 5^4$, $2^1 3^1 4^5 7^1$, $2^1 3^1 4^4 5^1 6^1$, $2^1 3^1 4^3 5^3$, $2^1 4^6 6^1$, $2^1 4^5 5^2$, $3^5 4^1 6^1 7^1$, $3^4 4^2 5^1 7^1$, $3^4 4^2 6^2$, $3^4 4^1 5^2 6^1$, $3^3 4^4 7^1$, $3^3 4^3 5^1 6^1$, $3^3 4^2 5^3$, $3^2 4^5 6^1$, $3^2 4^4 5^2$, $3^1 4^6 5^1$, 4^8

L_9 : $1^1 3^1 4^3 5^2 6^1 8^1$, $1^1 3^1 4^3 5^2 7^2$, $1^1 3^1 4^3 5^1 6^2 7^1$, $1^1 3^1 4^2 5^4 8^1$, $1^1 3^1 4^2 5^3 6^1 7^1$, $1^1 3^1 4^2 5^2 6^3$, $1^1 3^1 4^1 5^5 7^1$, $1^1 3^1 4^1 5^4 6^2$, $1^1 4^5 5^1 6^1 8^1$, $1^1 4^5 5^1 7^2$, $1^1 4^4 5^3 8^1$, $1^1 4^4 5^2 6^1 7^1$, $1^1 4^4 5^1 6^3$, $1^1 4^3 5^4 7^1$, $1^1 4^3 5^3 6^2$, $1^1 4^2 5^5 6^1$, $1^1 4^1 5^7$, $2^2 4^3 5^2 6^1 8^1$, $2^2 4^3 5^2 7^2$, $2^2 4^3 5^1 6^2 7^1$, $2^2 4^2 5^4 8^1$, $2^2 4^2 5^3 6^1 7^1$, $2^2 4^2 5^2 6^3$, $2^2 4^1 5^5 7^1$, $2^2 4^1 5^4 6^2$, $2^1 3^2 4^3 5^1 7^1 8^1$, $2^1 3^2 4^3 6^2 8^1$, $2^1 3^2 4^3 6^1 7^2$, $2^1 3^2 4^2 5^2 6^1 8^1$, $2^1 3^2 4^2 5^2 7^2$, $2^1 3^2 4^2 5^1 6^2 7^1$, $2^1 3^2 4^2 6^4$, $2^1 3^2 4^1 5^4 8^1$, $2^1 3^2 4^1 5^3 6^1 7^1$, $2^1 3^2 4^1 5^2 6^3$, $2^1 3^1 4^5 7^1 8^1$, $2^1 3^1 4^4 5^1 6^1 8^1$, $2^1 3^1 4^4 5^1 7^2$, $2^1 3^1 4^4 6^2 7^1$, $2^1 3^1 4^3 5^3 8^1$, $2^1 3^1 4^3 5^2 6^1 7^1$, $2^1 3^1 4^3 5^1 6^3$, $2^1 3^1 4^2 5^4 7^1$, $2^1 3^1 4^2 5^3 6^2$, $2^1 3^1 4^1 5^5 6^1$, $2^1 4^6 6^1 8^1$, $2^1 4^6 7^2$, $2^1 4^5 5^2 8^1$, $2^1 4^5 5^1 6^1 7^1$, $2^1 4^5 6^3$, $2^1 4^4 5^3 7^1$, $2^1 4^4 5^2 6^2$, $2^1 4^3 5^4 6^1$, $2^1 4^2 5^6$, $3^4 4^2 5^1 7^1 8^1$, $3^4 4^2 6^2 8^1$, $3^4 4^2 6^1 7^2$, $3^4 4^1 5^2 6^1 8^1$, $3^4 4^1 5^2 7^2$, $3^4 4^1 5^1 6^2 7^1$, $3^3 4^4 7^1 8^1$, $3^3 4^3 5^1 6^1 8^1$, $3^3 4^3 5^1 7^2$, $3^3 4^3 6^2 7^1$, $3^3 4^2 5^3 8^1$, $3^3 4^2 5^2 6^1 7^1$, $3^3 4^2 5^1 6^3$, $3^3 4^1 5^4 7^1$, $3^3 4^1 5^3 6^2$, $3^2 4^5 6^1 8^1$, $3^2 4^5 7^2$, $3^2 4^4 5^2 8^1$, $3^2 4^4 5^1 6^1 7^1$, $3^2 4^4 6^3$, $3^2 4^3 5^3 7^1$, $3^2 4^3 5^2 6^2$, $3^2 4^2 5^4 6^1$, $3^2 4^1 6^6$, $3^1 4^6 5^1 8^1$, $3^1 4^6 6^1 7^1$, $3^1 4^5 5^2 7^1$, $3^1 4^5 5^1 6^2$, $3^1 4^4 5^3 6^1$, $3^1 4^3 5^5$, $4^8 8^1$, $4^7 5^1 7^1$, $4^7 6^2$, $4^6 5^2 6^1$, $4^5 5^4$

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Received 20 January 2015