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Cycle double covers and long circuits of graphs

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The 5-Cycle Double Cover Conjecture claims that every bridgeless graph has a cycle double cover which consists of at most 5 cycles. In this paper, we prove that if a cubic graph has a long circuit, then it has a 5-cycle double cover. Our main theorem partially strengthens some previously known results.

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1. Introduction

We follow [22] for notations and terminology not defined in this paper. A connected 2-regular graph is called a *circuit* and a *cycle* is a graph with even degree for each vertex. A bridgeless cubic graph is a *snark* if it is not edge-3-colorable.

A family $\mathcal{F}$ of cycles of a graph G is called a *cycle double cover* of G if each edge of G is contained in exactly two cycles of $\mathcal{F}$. A *k-cycle double cover* of G is a cycle double cover consisting of at most k cycles.

The following Cycle Double Cover Conjecture, due to Szekeres and Seymour, is one of the most famous open problems in graph theory.

Conjecture 1.1 (Szekeres [16] and Seymour [15]). *Every bridgeless graph has a cycle double cover.*

The Strong Cycle Double Cover Conjecture, due to Seymour, allows one to include any specified circuit in the cover.

Conjecture 1.2 (Seymour, see [4] p. 237, and [5] and also [7]). *For every bridgeless G and every circuit C of G, there is a cycle double cover $\mathcal{F}$ of G with $C \in \mathcal{F}$.*

The 5-Cycle Double Cover Conjecture was proposed by Celmins and Preissmann:

Conjecture 1.3 (Celmins [3] and Preissmann [14]). *Every bridgeless graph has a 5-cycle double cover.*

Most recently, Hoffmann-Ostenhof proposed the Strong 5-Cycle Double Conjecture for cubic graphs:

Conjecture 1.4 (Hoffmann-Ostenhof [11]). *Let G be a bridgeless cubic graph and C be a circuit of G. Then G has a 5-cycle double cover $\mathcal{F}$ with $C \subseteq C_1 \in \mathcal{F}$.*

Moreover, Hoffmann-Ostenhof got a necessary and sufficient condition for cubic graphs to have 5-cycle double covers and also for cubic graphs to have strong 5-cycle double covers. These results are further generalized to bridgeless graphs by Xu [19].

Theorem 1.5 (Hoffmann-Ostenhof [11]). *Let G be a cubic graph and C_0 be a cycle of G. Then G has a 5-cycle double cover if and only if G has a matching M such that $G-M$ has a nowhere-zero 4-flow and G contains two cycles C_1 and C_2 with $E(C_1) \cap E(C_2) = M$.*

The Cycle Double Cover Conjecture has been verified for many families of graphs, such as, Petersen-minor free graphs [1], triangularly connected graphs [18], graphs with hamiltonian paths [7] and graphs with certain spanning subgraphs [8, 9, 21]. For 5-Cycle Double Cover Conjecture, there are only a few known results. Cubic graphs with a hamiltonian path and cubic graphs with oddness two have been verified to have 5-cycle double covers [12]. Readers are referred to [23] for a comprehensive discussion on circuit covers of graphs. Recently, Cycle Double Cover Conjecture has been studied for bridgeless cubic graphs with long circuits.

Theorem 1.6 (Fleischner and Häggvist [6]). *Let G be a bridgeless cubic graph with a circuit C. If the length of C is at least $n-4$ and $G-C$ is connected, then G has a cycle double cover containing C.*

Theorem 1.7 (Ye and Zhang [20]). *Let G be a bridgeless cubic graph with a circuit of length at least $n-7$. Then G has a cycle double cover.*

The above result is improved by Hägglund and Markström [10] to $n-9$, and later by Brinkmann et al. [2] as the following. Both of their proofs relied on computer search.

Theorem 1.8 (Brinkmann et al. [2]). *Let G be a bridgeless cubic graph with a circuit C of length at least $n-10$, then G has a cycle double cover.*

Theorem 1.9 (Brinkmann et al. [2]). *Let G be a bridgeless cubic graph with a circuit C of length at least $n-8$, then G has a 6-cycle double cover.*

In this paper, we study 5-cycle double covers of bridgeless cubic graphs with a long circuit. The following is our main result which partially strengthens Theorem 1.7, Theorem 1.8 and Theorem 1.9.

Theorem 1.10. *Let G be a bridgeless cubic graph with n vertices. If G has a circuit C of length at least $n-3$, then G has a 5-cycle double cover.*

2. Proof of the main result

Let G be a cubic graph. An edge cut S is *cyclic* if S separates G into two components each of which contains a circuit. Brinkmann et al. got the following result.

Proposition 2.1 (Brinkmann et al. [2])**.** *Let G be a snark and P be the Petersen graph. Then:*
(1) $|V(G)| \geq 10$ and P is the unique snark of 10 vertices;
(2) G has a cyclic edge-cut of size at most three if $G \neq P$ and $|V(G)| < 18$.

By Proposition 2.1, we know that Petersen graph is the only snark of size at most 16 without a cyclic 3-edge-cut.

Proposition 2.2. *Let P be the Petersen graph. Then P has no circuit of length 7 and every circuit of length at least 8 has at least two chords.*

Proof. Let P be the Petersen graph. Then the girth of P is five. Let C be a circuit of P.

First, we show that $|C| \neq 7$. Otherwise, if $|C| = 7$, then C has no chord since any chord of C will create a circuit of length at most four, this is impossible because the girth of P is 5. Let $E_P(C, P-C)$ be the edges joining vertices of $V(C)$ to vertices of $V(P)-V(C)$. Then $|E_P(C, P-C)| = 7$. Note that $2|E(P-C)| + |E_P(C, P-C)| = 3|V(P-C)| = 9$. That implies that $E(P-C)$ has only one edge. Then $P-C$ has an isolated vertex v. Assume the neighbors of v on C are v_0, v_1 and v_2 in clockwise order. Let $P_{i,j}$ be the path of C joining v_i and v_j but not containing v_k where $k \in \mathbb{Z}_3 \setminus \{i,j\}$. Then one of $|P_{i,j}| \leq 2$ since $|P_{0,1}| + |P_{1,2}| + |P_{2,0}| = 7$, say $|P_{0,1}|$. Then $P_{01} + v_1v + vv_0$ is a circuit of length at most four, a contradiction. This implies that P does not contain a circuit of length 7.

Now assume $|C| \geq 8$. Note that P does not have a hamiltonian circuit. So $8 \leq |C| \leq 9$. If $|C| = 8$, a similar argument as above will show that $P-C$ has an edge. And hence $|E_P(C, P-C)| = 4$ when $|C| = 8$, and $|E_P(C, P-C)| = 3$ when $|C| = 9$. So the number of chords of C is

$$\frac{|C| - |E_P(C, P-C)|}{2} \geq 2.$$

This completes the proof. □

There is a close relationship between nowhere-zero 4-flows and cycle double covers of graphs.

Lemma 2.3 (See Theorem 3.5.6 in [22])**.** *Let G be a graph and C_0 be a cycle of G. Then G admits a nowhere-zero 4-flow if and only if G has a 4-cycle double cover $\mathcal{F}$ such that $C_0 \in \mathcal{F}$.*

The following lemma is well-known.

Lemma 2.4. (a) (Tutte [17]) *A bridgeless cubic graph admits a nowhere-zero 4-flow if and only if it is edge-3-colorable.*

(b) (Jaeger [13]) *If a graph G is Hamiltonian, the G admits a nowhere-zero 4-flow.*

Now, we are ready to prove the main theorem.

Proof of Theorem 1.10. By Lemma 2.4 and Lemma 2.3, we may assume that G is a snark. If $|C| = n$, then G is Hamiltonian. By Lemma 2.4, G is edge-3-colorable, a contradiction. So we may assume that $n-3 \leq |C| \leq n-1$. For convenience, for any graph H, we use $\overline{H}$ for the graph obtained from H by suppressing all vertices of degree two.

Let M be the set of all chords of C. Then M is a matching since G is cubic. Let $G_0 := C \cup M$. Then $\overline{G_0}$ is a cubic graph with a Hamiltonian circuit C_0 obtained from C. By Lemma 2.4, $\overline{G_0}$ has an edge-3-coloring $\overline{c_0} : E(\overline{G_0}) \to \mathcal{Z}_3$ such that $M := \overline{c_0}^{-1}(0)$. The coloring $\overline{c_0}$ of $\overline{G_0}$ can be extended to an edge-coloring c_0 of G_0 such that every colored edge e of $\overline{G_0}$ is a monotone-colored path of G_0 which is the subdivision of e. Let $C_i := c_0^{-1}(i) \cup M$ for $i = 1$ and 2. Then C_1, C_2 are two cycles of G with $E(C_1) \cap E(C_2) = M$. Clearly, $\{C_1, C_2\}$ covers each edge of C exactly once and each edge of M exactly twice.

By Theorem 1.5 and Lemma 2.4-(a), it suffices to prove that $G - M$ has a nowhere-zero 4-flow, i.e. suppressed cubic graph $G' = \overline{G - M}$ is edge-3-colorable. Suppose, to the contrary, that G' is not edge-3-colorable. Note that $|C| \geq n - 3$ and C is chordless. It follows that $|G'| \leq 3 + 3 * 3 = 12$.

Claim: *G' is not the Petersen graph.*

Suppose that G' is the Petersen graph. Since C is chordless, C has length at most six by Proposition 2.2. So $|V(G')| \leq |C| + 3 = 9$. This contradicts the fact that Petersen graph has ten vertices.

By the Claim and Proposition 2.1, G' has a cyclic edge-cut S of size at most three. Note that G' is bridgeless, then $2 \leq |S| \leq 3$. Let G_1 and G_2 be two components of $G' - S$. If $S \cap E(C) = \emptyset$, assume that $C \subseteq G_1$. Since S is a cyclic edge-cut, G_2 contains a circuit. Note that $|C| \geq n-3$, then $|V(G_2)| \leq 3$. It follows that C is a spanning subgraph of G_1, $|V(G_2)| = 3$ and G_2 is a triangle. This implies that G' has exactly 6 vertices. By Propositon 2.1, G' is not a snark, a contradiction.

Hence $S \cap E(C) \neq \emptyset$. Since G' is not 3-edge-colorable, at least one of $\overline{G'/G_1}$ and $\overline{G'/G_2}$ is not 3-edge-colorable, say $\overline{G'/G_2}$. Since S is a cyclic edge-cut, and G_2 has a circuit and hence has at least three vertices.

If $|S| = 2$, then $|V(\overline{G'/G_2})| = |V(G')| - |V(G_2)| \leq 12 - 3 = 9$. By Proposition 2.1, $\overline{G'/G_2}$ is 3-edge-colorable, a contradiction.

So we may assume that $|S| = 3$. Then G'/G_2 has no vertices of degree two and hence $\overline{G'/G_2} = G'/G_2$. It follows that $|V(\overline{G'/G_2})| = |V(G')| - |V(G_2)| + 1 \leq 10$. By Proposition 2.1, G_2 has exactly 3 vertices (thus a triangle) and $\overline{G/G_2}$ is a snark with 10 vertices. By Proposition 2.1, $\overline{G/G_2}$ is the Petersen graph.

Let C' be the circuit of C by contracting the part of G_2. Then C' has at most one chord which is the edge in $S \setminus E(C)$, a contradiction to Proposition 2.2. This completes the proof. □

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