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A note on the circuit double cover of infinite graphs

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To Adrian who remains ageless (though finite)

We consider an infinite analogue of the cycle double cover conjecture. After a brief survey, we make some new observations and propose a slightly less restrictive version of the conjecture.

Keywords and phrases: Cycle double cover, nowhere-zero 4-flow, circuit double cover, allowed subgraph double cover.

1. Introduction

We begin with some definitions and notation and state some previous results that will be needed (theorems with letter names). For terminology and notation not defined, see, for example, [5]. Lower case k and d are positive integers throughout and, as set theorists know, ω is the first infinite ordinal, hence the first infinite cardinal, and can be thought of as $\mathbb{N}$, the set of natural numbers (0 is natural). A locally finite graph is an infinite graph in which every vertex has a finite degree. A ray is a connected graph with exactly one vertex of degree one, called the origin of the ray, and all other vertices of degree two. A v-ray, R_v, is a ray whose origin is the vertex v. A circuit is a non-empty connected 2-regular graph. A finite circuit is called a cycle and an infinite one a double ray. A bridge in a connected graph is an edge whose removal disconnects the graph; a trivial bridge is an edge with an end vertex of degree 1. A graph is even if all its vertices have even or infinite degrees (not to be thought of as a counterpart to odd components which have and odd number of vertices each).

A cycle double cover, (CDC), of a finite graph G is a collection of its cycles (not necessarily distinct) such that each edge of G lies in exactly two of the cycles. A k-CDC of a finite graph G is a collection $\mathcal{C}$ of its even subgraphs such that each edge of G lies in exactly two of the subgraphs and $|\mathcal{C}| = k$. A bridgeless graph admits a strong CDC if for every cycle C of G,

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there exists a CDC containing C. The following well known conjecture has been the subject of much research. It is often attributed to Seymour and, independently, to Szekeres, but had been made before, perhaps by Tutte, and is sometimes referred to as "folklore". Given Zhang's book [20], we will not give here the history and the many related problems and results beyond those needed here.

Conjecture 1. (Finite cycle double cover conjecture) *Every finite bridgeless graph has a* CDC*.*

Stronger conjectures were made by Seymour.

Conjecture 2. (Strong cycle double cover conjecture) *Every finite bridgeless graph admits a strong* CDC*.*

Let $G = (V, E)$ be a finite graph and let $p : E \longrightarrow \mathbb{N}$ be a count function. A p-faithful cycle cover of G is a family of cycles such that each edge e lies in $p(e)$ of the cycles. If $p(e)$ is even for all $e \in E$, call p *even.* For $F \subseteq E$, set $p(F) = \sum_{f \in F} p(f)$. Call p admissible if for each finite cut F, $p(F)$ is even and $p(f) \leq \frac{p(F)}{2}$ for each $f \in F$.

Conjecture 3. (Faithful cycle cover conjecture) *Let $G = (V, E)$ be a finite graph and $p : E \longrightarrow \mathbb{N}$ be an even admissible function. Then G has a p-faithful cycle cover.*

The concept of cycle double cover is related to that of nowhere-zero flow in graphs. Let G be a finite simple graph and (D, f) be an ordered pair where D is an orientation of $E(G)$ and $f : E(G) \longrightarrow \mathbb{Z}$ is a weight function. For each $v \in V(G)$, denote

$$f^+(v) = \sum_{vu \in D} f(vu) \qquad \text{and} \qquad f^-(v) = \sum_{uv \in D} f(uv).$$

An integer flow of G is an ordered pair (D, f) such that for every vertex $v \in V(G)$, $f^+(v) = f^-(v)$. A nowhere-zero k-flow of G is an integer flow (D, f) such that $0 < |f(e)| < k$, for every edge $e \in E(G)$ and is denoted by k-NZF [18].

Theorem A. [14] *If every edge of a finite graph G is contained in a cycle of length at most* 4*, then G admits a* 4-NZF*.*

Theorem B. [19] *A finite graph G admits a* 4-NZF *if and only if for each cycle C of G, G has a* 4-CDC *containing C.*

Theorem C. [19] *Let G be a graph admitting a* 4-NZF *and e be an edge of G such that $G' = G \setminus \{e\}$ is bridgeless. Then G' has a* 5-CDC*.*

Theorem D. [15] *Let G be a bridgeless graph with $o = o(G)$ odd degree vertices. Then we can add at most $\left\lceil \frac{1}{2} \left\lfloor \frac{o}{5} \right\rfloor \right\rceil$ edges such that the new graph admits a* 4-NZF*.*

By Theorems C and D, we have the following corollary.

Corollary 1. *Let G be a bridgeless graph with $o = o(G)$ odd degree vertices.*

- *If $o(G) = 2$ or 4 then G admits a* 4-NZF*.*
- *If $6 \leq o(G) \leq 14$ then G has a* 5-CDC*.*

2. Infinite graphs

The CDC conjecture has also been considered for infinite graphs, often for circuits rather than cycles. In finite graphs, *bridgeless* is equivalent to *every edge lies in a cycle*, but of course this is not the case for infinite graphs. The notion needed is *every edge lies in a circuit.*

Definition 1. [4] *A* circuit double cover*, $\mathcal{C}$, of an infinite graph G, is a collection of its circuits such that every edge of G is covered exactly twice by $\mathcal{C}$.*

Laviolette observed that if the edge set of a graph can be partitioned so that each class of the partition induces a bridgeless connected graph then it is sufficient to consider these induced subgraphs. He defined a decomposition of a graph G as an equivalence relation on the set $E(G)$ of its edges such that each class of the relation induces a connected subgraph. These subgraphs are called fragments. A decomposition whose fragments are all κ-edge-connected for some (finite or infinite) cardinal κ, is said to be κ-edge-connected, and a decomposition whose fragments are all of cardinality less than or equal to α, for some infinite cardinal α, is called an α-decomposition.

Laviolette, Corollary 3.2 of [13], proved that a bridgeless graph can always be decomposed into edge-disjoint bridgeless countable subgraphs.

Theorem E. [13] *Every bridgeless graph admits a 2-edge-connected ω-decomposition.*

A bond is non-empty edge cut which is minimal with respect to inclusion.

Definition 2. *An ω-decomposition $\mathcal{D}$ of G is said to be* bond-faithful *if*
(i) *any countable bond of G is contained in some fragment of $\mathcal{D}$;*
(ii) *any finite bond of a fragment of $\mathcal{D}$ is also a bond in G.*

Theorem F. [13] *Every graph has a bond-faithful ω-decomposition.*

This leads to the following.

Theorem G. [11] *If every countable bridgeless graph has a circuit double cover, then so does every infinite bridgeless graph.*

The following result extends that of the finite case.

Theorem H. [11] *If every infinite* 3*-regular bridgeless graph has a circuit double cover, then so does every infinite bridgeless graph.*

Locally finite graphs were investigated by Bruhn et al. mostly in the context of the p-faithful cycle cover conjecture.

Conjecture 4. Infinite faithful cycle cover conjecture *Let $G = (V, E)$ be a graph and $p : E \longrightarrow \mathbb{N}$ be even. Suppose further such that for each finite cut F, $p(F)$ is even and $p(f) \leq \frac{p(F)}{2}$ for each $f \in F$. Then G has a p-faithful circuit cover.*

Observe that if one of the conjectures is true for infinite graphs, it is also true for finite graphs. Indeed, it is enough that the conjecture hold for countably infinite graphs. To see this, suppose that a cover conjecture holds for infinite graphs. Given a finite graph G, fix a vertex u in it and create an infinite graph by identifying the vertices u in infinitely many otherwise disjoint copies of G. A required cover of the infinite graph induces one of G. Bruhn, Diestel, Stein show that one can go the other way for locally finite infinite graphs for the infinite faithful cycle cover conjecture.

Theorem I. [4] *Let G be a locally finite graph and $p : E(G) \longrightarrow \mathbb{N}$ an even admissible count function. If the finite faithful cover conjecture is true then G has a p-faithful cycle cover.*

The infinite faithful cover conjecture fails if *circuit* is replaced by *cycle* for infinite locally finite graphs but is equivalent to the finite version if double rays are allowed. It is curious that no example is known at this time where double rays are really needed for a cycle double cover to exist in an infinite graph.

Bruhn, Diestel, Stein also extended results for finite graphs of Alspach, Goddyn, Zhang. [1]. Take any subgraph of an infinite graph G, and contract some (possibly infinitely many) of its edges; the resulting graph will be called a minor of G.

Theorem J. [4] *A locally finite graph that has no Petersen minor, with an admissible count function p has a p-faithful circuit cover.*

Theorem K. [4] *If every finite bridgeless graph has a cycle double cover, then so does every infinite locally finite bridgeless graph.*

Theorem L. [4] *Let G be an infinite bridgeless locally finite graph not containing the Petersen graph as a minor. Then G has a circuit double cover.*

It is now tempting to make the following conjecture.

Conjecture 5. (Infinite circuit double cover conjecture) *Every infinite bridgeless graph has a circuit double cover.*

A circuit double cover uses double rays, but it is not hard to show that the finite cycle double cover conjecture is equivalent to the infinite circuit double cover conjecture.

Theorem 1. *The finite cycle double cover conjecture and the infinite circuit double cover conjecture are equivalent.*

Proof. Suppose that every finite bridgeless graph has a CDC. By Theorem K, every infinite 3-regular bridgeless graph has a circuit double cover and by Theorem H, every bridgeless infinite graph admits a circuit double cover.

Conversely, assume that the infinite circuit double cover conjecture is true. Let G be a bridgeless finite graph and let H be an infinite graph obtained by joining infinitely many copies of G at some common vertex v, $v \in V(G)$. Obviously, H is an infinite bridgeless graph and so admits a circuit double cover. It can be easily seen that every circuit double cover of H contains cycle double covers of (copies of) G. □

The above observation suggests that we consider the existence of circuit double covers in infinite bridgeless 3-regular graphs.

A k-factor of a graph G, $\mathcal{F}$, is a set of edges that induces k-regular spanning subgraph of G. It is easy to check that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two 1-factors of G, with $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$, then $\mathcal{F}_1 \cup \mathcal{F}_2$ has a partition into cycles and double rays.

Observation 1. *Let G be an infinite d-regular $(d \geq 3)$ bridgeless graph. If G has chromatic index d, then G has a circuit double cover.*

Proof. Suppose that G is a d-regular bridgeless d-edge-colourable graph. Let $M_1, M_2, \ldots, M_d$ be colour classes of $E(G)$. Each M_i is a 1-factor of G, $i = 1, \ldots, d$, and so $\{M_1 \cup M_2, M_2 \cup M_3, \ldots, M_{d-1} \cup M_d, M_d \cup M_1\}$ is a circuit double cover of G. □

Using a technique similar to that of Theorem K and König's lemma we prove the following theorems. For completeness, we recall the lemma in a perhaps less known version.

Theorem M. [5] (König's infinity lemma) *Let $V_0, V_1, \dots$ be an infinite sequence of disjoint non-empty finite sets, and let G be a graph on their union. Assume that every vertex v in a set V_n with $n \geq 1$ has at least one neighbor in V_{n-1}. Then G contains a ray $v_0 v_1 \dots$ with $v_n \in V_n$ for all n.*

We will also need the next notion and result. A family $(G_i)_{i \in I}$ of subgraphs of an infinite graph G is called thin if no vertex of G lies in infinitely many of the G_i's. Let the sum $\sum_{i \in I} G_i$ of this family be the subgraph of G consisting of all edges that lie in G_i for an odd number of indices i (and the vertices incident with these edges), and let the cycle space $\mathcal{C}(G)$ of G be the set of all sums of thin families of cycles [7].

Theorem N. [7] *Every element of the cycle space of an infinite graph is the edge-disjoint union of cycles.*

Theorem 2. *Let G be a connected locally finite graph. If every edge of G lies in a cycle of length at most 4, then G has a circuit double cover.*

Proof. Since G is a connected locally finite graph, G is countable. Let $V(G) = \{v_1, v_2, \dots\}$ and $G_n = G[\{v_1, \dots, v_n\}]$. We define W_n as the set of all families $\mathcal{E}$ of edge set $E \subset E(G_n)$ such that

(i) every edge $e \in G_n$ lies in exactly two members of $\mathcal{E}$; and
(ii) for every $E \in \mathcal{E}$ there is a finite cycle $C \subset E(G)$ with $E(C \cap G_n) = E$.

The set W_n is nonempty and finite. Consider the finite graph obtained by contracting the components of $G \setminus G_n$ to one vertex each, keeping parallel edges but deleting loops. Subdividing the parallel edges, we obtain a simple finite graph G'_n. It can be easily seen that every edge of G'_n lies in a cycle of length at most 4. Therefore, by Theorem B, G'_n has a cycle double cover, and the corresponding edges in G satisfy (i) and (ii). Define a new graph H on $\bigcup_{n=1}^{\infty} W_n$. For $n \geq 2$, let $\mathcal{E} \in W_n$ be adjacent to $\mathcal{E}' \in W_{n-1}$ if and only if

(iii) for each $E' \in \mathcal{E}'$ there is an $E \in \mathcal{E}$ such that $E \cap E(G_{n-1}) = E'$.

Observe that for $n \geq 2$, every vertex in W_n has a neighbor in W_{n-1}. By the infinity Lemma (Theorem M), there is a ray $\mathcal{E}_1 \mathcal{E}_2 \dots$ in H with $\mathcal{E}_i \in W_i$, for $i \geq 1$. For every nonempty element $E_1 \in \mathcal{E}_1$ by (iii), there is a unique ascending chain $E_1 \subset E_2 \subset \dots$ with $E_i \in \mathcal{E}_i$, for all $i \geq 1$. Let $\mathcal{F}$ be the family consisting of the unions of such ascending chains. Thus $\mathcal{F}$ is a family of element of the cycle space of G such that every edge of G lies in exactly two members of this family. By Theorem N, we can modify this into a circuit double cover. □

Let K_ω be the complete graph of order $\aleph_0$. Every edge of K_ω lies in a cycle of length at most 4, but it is not a locally finite graph. In the following theorem, we show that K_ω admits a circuit double cover.

Theorem 3. *K_ω has a circuit double cover.*

Proof. We will give several proofs of this fact.

1. (double rays) Without loss of generality, assume that the vertex set of K_ω is $\mathbb{Z}$. For $n \in \mathbb{N}^{>0}$ let $D_n = \{ij \in E(K_\omega) : j = i + n\}$ be the edge set of the graph on $\mathbb{Z}$. Clearly each edge of K_ω lies in exactly one D_n. Each set D_n can be decomposed into n double rays by the equivalence relation defined on it by setting ij equivalent to $i'j'$ if $i' - i \equiv 0 \pmod{n}$. Let R_n be the set of the n double rays of D_n. Two copies of $R = \cup_{n\in\mathbb{N}^{>0}} R_n$ constitute a double cover of K_ω by double rays.

2. (cycles) Assume again that the vertex set of K_ω is $\mathbb{Z}$. We will construct a set of triangles that will give a double cover of K_ω. Let $t_0 = (i_0, j_0, i_0 + j_0) = (1, 2, 3)$, $T_0 = \{t_0\}$ and let $D_0 = \{1, 2, 3\}$. Having defined t_n, T_n and D_n, let $t_{n+1} = (i_{n+1}, j_{n+1}, i_{n+1} + j_{n+1})$ defined by $i_{n+1} = \min\ \mathbb{N}^{>0} \setminus D_n$, $j_{n+1} = \min\ \mathbb{N}^{>0} \setminus (D_n \cup \{i\})$. Set $D_{n+1} = D_n \cup \{i_{n+1}, j_{n+1}, i_{n+1} + j_{n+1}\}$. Let also $T_{n+1} = T_n \cup \{(i_{n+1}, j_{n+1}, i_{n+1} + j_{n+1})\}$. Let $D = \cup_{n\in\mathbb{N}^{>0}} D_n$ and $T = T_\omega = \cup_{n\in\mathbb{N}^{>0}} T_n$. Observe that, by construction, $D = \mathbb{N}^{>0}$ and that, for each n, $i_n + j_n \notin D_{n-1}$. The latter because $i_n > i, j_n > j$ for any i, j such that $(i, j, i + j) \in T_{n-1}$. The elements of T define 3-cycles that cover each edge of K_ω: for each $i \in \mathbb{Z}$, let $\mathcal{C}_i = \{(i, i + i_n, i + i_n + j_n) : n \in \mathbb{N}^{>0}\}$. Taking $\mathcal{C} = \cup_{i\in\mathbb{Z}} \mathcal{C}_i$ twice gives a cycle double cover.

3. The above can be combined and cycle lengths mixed with a bit of care to get mixed double covers.

4. (general) Let $G = K_\omega$ and $V(K_\omega) = \{v_1, v_2, \ldots\}$ and $G_n = G[\{v_1, \ldots, v_n\}]$. We define W_n as the set of all families $\mathcal{E}$ of edge set $E \subset E(G_n)$ such that

(i) every edge $e \in G_n$ lies in exactly two members of $\mathcal{E}$; and
(ii) for every $E \in \mathcal{E}$ there is a finite cycle $C \subset E(K_\omega)$ with $E(C \cap G_n) = E$.

The finite subgraph $G_{n+1} = K_{n+1}$ has a cycle double cover, and the corresponding edges in K_ω satisfy (i) and (ii). Therefore, W_n is nonempty. Since G_n is a finite graph, W_n is finite. Define a new graph H on $\bigcup_{n=1}^{\infty} W_n$. For $n \geq 2$, let $\mathcal{E} \in W_n$ be adjacent to $\mathcal{E}' \in W_{n-1}$ if and only if

(iii) for each $E' \in \mathcal{E}'$ there is an $E \in \mathcal{E}$ such that $E \cap E(G_{n-1}) = E'$.

Observe that for $n \geq 2$, every vertex in W_n has a neighbor in W_{n-1}. By the infinity Lemma (Theorem M), there is a ray $\mathcal{E}_1\mathcal{E}_2\dots$ in H with $\mathcal{E}_i \in W_i$, for $i \geq 1$. For every nonempty element $E_1 \in \mathcal{E}_1$ by (iii), there is a unique ascending chain $E_1 \subset E_2 \subset \dots$ with $E_i \in \mathcal{E}_i$, for all $i \geq 1$. Let $\mathcal{F}$ be the family consisting of the unions of such ascending chains. Now $\mathcal{F}$ is a family of element of the cycle space of G such that every edge of G lies in exactly two members of this family. By Theorem N, we can modify this into a circuit double cover. □

One of the original motivations for studying finite cycle double covers is that if every vertex of a finite graph has an even degree, then it admits a cycle decomposition. In this part, we consider some results of Laviolette about the cycle and circuit decomposition of infinite graphs. Then we conclude the same result for infinite graph with even degrees.

Definition 3. [12] *A* cycle decomposition *of a graph G is a partition of $E(G)$ into finite connected 2-regular subgraphs (cycles), and a* circuit decomposition *of G is a partition of $E(G)$ into finite and infinite connected 2-regular subgraphs (cycles and double rays).*

Theorem O. [12] *A graph G has a cycle decomposition if and only if every finite subgraph of G is contained in a finite even subgraph of G.*

Since two copies of every cycle decomposition is a cycle double cover, we have the following corollary.

Corollary 2. *Let G be a graph. If every finite subgraph of G is contained in a finite even subgraph of G, then G has a cycle double cover.*

The following statements are a generalization of the cycle decomposition and the cycle double cover of finite even graphs to locally finite even graphs, respectively.

Theorem P. [12] *A locally finite graph has a circuit decomposition if and only if it is even.*

Corollary 3. *Every locally finite even graph has a circuit double cover.*

Proposition 1. *Let G be a locally finite graph. If G has a circuit cover $\mathcal{C}$ such that every edge of G is covered at most twice by elements of $\mathcal{C}$, then G admits a circuit double cover.*

Proof. Let H be a subgraph of G that consists of edges which covered exactly once by elements of $\mathcal{C}$. Since for every vertex v of G, $\deg(v)$ is finite, the degree of v in every element of $\mathcal{C}$ is 0 or 2, and $\mathcal{C}$ is a $(1,2)$-circuit cover of

G, we have H is a locally finite even graph. Therefore by Theorem P, H has a circuit decomposition $\mathcal{C}'$. Thus $\mathcal{C}'' = \mathcal{C} \cup \mathcal{C}'$ is a circuit double cover of G. □

Note that every vertex of a locally finite even graph has an even degree. Thus, we consider the locally finite graphs with just one vertex of odd degree.

Theorem 4. *Let G be a locally finite graph with exactly one vertex of odd degree, then G has a circuit double cover.*

Proof. Note that a locally finite graph with exactly one vertex of odd degree is such that every edge of G lies in a circuit and $\delta(G) \geq 3$. Suppose that $v \in V(G)$ is the only odd vertex of G. If v lies in a double ray, $R = R_1 v R_2$, where R_1 and R_2 are two vertex disjoint v-rays, then by Theorem P, $G_i = G \backslash E(R_i)$ has a circuit decomposition $\mathcal{C}_i$, for $i = 1, 2$. Therefore, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{R\}$ is a circuit double cover of G. If v does not lie in a double ray, then v lies in a cycle. Let G_0 be the maximal induced subgraph of G such that no edge of G_0 lies in a double ray. Obviously, G_0 is a finite bridgeless graph with two odd degree vertices. By Corollary 1, G_0 has a cycle double cover. Let $G_1 = G \setminus G_0$. $V(G_0) \cap V(G_1) = \{u\}$. Therefore, G_1 is a locally finite graph such that every edge of G_1 lies in a circuit and $\delta(G_1) \geq 3$. Since u lies in a double ray, by the first part of this proof, G_1 admits a circuit double cover. Therefore, G has a circuit double cover. □

In the proof of Lemma 2.2 in [10], Laviolette proved that if v is the only vertex of odd or infinite degree in G, then the maximal set of pairwise edge disjoint circuits and v-rays is a decomposition of the edge set of G. Note that the union of two edge disjoint v-rays is a collection of cycles and at most one double ray.

Theorem 5. *Let G be an infinite graph such that every vertex of G has even degree except $v \in V(G)$ where it has an infinite countable degree and v lies in a double ray. Then G admits a circuit double cover.*

Proof. Let R be a double ray containing v and $G' = G \setminus E(R)$. Assume that $\mathcal{F}$ is the maximal set of pairwise edge disjoint circuits and v-rays in G'. By the proof of Lemma 2.2 in [10], $\mathcal{F}$ is a decomposition of $E(G')$. Suppose that $\mathcal{R}$ is the set of all v-rays in $\mathcal{F}$. Let $\mathcal{R} = \{R_1, R_2, \ldots, R_n\}$ be a finite set. If $n \geq 2$, then $\mathcal{C} = (\mathcal{F} \setminus \mathcal{R}) \bigcup (\mathcal{F} \setminus \mathcal{R}) \bigcup \{R, R, R_1 \cup R_2, R_2 \cup R_3, \ldots, R_{n-1} \cup R_n, R_n \cup R_1\}$ is a circuit double cover of G. If $n = 1$, then $\mathcal{C} = (\mathcal{F} \setminus \mathcal{R}) \bigcup (\mathcal{F} \setminus \mathcal{R}) \bigcup \{R, R_1 \cup R_r, R_l \cup R_1\}$ is a circuit double cover of G, where $R = R_l v R_r$. Otherwise, $\mathcal{R} = \{R_1, R_2, \ldots\}$. Thus, $\mathcal{C} = (\mathcal{F} \setminus \mathcal{R}) \bigcup (\mathcal{F} \setminus \mathcal{R}) \bigcup \{R, R, R_1 \cup R_2, R_2 \cup R_3, R_3 \cup R_4, \ldots\}$ is a circuit double cover of G. □

By the first case of the proof of Theorem 4, we prove the following theorem.

Theorem 6. *Let G be a locally finite graph such that every edge of G lies in a circuit and $\delta(G) \geq 3$. If G has $v_1, \ldots, v_k$ vertices of odd degree and there are k disjoint edge double rays $R_1, \ldots, R_k$ such that $v_i \in V(R_i)$ for $i = 1, \ldots, k$, then G admits a circuit double cover.*

Proof. Suppose that v_i lies in a double ray, $R_i = R_{i_1} v_i R_{i_2}$, where R_{i_1} and R_{i_2} are two edge disjoint v_i-rays, for $i = 1, \ldots, k$. Thus by Theorem P, $G_j = G \setminus (\bigcup_{1 \leq i \leq k} \{R_{i_j}\})$ has a circuit decomposition $\mathcal{C}_j$, for $j = 1, 2$. Therefore, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup (\bigcup_{1 \leq i \leq k} \{R_i\})$ is a circuit double cover of G. □

3. Hamiltonian locally finite graphs

Nash-Williams [16] addressed the problem and proposed spanning double rays as infinite analogs of Hamilton circuits. He noticed that for a spanning double ray to exist the graph needs to be 3-indivisible. A graph is k-indivisible if the deletion of finitely many vertices leaves at most $k - 1$ infinite components.

The restriction to 3-indivisible graphs is a quite serious one that at first appears unavoidable. Yet, while double rays are the obvious first choice for an infinite analog of cycles there is a more subtle alternative, which was introduced by Diestel and Kühn [6, 7]. They call the homeomorphic image C of the unit circle in the Freudenthal compactification of a locally finite graph G a circle; the subgraph $C \cap G$ is then a hypercycle (for finite graphs this definition coincides with the cycle). A Hamilton hypercycle is a hypercycle that contains every vertex of G. An infinite graph G is called Hamiltonian, if it contains a Hamiltonian hypercycle. In a series of papers it has been shown that this notion is very successful and more suitable than double rays. These circles overcome the restriction to 3-indivisible graphs. Indeed, there is an infinite Hamilton hypercycle in a graph that is not k-indivisible for any k. The example is due to Diestel and Kühn [6].

Theorem Q. [3] *Let G be a Hamiltonian locally finite graph. Then there are Z_1 and $Z_2 \in \mathcal{C}(G)$ with $E(G) = Z_1 \cup Z_2$.*

By Theorem P, $Z_1 \Delta Z_2$ has a circuit decomposition. Thus, we can conclude the following theorem.

Theorem 7. *Every Hamiltonian locally finite graph admits a circuit double cover.*

Proof. Let G be a Hamiltonian locally finite graph. By Theorem Q, there are Z_1 and $Z_2 \in \mathcal{C}(G)$ with $E(G) = Z_1 \cup Z_2$. By Theorem P, $Z_1 \Delta Z_2$ has a circuit decomposition. Therefore, $Z_1 \cup Z_2 \cup (Z_1 \Delta Z_2)$ is a circuit double cover of G. □

4. Allowed subgraph double cover

In this section we define the notion of an *allowed* graph.

Theorem 8. *Let G be a finite bridgeless graph with a strong* CDC*. If H is an infinite graph obtained from G, u-ray, R_u, and v-ray, R_v, by joining at vertex u and vertex v, respectively, for some $u, v \in V(G)$, then H has a circuit double cover.*

Proof. Since G is bridgeless, there exists a cycle C in G such that $u, v \in V(C)$. Thus, G has a CDC, $\mathcal{C}'$, which contains C. Suppose that $C = uw_1w_2 \dots w_kvw_{k+1}w_{k+2} \dots w_{k+l}u$. Let $R_1 = R_uuw_1w_2 \dots w_kvR_v$ and $R_2 = R_vvw_{k+1}w_{k+2} \dots w_{k+l}uR_u$. Therefore, $\mathcal{C} = (\mathcal{C}' \setminus \{C\}) \bigcup \{R_1, R_2\}$ is a circuit double cover of H. □

In the general case, if G has a CDC, $\mathcal{C}_G$, and $C \in \mathcal{C}_G$ is a cycle, $\{u_1, \dots, u_k\} \subset V(C)$, and H is an infinite graph obtained from G and u_i-ray, R_{u_i}, by joining at vertex u_i, $i = 1, \dots, k$, then H has a circuit double cover.

Theorem 9. *Let G be an infinite graph and $\mathcal{C}$ be a circuit double cover of G with double ray $R \in \mathcal{C}$. If H is an infinite graph obtained from G and u-ray, R_u, by joining at vertex u and $u \in V(R)$, then H has a circuit double cover.*

Proof. Suppose that $\mathcal{C}$ be a circuit double cover of G with double ray $R \in \mathcal{C}$ where $R = \dots w_3w_2w_1uv_1v_2v_3 \dots$. Let $R_1 = R_uuv_1v_2v_3 \dots$ and $R_2 = \dots w_3w_2w_1uR_u$. Therefore, $\mathcal{C}_H = (\mathcal{C} \setminus \{R\}) \bigcup \{R_1, R_2\}$ is a circuit double cover of H. □

Since every edge of a d-regular infinite tree, T_d, lies in a double ray and T_d has chromatic index d, the following follows from the Observation 1.

Corollary 4. *Every d-regular infinite tree, T_d, has a circuit double cover.*

We observed that in general we can cover the edge set of each tree with the double rays and some allowed paths and rays, twice. In fact, we want to use the leaves of tree just as the first edge of paths or rays and every ray or path begins with a leaf.

Definition 4.
(i) *An* allowed subgraph *is either a double ray or a v-ray where* $\deg_G(v) = 1$ *or an uv-path where* $\deg_G(u) = \deg_G(v) = 1$ *or a finite cycle.*

(ii) *An* allowed subgraph double cover (ASDC) *of a graph G is a collection of its allowed subgraphs such that each edge of G lies in exactly two of the allowed subgraphs.*

Theorem 10. *Every tree admits an* ASDC*.*

Proof. By Theorem F, assume that T is a countable tree and $v \in V(T)$ with degree at least 2. Consider the rooted tree of T with the root v. We have two cases.
(i) $\deg(v)$ is finite. Let $N(v) = \{u_1, u_2, \ldots, u_k\}$.
Define $\mathcal{C}_1 = \{u_1vu_2, u_2vu_3, \ldots, u_{k-1}vu_k, u_kvu_1\}$.
(ii) $\deg(v)$ is infinite. Let $N(v) = \{u_1, u_2, \ldots\}$.
Define $\mathcal{C}_1 = \{u_ivu_{i+2} : i \in \mathbb{N}\} \cup \{u_1vu_2\}$.
For each level L_ℓ (the set of vertices at distance ℓ from v) we define

$$\mathcal{C}_l = \bigcup_{u \in L_\ell} \mathcal{P}_u,$$

where $\mathcal{P}_u$ is defined as follows. By the definition of C_i, we know that there are two paths P_1 and P_2 in $\mathcal{C}_{\ell-1}$ that begin with the vertex u. We have two cases.
(i) $\deg(u)$ is finite. Let $N(u) = \{w_1, w_2, \ldots, w_k\}$.
Define $\mathcal{P}_u = \{w_1uP_1, w_1uw_2, w_2uw_3, \ldots, w_{k-1}uw_k, w_kuP_2\}$.
(ii) $\deg(u)$ is infinite. Let $N(u) = \{w_1, w_2, \ldots\}$.
Define $\mathcal{P}_u = \{w_iuw_{i+2} : i \in \mathbb{N}\} \cup \{w_1uP_1, w_2uP_2\}$. Clearly, $\mathcal{C} = \cup_{i \in \mathbb{N}} \mathcal{C}_i$ is an ASDC of T. □

Since for an infinite tree with minimum degree at least 2, every ASDC is a circuit double cover, we have the following corollary to Theorem 10.

Corollary 5. *Every infinite tree T with $\delta(T) \geq 2$ has a circuit double cover.*

The above discussion motivates us to present the following conjecture.

Conjecture 6. (Infinite allowed subgraph double cover conjecture) *Let G be an infinite graph such that every edge of G lies in an allowed subgraph. Then G admits an* ASDC*.*

5. Finite allowed subgraph double cover

In this section we consider allowed subgraph double covers for finite graphs.

Definition 5. *Call a finite graph G* semi-bridgeless *if for every bridge $e \in E(G)$, each component of $G \setminus e$ with at least 2 vertices has a vertex v with degree 1 in G.*

Note that in the finite case, each allowed subgraph is either a cycle or an uv-path where $\deg(u) = \deg(v) = 1$

Conjecture 7. (Finite allowed subgraph double cover conjecture) *Every semi-bridgeless finite graph admits an* ASDC.

Theorem 11. *The finite allowed subgraph double cover conjecture and the finite cycle double cover conjecture are equivalent.*

Proof. Obviously, every ASDC of a bridgeless graph G is a CDC of G, and vice versa. Assume that the finite cycle double cover conjecture is true. Let G be a semi-bridgeless graph. Let $V_1 = \{v_1, v_2, \dots, v_l\}$ be the set of all of the vertices of degree 1, and $G' = G \cup \{v_1w, v_2w, \dots, v_lw\}$, where w is a new vertex. Since G' is bridgeless, it admits a CDC, $\mathcal{C}$. Obviously, $\mathcal{C} \setminus \{w\}$ is an ASDC of G. □

Theorem 12. *The minimal counterexamples to the* ASDC *conjecture and the* CDC *conjecture are the same snark.*

Proof. Let G_1 and G_2 be the minimal counterexamples to the ASDC conjecture and the CDC conjecture, respectively, and each G_i has n_i vertices and e_i edges, $i = 1, 2$. Since every CDC of a bridgeless graph is also an ASDC, $n_1 + e_1 \leq n_2 + e_2$. Suppose that G_1 is a semi-bridgeless graph with l_1 vertices of degree one. We identify the vertices of degree one to a new vertex w. The new graph G'_1 is bridgeless with $n_1 - l_1 + 1$ vertices and e_1 edges. Is is easy to see that every CDC produce an ASDC of G_1. Therefore, G'_1 is a counterexample to the CDC conjecture and $n_2 + e_2 \leq n_1 - l_1 + 1 + e_1$. So $l_1 = 0$ and G_1 is bridgeless. G_1 is also the minimal counterexample to the CDC conjecture. □

A stronger version of the finite cycle double cover conjecture was made by Bondy.

Conjecture 8. [2] (Small cycle double cover conjecture) *Every finite bridgeless graph of order n admits a* CDC *with at most $n - 1$ cycles.*

It is easy to check that the small cycle double cover conjecture and the following conjecture are equivalent.

Conjecture 9. (Small allowed subgraph double cover conjecture) *Every semi-bridgeless finite graph of order n with k vertices of degree one admits an* ASDC *with at most $n - k$ allowed subgraphs.*

Since the finite CDC conjecture and the finite ASDC conjecture are equivalent, by using a techniques similar to that of the proof of Theorem K we have the following theorem.

Theorem 13. *If the finite cycle double cover conjecture is true then every locally finite graph whose every edge lies in an allowed subgraph admits an* ASDC*.*

Theorem 14. *Let G be a finite or locally finite bridgeless graph and G' be a graph obtained from G by adding a new vertex a_v and joining to v for every vertex $v \in V(G)$. Then G' has an* ASDC*.*

Proof. Let G'' be a bridgeless graph obtained from G' by identifying the vertices $\{a_v \mid v \in V(G)\}$ to a new vertex w. Every edge of G'' lies in a cycle of length 3. Therefore, by Theorems A and 2, G'' has a circuit double cover and G' admits an ASDC. □

Theorem 15. *Every finite tree admits an* ASDC*.*

Proof. Let T be a finite tree and G be a bridgeless graph obtained from T by joining a new vertex w to all vertices of T with degree one (the leaves). Since G is a planar graph, it has a finite CDC, $\mathcal{C}$. Assume that $\mathcal{C}'$ is a collection that obtained by deleting the vertex w from the cycles of $\mathcal{C}$. Obviously, the collection $\mathcal{C}'$ is an ASDC of T. □

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