Circular chromatic Ramsey number

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Let $\chi_c(H)$ denote the circular chromatic number of a graph H. For graphs F and G, the *circular chromatic Ramsey number* $R_{\chi_c}(F, G)$ is the infimum of $\chi_c(H)$ over graphs H such that every red/blue edge-coloring of H contains a red copy of F or a blue copy of G. We characterize $R_{\chi_c}(F, G)$ in terms of a Ramsey problem for the families of homomorphic images of F and G. Letting $z_t = 3 - 2^{-t}$, we prove that $z_{t-1} < \chi_c(G) \leq z_t$ implies $2z_{t-1} \leq R_{\chi_c}(G, G) \leq 2z_t$. For integer k with $k > 1$, there is a graph G with $\chi_c(G) \geq k$ and $R_{\chi_c}(G, G) \leq k(k-1)$. Our most difficult result is $R_{\chi_c}(F, G) = 4$ when $\chi_c(F), \chi_c(G) \in (2, \frac{5}{2}]$. As a consequence, no graph G satisfies $4 < R_{\chi_c}(G, G) < 5$. We also prove $\frac{14}{3} \leq R_{\chi_c}(C_3, C_5) \leq 5$ and $4 \leq R_{\chi_c}(C_3, C_7) \leq \frac{9}{2}$.

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1. Introduction

Given families $\mathcal{F}$ and $\mathcal{G}$ of graphs, classical graph Ramsey theory considers graphs H such that every red/blue edge-coloring of H contains a red graph from $\mathcal{F}$ or a blue graph from $\mathcal{G}$; we then write $H \to (\mathcal{F}, \mathcal{G})$ and say that H *arrows* $(\mathcal{F}, \mathcal{G})$. The *Ramsey number* $R(\mathcal{F}, \mathcal{G})$ is $\min\{|V(H)|\colon H \to (\mathcal{F}, \mathcal{G})\}$. (When $\mathcal{F}$ or $\mathcal{G}$ is a single graph, we drop set braces.)

For any mononotone graph parameter ρ, the ρ-Ramsey number of $(\mathcal{F}, \mathcal{G})$, written $R_\rho(\mathcal{F}, \mathcal{G})$, is $\inf\{\rho(H)\colon H \to (\mathcal{F}, \mathcal{G})\}$. Besides the number of vertices, the notion has been studied with ρ being the clique number [4, 11], chromatic number [2, 17], number of edges (yielding the "size Ramsey number") [3, 14],

and maximum degree [8, 10]. In this paper we let ρ be the "circular chromatic number".

The *circular chromatic number* of a graph G, written $\chi_c(G)$, is the infimum of p/q such that the vertices of H can be assigned congruence classes modulo p so that the classes of adjacent vertices differ by at least q. Formally, for $p, q \in \mathbb{N}$ with $p \geq 2q$, the *generalized complete graph* or *circular clique* $K_{p:q}$ has vertex set $\{v_i \colon 0 \leq i \leq p-1\}$ and edge set $\{v_i v_j \colon q \leq |i-j| \leq p-q\}$. A *homomorphism* from G to H is a map $\phi \colon V(G) \to V(H)$ that preserves edges. When a homomorphism exists from G to $K_{p:q}$, we say that G is (p,q)-*colorable*, so $\chi_c(G) = \inf\{p/q \colon G \text{ is } (p,q)\text{-colorable}\}$. If G is finite, then $\chi_c(G)$ is rational and the infimum is the minimum [1, 16].

Since a $(p,1)$-coloring is just an ordinary proper p-coloring, $\chi_c(G) \leq \chi(G)$. It is also well known that always $\chi_c(G) > \chi(G) - 1$ [1, 16]. Hence in fact $\chi(G) = \lceil \chi_c(G) \rceil$, meaning that the circular chromatic number is a refinement of the chromatic number.

Letting $\rho = \chi_c$, we study the *circular chromatic Ramsey number:*

$$R_{\chi_c}(\mathcal{F}, \mathcal{G}) = \inf\{\chi_c(H) \colon H \to (\mathcal{F}, \mathcal{G})\}.$$

Here the infimum is needed; the minimum may not exist.

Circular chromatic Ramsey number also arises from a different extension of Ramsey number. In studying $R(\mathcal{F}, \mathcal{G})$, we seek monochromatic copies of graphs in $\mathcal{F}$ or $\mathcal{G}$. If instead we are content with a monochromatic homomorphic image of such a graph when coloring $E(K_n)$, then the least n that suffices is $R_\chi(\mathcal{F}, \mathcal{G})$, by a result of [2] that we will state later. If we extend the options for the host graph to all $K_{p:q}$, then the resulting value is $R_{\chi_c}(\mathcal{F}, \mathcal{G})$.

In Section 2, we discuss the basic properties of R_{χ_c} and characterize $R_{\chi_c}(F, G)$ in terms of which circular cliques arrow the families of homomorphic images of F and G. Section 3 begins with easy arguments for small graphs analogous to the simplest Ramsey problems, obtaining $R_{\chi_c}(K_3, K_p) = R(K_3, K_p)$ for $p \in \{3, 4, 5\}$. However, $R_{\chi_c}(K_4, K_4) \leq 17.5 < 18 = R(K_4, K_4)$. We then obtain bounds on $R_{\chi_c}(G)$ whenever $2 \leq \chi_c(G) \leq 3$. Letting $z_t = 3 - 2^{-t}$, we show that $z_{t-1} < \chi_c(G) \leq z_t$ implies $2z_{t-1} \leq R_{\chi_c}(G) \leq 2z_t$. This yields non-integer values of R_{χ_c} between 5 and 6 when $t \geq 2$. We also obtain $R_{\chi_c}(G) = 6$ when $\chi_c(G) = 3$.

For integer k with $k > 2$, in Section 4 we show the existence of a graph G_k with $\chi_c(G_k) \geq k$ and $R_{\chi_c}(G_k, G_k) \leq k(k-1)$. This supports an analogue of the conjecture from [2] that $R_\chi(k) = (k-1)^2 + 1$, where $R_\chi(k) = \inf\{R_\chi(G, G) \colon \chi(G) = k\}$, which was proved by Zhu [19]. This result shows that the values of $R_{\chi_c}(G, G)$ vary exponentially when $\chi_c(G)$ is

fixed, since $R_{\chi_c}(K_k, K_k) \geq R_\chi(K_k, K_k) - 1 = R(k, k) - 1 > O(k2^{k/2})$, using the well-known lower bound on classical Ramsey numbers.

In Section 5 we prove our most difficult result: $R_{\chi_c}(G, G) = 4$ whenever $2 \leq \chi_c(G) \leq 5/2$. Combined with the observations of Section 2, we conclude that if each of $\mathcal{F}$ and $\mathcal{G}$ contains no bipartite graph but contains a graph with circular chromatic number at most $5/2$ (such as an odd cycle of length at least 5), then $R_{\chi_c}(\mathcal{F}, \mathcal{G}) = 4$. Furthermore, there is no graph G such that $4 < R_{\chi_c}(G, G) < 5$.

For the "non-diagonal" case, in Section 6 we prove $\frac{14}{3} \leq R_{\chi_c}(C_3, C_5) \leq 5$ and $4 \leq R_{\chi_c}(C_3, C_7) \leq \frac{9}{2}$. In contrast to having $R_{\chi_c}(C_{2k+1}, C_{2k+1}) = 4$ whenever $k \geq 2$, we now have different values for $R_{\chi_c}(C_3, C_5)$ and $R_{\chi_c}(C_3, C_7)$, though we do not yet know either value.

2. General observations

The classical bounds $\chi(H) - 1 < \chi_c(H) \leq \chi(H)$ yield analogous bounds on $R_{\chi_c}(\mathcal{F}, \mathcal{G})$.

Proposition 2.1. $R_\chi(\mathcal{F}, \mathcal{G}) - 1 \leq R_{\chi_c}(\mathcal{F}, \mathcal{G}) \leq R_\chi(\mathcal{F}, \mathcal{G})$.

Proof. Since $\chi_c(H) \leq \chi(H)$ when $H \to (\mathcal{F}, \mathcal{G})$, also $R_{\chi_c}(\mathcal{F}, \mathcal{G}) \leq R_\chi(\mathcal{F}, \mathcal{G})$. For the other inequality, if $k \leq R_{\chi_c}(\mathcal{F}, \mathcal{G}) < k + 1$ for some integer k, then $H \to (\mathcal{F}, \mathcal{G})$ for some graph H with $k \leq \chi_c(H) < k + 1$. Since $k \leq \chi(H) \leq k + 1$, we have $R_\chi(\mathcal{F}, \mathcal{G}) \leq k + 1$, and hence $R_\chi(\mathcal{F}, \mathcal{G}) \leq R_{\chi_c}(\mathcal{F}, \mathcal{G}) + 1$. $\square$

Early work on the circular chromatic number asked when $\chi_c(H)$ equals $\chi(H)$ or is close to $\chi(H) - 1$. We ask the same question for the bounds on $R_{\chi_c}(\mathcal{F}, \mathcal{G})$ in Proposition 2.1.

Like the chromatic Ramsey number in [2], the circular chromatic Ramsey number can be phrased as a classical graph Ramsey problem using homomorphisms. If there is a homomorphism from H to H', then H' need not contain every subgraph in H, but it contains a homomorphic image of every subgraph in H. The resulting lemma will help us rephrase R_{χ_c} using circular cliques. Given a family $\mathcal{F}$, let $\mathrm{Hom}(\mathcal{F})$ denote the family of all homomorphic images of graphs in $\mathcal{F}$ via surjective maps.

Lemma 2.2. *If $H \to (\mathcal{F}, \mathcal{G})$ and there is a homomorphism $\phi\colon H \to H'$, then $H' \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))$.*

Proof. Let f' be a red/blue edge-coloring of H'. For $e \in E(H)$, let $f(e) = f'(\phi(e))$. Under f, in H there is a red subgraph belonging to $\mathcal{F}$ or a blue subgraph belonging to $\mathcal{G}$. This subgraph maps under ϕ to a subgraph of H' in $\mathrm{Hom}(\mathcal{F})$ or $\mathrm{Hom}(\mathcal{G})$ that has the specified color under f'. Hence $H' \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))$. $\square$

Burr, Erdős, and Lovász [2] proved

$$R_\chi(\mathcal{F}, \mathcal{G}) = R(\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G})) = \inf\{n\colon K_n \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))\}.$$

They used the bipartite version of Ramsey's Theorem and the *blowup* $J[n]$ of a graph J, defined to be the graph obtained from J by expanding each vertex into an independent set of size n (and each edge into a copy of the complete bipartite graph $K_{n,n}$). Their idea applies also to R_{χ_c}, yielding the following result.

Theorem 2.3. *If $\mathcal{F}$ and $\mathcal{G}$ are finite, then*

$$R_{\chi_c}(\mathcal{F}, \mathcal{G}) = \inf\{p/q\colon K_{p:q} \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))\}.$$

Proof. Whenever $H \to (\mathcal{F}, \mathcal{G})$ with $\chi_c(H) = p/q$, Lemma 2.2 yields $K_{p:q} \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))$. Thus $\inf\{p/q\colon K_{p:q} \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))\} \leq R_{\chi_c}(\mathcal{F}, \mathcal{G})$.

For the reverse inequality, suppose $K_{p:q} \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))$. Let $H = K_{p:q}[n]$, where n is the largest number of vertices among graphs in $\mathcal{F}$ or $\mathcal{G}$. When N is sufficiently large, every 2-edge-coloring of $H[N]$ contains a copy of H in which each copy of $K_{n,n}$ corresponding to a single edge of $K_{p:q}$ is monochromatic. This is proved by iterating the bipartite version of Ramsey's Theorem, as in [2]. The resulting 2-edge-coloring of $K_{p:q}$ has a red copy of some graph in $\mathrm{Hom}(\mathcal{F})$ or a blue copy of some graph in $\mathrm{Hom}(\mathcal{G})$, since $K_{p:q} \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))$. In the 2-edge-coloring of H, this subgraph expands into a monochromatic copy of the corresponding graph in $\mathcal{F}$ or $\mathcal{G}$ with the right color. Hence $H[N] \to (\mathcal{F}, \mathcal{G})$. Since $\chi_c(H[N]) = p/q$, we have $R_{\chi_c}(\mathcal{F}, \mathcal{G}) \leq \inf\{p/q\colon K_{p:q} \to (\mathrm{Hom}(\mathcal{F}), \mathrm{Hom}(\mathcal{G}))\}$. □

Lemma 2.4. *If each graph in $\mathcal{F}'$ contains a homomorphic image of some graph in $\mathcal{F}$, and similarly for $\mathcal{G}'$ and $\mathcal{G}$, then $R_{\chi_c}(\mathcal{F}, \mathcal{G}) \leq R_{\chi_c}(\mathcal{F}', \mathcal{G}')$.*

Proof. Given any $p, q \in \mathbb{N}$ with $p/q \geq R_{\chi_c}(\mathcal{F}', \mathcal{G}')$, let f be a red-blue coloring of $K_{p:q}$. By Theorem 2.3 and symmetry, we may assume that f on $K_{p:q}$ yields a red copy of some graph F'' in $\mathrm{Hom}(\mathcal{F}')$. By definition, there exists $F' \in \mathcal{F}'$ that admits a homomorphism onto F''.

By hypothesis, there exists $F \in \mathcal{F}$ that admits a homomorphism to F'. By composition, the red copy of F'' contains a homomorphic image of F. Thus $K_{p:q} \to (\mathrm{Hom}(\mathcal{F}, \mathcal{G}))$. We conclude $R_{\chi_c}(\mathcal{F}, \mathcal{G}) \leq p/q$. Since this holds whenever $p/q \geq R_{\chi_c}(\mathcal{F}', \mathcal{G}')$, we have $R_{\chi_c}(\mathcal{F}, \mathcal{G}) \leq R_{\chi_c}(\mathcal{F}', \mathcal{G}')$. □

When $\mathcal{F} = \mathcal{G} = \{G\}$, we write $H \to G$ for $H \to (G, G)$ and $R_\rho(G)$ for $R_\rho(G, G)$.

Corollary 2.5. *If a graph G' contains a homomorphic image of a graph G, then $R_{\chi_c}(G) \leq R_{\chi_c}(G')$.*

Proof. This statement is just the special case of Lemma 2.4 when $\mathcal{F} = \mathcal{G}$ and $\mathcal{F}' = \mathcal{G}'$ and each family consists of a single graph. $\square$

Corollary 2.5 yields $R_{\chi_c}(G) \leq R_{\chi_c}(K_{p:q})$ when $\chi_c(G) = p/q$, so we study $R_{\chi_c}(K_{p:q})$ to obtain upper bounds on $R_{\chi_c}(G)$, especially when $\chi_c(G)$ is small. In Section 3, we compute circular chromatic Ramsey numbers for small complete graphs and obtain bounds for some circular cliques. Since $C_{2k+1} \cong K_{(2k+1):2}$, this theme continues when we consider odd cycles and pairs of odd cycles in Sections 5 and 6.

Theorem 5.4 proves $R_{\chi_c}(C_5) = 4$. Using the general observations above, this result implies the following.

Corollary 2.6. $R_{\chi_c}(\mathcal{F}, \mathcal{G}) \leq 4$ *whenever* $\mathcal{F}$ *and* $\mathcal{G}$ *each contain some graph with circular chromatic number at most* $\frac{5}{2}$. *Equality holds when neither* $\mathcal{F}$ *nor* $\mathcal{G}$ *contains a bipartite graph.*

Proof. Every graph with circular chromatic number at most 5/2 admits a homomorphism into C_5. Thus $\mathcal{F}' = \mathcal{G}' = \{C_5\}$ in Lemma 2.4 yields the upper bound. For the lower bound, every graph with chromatic number at most 4 decomposes into two bipartite graphs. $\square$

From the characterization of $R_\chi(G)$ in [2], it follows that $R_\chi(G) \in \{5, 6\}$ when G is 3-chromatic, and the value is 5 if and only if G is $(5, 2)$-colorable. On the other hand, $2 < \chi_c(G) \leq \frac{5}{2}$ implies $R_{\chi_c}(G) = 4$. When $\frac{5}{2} < \chi_c(G) \leq 3$, we have $R_\chi(G) = 6$. Since Proposition 2.1 yields $R_\chi(G) - 1 \leq R_{\chi_c}(G) \leq R_\chi(G)$, it follows that $\frac{5}{2} < \chi_c(G) \leq 3$ implies $5 \leq R_{\chi_c}(G) \leq 6$, and hence there are no circular chromatic Ramsey numbers between 4 and 5. However, we show in Section 3 that there are such values between 5 and 6, using the following remark.

Remark 2.7. *Upper and lower bounds.* By Theorem 2.3, $K_{p:q} \to \text{Hom}(G)$ implies $R_{\chi_c}(G) \leq p/q$. Also, if $K_{p:q} \to \text{Hom}(G)$ and $p'/q' \geq p/q$, then $K_{p':q'} \to \text{Hom}(G)$, by Lemma 2.2. The homomorphic images of G all have circular chromatic number at least $\chi_c(G)$ and include all $K_{r:s}$ with $r/s \geq \chi_c(G)$. Therefore, if there exists $K_{r:s}$ having a red/blue edge-coloring such that each color class has circular chromatic number less than $\chi_c(G)$, then $R_{\chi_c}(G) \geq r/s$. We use this method in Section 3 to obtain upper and lower bounds showing that if $\frac{11}{4} \leq \chi_c(G) < 3$, then $5.5 < R_{\chi_c}(G) < 6$.

3. Bounds and small values

Using the known colorings that establish lower bounds for $R(K_3, G)$, we prove $R_{\chi_c}(K_3, G) = R(K_3, G)$ for $G \in \{K_3, K_4, K_5\}$. Since $R_{\chi_c}(F, G) \leq R(F, G)$, we need only discuss the lower bounds. Let an $(\mathcal{F}, \mathcal{G})$-*avoidance* on H be a red/blue edge-coloring of H having no red subgraph in $\mathcal{F}$ and no blue subgraph in $\mathcal{G}$.

The *length* of an edge $v_i v_j$ in $K_{p:q}$ is $\min\{|i - j|, p - |i - j|\}$. Recall that $\mathrm{Hom}(K_n) = \{K_n\}$.

Proposition 3.1. $R_{\chi_c}(K_3, K_3) = R(K_3, K_3) = 6$.

Proof. It suffices to find for all q a (K_3, K_3)-avoidance on $K_{(6q-1):q}$. Assign red to the edges with lengths $q, \ldots, 2q-1$, and blue to the edges with lengths $2q, \ldots, \lfloor \frac{(6q-1)}{2} \rfloor$.

The lengths of the edges of a triangle in $K_{(6q-1):q}$ are a, b, c with $a + b + c = 6q - 1$ or $a + b = c$. For red edges of any lengths a, b, c, we have $a+b+c \leq 6q-3 < 6q-1$ and $a+b \geq 2q > c$. For blue edges of these lengths, we have $a + b + c \geq 6q > 6q - 1$ and $a + b \geq 4q > c$. With a contradiction in each case, there is no monochromatic triangle. □

Proposition 3.2. $R_{\chi_c}(K_3, K_4) = R(K_3, K_4) = 9$.

Proof. It suffices to find for all q a (K_3, K_4)-avoidance on $K_{(9q-1):q}$. Assign blue to all edges with lengths $q, \ldots, 3q-1$ and red to the edges with lengths $3q, \ldots, \lfloor \frac{(9q-1)}{2} \rfloor$.

The lengths of the three edges of a triangle in $K_{(9q-1):q}$ are a, b, c with $a \leq b \leq c$ so that $a + b + c = 9q - 1$ or $a + b = c \leq (9q - 1)/2$. Since the lengths of three red edges sum to at least $9q$, and two sum to at least $6q$, there is no red triangle.

Consider the four vertices of a 4-clique in $K_{(9q-1):q}$ in cyclic order of indices. The edge joining two opposite vertices u and v forms triangles with each of the two remaining vertices, x and y. In one of these triangles, the three lengths sum to $9q - 1$. Since three blue lengths sum to at most $9q - 3$, there is no blue copy of K_4. □

The particular coloring in Proposition 3.2 was noticed by Daniel Cranston. It is slightly simpler than our original coloring.

Proposition 3.3. $R_{\chi_c}(K_3, K_5) = R(K_3, K_5) = 14$.

Proof. It suffices to find for all q a (K_3, K_5)-avoidance on $K_{(14q-1):q}$. Assign red to all edges with lengths $2q, \ldots, 4q-1$ and blue to all other edges.

Since $12q - 3 < 14q - 1$, a red triangle can occur only by using two red lengths that sum to a red length, but $2q + 2q > 4q - 1$ prevents this.

It remains to prohibit blue 5-cliques. Blue edges include short edges with lengths from q to $2q - 1$ and long edges with lengths at least $4q$. Consider the separations among consecutive vertices of a blue 5-clique when viewed in cyclic order of subscripts. The separations must sum to $14q - 1$ using blue lengths. The sum of two short blue lengths is between $2q$ and $4q - 2$, all of which are red lengths, so a blue 5-clique cannot have consecutive short separations. Hence it has at most two short separations. Since the long separations are at least $4q$ each, the sum of the five separations in a blue 5-clique is at least $14q$, a contradiction. □

The behavior of $R_{\chi_c}(K_4, K_4)$ is quite different. Although $R(K_4, K_4) = 18$, we will prove $R_{\chi_c}(K_4, K_4) \leq 17.5$. Greenwood and Gleason [6] showed $R(K_4, K_4) = 18$ by coloring K_{17} with no monochromatic 4-clique, using cyclic symmetry. Viewed as $K_{17:1}$, the edges of lengths $1, 2, 4, 8$ are one color, and those of lengths $2, 5, 6, 7$ are the other color. The coloring has monochromatic 4-cycles, but their chords do not both have that color. In fact, there is only one graph on 17 vertices having no set of four pairwise adjacent or pairwise nonadjacent vertices; the survey of Radziszowski [13] states that this was proved by Kalbfleisch [9]. We use that each color class in the (K_4, K_4)-avoidance on K_{17} is 4-regular. The next lemma will enable us to use this.

Lemma 3.4. *If the red and blue subgraphs in every $(\mathcal{F}, \mathcal{G})$-avoidance on $K_{p:r}$ are regular, then every $(\mathcal{F}, \mathcal{G})$-avoidance on $K_{(2p+1):2r}$ is constant on consecutive pairs of length classes, each pair consisting of all edges with lengths $2i$ or $2i + 1$ for some i in $\{r, \ldots, \lfloor p/2 \rfloor\}$.*

Proof. The subgraph of $K_{(2p+1):2r}$ induced by $\{v_0, v_2, \ldots, v_{2p-2}\}$ is isomorphic to $K_{p:r}$; call it K. The length of the edge from the first to the last of these vertices is 3 instead of 2. Substituting v_{2p} for v_0 yields another copy K' of $K_{p:r}$. Given that every $(\mathcal{F}, \mathcal{G})$-avoidance on $K_{p:r}$ is regular, any edge $v_0 v_{2i}$ lost from K by deleting v_0 must be replaced by the edge $v_{2p} v_{2i}$ having the same color, for $r \leq i \leq p - r$.

Now fix $i \in \{r, \ldots, \lfloor p/2 \rfloor\}$. Replace $\{v_{2j} : 1 \leq j \leq i - 1\}$ in K' with $\{v_{2j-1} : 1 \leq j \leq i - 1\}$ to obtain L_i, and replace v_{2i} with v_{2i-1} in L_i to form L_i'. By the same argument as before, $v_{2p} v_{2i}$ and $v_{2p} v_{2i-1}$ have the same color. In particular, $v_{2p} v_{2i-1}$ and $v_0 v_{2i}$ have the same color. Having shown that consecutive edges of length $2i$ have the same color, we conclude that all edges of length $2i$ have the same color. Also, since the consecutive edges

were linked by an edge of length $2i + 1$ with that color, the edges of length $2i + 1$ have the same color as the edges of length $2i$. □

Proposition 3.5. $R_{\chi_c}(K_4, K_4) \leq 17.5$.

Proof. If some vertex in a red/blue edge-coloring of K_{17} has nine neighbors of the same color, then $R(K_3, K_4) = 9$ yields a monochromatic copy of K_4. Hence in every (K_4, K_4)-avoidance on $K_{17:1}$, every vertex is incident to eight edges of each color. By Lemma 3.4, a (K_4, K_4)-avoidance f on $K_{35:2}$ must be constant on pairs of consecutive distance classes.

We may assume that the pair $\{8, 9\}$ is red. Hence the cycle $[v_0, v_9, v_{17}, v_{26}]$ is all red. Both chords have length 17, so the pair $\{16, 17\}$ is blue.

If $\{2, 3\}$ is red, then $[0, 3, 5, 8] \Rightarrow \{4, 5\}$ is blue, $[0, 5, 17, 22] \Rightarrow \{12, 13\}$ is red, $[0, 2, 10, 12] \Rightarrow \{10, 11\}$ is blue, and $[0, 3, 9, 12] \Rightarrow \{6, 7\}$ is blue, leaving $[0, 7, 17, 24]$ as a blue copy of K_4.

If $\{2, 3\}$ is blue, then $[0, 2, 17, 19] \Rightarrow \{14, 15\}$ is red, $[0, 3, 16, 19] \Rightarrow \{12, 13\}$ is red, $[0, 6, 14, 20] \Rightarrow \{6, 7\}$ is blue, $[0, 7, 17, 24] \Rightarrow \{10, 11\}$ is red, and $[0, 5, 15, 20] \Rightarrow \{4, 5\}$ is blue, leaving $[0, 2, 4, 6]$ as a blue copy of K_4.

Hence there is no (K_4, K_4)-avoidance on $K_{35:2}$. □

Question 3.6. What is the value of $R_{\chi_c}(K_4, K_4)$?

Possibly $R_{\chi_c}(K_4, K_4) = 17$, which would be proved by showing $K_{17q+1:q} \to K_4$ for infinitely many q. Each such q yields an upper bound; Proposition 3.5 does this for $q = 2$.

Next we use colorings where the color of an edge in $K_{p:q}$ is determined only by its length to obtain upper and lower bounds on $R_{\chi_c}(G)$ for all G with $2 < \chi_c(G) \leq 3$. From Proposition 3.1, Remark 2.7, and the fact that 4-colorable graphs decompose into two bipartite graphs, we already know $4 \leq R_{\chi_c}(G) \leq 6$, but now we improve the bounds. They become more accurate when $\chi_c(G)$ is close to 3, and indeed $\lim_{\chi_c(G) \to 3^-} R_{\chi_c}(G) = 6$. In Section 5 we prove $R_{\chi_c}(G) = 4$ whenever $2 < \chi_c(G) \leq \frac{5}{2}$.

Lemma 3.7. *In $K_{(6q-1):q}$, the edges of lengths q through $2q - 1$ form a subgraph isomorphic to $K_{(6q-1):2q}$.*

Proof. Redraw the graph by putting v_i in position $2i$ (modulo $6q - 1$) in the ordering of the vertices around a circle. The edge to v_j from v_i now has length $2(j-i)$. Thus edges of lengths $q, q+1, \ldots, 2q-1$ from v_i become edges of lengths $2q, 2q+2, \ldots, 4q-2$, while edges of lengths $-q, -q-1, \ldots, -2q+1$ become edges of lengths $-2q, -2q-2, \ldots, -4q+2$, which equal $4q-1, 4q-3, \ldots, 2q + 1$. Thus in the redrawing two vertices are adjacent if and only if they are separated by at least $2q$ positions, as desired. □

Since no vertex can have three incident edges of one color, every (K_3, K_3)-avoidance on K_5 has complementary monochromatic 5-cycles. Again we take advantage of regularity.

Lemma 3.8. *Let* $G_t = K_{(3 \cdot 2^t - 1):2^t}$ *and* $H_t = K_{(3 \cdot 2^t - 1):2^{t-1}}$. *Let* f *be a* (K_3, K_3)-*avoidance on* H_t. *For* $t \geq 1$, *each color class is isomorphic to* G_t. *For* $t \geq 2$, *all edges with length at least* 2^t *have one color, and all shorter edges have the other color.*

Proof. Let $n = 3 \cdot 2^t - 1$. For $t = 1$, we have observed that each color class in a (K_3, K_3)-avoidance on $K_{5:1}$ is a 5-cycle. For $t \geq 2$, note that the edges of length at least 2^t form G_t, and by Lemma 3.7 the subgraph consisting of the shorter edges is also isomorphic to G_t.

Hence it suffices to prove the partitioning statement for $t \geq 2$; we use induction on t. Explicitly for $K_{5:1}$ when $t = 2$, or by the induction hypothesis when $t > 2$, every (K_3, K_3)-avoidance on H_{t-1} is regular. By Lemma 3.4, f is constant on length classes in H_t, with edges of lengths $2i$ and $2i+1$ having the same color. For $t = 2$, this suffices: the coloring of $K_{11:2}$ gives edges of lengths 2 and 3 the same color and gives edges of lengths 4 and 5 the same color. The two sets must have different colors, completing the description.

For $t > 2$, we must strengthen the statement about which classes have the same color. Let H be the subgraph of H_t induced by $\{v_{2j}: 0 \leq j \leq 3 \cdot 2^{t-1} - 2\}$. Note that $H \cong H_{t-1}$, even though v_{n-3} and v_0 are separated by three positions instead of two. Also f restricts to a (K_3, K_3)-avoidance on H. By the induction hypothesis, all edges of lengths 2^{t-2} through $2^{t-1} - 1$ in this copy H of H_{t-1} have one color, say red, and the longer edges are all blue. Edges of length i with respect to H have length $2i$ or $2i+1$ in the given numbering of H_t, and there is at least one of length $2i + 1$. Since the edges of length $2i$ or $2i + 1$ all have the same color, we conclude that all edges of length 2^{t-1} through $2^t - 1$ are red, and the longer edges are all blue. $\quad\square$

In the special case $\mathcal{F} = \mathcal{G} = \{G\}$, Corollary 2.6 states that $R_{\chi_c}(G) = 4$ when $2 < \chi_c(G) \leq \frac{5}{2}$. This is a stronger statement than would be provided by the statement of the next result for the omitted case $t = 1$. The theorem provides upper and lower bounds for $R_{\chi_c}(G)$ when $2.5 < \chi_c(G) < 3$.

Theorem 3.9. *Let* $z_t = 3 - 2^{-t}$. *If* $z_{t-1} < \chi_c(G) \leq z_t$ *with* $t \geq 2$, *then* $2z_{t-1} \leq R_{\chi_c}(G) \leq 2z_t$.

Proof. For $t \geq 1$, let $G_t = K_{(3 \cdot 2^t - 1):2^t}$ and $H_t = K_{(3 \cdot 2^t - 1):2^{t-1}}$, so $\chi_c(G_t) = z_t$ and $\chi_c(H_t) = 2z_t$. By Lemma 3.8, every 2-coloring of $E(H_t)$ has a monochromatic triangle (which is a homomorphic image of G_t) or has both color classes isomorphic to G_t. Hence $H_t \to \mathrm{Hom}(G_t)$, which yields $R_{\chi_c}(G_t) \leq$

$\chi_c(H_t) = 2z_t$. Also, since G_t is $K_{p:q}$ with $p/q = z_t$, it contains a homomorphic image of every graph with circular chromatic number at most z_t, and hence Corollary 2.5 yields $R_{\chi_c}(G) \le R_{\chi_c}(G_t) \le 2z_t$.

For the lower bound, Lemma 3.8 yields a 2-coloring of $E(H_{t-1})$ such that each color class is isomorphic to G_{t-1}. Since $\chi_c(G_t) > \chi_c(G_{t-1})$, neither class contains a homomorphic image of G_t. Therefore, $H_{t-1} \not\to \mathrm{Hom}(G_t)$. Note that H_{t-1} is $K_{r:s}$ with $r/s = 2z_{t-1}$. We have given $K_{r:s}$ a red/blue edge-coloring such that each color class has circular chromatic number z_{t-1}, which by hypothesis is less than $\chi_c(G)$. By Remark 2.7, $R_{\chi_c}(G) \ge r/s = 2z_{t-1}$. $\square$

A slightly different argument about length classes proves for $q \in \{3, 4\}$ that $R_{\chi_c}(G) \le 5 + \frac{1}{q}$ when $\chi_c(G) \le \frac{5}{2} + \frac{1}{2q}$. This refines the bounds when $\frac{5}{2} < \chi_c(G) \le \frac{8}{3}$ (the case $q = 2$ is already the case $t = 2$ in Theorem 3.9).

When $\chi_c(G)$ gets all the way to 3, we can determine $R_{\chi_c}(G)$ exactly.

Corollary 3.10. *If $\chi_c(G) = 3$, then $R_{\chi_c}(G) = 6$.*

Proof. We use the notation of the statement and proof of Theorem 3.9. For $t \ge 2$, let G'_t be a graph such that $\chi_c(G') = z_t$ and $G \in \mathrm{Hom}(G'_t)$. Such a graph G'_t always exists. For example, Theorem 4.2, which we will state shortly, allows us to use $G'_t = G \times G_t$, where $\times$ is the categorical product defined in Section 4. Thus G is a homomorphic image of G'_t. By Corollary 2.5 and Theorem 3.9, we have $6 = R_{\chi_c}(K_3) \ge R_{\chi_c}(G) \ge R_{\chi_c}(G'_t) \ge 2z_{t-1}$. Since $\lim_{t\to\infty} z_t = 3$, this yields $R_{\chi_c}(G) = 6$. $\square$

4. Extremal problem when $\chi_c \ge z$

Since $K_{p:q} \to K_{r:s}$ immediately yields $R_{\chi_c}(G) \le p/q$ for all G such that $\chi_c(G) \le r/s$, one naturally wonders how small R_{χ_c} can be among graphs with the same circular chromatic number. Note first that $\chi_c(G)$ does not determine $R_{\chi_c}(G)$. For example, K_4 and the graph G obtained from C_5 by adding one vertex joined to the other five both have circular chromatic number 4. However, in [12] it is shown that $R_\chi(G) = 14$, so $13 \le R_{\chi_c}(G) \le 14$, while our Proposition 3.5 shows $17 \le R_{\chi_c}(K_4) \le 17.5$.

Let $R_{\chi_c}(z) = \inf\{R_{\chi_c}(G): \chi_c(G) \ge z\}$ and $R_\chi(k) = \inf\{R_\chi(G): \chi(G) = k\}$. Using the result of Zhu [19] that $R_\chi(k) = (k-1)^2 + 1$ (conjectured in [2]),

$$R_{\chi_c}(k) = \inf\{R_{\chi_c}(G): \chi_c(G) \ge k\} \le \inf\{R_\chi(G): \chi_c(G) \ge k\}$$
$$\le \inf\{R_\chi(G): \chi(G) \ge k+1\} = k^2 + 1.$$

We use $\chi_c(G) \ge z$ instead of $\chi_c(G) = z$ in the definition of $R_{\chi_c}(z)$ because it is not clear that $\inf\{R_{\chi_c}(G): \chi_c(G) = k\} \le \inf\{R_\chi(G): \chi(G) = k\}$.

We prove the stronger inequality $R_{\chi_c}(k) \leq k(k-1)$ using the method of [19]. The *fractional chromatic number* of a graph G, written $\chi_f(G)$, is the linear programming relaxation of the chromatic number. That is, $\chi_f(G)$ is the minimum sum of weights on the independent sets in G such that each vertex receives total weight at least 1. It is well known that $\chi_f(G) \leq \chi_c(G)$ (as noted in [18], it is easy to obtain a weighting with total weight p/q from a (p,q)-coloring of G).

The *categorical* or *direct* product $G \times H$ of graphs G and H has vertex set $V(G) \times V(H)$, with (u,v) and (u',v') adjacent if and only if $uu' \in E(G)$ and $vv' \in E(H)$. By coloring $G \times H$ according to a proper coloring ϕ of one factor (for example, let the color of (u,v) be $\phi(u)$ for all v), always $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$, and the corresponding inequalities for χ_c and χ_f also hold. Hedetniemi [7] conjectured that always $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$. Zhu [19] proved the equality for χ_f, and Tardif [15] proved the equality for χ_c when the minimum over the factors is at most 4.

Theorem 4.1 ([19]). *Always $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$.*

Theorem 4.2 ([15]). *If $z \leq 4$ and $\chi_c(G), \chi_c(H) \geq z$, then $\chi_c(G \times H) \geq z$.*

Iterating the product yields $\chi_f(G_1 \times \cdots \times G_t) = \min\{\chi_f(G_1), \ldots, \chi_f(G_t)\}$. Similarly, $\chi_c(G_1 \times \cdots \times G_t) = \min\{\chi_c(G_1), \ldots, \chi_c(G_t)\}$ if the minimum is at most 4.

Lemma 4.3. *If every 2-edge-coloring of a graph H contains a monochromatic subgraph with fractional chromatic number at least z, then there exists a graph G with $\chi_f(G) \geq z$ such that $H \to \mathrm{Hom}(G)$. When $z \leq 4$, the same statement holds for circular chromatic number.*

Proof. Let t be the number of 2-edge-colorings of H. Let G_i be a graph with fractional chromatic number at least z that occurs as a monochromatic subgraph in the ith coloring. Let $G = G_1 \times \cdots \times G_t$. Each G_i is a homomorphic image of G, obtained by mapping the independent sets having a fixed value in the ith coordinate into the corresponding vertices in G_i. Hence $H \to \mathrm{Hom}(G)$, by construction.

When $\chi_f(G_i) \geq z$ for each i, the conclusion $\chi_f(G) \geq z$ follows from Theorem 4.1. When $z \leq 4$ and $\chi_c(G_i) \geq z$ for each i, the conclusion $\chi_c(G) \geq z$ follows from Theorem 4.2. $\square$

In the next result, the comment about circular chromatic number improves the upper bound on $R_{\chi_c}(z)$ when $z \leq 4$, because a monochromatic subgraph with $\chi_c \geq z$ may be forced by $K_{p:q}$ with smaller p/q than needed for $\chi_f \geq z$, since always $\chi_c \geq \chi_f$.

Lemma 4.4. *If every* 2-*edge-coloring of* $K_{p:q}$ *contains a monochromatic subgraph with fractional chromatic number at least* z, *then* $R_{\chi_c}(z) \leq p/q$; *that is, there exists a graph* G *with* $\chi_c(G) \geq z$ *and* $R_{\chi_c}(G) \leq p/q$. *When* $z \leq 4$, *the same conclusion follows also when the monochromatic subgraphs are only required to have circular chromatic number at least* z.

Proof. By Lemma 4.3, there exist a graph G with $\chi_f(G) \geq z$ and $K_{p:q} \to$ Hom(G). Since $\chi_c(G) \geq \chi_f(G)$ and $R_{\chi_c}(G) = \inf\{p/q\colon K_{p:q} \to \text{Hom}(G)\}$, the claim follows. When $z \leq 4$, applying the second statement of Lemma 4.3 yields $R_{\chi_c}(z) \leq p/q$ directly.   □

Theorem 4.5. $R_{\chi_c}(k) \leq k(k-1)$ *for* $k \in \mathbb{N} - \{1\}$.

Proof. Let R and B be the spanning subgraphs formed by the color classes in a red/blue edge-coloring of $K_{k(k-1)}$. If R has a clique of size k, then $\chi_f(R) \geq k$; otherwise, B has independence number at most $k-1$, and then $\chi_f(B) \geq k(k-1)/(k-1) = k$. Since $K_{k(k-1)} = K_{k(k-1):1}$, the claim follows from Lemma 4.4.   □

Note that although Theorem 4.5 applies only to integers, it is consistent with the bounds obtained in Theorem 3.9.

Question 4.6. Is it true for all $z \in \mathbb{R}$ with $z > 3$ that $R_{\chi_c}(z) \leq z(\lceil z \rceil - 1)$? If true, is the bound sharp?

5. Odd cycles

When $k = 1$, Theorem 3.9 yields $4 \leq R_{\chi_c}(G) \leq 5$ when $2 \leq \chi_c(G) \leq \frac{5}{2}$. We prove $R_{\chi_c}(G) = 4$ for all such graphs. As observed in Corollary 2.6, this yields $R_{\chi_c}(\mathcal{F}, \mathcal{G}) = 4$ whenever $\mathcal{F}$ and $\mathcal{G}$ both consist of nonbipartite graphs and have a member with circular chromatic number at most $\frac{5}{2}$.

Our main task is proving $R_{\chi_c}(C_5) = 4$. Since C_3 and C_5 are homomorphic images of C_5, we do this by proving for $q \geq 1$ that every 2-edge-coloring of $K_{4q+1:q}$ has a monochromatic 3-cycle or a monochromatic 5-cycle.

Let $K_{p:q}^-$ denote $K_{p:q} - v_0 v_q$ (deleting a shortest edge). We call the endpoints of the edge that was deleted the *special pair*. Let a 3, 5-*free coloring* of a graph G be a 2-edge-coloring having no monochromatic 3-cycle or 5-cycle.

Lemma 5.1. *Every* 3, 5-*free coloring of* $K_{5:1}^-$ *has monochromatic paths of length* 2 *in both colors joining* v_0 *and* v_4, *the endpoints of the missing edge.*

Proof. Consider a 3, 5-free coloring. There are nine edges; let red be the larger class. Each color class must be bipartite.

There are five or six red edges, since the maximum number of edges in a bipartite subgraph of K_5 is 6, achieved only by $K_{2,3}$. To have at least five red edges, the partite sets of the red graph must have sizes 2 and 3. Hence the red graph is $K_{2,3}$ with at most one edge deleted.

Since the blue graph must not contain a triangle, the partite set of size 3 must contain $\{v_0, v_4\}$. Now there is a blue path joining them through the third vertex of that part. There is a red path joining them via the other partite set, because there are two such possible paths and at most one edge was deleted from $K_{2,3}$ to form the red graph. $\qquad \square$

Lemma 5.2. *In* $V(K^-_{4q+1:q})$, *let* $S = \{v_0, v_q, v_{2q+1}, v_{3q+1}\}$ *and let* $T = \{v_0, v_q, v_{2q}, v_{3q+1}\}$. *Both* $K^-_{4q+1:q} - S$ *and* $K^-_{4q+1:q} - T$ *are isomorphic to* $K^-_{4(q-1)+1:q-1}$, *with* $\{v_{q+1}, v_{2q}\}$ *being the special pair when* S *is deleted and* $\{v_{2q+1}, v_{3q}\}$ *being the special pair when* T *is deleted.*

Proof. The vertices of S or T are spaced by $q, q, q, q+1$ (in cyclic order) along the indexing. Hence when S or T is deleted, any two vertices at least $q-1$ steps apart in the new indexing were separated by a deleted vertex and hence were at least q steps apart in the old indexing, except the pair $\{v_{q+1}, v_{2q}\}$ in the first case and the pair $\{v_{2q+1}, v_{3q}\}$ in the second case. Hence the edges are those of $K^-_{4(q-1)+1:q-1}$, with the special pairs as specified. $\qquad \square$

When $q = 2$, the special pairs in the two resulting subgraphs in Lemma 5.2 are $\{v_3, v_4\}$ and $\{v_5, v_6\}$. In the inductive proof of the main theorem, we will combine Lemma 5.2 with the following technical result about these two pairs in $K^-_{9:2}$. Write a path or cycle with vertices $v_1, \ldots, v_n$ in order as $\langle v_1, \ldots, v_n \rangle$ or $[v_1, \ldots, v_n]$, respectively.

Lemma 5.3. *Any* $3, 5$-*free coloring of* $K^-_{9:2} - \{v_1, v_8\}$ *having no monochromatic* v_3, v_4-*path or* v_5, v_6-*path of length 3 has monochromatic* v_0, v_2-*paths of length* 2 *in both colors.*

Proof. Let $G' = K^-_{9:2} - \{v_1, v_8\}$ and $G = G' - v_0$, shown in bold in Figure 1. Let G_r and G_b be the red and blue color classes of G under the given $3, 5$-free coloring. Since G has only six vertices, G_r and G_b are bipartite. We prove first that v_3 and v_4 are in the same partite set in each of G_r and G_b, as are v_5 and v_6. By symmetry, it suffices to forbid v_3 and v_4 being in opposite parts in G_r.

By hypothesis there is no red v_3, v_4-path of length 3, so being in opposite parts requires a spanning v_3, v_4-path P in G_r. After v_3, the next vertex u must be one of $\{v_5, v_6, v_7\}$. In each case, we obtain a contradiction. If $u = v_5$, then $P = \langle v_3, v_5, v_7, v_2, v_6, v_4 \rangle$, but then $\langle v_5, v_7, v_2, v_6 \rangle$ is a forbidden red

v_5, v_6-path of length 3. If $u = v_6$, then $P = \langle v_3, v_6, v_2, v_5, v_7, v_4 \rangle$. To avoid completing red odd cycles with edges of P, both $v_3 v_7$ and $v_7 v_2$ must be blue. Now there are v_3, v_2-paths of length 2 in both colors, and one extends along $v_2 v_4$ to complete a monochromatic v_3, v_4-path of length 3. If $u = v_7$, then $P = \langle v_3, v_7, v_5, v_2, v_6, v_4 \rangle$. To avoid completing red odd cycles with edges of P, all of $\{v_7 v_2, v_2 v_4, v_4 v_7\}$ must be blue, which completes a blue 3-cycle.
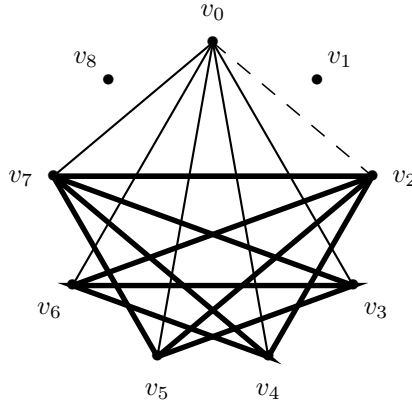

Figure 1: The graphs G' and G in Lemma 5.3.

Now each of $\{v_3, v_4\}$ and $\{v_5, v_6\}$ lies in one partite set in both G_r and G_b. Since $\{v_2, v_4, v_6\}$ and $\{v_3, v_5, v_7\}$ form triangles, putting all of v_3, v_4, v_5, v_6 into the same part in G_r or G_b forces v_2 and v_7 into the other part. Similarly, since $\{v_2, v_7\}$ cannot lie in the same part with v_4 or v_5, putting $\{v_3, v_4\}$ and $\{v_5, v_6\}$ into opposite parts forces v_2 and v_7 into opposite parts. Hence each of the resulting bipartions R and B of the indices has three possibilities: $(3456|27)$, $(347|562)$, and $(342|567)$. Since the edges within a partite set get the other color, each choice for R restricts the choice for B. Since the two subgraphs cannot have the same bipartition, by symmetry there remain three cases. In each case we study G' to obtain the monochromatic v_0, v_2-paths of length 2 in both colors.

Case 1. $R = (3456|27)$, $B = (347|562)$. If $v_0 v_7$ is red, then avoiding $[v_0, v_7, v_4]$ in red makes $v_0 v_4$ blue. Now avoiding $[v_0, v_4, v_6, v_3, v_5]$ in blue makes $v_0 v_5$ red, so $\langle v_0, v_5, v_2 \rangle$ is red. Avoiding $[v_4, v_7, v_0, v_5, v_2]$ in red makes $v_4 v_2$ blue, so $\langle v_0, v_4, v_2 \rangle$ is blue.

If $v_0 v_7$ is blue, then $\langle v_0, v_7, v_2 \rangle$ is blue. Avoiding $\langle v_0, v_5, v_2 \rangle$ and $\langle v_0, v_6, v_2 \rangle$ in red would make $v_0 v_5$ and $v_0 v_6$ blue. Avoiding $[v_0, v_4, v_6]$ in blue then

makes v_0v_4 red. Avoiding $[v_2, v_7, v_0, v_6, v_4]$ in blue makes v_4v_2 red, and now $\langle v_0, v_4, v_2 \rangle$ is red.

Case 2. $R = (3456|27)$, $B = (342|567)$. If v_0v_7 is red, then avoiding $[v_0, v_7, v_5]$ in red makes v_0v_5 blue. Now avoiding $[v_0, v_5, v_3, v_6, v_4]$ in blue makes v_0v_4 red, and hence $\langle v_0, v_4, v_2 \rangle$ is red. Avoiding $[v_5, v_7, v_0, v_4, v_2]$ in red makes v_5v_2 blue, so $\langle v_0, v_5, v_2 \rangle$ is blue.

If v_0v_7 is blue, then $\langle v_0, v_7, v_2 \rangle$ is blue. Avoiding $\langle v_0, v_4, v_2 \rangle$ in red would make v_0v_4 blue, and then avoiding $[v_0, v_4, v_6]$ in blue makes v_0v_6 red. Avoiding $[v_2, v_7, v_0, v_4, v_6]$ in blue makes v_2v_6 red, and now $\langle v_0, v_6, v_2 \rangle$ is red.

Case 3. $R = (342|567)$, $B = (347|562)$. If v_0v_7 is red, then avoiding $[v_0, v_7, v_4]$ in red makes v_0v_4 blue, so $\langle v_0, v_4, v_2 \rangle$ is blue. Now avoiding $[v_0, v_4, v_2, v_7, v_5]$ in blue makes v_2v_7 or v_0v_5 red, so $\langle v_0, v_7, v_2 \rangle$ or $\langle v_0, v_5, v_2 \rangle$ is red.

If v_0v_7 is blue, then avoiding $[v_0, v_7, v_5]$ in blue makes v_0v_5 red, so $\langle v_0, v_5, v_2 \rangle$ is red. Avoiding $[v_0, v_5, v_2, v_7, v_4]$ in red makes v_2v_7 or v_0v_4 blue, so $\langle v_0, v_7, v_2 \rangle$ or $\langle v_0, v_4, v_2 \rangle$ is blue. $\qquad\square$

Theorem 5.4. $R_{\chi_c}(C_5) = 4$.

Proof. It suffices to show $K_{4q+1:q} \to \{C_3, C_5\}$ for $q \geq 1$. We use induction on q to prove that every $3, 5$-free coloring of $K^-_{4q+1:q}$ contains monochromatic v_0, v_q-paths of length 2 in both colors. Adding the edge v_0v_q then completes a monochromatic triangle. Lemma 5.1 proves the case $q = 1$.

For $q > 1$, let $G = K^-_{4q+1:q}$, and consider a $3, 5$-free coloring of G. Let $S = \{v_0, v_q, v_{2q+1}, v_{3q+1}\}$ and $T = \{v_0, v_q, v_{2q}, v_{3q+1}\}$. By Lemma 5.2, both $G - S$ and $G - T$ are isomorphic to $K^-_{4(q-1)+1:q-1}$, with special pairs $\{v_{q+1}, v_{2q}\}$ and $\{v_{2q+1}, v_{3q}\}$, respectively. By the induction hypothesis, there are monochromatic v_{q+1}, v_{2q}-paths and v_{2q+1}, v_{3q}-paths of length 2 in both colors. A monochromatic v_{q+1}, v_{2q}-path or v_{2q+1}, v_{3q}-path of length 3 in G then completes a monochromatic closed odd walk of length 5, which yields a monochromatic 3-cycle or 5-cycle, so there is no such path for either pair.

The subgraph of G induced by $\{v_0, v_q, v_{q+1}, v_{2q}, v_{2q+1}, v_{3q}, v_{3q+1}\}$ is now isomorphic to $K^-_{9:2} - \{v_1, v_8\}$, with vertices representing $v_0, v_2, v_3, v_4, v_5, v_6, v_7$ in order. By Lemma 5.3, there are monochromatic v_0, v_q-paths of length 2 in both colors. $\qquad\square$

6. Non-diagonal Ramsey numbers for cycles

In this section we prove $4 \leq R_{\chi_c}(C_3, C_7) \leq \frac{9}{2} < \frac{14}{3} \leq R_{\chi_c}(C_3, C_5) \leq 5$.

For $R_{\chi_c}(C_3, C_5)$, we first found a lower bound of $\frac{9}{2}$ using a red/blue coloring of $K_{9:2}$ that was not constant on distance classes. In fact, this coloring arises by the natural homomorphism from $K_{9:2}$ into the coloring of $K_{14:3}$ in Theorem 6.1 (found by Daniel Cranston). It is not yet known whether distance-invariant optimal lower-bound colorings can always be found for $R_{\chi_c}(F, G)$, although certainly such colorings are much easier to study than more general colorings.

Theorem 6.1. $\frac{14}{3} \leq R_{\chi_c}(C_3, C_5) \leq 5$.

Proof. For the upper bound, it suffices to show $K_{5:1} \to (\mathrm{Hom}(C_3), \mathrm{Hom}(C_5))$. Note that $\mathrm{Hom}(C_5) = \{C_3, C_5\}$. We have remarked that the only red/blue-coloring of K_5 having no monochromatic triangle has monochromatic 5-cycles in both colors.

For the lower bound, we color $E(K_{14:3})$. Edges of lengths 3 and 4 are blue; edges of lengths 5, 6, and 7 are red. Since the lengths of two red edges sum to at least 10 and the lengths of three sum to at least 15, there is no red triangle. The lengths of two blue edges sum to at least 6 and the lengths of three sum to at most 12, so there is no blue triangle.

Now consider blue 5-cycles. The cycle may take steps in both directions, but the total net movement in one direction must be a multiple of 14. With five steps in the same direction, the sum is at least 15 and at most 20. With four edges in one direction (total from 12 to 16) and one in the other (length 3 or 4), the net movement is at least 8 and at most 13, not a multiple of 14. Similarly, three edge in one direction (total from 9 to 12) and two in the other (total from 6 to 8) yield net movement at least 1 and at most 6, again not a multiple of 14. Hence there is no blue 5-cycle. □

Other colorings of $E(K_{14:3})$ also establish this lower bound. One can make the edges of lengths 3 and 4 red, the edges of lengths 6 and 7 blue, and alternate red and blue along the 14-cycle formed by the edges of length 5. Both this and the coloring in Theorem 6.1 have 35 red edges and 28 blue edges, but they are not isomorphic. There is also a messy coloring that is not distance-invariant. This multiplicity of colorings suggests that there could be better colorings, but we show next that they will be hard to find.

Remark 6.2. The construction in Theorem 6.1 does not generalize to a larger lower bound for $R_{\chi_c}(C_3, C_5)$ or a smaller lower bound for $R_{\chi_c}(C_3, C_{2k+1})$. Consider a red/blue coloring of $E(K_{p:q})$ with $p/q > 4$ such that all edges with lengths from q to r are blue, all edges with lengths from $r+1$ to $\lfloor p/2 \rfloor$ are red, no triangle is red, and no odd cycle of length at most $2k+1$ is blue. In Theorem 6.1 we have the case $(p, q, r) = (14, 3, 4)$ and $k = 2$.

To avoid red triangles, we must have $r+1 > p/3$. Avoiding blue triangles requires $r < p/3$. Hence $p \in \{3r+1, 3r+2\}$. To avoid creating blue 5-cycles that take three steps forward and two steps back, we need $3q > 2r$. Now $2p \le 6r + 4 \le 9q + 1$. However, we must also avoid blue 5-cycles with four steps forward and one step back for net movement p. The smallest such total is $4q - r$, the largest is $4r - q$, and all values between them are achievable. With $q = r - s$, these values range from $3r - 3s$ to $3r + s$.

Since $p \in \{3r+1, 3r+2\}$, the range includes p unless $s = 1$ and $p = 3r+2$. Now also $q = r - 1$, so $p/q = 3 + 5/q$. Since we want $4 < p/q < 5$, the only possible instances are $(p,q,r) = (14,3,4)$ (as in Theorem 6.1) and $(p,q,r) = (17,4,5)$. Since $\frac{17}{4} < \frac{14}{3}$, the second construction would be of interest only for $R_{\chi_c}(C_3, C_7)$. However, the construction contains a blue 7-cycle in that case, since $6 \cdot 5 + 4 = 2 \cdot 17$.

Making the short edges red and long edges blue also does not succeed.

We next give an upper bound for $R_{\chi_c}(C_3, C_7)$. Let a *good coloring* of a graph be a red/blue edge-coloring having no red triangle and no blue triangle or 5-cycle. Recall from Section 5 that the endpoints of the missing edge in $K_{5:1}^-$ are v_0 and v_4, cyclically indexed.

Lemma 6.3. *Good colorings of $K_{5:1}^-$ are of two types. One has monochromatic paths of length 2 in both colors joining v_0 and v_4. The other consists of a red 5-cycle and a spanning blue path with endpoints v_0 and v_4. edge.*

Proof. By Lemma 5.1, every good coloring having no monochromatic 3-cycle or 5-cycle is of the first type. Hence we need only consider colorings that have a red 5-cycle. Since v_0 and v_4 are not adjacent, they are not consecutive on the cycle. Since v_1, v_2, and v_3 are in the same isomorphism class, we may assume by symmetry that the cycle visits v_0, v_1, v_4, v_2, v_3 in order. Any red chord creates a red triangle, so the remaining four edges must be blue, yielding the blue path with vertices v_0, v_2, v_1, v_3, v_4 in order. $\square$

Theorem 6.4. $R_{\chi_c}(C_3, C_7) \le \frac{9}{2}$.

Proof. It suffices to show that every red/blue coloring of $K_{9:2}$ has a red triangle or a blue odd cycle with length at most 7. Consider a coloring that avoids this.

If the edges of length 2 are all red, then the edges of length 4 form a blue 9-cycle. Any additional blue edge would be a chord yielding a shorter blue odd cycle, so the edges of length 3 are all red and form triangles.

Hence we may assume by symmetry that the edge $v_0 v_2$ is blue. Consider the copies of $K_{5:1}^-$ induced by $\{v_1, v_3, v_5, v_7, v_0\}$ and $\{v_2, v_4, v_6, v_8, v_1\}$; call them Q^- and Q^+, respectively. Note that Q^- and Q^+ share only v_1. By Lemma 6.3, if either Q^- or Q^+ has a good coloring of the first type, then

the union of the guaranteed blue v_0, v_1-path in Q^-, the guaranteed blue v_1, v_2-path in Q^+, and the edge $v_2 v_0$ is a blue 5-cycle or 7-cycle.

If both Q^- and Q^+ have good colorings of the second type, then the union of the guaranteed blue paths with the edge $v_2 v_0$ is a blue 9-cycle. Again any additional blue edge would yield a shorter blue odd cycle, so the blue graph is precisely a 9-cycle.

If both copies have blue paths of length 4, then G has a blue 9-cycle and any additional blue edge is a chord and completes a shorter cycle. Therefore, the blue graph is C_9. Edges joining $V(Q^-)$ and $V(Q^+)$ must now all be red. These include $v_0 v_4$, $v_4 v_7$, $v_0 v_6$, and $v_6 v_3$. To avoid red triangles, $v_0 v_7$ and $v_0 v_3$ must be blue. Now v_0 has two incident blue edges other than $v_0 v_2$, which contradicts the blue graph being C_9.   □

The difficulty of extending the construction of Theorem 6.1 via Remark 6.2 suggests perhaps $R_{\chi_c}(C_3, C_5) = \frac{14}{3}$. The same difficulty, coupled with Theorem 6.4, suggests perhaps $R_{\chi_c}(C_3, C_{2k+1}) = 4$ for $k \geq 3$.

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