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Incidence coloring of planar graphs without adjacent small cycles

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Dedicated to Adrian Bondy on the occasion of his 70^{th} birthday

An incidence of an undirected graph G is a pair (v, e) where v is a vertex of G and e an edge of G incident with v. Two incidences (v, e) and (w, f) are adjacent if one of the following holds: (i) $v = w$, (ii) $e = f$ or (iii) $vw = e$ or f. An incidence coloring of G assigns a color to each incidence of G in such a way that adjacent incidences get distinct colors. In 2012, Yang [15] proved that every planar graph has an incidence coloring with at most $\Delta + 5$ colors, where Δ denotes the maximum degree of the graph. In this paper, we show that $\Delta + 4$ colors suffice if the graph is planar and without a C_3 adjacent to a C_4. Moreover, we prove that every planar graph without C_4 and C_5 and maximum degree at least 5 admits an incidence coloring with at most $\Delta + 3$ colors.

1. Introduction

Let G be a graph without loops and multiple edges. Let $V(G)$ and $E(G)$ be its vertex and edge set respectively. We denote by $\Delta(G)$ the maximum degree of G.

A *proper edge-coloring* of a graph $G = (V, E)$ is an assignment of colors to the edges of the graph such that no two adjacent edges use the same color. A *strong edge-coloring* (called also distance 2 edge-coloring) of G is a proper edge-coloring where each color class induces a matching. We denote by $\chi'_s(G)$ the strong chromatic index of G which is the smallest integer k such that G can be strong edge-colored with k colors.

An *incidence* in G is a pair (v, e) with $v \in V(G)$ and $e \in E(G)$, such that v and e are incident. The set of all incidences in G is denoted by $I(G)$, where

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$$I(G) = \{(v, e) \in V(G) \times E(G) : \text{edge } e \text{ is incident to } v\}.$$

Two incidences (u, e) and (v, f) are adjacent if one of the following holds:

$$i)\ u = v,\ ii)\ e = f, \text{ and } iii) \text{ the edge } uv = e \text{ or } uv = f.$$

An *incidence k-coloring* of a graph G is defined as a function ϕ from $I(G)$ to a set of colors $C = \{1, 2 \cdots, k\}$, such that adjacent incidences are assigned with distinct colors. The minimum cardinality k for which G has an incidence k-coloring is the incidence chromatic number $\chi_i(G)$ of G.

An alternate way of looking at the incidence chromatic number of a graph G is to consider the bipartite graph $G' = (X \cup Y, E)$, obtained from G such that, $X = V(G)$, $Y = E(G)$ and $E(G') = \{(v, e), v \in V(G), e \in E(G), v \textit{ is incident with } e\}$. Each edge of G' corresponds to an incidence of G; therefore, any incidence coloring of G corresponds to a strong edge coloring of G'.

$$\chi_i(G) = \chi'_s(G').$$

The notion of incidence coloring was introduced by Brualdi and Massey [3] in 1993. They proved the following theorem:

Theorem 1 (Brualdi and Massey [3])**.** *For every graph G, $\Delta(G) + 1 \leq \chi_i(G) \leq 2\Delta(G)$.*

And they proposed the Incidence Coloring Conjecture, which states that:

Conjecture 1 (Brualdi and Massey [3])**.** *For every graph G, $\chi_i(G) \leq \Delta(G) + 2$.*

However, in 1997, by observing that the concept of incidence coloring is a particular case of directed star arboricity introduced by Algor and Alon [1], Guiduli [6] disproved the Incidence Coloring Conjecture showing that Paley graphs have an incidence chromatic number at least $\Delta + \Omega(\log \Delta)$. He also improved the upper bound proposed by Brualdy and Massey in Theorem 1.

Theorem 2 (Guiduli [6])**.** *For every graph G, $\chi_i(G) \leq \Delta(G) + O(\log \Delta(G))$.*

The incidence coloring of graphs has been extensively studied. Most of authors consider the values of $\chi_i(G)$ on particular classes of graphs (tree [3], cubic graphs [7, 10, 14], Halin graphs [12], k-degenerated graphs [4], K_4-minor free graph [4], outerplanar graphs [13], regular graphs and complement graphs [11], pseudo-Halin graphs [8], the powers of cycles [9], graphs with maximum degree 3 [7]).

In [4], Hosseini Dolama, Sopena and Zhu gave an upper bound of $\chi_i(G)$ for planar graph.

Theorem 3 (Hosseini Dolama *et al.* [4]). *For every planar graph G, $\chi_i(G) \leq \Delta(G) + 7$.*

This last result was improved by Yang in [15], (paper written in 2007) by using the link between the incidence chromatic number, the star arboricity and the chromatic index of a graph:

Theorem 4 (Yang [15]). *For every planar graph G, $\chi_i(G) \leq \Delta(G) + 5$, if $\Delta(G) \neq 6$ and $\chi_i(G) \leq 12$, if $\Delta(G) = 6$*

An interesting question is to see how the incidence chromatic number behaves for sparse planar graphs. Recall that the girth of a graph is the length of a shortest cycle in this graph. For instance, we collect results concerning the incidence chromatic number of planar graphs in the following lemma:

Lemma 1.

1. $\chi_i(G) \leq \Delta(G) + 4$ *for every triangle free planar graph G. [5]*
2. $\chi_i(G) \leq \Delta(G) + 3$ *for every planar graph G with girth $g \geq 6$. [5]*
3. $\chi_i(G) \leq \Delta(G) + 2$ *for every planar graph G with girth $g \geq 6$ and $\Delta(G) \geq 5$. [5]*
4. $\chi_i(G) \leq \Delta(G) + 2$ *for every planar graph G with girth $g \geq 11$. [5]*
5. $\chi_i(G) = \Delta(G) + 1$ *for every planar graph G with girth $g \geq 14$ and $\Delta(G) \geq 4$. [2]*

Our mains results in this paper improve the upper bound in Theorem 3 and in Theorem 4 for some classes of planar graphs. We denote by C_k a cycle of length k ($k \in \mathbb{N}, k \geq 3$). In particular, we show the following.

Theorem 5.

1. $\chi_i(G) \leq \Delta(G) + 4$ *for every planar graph G without a C_3 adjacent to a C_4.*
2. $\chi_i(G) \leq \Delta(G) + 3$ *for every planar graph G without C_4 and C_5 when $\Delta(G) \neq 4$, and $\chi_i(G) \leq 8$ if $\Delta(G) = 4$.*

From this first item of the previous Theorem we easely deduce:

Corollary 1. $\chi_i(G) \leq \Delta(G) + 4$ *for every planar graph G without a C_4.*

Before proving our results we introduce some notations.

2. Notation

Let G be a planar graph. We use $V(G)$, $E(G)$, and $F(G)$ to denote, respectively, the set of vertices, edges, and faces of G. Let $d(v)$ denote the degree of a vertex v in G and $r(f)$ the degree of a face f in G. A vertex of degree k is called a k-vertex. A k^+-vertex (respectively, k^--vertex) is a vertex of degree at least k (respectively, at most k). A $(l_1, \cdots, l_k)$-vertex is a k-vertex having k-neighbors $x_1, \cdots, x_k$ such that $d(x_i) = l_i$ for $i \in \{1, \cdots, k\}$. We will also use for l_i the notation l_i^+ (respectively l_i^-), if x_i is a vertex of degree at least l_i (respectively at most l_i). We use the same notations for faces: a k-face (respectively, k^+-face, k^--face) is a face of degree k (respectively, at least k, at most k). A k-face having the boundary vertices x_1, x_2, ..., x_k in the cyclic order is denoted by $[x_1x_2...x_k]$. A $(l_1, \cdots, l_k)$-face is a k-face $[x_1x_2...x_k]$ such that $d(x_i) = l_i$ for $i \in \{1, \cdots, k\}$. We will also use for l_i the notation l_i^+ (respectively l_i^-), if x_i is a vertex of degree at least l_i (respectively at most l_i). A (k_1, k_2, k_3)-triangle is a 3-face $[x_1x_2x_3]$ with $d(x_1) = k_1$, $d(x_2) = k_2$ and $d(x_3) = k_3$. As above, we will use for k_i the notation k_i^+ (respectively k_i^-), if x_i is a vertex of degree at least k_i (respectively at most k_i). For a vertex $v \in V(G)$, let $n_i(v)$ denote the number of i-vertices adjacent to v for $i \geq 1$, and $m_i(v)$ the number of i-faces incident to v for $i \geq 1$.

If ϕ is an incidence coloring of a graph G, and S a set of incidences of G, then $\phi(S)$ denotes the set of colors used to color the incidences belonging to S.

Definition 1 (Hosseini Dolama *et al.* [4]). *Let G be a graph, for every vertex u of G we denote by I_u the set of incidences of the form (u, uw) and by A_u the set of incidences of the form (w, wu), for all neighbors w of u.*

1. *A partial incidence coloring ϕ' of G, is an incidence coloring only defined on some subset I of $I(G)$. For every uncolored incidence $(u, uv) \in I(G) \setminus I$, $F_G^{\phi'}(u, uv)$ is defined by the set of forbidden colors of (u, uv), that is:*
$$F_G^{\phi'}(u, uv) = \phi'(A_u) \cup \phi'(I_u) \cup \phi'(I_v),$$
2. *An incidence (k, l)-coloring of a graph G is a incidence k-coloring ϕ of G such that for every vertex $v \in V(G)$, $| \phi(A_v) | \leq l$.*

Remark 1. *It is easy to see that every incidence (k, l)-coloring is also an incidence (k', l)-coloring for any $k' > k$.*

The following observation will be used implicitly throughout.

Observation 1. *For every graph G with maximum degree $\Delta(G)$, by Theorem 1 and Definition 1, G admits an incidence $(2\Delta(G), \Delta(G))$-coloring.*

3. Proof of Theorem 5.1

We will prove the following stronger version of Theorem 5.1:

Theorem 6. *Every planar graph G without a C_3 adjacent to a C_4 admits an incidence $(k+4,4)$-coloring for every $k \geq \Delta(G)$, $k \in \mathbb{N}$. Therefore, $\chi_i(G) \leq \Delta(G)+4$.*

Observation 2. *We have to consider only $k \geq \Delta(G) \geq 5$ since otherwise we obtain by Theorem 1*

$$\chi_i(G) \leq 2\Delta(G) \leq \Delta(G)+4 \leq k+4.$$

3.1. Structural properties

We proceed by contradiction. Let H be a counterexample to the theorem that minimizes $|E(H)| + |V(H)|$. By hypothesis there exists $k \geq \max\{\Delta(G), 5\}$ such that H does not admit an incidence $(k+4,4)$-coloring. Let $k \geq \max\{\Delta(G), 5\}$ be the smallest integer such that H does not admit an incidence $(k+4,4)$-coloring. By using Remark 1, we must have $k = \max\{\Delta(G), 5\}$. Moreover by minimality it is easy to see that H is connected. H satisfies the following properties:

Lemma 2. *H does not contain:*

1. *1-vertices,*
2. *2-vertices,*
3. *a 3-vertex adjacent to a 4^--vertex,*
4. *a $(4^-, 4^-, \Delta^-)$-triangle,*
5. *a $(3,3,3,3,\Delta^-)$-vertex.*

For each of the parts of Lemma 2, we will suppose that the described configuration exists in H. Then we construct a graph H' obtained from H by deleting a certain number of vertices and edges. Due to the minimality of H, the graph H' admits an incidence $(k'+4,4)$-coloring ϕ' for any $k' \geq \max\{\Delta(H'), 5\}$. Since $\Delta(H) \geq \Delta(H')$, the set of integers k' contains the set of integers k. Hence for the value $k' = k$, H' admits an incidence $(k+4,4)$-coloring ϕ'. Finally, for each case, we will prove a contradiction by extending ϕ' to an incidence $(k+4,4)$-coloring ϕ of H.

Proof. We recall that $k \geq 5$, its implies that the minimum number of colors we can use is 9.

1. Suppose H contains a 1-vertex u and let v be its unique neighbor in H. Consider $H' = H - \{u\}$. By minimality of H, H' admits an incidence $(k+4,4)$-coloring ϕ'. We will extend ϕ' to an incidence $(k+4,4)$-coloring ϕ of H as follows.
   Since for all $w \in V(H')$, $\mid \phi'(A_w) \mid \leq 4$, we have $\mid F_H^{\phi'}(v,vu) \mid = \mid \phi'(I_v) \cup \phi'(A_v) \cup \phi'(I_u) \mid \leq \Delta(H) - 1 + 4 + 0 = \Delta(H) + 3 \leq k+3$, then there exists at least one color, say α, such that $\alpha \notin F_H^{\phi'}(v,vu)$. Hence, we set $\phi(v,vu) = \alpha$ and one can observe that $\mid \phi(A_u) \mid = 1 \leq 4$. According to $\mid \phi'(A_v) \mid \leq 4$, it suffices to set $\phi(u,uv) = \beta$ for any color β in $\phi'(A_v)$ and we are done. We have extended the coloring, a contradiction.
2. Suppose H contains a 2-vertex v and let u, w be the two neighbors of v in H. Consider $H' = H - \{uv\}$. Then by minimality of H, H' admits an incidence $(k+4,4)$-coloring ϕ'. We will extend ϕ' to an incidence $(k+4,4)$-coloring ϕ of H as follows. First, we uncolor the incidence (v,vw) and assume that $\phi'(w,wv) = \beta$. By a counting argument, there exists at least one color $\alpha \notin F_H^{\phi'}(u,uv)$. Then we color (u,uv) with α and $\mid \phi(A_v) \mid = 2 \leq 4$. For coloring the incidence (v,vu), we consider the following cases:
   (a) If $\mid \phi'(A_u) \mid = 4$ then we color (v,vu) with a color $\gamma \in \phi'(A_u) \setminus \{\beta\}$ (note that $\alpha \notin \phi'(A_u)$).
   (b) If $\mid \phi'(A_u) \mid \leq 3$ then we color (v,vu) with a color $\gamma \notin F_H^{\phi'}(v,vu)$ (note that we have three choices). One can observe that $\mid \phi(A_u) \mid \leq 4$.

   Now, we color the incidence (v,vw) as follows:
   (a) If $\mid \phi'(A_w) \mid = 4$ then we color (v,vw) with a color $\zeta \in \phi'(A_w) \setminus \{\alpha,\gamma\}$ and we have $\mid \phi(A_v) \mid = 2$.
   (b) If $\mid \phi'(A_w) \mid \leq 3$ then we color (v,vw) with a color $\zeta \notin F_H^{\phi'}(v,vw)$ and we have $\mid \phi(A_w) \mid \leq 4$ (note that we have two choices).

   So, we have extended the coloring, a contradiction.
3. Suppose H contains a 3-vertex u adjacent to a 4^--vertex v. Consider $H' = H - \{uv\}$. By minimality of H, H' admits an incidence $(k+4,4)$-coloring ϕ'. We will extend ϕ' to an incidence $(k+4,4)$-coloring ϕ of H as follows. As above, by a counting argument, it is easy to see that there exists at least one color $\alpha \notin F_H^{\phi'}(v,vu)$. Then we color the incidence (v,vu) with α and $\mid \phi(A_u) \mid \leq 3 \leq 4$. Now we color the incidence (u,uv) with a color $\beta \notin F_H^{\phi'}(u,uv) \cup \{\alpha\}$ ($\mid F_H^{\phi'}(u,uv) \mid \leq 7$), we have $\mid \phi(A_v) \mid \leq 4$. We have extended the coloring, a contradiction.

4. Suppose H contains a 3-face $[uvw]$ such that $d(u) \leq 4$, $d(v) \leq 4$. By minimality of H, the graph $H' = H - \{uv\}$ admits an incidence $(k+4, 4)$-coloring. We will extend ϕ' to an incidence $(k+4, 4)$-coloring ϕ of H as follows.

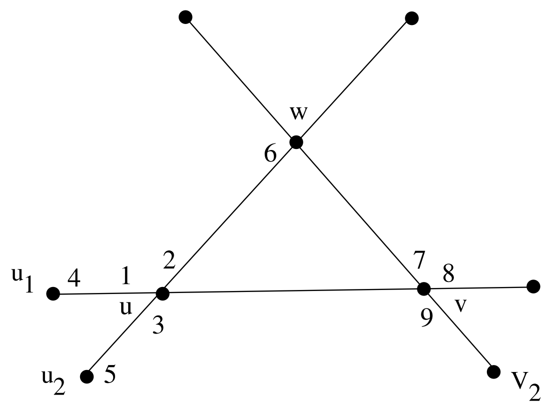

Figure 1: $\mid F_H^{\phi'}(u, uv) \mid = 9$.

First we color (u, uv).
It is easy to see that

$$\mid F_H^{\phi'}(u, uv) \mid = \mid \phi'(I_u) \cup \phi'(A_u) \cup \phi'(I_v) \mid \leq 3 + 3 + 3 = 9$$

- Assume that: $\mid F_H^{\phi'}(u, uv) \mid = 9$, then we are in the situation described in Figure 1. We replace 2 by 7 and we color (u, uv) with 2 ($7 \notin \phi(I_w)$ by definition).
  Now we consider $\phi'(A_u)$ and $\phi'(A_v)$. If there is a color a in $\phi'(A_u) \setminus \phi'(A_v)$. We color (v, vu) with a. If $\phi'(A_u) = \phi'(A_v)$, let b be the color of (w, wv), we interchange the colors of (w, wu) and (w, wv). Then we color (v, vu) with the color b.
- We assume now that $\mid F_H^{\phi'}(u, uv) \mid < 9$ and $\mid F_H^{\phi'}(v, vu) \mid < 9$. If there are two free colors for one of the two incidences, it is done. So we assume that $\mid F_H^{\phi'}(u, uv) \mid = 8$ and $\mid F_H^{\phi'}(v, vu) \mid = 8$ and the two incidences have the same free color.
  We have two cases:

  Case 1: $\mid \phi'(A_u) \mid = 2$

  Case 1.1 Without loss of generality, assume that $\phi'(u_1, u_1u) = \phi'(u_2, u_2u) = 4$, and that the only free color is 5.

(a) If we can replace 7 by 2 ($2 \notin \phi(I_w)$), then we color (u, uv) with 7 and (v, vu) by 5, it is done. Hence we cannot replace 7 by 2, it means that $2 \in \phi'(A_v \setminus (w, wv))$. Without loss of generality let $\phi'(v_1, v_1v) = 2$.

(b) If we can color (v, vu) with 6, then we color (u, uv) with 5 and it is done. Hence $6 \in \phi'(A_v)$, the only possibility is $\phi'(v_2, v_2v) = 6$.

(c) If we can color (v, vu) with 4, then we color (u, uv) with 5 and it is done. Hence $4 \in \phi'(A_v)$, then the only possibility is $\phi'(w, wv) = 4$.

So now we permute 6 and 4 around w and we color (v, vu) with 4 and (u, uv) with 5. It follows that we have extended the coloring, a contradiction.

Case 1.2 If $\phi'(u_1, u_1u) \neq \phi'(u_2, u_2u)$, Without loss of generality, we can assume that $\phi'(w, wu) = 4$ and that the only free color is 6.

(a) If we can replace 7 by 2 ($2 \notin \phi(I_w)$), then we color (u, uv) with 7 and (v, vu) by 6, it is done. Hence ve cannot replace 7 by 2, it means that $2 \in \phi'(A_v)$. Without loss of generality let $\phi'(v_1, v_1v) = 2$.

(b) If we can color (v, vu) with 4, then we color (u, uv) with 6 and it is done. Hence $4 \in \phi'(A_v)$, let $\phi'(v_2, v_2v) = 4$.

(c) If we can color (v, vu) with 5, then we color (u, uv) with 6 and it is done. Hence $5 \in \phi'(A_v)$, let $\phi'(w, wv) = 5$.

So now we permute 4 and 5 around w and we color (v, vu) with 5 and (u, uv) with 6. It follows that we have extended the coloring, a contradiction.

Case 2: $\mid \phi'(A_u) \mid = 3$

In this case one of the colors of (u_1, u_1u) or (u_2, u_2u) must be in $\phi'(I_v)$ or the color of (w, wu) is in $\phi'(I_v)$.

Case 2.1 One of the colors of (u_1, u_1u) or (u_2, u_2u) is in $\phi'(I_v)$. In our figure assume that 4 is the color of (v, vv_1). Then the free color is 8.

(a) If we can replace 7 by 2, then we color (u, uv) with 7 and (v, vu) by 8, it is done. Hence ve cannot replace 7 by 2, it means that $2 \in \phi'(A_v)$. Without loss of generality let $\phi'(v_1, v_1v) = 2$.

(b) If we can color (v, vu) with 6, then we color (u, uv) with 8 and it is done. Hence $6 \in \phi'(A_v)$, let $\phi'(v_2, v_2v) = 6$.

(c) If we can color (v, vu) with 5, then we color (u, uv) with 8 and it is done. Hence $5 \in \phi'(A_v)$, let $\phi'(w, wv) = 5$.

So now we permute 6 and 5 around w and we color (v, vu) with 5 and (u, uv) with 8. It follows that we have extended the coloring, a contradiction.

Case 2.2 The color of (w, wu) is in $\phi'(I_v)$. In our figure assume that 6 is the color of (v, vv_1). Then the free color is 8.

(a) If we can replace 7 by 2, then we color (u, uv) with 7 and (v, vu) by 8, it is done. Hence we cannot replace 7 by 2, it means that $2 \in \phi'(A_v)$. Without loss of generality let $\phi'(v_1, v_1v) = 2$.

(b) If we can color (v, vu) with 4, then we color (u, uv) with 8 and it is done. Hence $4 \in \phi'(A_v)$, let $\phi'(v_2, v_2v) = 4$ or $\phi'(w, wv) = 4$.

(c) If we can color (v, vu) with 5, then we color (u, uv) with 8 and it is done. Hence $5 \in \phi'(A_v)$, let $\phi'(w, wv) = 5$ (resp $\phi'(v_2, v_2v) = 5$) if $\phi'(v_2, v_2v) = 4$ (resp. $\phi'(w, wv) = 4$).

Now we replace 2 by 7 ($7 \notin \phi(I_w)$) then we color (u, uv) with 2 and (v, vu) by 8, it is done. It completes the proof.

5. Suppose H contains a $(3, 3, 3, 3, \Delta^-)$-vertex u. Let u_i for $i \in \{1, 2, 3, 4\}$ be the neighbors of u having a degree equal to 3 and v be the neighbor such that $d(v) \leq \Delta$. By minimality of H, the graph $H' = H - \{u\}$ admits an incidence $(k + 4, 4)$-coloring ϕ'. We will extend ϕ' to an incidence $(k + 4, 4)$-coloring ϕ of H as follows.

   - We have $\mid F_H^{\phi'}(v, vu) \mid = \mid \phi'(I_v) \cup \phi'(A_v) \cup \phi'(I_u) \mid \leq \Delta(H) - 1 + 4 = \Delta(H) + 3 \leq k + 3$

     There is one free color for (v, vu). Without loss of generality, we set $\phi(v, vu) = 1$. For (u, uv) we have 4 free colors $(\phi'(A_v))$.
   - We denote by L_i the list of available colors of (u_i, u_iu) for $i \in \{1, 2, 3, 4\}$ and by L'_i the list of available colors of (u, uu_i) for $i \in \{1, 2, 3, 4\}$. We denote by L_u the list available colors of (u, uv). By a computation as above it is easy to see that $\mid L'_i \mid \geq k + 1 \geq 6$ (we recall that (v, vu) is colored with 1) and $\mid L_i \mid \geq k \geq 5$.
   - Using a counting argument it is easy to see that there exists a color α belonging to at least 3 lists among the lists L_i, $i \in$

$\{1,2,3,4\}$. Without loss of generality. we assume that α belongs to L_i $i \in \{1,2,3\}$.

(a) If $\alpha = 1$, first we set $\phi(u_i, u_iu) = 1$ for $i \in \{1,2,3\}$. Next we color (u, uv) (we have the 4 colors of $\phi(A_v)$) and (u_4, u_4u) from the list L_4. Now we color (u, uu_i) for $i \in \{1,2,3,4\}$ in this order one after the other. We have extended the coloring, a contradiction.

(b) If $\alpha \neq 1$. Without loss of generality, we assume that $\alpha = 2$, we color (u_i, u_iu) for $i \in \{1,2,3\}$ by 2. Next we color (u, uv) with a color belonging to $\phi'(A_v)$ different from 2 and 1, we set $\phi(u, uv) = 3$. Then we color (u, uu_i) for $i \in \{1,2,3,4\}$ in this order one after the other. We have enough colors in each list of each incidence (u, uu_i), $i \in \{1,2,3,4\}$. Without loss of generality, we set $\phi(u, uu_i) = i + 3$ for $i \in \{1,2,3,4\}$. If we can color properly the incidence (u_4, u_4u) we are done. If we cannot color (u_4, u_4u). It means that $L_4 = \{3,4,5,6,7\}$. Without loss of generality, we can assume that $\phi'(A_{u_4}) \cup \phi'(I_{u_4}) = \{1,2,8,9\}$. Assume that we can replace one of the colors of (u, uu_i), $i \in \{1,2,3\}$ by 8 or 9 without destroying the incidence coloring (let say 8), then we color the corresponding incidence by 8 (let say $\phi(u, uu_1) = 8$), and we color (u_4, u_4u) with 4. We are done. Hence $\phi(I_{u_i}) = \{2,8,9\}$, $i \in \{1,2,3\}$. We recall that $\mid L_i \mid \geq 5$,hence by the previous argument, $L_i \subset \{1,2,3,4,5,6,7\}$, for $i \in \{1,2,3\}$.

- If there exists an other color $\beta \notin \{1,2\}$ belonging to $\cap_{i=1}^{i=3} L_i$, this color belongs also to L_4. Then we color (u_i, u_iu) for $i \in \{1,2,3,4\}$ with β, (u, uv) with a color different from β. Next we color (u, uu_i), $i \in \{1,2,3,4\}$ one after the other, by the way we extend the coloring a contradiction.
- If it is not the case, then it is easy to see that each element of $\{1,3,4,5,6,7\}$ belongs to exactly two lists L_i. Without loss of generality, we assume that $1 \in L_1 \cap L_2$. We color (u_1, u_1u) and (u_2, u_2u) with 1. We recall that $\mid L_i \mid \geq 5$, we take any color of $L_3 \cap L_4$ to color (u_3, u_3u) and (u_4, u_4u), let say 3. Then we color (u, uv) with a color different from 3. Next we color (u, uu_i), $i \in \{1,2,3,4\}$ one after the other, by the way we extend the coloring, a contradiction. This completes the proof. □

3.2. Discharging procedure

Euler's formula $|V(H)|-|E(H)|+|F(H)|=2$ can be rewritten as $(6|E(H)|-10|V(H)|)+(4|E(H)|-10|F(H)|)=-20$. Using the relation $\sum_{v\in V(H)} d(v) = \sum_{f\in F(H)} r(f) = 2|E(H)|$, we get that:

$$(1) \qquad \sum_{v\in V(H)} (3d(v)-10) + \sum_{f\in F(H)} (2r(f)-10) = -20$$

We define the weight function $\omega : V(H) \cup F(H) \longrightarrow \mathbb{R}$ by $\omega(x) = 3d(x)-10$ if $x \in V(H)$ and $\omega(x) = 2r(x)-10$ if $x \in F(H)$. It follows from Equation (1) that the total sum of weights is equal to -20. In what follows, we will define discharging rules (R1) to (R8) and redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced. However, the total sum of weights is kept fixed when the discharging is finished. Nevertheless, we will show that $\omega^*(x) \geq 0$ for all $x \in V(H) \cup F(H)$. This will lead us to the following contradiction:

$$0 \leq \sum_{x\in V(H)\cup F(H)} \omega^*(x) = \sum_{x\in V(H)\cup F(H)} \omega(x) = -20 < 0$$

and hence will demonstrate that such a counterexample cannot exist.

The discharging rules are defined as follows:

(R1) Every k-vertex, for $k \geq 5$, gives $\frac{1}{3}$ to each adjacent 3-vertex.
(R2) Every 4-vertex gives 1 to each incident 3-face.
(R3) Every k-vertex, for $k \geq 5$, gives 2 to each incident 3-face.
(R4) Every 4-vertex gives $\frac{1}{2}$ to each incident 4-face.
(R5) Every k-vertex, for $k \geq 5$, gives 1 to each incident 4-face: $(5^+,3,5^+,3)$.
(R6) Every k-vertex, for $k \geq 5$, gives $\frac{3}{4}$ to each incident 4-face: $(5^+,3,5^+,4)$.
(R7) Every k-vertex, for $k \geq 5$, gives $\frac{2}{3}$ to each incident 4-face: $(5^+,5^+,5^+,3)$.
(R8) Every k-vertex, for $k \geq 5$, gives $\frac{1}{2}$ to each incident 4-face: $(5^+,4^+,4^+,4^+)$.

Since H does not contain a C_4 adjacent to a C_3, by hypothesis, the following fact is easy to observe and will be frequently used throughout the proof without further notice.

Observation 3. *H does not contain the following structures:*

1. *adjacent 3-cycles,*
2. *a 4-cycle adjacent to a 3-cycle.*

One can easily derive the following observation.

Observation 4. *Let v be a k-vertex with $k \geq 3$ then $m_3(v) \leq \lfloor \frac{k}{2} \rfloor$ and we have the following cases by Observation 3:*

- *If $m_3(v) = \lfloor \frac{k}{2} \rfloor$ then $m_4(v) = 0$.*
- *If $1 \leq m_3(v) < \lfloor \frac{k}{2} \rfloor$ then $m_4(v) \leq d(v) - 2 \times m_3(v) - 1$.*
- *If $m_3(v) = 0$ then $m_4(v) \leq d(v)$.*

Let $v \in V(H)$ be a k-vertex. By Lemma 2.1 and Lemma 2.2, $k \geq 3$. Consider the following cases:

Case $k = 3$. Observe that $\omega(v) = -1$. By Lemma 2.3, v has three neighbors both of degree at least 5. Then, by (R1), we have: $\omega^*(v) = -1 + 3 \times \frac{1}{3} = 0$.

Case $k = 4$. Observe that $\omega(v) = 2$. By Lemma 2.3, v has four neighbors both of degree at least 4. By Observation 4, we have the following cases:

- If $m_3(v) = 2$ then $m_4(v) = 0$. Hence, by (R2), we have: $\omega^*(v) \geq 2 - 2 \times 1 = 0$.
- If $m_3(v) = 1$ then $m_4(v) \leq 1$. Hence, by (R2) and (R4), we have: $\omega^*(v) \geq 2 - 1 \times 1 - 1 \times \frac{1}{2} > 0$.
- If $m_3(v) = 0$ then $m_4(v) \leq 4$. Hence, by (R4), we have: $\omega^*(v) \geq 2 - 4 \times \frac{1}{2} = 0$.

Case $k = 5$. Observe that $\omega(v) = 5$. By Lemma 2.5, v has at most 3 neighbors whose degrees are all of 3. By Observation 4, we have the following cases:

- If $m_3(v) = 2$ then $m_4(v) = 0$. Hence, by (R1) and (R3), we have: $\omega^*(v) \geq 5 - 2 \times 2 - 3 \times \frac{1}{3} = 0$.
- If $m_3(v) = 1$ then $m_4(v) \leq 2$. In the worst-case v is incident to two $(5, 3, 5^+, 3)$-faces Hence, by (R1), (R3) and (R5), we have: $\omega^*(v) \geq 5 - 1 \times 2 - 2 \times 1 - 3 \times \frac{1}{3} = 0$.
- If $m_3(v) = 0$ then $m_4(v) \leq 5$. We have to consider several cases:
  (a) If $n_3(v) = 3$, then we have three cases (we always consider the worst-case, it is the case when v gives the biggest amount of charge):
    (i) v is incident to two $(5, 3, 5^+, 3)$-faces, two $(5, 3, 5^+, 4)$-faces and one $(5, 4^+, 4^+, 4^+)$-face. Hence, by (R1), (R5), (R6) and (R8), we have: $\omega^*(v) \geq 5 - 2 \times 1 - 2 \times \frac{3}{4} - 3 \times \frac{1}{3} - \frac{1}{2} = 0$.

(ii) v is incident to two $(5,3,5^+,3)$-faces and three $(5^+,5^+,5^+,3)$-faces. Hence, by (R1), (R5) and (R7), we have: $\omega^*(v) \geq 5 - 1 \times 2 - 3 \times \frac{2}{3} - 3 \times \frac{1}{3} = 0$.

(iii) v is incident to one $(5,3,5^+,3)$-face and four $(5,3,5^+,4)$-faces. Hence, by (R1), (R5) and (R6), we have: $\omega^*(v) \geq 5 - 1 \times 1 - 4 \times \frac{3}{4} - 3 \times \frac{1}{3} = 0$.

(b) If $n_3(v) = 2$, then we have 3 cases to consider:

(i) v is incident to one $(5,3,5^+,3)$-face, and four $(5,3,5^+,4^+)$-faces. By (R1), (R5) and (R6), we have: $\omega^*(v) \geq 5 - 1 \times 1 - 4 \times \frac{3}{4} - 2 \times \frac{1}{3} = \frac{1}{3} \geq 0$.

(ii) v is incident to four $(5,3,5^+,4^+)$-faces and one $(5,4^+,4^+,4^+)$-face. Hence, by (R1), (R6) and (R8), we have: $\omega^*(v) \geq 5 - 4 \times \frac{3}{4} - 1 \times \frac{1}{2} - 2 \times \frac{1}{3} = \frac{5}{6} \geq 0$.

(iii) v is incident to two $(5,3,5^+,4^+)$-faces and three $(5,5^+,5^+,3)$-faces. Hence, by (R1), (R6) and (R7), we have: $\omega^*(v) \geq 5 - 2 \times \frac{3}{4} - 3 \times \frac{2}{3} - 2 \times \frac{1}{3} = \frac{5}{6} \geq 0$.

(c) If $n_3(v) = 1$, in the worst-case v may give $1/3$ to the neighbor of degree 3 and 5 times $3/4$ to the 5 incident faces. Hence, $\omega^*(v) \geq 5 - 1 \times \frac{1}{3} - 5 \times \frac{3}{4} = \frac{11}{12} \geq 0$.

(d) If $n_3(v) = 0$, in the worst-case v can be incident to five $(5,5^+,5^+,3)$-faces. Then by (R7), we have: $\omega^*(v) \geq 5 - 5 \times \frac{2}{3} = \frac{5}{3} \geq 0$.

Case $k = 6$. Observe that $\omega(v) = 8$. By Observation 4, we have the following cases:

- If $m_3(v) = 3$ then $m_4(v) = 0$. By Lemma 2.3 and Lemma 2.4, v is adjacent to at most three 3-vertices. Hence, by (R1) and (R3), we have: $\omega^*(v) \geq 8 - 3 \times 2 - 3 \times \frac{1}{3} = 1 > 0$.
- If $m_3(v) = 2$ then $m_4(v) \leq 1$. By Lemma 2.3 and Lemma 2.4, v is adjacent to at most four 3-vertices and incident to at most one $(5^+,3,5^+,3)$-face. Hence, by (R1), (R3) and (R5), we have: $\omega^*(v) \geq 8 - 2 \times 2 - 1 \times 1 - 4 \times \frac{1}{3} = \frac{5}{3} > 0$.
- If $m_3(v) = 1$ then $m_4(v) \leq 3$. By Lemma 2.3 and Lemma 2.4, v is adjacent to at most five 3-vertices and incident to at most three $(5^+,3,5^+,3)$-faces. Hence, by (R1), (R3) and (R5), we have: $\omega^*(v) \geq 8 - 1 \times 2 - 3 \times 1 - 5 \times \frac{1}{3} = \frac{4}{3} > 0$.
- If $m_3(v) = 0$ then $m_4(v) \leq 6$. v can be incident to six $(5^+,3,5^+,3)$-faces. Hence, by (R1) and (R5), we have: $\omega^*(v) \geq 8 - 6 \times 1 - 6 \times \frac{1}{3} = 0$.

Case $k = 7$. Observe that $\omega(v) = 11$. By Observation 4, we have the following cases:

- If $m_3(v) = 3$ then $m_4(v) = 0$. By Lemma 2.3 and Lemma 2.4, v is adjacent to at most four 3-vertices. Hence, by (R1) and (R3), we have: $\omega^*(v) \geq 11 - 3 \times 2 - 4 \times \frac{1}{3} = \frac{11}{3} > 0$.
- If $m_3(v) = 2$ then $m_4(v) \leq 2$. By Lemma 2.3 and Lemma 2.4, v is adjacent to at most five 3-vertices and incident to at most two $(5^+, 3, 5^+, 3)$-face. Hence, by (R1), (R3) and (R5), we have: $\omega^*(v) \geq 11 - 2 \times 2 - 2 \times 1 - 5 \times \frac{1}{3} = \frac{10}{3} > 0$.
- If $m_3(v) = 1$ then $m_4(v) \leq 4$. By Lemma 2.3 and Lemma 2.4, v is adjacent to at most six 3-vertices and incident to at most four $(5^+, 3, 5^+, 3)$-faces. Hence, by (R1), (R3) and (R5), we have: $\omega^*(v) \geq 11 - 1 \times 2 - 4 \times 1 - 6 \times \frac{1}{3} = 3 > 0$.
- If $m_3(v) = 0$ then $m_4(v) \leq 7$. v can be incident to seven $(5^+, 3, 5^+, 3)$-faces and seven 3-vertices. Hence, by (R1) and (R5), we have: $\omega^*(v) \geq 11 - 7 \times 1 - 7 \times \frac{1}{3} = \frac{5}{3} > 0$.

Case $k \geq 8$. Observe that $\omega(v) = 3k - 10$. By Observation 4, we have the following cases:

- If $m_3(v) = \lfloor \frac{k}{2} \rfloor$ then $m_4(v) = 0$. Hence, by (R1) and (R3):

$$\begin{aligned}\omega^*(v) &= 3k - 10 - 2 \times m_3(v) - \frac{1}{3} \times n_3(v) \\ &\geq 3k - 10 - 2 \times \left\lfloor \frac{k}{2} \right\rfloor - \frac{1}{3} \times k \\ &\geq \frac{5}{3}k - 10 > 0\end{aligned}$$

- If $1 \leq m_3(v) \leq \lfloor \frac{k}{2} \rfloor - 1$ then $m_4(v) \leq k - 3$. Hence, by (R1), (R3) and (R5), we have:

$$\begin{aligned}\omega^*(v) &= 3k - 10 - 2 \times m_3(v) - 1 \times m_4(v) - \frac{1}{3} \times n_3(v) \\ &\geq 3k - 10 - 2 \times \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 \right) - (k - 3) \times 1 - \frac{1}{3} \times k \\ &\geq \frac{2}{3}k - 5 > 0\end{aligned}$$

- If $m_3(v) = 0$ then $m_4(v) \leq k$. Hence, by (R1) and (R5), we have: $\omega^*(v) \geq 3k - 10 - k \times 1 - k \times \frac{1}{3} = \frac{5}{3}k - 10 > 0$.

Let $f \in F(H)$ be a k-face.

Case $k = 3$. Observe that $\omega(f) = -4$. Suppose $f = [rst]$. Consider the following cases:

(a) Suppose $d(r) = 3$. Then, by Lemma 2.3, r is the unique 3-vertex and $d(s) \geq 5$ and $d(t) \geq 5$. Hence, by (R3), we have: $\omega^*(f) = -4 + 2 \times 2 = 0$

(b) Suppose now $d(r) \geq 4$, $d(s) \geq 4$ and $d(t) \geq 4$. By Lemma 2.4, at least two of the three vertices r, s and t is a 5^+-vertex. Assume that $d(s) \geq 5$ and $d(t) \geq 5$. Then, by (R2) and (R3), we have: $\omega^*(f) \geq -4 + 2 \times 2 + 1 \times 1 = 1 \geq 0$.

Case $k = 4$. The initial charge of f is $\omega(f) = -2$. By Lemma 2.3, at most two 3-vertices are incident to the 4-face. Suppose $f = [rstu]$. Consider the following cases:

(a) Suppose $d(r) = d(t) = 3$. Then, by Lemma 2.3, $d(s) \geq 5$ and $d(u) \geq 5$. Hence, by (R5), we have: $\omega^*(f) = -2 + 2 \times 1 = 0$

(b) Suppose now $d(r) = 3$. Then, by Lemma 2.3, $d(s) \geq 5$ and $d(u) \geq 5$. Moreover, assume $d(t) = 4$. Then, by (R4) and (R6), we have: $\omega^*(f) \geq -2 + 2 \times \frac{3}{4} + 1 \times \frac{1}{2} = 0$. If $d(t) \geq 5$, by (R7) we have $\omega^*(f) \geq -2 + 3 \times \frac{2}{3} = 0$.

(c) Assume $d(r) \geq 4$, $d(s) \geq 4$, $d(t) \geq 4$ and $d(u) \geq 4$. Then, by (R4), we have: $\omega^*(f) \geq -2 + 4 \times \frac{1}{2} = 0$.

Case $k \geq 5$. The initial charge of f is $\omega(f) = 2k - 10 \geq 0$ and it remains unchanged during the discharging process. Hence, $\omega(v) = \omega^*(v) = 2k - 10 \geq 0$.

After performing the discharging procedure the new weights of all faces and vertices are positive and therefore, H cannot exist. This completes the proof of Theorem 5.1

4. Proof of Theorem 5.2

We will prove the following stronger version of Theorem 5.2:

Theorem 7. *Every planar graph G without C_4 and C_5 admits an incidence $(k+3, 3)$-coloring for every $k \geq \Delta(G) \geq 5$, $k \in \mathbb{N}$. Therefore, $\chi_i(G) \leq \Delta(G) + 3$.*

Observation 5. *We consider only $k \geq \Delta(G) \geq 5$. If $\Delta(G) < 4$, we obtain by Theorem 1*

$$\chi_i(G) \leq 2\Delta(G) \leq \Delta(G) + 3 \leq k + 3.$$

4.1. Structural properties

We proceed by contradiction. Let H be a counterexample to the theorem that minimizes $|E(H)|+|V(H)|$. By hypothesis there exists $k \geq \max\{\Delta(G), 5\}$ such that H does not admit an incidence $(k+3,3)$-coloring. Let $k \geq \max\{\Delta(G), 5\}$ be the smallest integer such that H does not admit an incidence $(k+3,3)$-coloring. By using Remark 1, we must have $k = \max\{\Delta(G), 5\}$. Moreover by minimality it is easy to see that H is connected.

H satisfies the following properties:

Lemma 3. *H does not contain:*

1. *1-vertices,*
2. *2-vertices,*
3. *a 3-vertex adjacent to a 3-vertex,*
4. *a $(3,4,4)$-triangle.*

Proof. First, we will suppose by contradiction that the described configuration exists in H. Then we consider a graph H' obtained from H by deleting an edge or a vertex from H. The graph H' does not contain a C_4 neither a C_3. Due to the minimality of H, the graph H' admits an incidence $(k'+3,3)$-coloring for any $k' \geq \max\{\Delta(H'), 5\}$. Since $\Delta(H) \geq \Delta(H')$, the set of integers k' contains the set of integers k. Hence for the value $k' = k$, H' admits an incidence $(k+3,3)$-coloring ϕ'. Finally, for each cases, we will prove a contradiction by extending ϕ' to an incidence $(k+3,3)$-coloring ϕ of H.

1. By using the same method as in the proof of Theorem 5.1 and Lemma 2.1, it is easy to prove Lemma 3.1.
2. By using the same method as in the proof of Theorem 5.1 and Lemma 2.2, it is easy to prove Lemma 3.2.
3. Suppose H contains a 3-vertex u adjacent to a 3-vertex v. By minimality of H, the graph $H' = H \setminus \{uv\}$ has an incidence $(k+3,3)$-coloring ϕ'. We recall that we have at least 8-colors and $\mid \phi'(A_u) \mid \leq 2$ and $\mid \phi'(A_v) \mid \leq 2$. Note that

$$\mid F_H^{\phi'}(u, uv) \mid = \mid \phi'(I_u) \cup \phi'(A_u) \cup \phi'(I_v) \mid \leq 2+2+2 = 6$$

   We have at least $k-3 \geq 2$ free colors for (u, uv). Choose a color for (u, uv), then for (v, vu) by using the same calculation at most 7 colors are forbidden for this incidence. We have at least $k-4 \geq 1$ free colors. So we can extend the coloring, a contradiction.

4. Suppose that H contains a $(3,4,4)$-triangle. Let u be the vertex of degree equal to 3 and v, w the two vertices of degree equal to 4. Let t be the neighbor of u different from v and w. By minimality of H, the graph $H' = H - \{uv, uw\}$ admits an incidence $(k+3,3)$-coloring. We will extend ϕ' to an incidence $(k+3,3)$-coloring ϕ of H as follows. We recall that we have at least 8 colors. We have:
   - $\mid F_H^{\phi'}(v, vu) \mid = \mid \phi'(I_v) \cup \phi'(A_v) \cup \phi'(I_u) \mid \leq 3+3+1 = 7$. Hence we have one available color for (v, vu).
   - In the same way, we have one available for (w, wu).
   - We have 3 available colors for (u, uv) belonging to $\phi'(A_v)$ and 3 available colors for (u, uw) belonging to $\phi'(A_w)$.

   Without loss of generality, we assume that we are in the situation described in Figure 2.

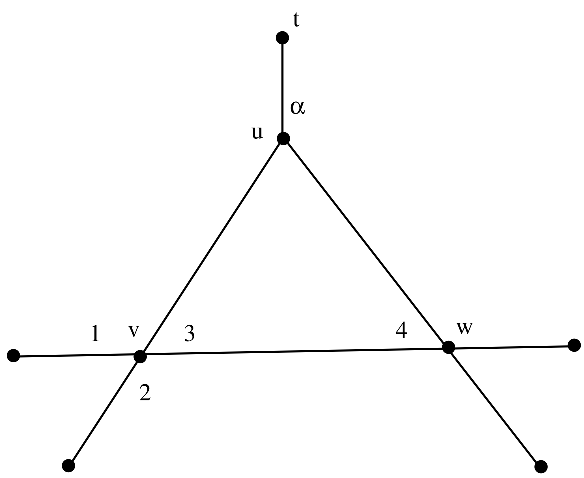

Figure 2: $\mid F_H^{\phi'}(u, uv) \mid \leq 7$.

First, we can assume that $\phi'(u, ut) = \alpha \notin \{3, 4\}$, because if it is not the case we can recolor (u, ut) with a color different from 3 and 4 (it is possible because we have 3 available colors for (u, ut) belonging to $\phi'(A_t)$).
Now, we color (v, vu) with the available color (say a) and (w, wu) with one available color (say b). We consider two cases.

- Assume that we can color (u, uv) with 4. If we can color (u, uw) with 3, we are done. If we cannot color (u, uw) with 3. It means that $\phi'(t,tu)=3$ (note that $\alpha\neq 3$ and $3\notin\phi'(I_w)$). If we cannot color (u,uw) with an other color of $\phi'(A_w)$ we have to $\phi'(A_w)=\{3,a,\alpha\}$. Then we permute 3 and a in $\phi'(I_v)$ and we color (u,uw) with a color a, we are done.

- Assume that we cannot color (u,uv) with 4, it means that $\phi'(t,tu)=4$ (note that $\alpha\neq 4$). If we cannot color (u,uv) with an other color of $\phi'(A_v)$ (if it is the case then we can color (u,uw) with 3, we are done), it means that $\phi'(A_v)=\{4,b,\alpha\}$. Then we permute 4 and b in $\phi'(I_w)$ and we color (u,uv) with a color b and (u,uv) with color 3, we are done.

We have extended the coloring for all the cases, a contradiction. □

4.2. Discharging procedure

Euler's formula $|V(H)|-|E(H)|+|F(H)|=2$ can be rewritten as $(4|E(H)|-6|V(H)|)+(2|E(H)|-6|F(H)|)=-12$. Using the relation $\sum_{v\in V(H)}d(v)=\sum_{f\in F(H)}r(f)=2|E(H)|$ we get that:

$$\sum_{v\in V(H)}(2d(v)-6)\;+\;\sum_{f\in F(H)}(r(f)-6)\;=\;-12 \tag{2}$$

We define the weight function $\omega: V(H)\cup F(H)\longrightarrow\mathbb{R}$ by $\omega(x)=2d(x)-6$ if $x\in V(H)$ and $\omega(x)=r(x)-6$ if $x\in F(H)$. It follows from Equation (2) that the total sum of weights is equal to -12. In what follows, we will define discharging rules (R1) and (R2). Next we redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced. However, the total sum of weights is kept fixed when the discharging is finished. Nevertheless, we will show that $\omega^*(x)\geq 0$ for all $x\in V(H)\cup F(H)$. This will lead us to the following contradiction:

$$0\;\leq\sum_{x\in V(H)\cup F(H)}\omega^*(x)\;=\sum_{x\in V(H)\cup F(H)}\omega(x)\;=\;-12\;<\;0$$

and hence will demonstrate that such a counterexample cannot exist.

The discharging rules are defined as follows:

(R1) Every 4-vertex, gives 1 to each incident 3-face.
(R2) Every k-vertex, for $k\geq 5$, gives 2 to each incident 3-face.

Let $v \in V(H)$ be a k-vertex.

By Lemma 3.1 and Lemma 3.2, $k \geq 3$. We recall that since H does not contain C_4, there are no adjacent 3-faces. Consider the following cases:

Case $k = 3$. Observe that $\omega(v) = 0$. v does not give anything and does not get anything. We have $\omega^*(v) = \omega(v) = 0$.
Case $k = 4$. $\omega(v) = 2$. It is easy to see that v is incident to at most two 3-faces. Then, by (R1) we have $\omega^*(v) \geq 2 - 2 \times 1 = 0$.
Case $k \geq 5$. Observe that $\omega(v) = 2k-6$. It is easy to see that $m_3(v) \leq \lfloor \frac{k}{2} \rfloor$. Hence, by (R2), we have: $\omega^*(v) \geq 2k - 6 - 2 \times \lfloor \frac{k}{2} \rfloor \geq 0$.

Let $f \in F(H)$ be a k-face.

Case $k = 3$. Observe that $\omega(f) = -3$. Suppose $f = [rst]$. Consider the following cases:

(a) Suppose $d(r) = 3$. Then, by Lemma 3.3, r is the unique 3-vertex and by Lemma 3.4, $d(s) \geq 4$ and $d(t) \geq 5$. Hence, by (R1) and (R2), we have $\omega^*(f) \geq -3 + 1 \times 1 + 1 \times 2 = 0$

(b) Suppose now $d(r) \geq 4$, $d(s) \geq 4$ and $d(t) \geq 4$.Then, by (R1) and (R2), $\omega^*(f) \geq -3 + 3 \times 1 = 0$.

Case $k \geq 6$. The face is not involved in the discharging procedure. $\omega(f) = \omega^*(f) \geq 0$.

After performing the discharging procedure the new weights of all faces and vertices are positive and therefore, H cannot exist. This completes the proof of Theorem 5.2

Question 1. *By Theorem 1, the bound of Theorem 5.2 is true for* $\Delta(G) \leq 3$, $\chi_i(G) \leq \Delta(G) + 3$. *In Theorem 5.2 for* $\Delta(G) = 4$ *we have* $\chi_i(G) \leq 8$, *can we prove* $\chi_i(G) \leq 7$*?*

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