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Induced forests in bipartite planar graphs

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Akiyama and Watanabe conjectured that every simple planar bipartite graph on n vertices contains an induced forest on at least $5n/8$ vertices. We apply the discharging method to show that every simple bipartite planar graph on n vertices contains an induced forest on at least $\lceil (4n+3)/7 \rceil$ vertices.

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1. Introduction

In this paper, we consider simple graphs only. Clearly, every bipartite graph contains an independent set of size at least half of its vertices. It is natural to ask under what conditions can we find considerably larger sparse induced subgraphs, for example, induced forests? The study of the maximum size of induced forests was initiated by Erdős, Saks, and Sós in 1986 [10]. Later, Matoušek and Šámal [12], and also Fox, Loh, and Sudakov [9] studied large induced trees in triangle-free graphs and K_r-free graphs, respectively.

For a graph G, let $|G| = |V(G)|$ and let $a(G)$ denote the largest number of vertices of an induced forest in G. For later convenience, we use $A(G)$ to denote an induced forest in G of size $a(G)$. Albertson and Berman [2] (also see Albertson and Haas in [3]) conjectured in 1979 that $a(G) \geq |G|/2$ for any planar graph G. For bipartite planar graphs, Akiyama and Watanabe [1] made the following in 1987

Conjecture 1.1. *If G is a bipartite planar graph, then $a(G) \geq 5|G|/8$.*

The bound in Conjecture 1.1 is tight with Q_3 (the 3-cube), and more examples can be constructed, for example, by adding a matching between two 4-cycles in two Q_3's.

Planar graphs have average degree strictly less than 6. Alon [4] considered bipartite graphs G with average degree at most $d \geq 1$, and showed that

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$a(G) \geq (\frac{1}{2} + e^{-bd^2})|G|$, for some absolute constant $b > 0$. Conlon *et al.* [7] improved Alon's bound to $(\frac{1}{2} + d^{-b'd})|G|$, for some constant $b' > 0$. Since the average degree of any bipartite planar graph is less than 4, the above results give a nontrivial bound for Conjecture 1.1.

There has been some recent activities on Conjecture 1.1. It is shown in [13] (also see [11]) that if G is a triangle-free planar graph then $a(G) \geq (17|G| + 24)/32$, which is improved to $(6|G| + 7)/11$ in [8]. In this paper, we prove the following

Theorem 1.2. *Let G be a bipartite planar graph. Then $a(G) \geq \lceil (4|G| + 3)/7 \rceil$.*

In our proof of Theorem 1.2, we apply the discharging method. Suppose Theorem 1.2 is false, and let G be a counterexample with $|G|$ minimum. Using the discharging technique, we force some small configurations, which are reducible in the sense that after certain operations we can use an induced forest from a smaller graph to construct an induced forest in G. Often such operations involve the identification of vertices, which may result in multiple edges; we remove all but one such edges after the identification. Note that we always identify vertices in the same color class of the bipartite graph G. Hence, there will be no loop after the identification.

We need some notations and terminologies. Let $v \in V(G)$ and $X, Y \subseteq V(G)$. $N(v)$ denotes the set of neighbors of v, and $G[X]$ denotes the induced subgraph of G on X. We define $G - v := G[V(G) - \{v\}]$, $G - X := G[V(G) - X]$, $G[X + v] := G[X \cup \{v\}]$ and $G[X + Y] := G[X \cup Y]$. Let n be a positive integer. We denote V_n, $V_{\leq n}$, $V_{\geq n}$ the set of vertices of degree exactly n, at most n, and at least n, respectively. We call a vertex v in G is a n-vertex (n^+-vertex, n^--vertex, respectively) if $v \in V_n$ ($v \in V_{\geq n}$, $v \in V_{\leq n}$, respectively) If G is a planar graph and $v_1, v_2, ..., v_k$ are vertices of G incident with a common face F, then $G/v_1v_2...v_k$ denotes the simple plane graph obtained from G by identifying $v_1, v_2, ..., v_k$ in F as a new vertex w. We define $G/\{v_1v_2, ..., v_{k-1}v_k\} = (G/v_1v_2)/\{v_3v_4, ..., v_{k-1}v_k\}$. $G + v_1v_2$ denotes the simple plane graph obtained from G by adding the edge v_1v_2 in F if $v_1v_2 \notin E(G)$. $X \triangle Y$ denotes the symmetric difference between X and Y. A *separation* in a graph G consists of a pair of subgraphs G_1, G_2, denoted as (G_1, G_2), such that $E(G_1) \cup E(G_2) = E(G)$, $E(G_1 \cap G_2) = \emptyset$, $G_1 \nsubseteq G_2$, and $G_2 \nsubseteq G_1$. $e(X)$ denotes the number of edges in $G[X]$ and $e(X, Y)$ denotes the number of edges of G between vertices in X and vertices in Y.

The rest of the paper is organized as follows. In Section 2, we present some inequalities that we use, which can be established by considering remainders modular 7. We also set up some notation for a minimum counterexample G of Theorem 1.2, and prove some basic properties about G.

In Section 3, we derive information about the structures around a vertex of degree 2 in G. In Section 4, we work on the neighbors of a degree 3 vertex. In Section 5 and 6, we deal with two forbidden configurations around a 3-vertex. In Section 7, we work with degree 5 and 6 vertices. We prove Theorem 1.2 in Section 8 by giving discharging rules based on the structural information obtained in the previous sections.

2. Useful inequalities and the minimum counterexample

We begin with some inequalities that will be used frequently throughout the paper.

Lemma 2.1. *Let $a_1, a_2 \geq 1$ be integers such that $a_1 + a_2 = n + 3 - k$, with $k \leq 8$. Then* $\max\{\lceil (4a_1+3)/7\rceil + \lceil (4a_2+3)/7\rceil + 2, \lceil (4a_1-1)/7\rceil + \lceil (4a_2-1)/7\rceil + 3\} \geq \lceil (4n+3)/7\rceil$.

Proof. Note the symmetry between a_1 and a_2. If $4a_1 + 3 \equiv 0 \mod 7$ then $\lceil (4a_1-1)/7\rceil + \lceil (4a_2-1)/7\rceil + 3 \geq (4a_1 - 1 + 4)/7 + (4a_2 - 1)/7 + 3 = (4n + 3 - 4k + 32)/7 \geq (4n + 3)/7$.

So we may assume $4a_i + 3 \not\equiv 0 \mod 7$ for $i = 1, 2$. Let $4a_i + 3 \equiv r_i \mod 7$ with $1 \leq r_i \leq 6$ for $i = 1, 2$. If $r_1 \neq 6$ or $r_2 \neq 6$ then $\lceil (4a_1+3)/7\rceil + \lceil (4a_2+3)/7\rceil + 2 \geq (4a_1+3)/7 + (4a_2+3)/7 + 2 + 3/7 = (4n+3-4k+32)/7 \geq (4n + 3)/7$.

So assume $r_1 = r_2 = 6$. Then $\lceil (4a_1-1)/7\rceil + \lceil (4a_2-1)/7\rceil + 3 \geq (4a_1 - 1 + 5)/7 + (4a_2 - 1 + 5)/7 + 3 = (4n + 3 - 4k + 38)/7 > (4n + 3)/7$.

Therefore, the conclusion holds since the left hand side of the inequality is an integer. □

With similar, but more involved arguments, we have the following inequalities. We leave out the details.

Lemma 2.2. *Let $a, a_1, a_2, ..., a_k, c, n$ be positive integers where $k \geq 1$. Let L be a set of integers and b_j be a positive integer for all $j \in L$.*

(1) If $(4a+3)/7 + \sum_{i=1}^{k} (4a_i+3)/7 + \sum_{j\in L} (4b_j+3)/7 + c - k \geq (4n+3-3k)/7$,

then $\max_{\substack{A_i\in\{0,1\},\\ \forall i\in[k]}} \{\lceil (4(a - \sum_{i=1}^{k} A_i)+3)/7\rceil + \sum_{i=1}^{k} \lceil (4(a_i - A_i)+3)/7\rceil + \sum_{j\in L} \lceil (4b_j + 3)/7\rceil + c - \sum_{i=1}^{k} (1 - A_i)\} \geq \lceil (4n + 3)/7\rceil$.

(2) *If* $(4a+3)/7+(4a_1+3)/7+c-1 \geq (4n-1)/7$ *and* $(4a+3, 4a_1+3) \not\equiv (0,4),(4,0) \mod 7$*, then* $\max_{A_1\in\{0,1\}}\{\lceil(4(a-A_1)+3)/7\rceil + \lceil(4(a_1-A_1)+3)/7\rceil + c-(1-A_1)\} \geq \lceil(4n+3)/7\rceil$*;*

(3) *If* $(4a+3)/7+(4a_1+3)/7+c \geq (4n-1)/7$*, then* $\lceil(4a+3)/7\rceil+\lceil(4a_1+3)/7\rceil+c \geq \lceil(4n+3)/7\rceil$ *if* $(4a+3, 4a_1+3) \not\equiv (0,0),(0,6),(0,5),(0,4),(4,0),(6,5),(5,6),(5,0),(6,6),(6,0) \mod 7$*;*

(4) *If* $(4a+3)/7+\sum_{i=1}^{2}(4a_i+3)/7+c-2 \geq (4n-4)/7$*, then* $\max_{A_1,A_2\in\{0,1\}}\{\lceil(4(a-\sum_{i=1}^{2}A_i)+3)/7\rceil+\sum_{i=1}^{2}\lceil(4(a_i-A_i)+3)/7\rceil+c-\sum_{i=1}^{2}(1-A_i)\} \geq \lceil(4n+3)/7\rceil$*, unless* $(4a+3, 4a_1+3, 4a_2+3) \equiv (1,0,0),(4,0,4),(4,4,0),(0,4,4) \mod 7$*;*

(5) *If* $(4a+3)/7+\sum_{i=1}^{2}(4a_i+3)/7+c-2 \geq (4n-5)/7$*, then* $\max_{A_1,A_2\in\{0,1\}}\{\lceil(4(a-\sum_{i=1}^{2}A_i)+3)/7\rceil+\sum_{i=1}^{2}\lceil(4(a_i-A_i)+3)/7\rceil+c-\sum_{i=1}^{2}(1-A_i)\} \geq \lceil(4n+3)/7\rceil$*, unless* $(4a+3, 4a_1+3, 4a_2+3) \equiv (0,0,0)$, $(1,0,0)$, $(4,0,3),(4,3,0),(3,0,4),(4,0,4),(3,4,0),(4,4,0)$, $(1,6,0)$, $(1,0,6)$, $(0,3,4)$, $(0,4,3)$, $(0,4,4)$, $(6,4,4),(4,4,6),(4,6,4) \mod 7$*;*

(6) *If* $\sum_{i=1}^{k}(4a_i+3)/7+c \geq (4n+2)/7$*, then* $\sum_{i=1}^{k}\lceil(4a_i+3)/7\rceil+c \geq \lceil(4n+3)/7\rceil$*, unless* $4a_i+3 \equiv 0 \mod 7$ *for* $i \in [k]$*;*

(7) *If* $\sum_{i=1}^{k}(4a_i+3)/7+c \geq (4n+1)/7$*, then* $\sum_{i=1}^{k}\lceil(4a_i+3)/7\rceil+c \geq \lceil(4n+3)/7\rceil$*, unless there exists* $j \in [n]$ *such that* $4a_j+3 \equiv 0,6 \mod 7$ *and* $4a_i+3 \equiv 0 \mod 7$ *for* $i \in [k]-\{j\}$*;*

(8) *If* $(4a+3)/7+(4a_1+3)/7+c \geq 4n/7$*, then* $\lceil(4a+3)/7\rceil+\lceil(4a_1+3)/7\rceil+c \geq \lceil(4n+3)/7\rceil$ *unless* $(4a+3, 4a_1+3) \equiv (0,0),(0,6),(0,5),(5,0),(6,6)$, $(6,0) \mod 7$.

Note that in applications $a_1, a_2, ..., a_k, b_1, ..., b_l$ are the numbers of vertices in some subgraphs of a given graph, and A_i is the indicator function whether a vertex is included or not. Moreover, we have $k \leq 4$ and $l \leq 2$ in all applications.

We now set up some notation for the proof of Theorem 1.2. Throughtout the remainder of this paper, let G be a bipartite plane graph with $|G| = n$ such that

(i) $a(G) < \lceil(4n+3)/7\rceil$,
(ii) subject to (i), $|G|$ is minimum, and
(iii) subject to (ii), $|E(G)|$ is maximum.

Lemma 2.3. *G is a connected quadrangulation, $\delta(G) \geq 2$, and for each $v \in V_{\leq 3}$ we may choose $A(G)$ so that $v \in A(G)$.*

Proof. If G is disconnected, let $G_1, ..., G_k$ be the components of G (hence $k \geq 2$). By the choice of G, $a(G_i) \geq \lceil (4|G_i| + 3)/7 \rceil$ for $i \in [k]$. So $a(G) \geq \sum_{i=1}^{k} \lceil (4|G_i| + 3)/7 \rceil \geq \lceil (4n + 3)/7 \rceil$, a contradiction. So G is connected.

If G is not a quadrangulation, then G has a facial walk $a_1 a_2 ... a_k a_1$ with $k \geq 6$. By the choice of G, $a(G + a_1 a_4) \geq \lceil (4n + 3)/7 \rceil$. This implies that $a(G) \geq \lceil (4n + 3)/7 \rceil$, a contradiction. Thus G is a quadrangulation, and hence, $\delta(G) \geq 2$.

Now let $F = A(G)$ with $v \in V_{\leq 3} - V(F)$. By the maximality of $A(G)$, $N(v) \cap V(F) \neq \emptyset$. If $|V(F) \cap N(v)| \leq 2$, then let $w \in V(F) \cap N(v)$; if $|V(F) \cap N(v)| = 3$, then there exists $w \in V(F)$ such that no two vertices in $V(F) \cap N(v)$ are contained in the same component of $F - w$. Now $G[F - w + v]$ is a maximum induced forest in G containing v. $\square$

The following notation will be convenient when performing graph operations.

Notation 2.4. *Let $v \in V(G)$ and $U \subseteq N(v)$. Define $R_{v,U} := R^1_{v,U} \cup R^2_{v,U}$ where $R^1_{v,U} = \{\{r\} \subseteq N(v) - U : r \in V_{\leq 2}\}$ and $R^2_{v,U} = \{\{r_1, r_2\} \subseteq N(v) - U : r_1, r_2 \in V_3$ and r_1, r_2 are cofacial$\}$.*

Lemma 2.5. *For any $v \in V(G)$ and $U \subseteq N(v)$, if $R_1, R_2 \in R_{v,U}$, then $R_1 \cap R_2 \neq \emptyset$.*

Proof. First, assume that there exist distinct $\{x\}, \{y\} \in R^1_{v,U}$. Let $F' = A(G - \{v, x, y\})$. By the choice of G, $|F'| = a(G') \geq \lceil (4(n-3)+3)/7 \rceil$. Hence $G[F' + \{x, y\}]$ is an induced forest in G; so $a(G) \geq |F'| + 2 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Now assume there exist $\{x\} \in R^1_{v,U}, \{y, z\} \in R^2_{v,U}$. Let $w \in V(G)$ such that $vywzv$ is a facial cycle. Let $F' = A(G - \{v, x, y, z, w\})$. Then $|F'| \geq \lceil (4(n-5)+3)/7 \rceil$ by the choice of G. Clearly, $G[F' + \{x, y, z\}]$ is an induced forest in G; so $a(G) \geq |F'| + 3 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Finally, assume $\{x_1, x_2\}, \{y_1, y_2\} \in R^2_{v,U}$ with $\{x_1, x_2\} \cap \{y_1, y_2\} = \emptyset$. Let $x_3, y_3 \in V(G)$ such that $vx_1x_3x_2v$ and $vy_1y_3y_2v$ are facial cycles. Let $F' = A(G - \{v, x_1, x_2, x_3, y_1, y_2, y_3\})$. By the choice of G, $|F'| \geq \lceil (4(n - 7) + 3)/7 \rceil$. Now $G[F' + \{x_1, x_2, y_1, y_2\}]$ is an induced forest in G, implying $a(G) \geq |F'| + 4 \geq \lceil (4n + 3)/7 \rceil$, a contradiction. $\square$

Notation 2.6. *Let $v \in V(G)$ and $U \subseteq N(v)$, and let $R \in R_{v,U}$. We define $G * R = G - \{v, r\}$ if $R = \{r\}$, and $G * R = (G - v)/r_1 r_2$ if $R = \{r_1, r_2\}$. For $F \subseteq G * R$, define $F \cdot R = G[F + r]$ if $R = \{r\}$. If $R = \{r_1, r_2\}$ and*

$r \in F$ where r denotes the identification of r_1 and r_2, then define $F \cdot R = G[F - r + \{r_1, r_2\}]$.

Remark 2.7. *Let $v \in V(G)$ and $U \subseteq N(v)$. If $R = \{r_1, r_2\} \in R^2_{v,U}$ and r denotes the identification of r_1 and r_2, then by Lemma 2.3 there exists $F = A(G * R)$ such that $r \in F$.*

3. Structure around 2-vertices

The objective of this section is to prove the following lemma about neighbors of a 2-vertex in G. This will be used later for discharging rules.

Lemma 3.1. *For each $x \in V_2$, there exist $v_5, v_5' \in V_{\geq 5} \cap N(x)$ or there exist $v_4 \in V_{\leq 4} \cap N(x)$ and $v_6 \in V_{\geq 6} \cap N(x)$.*

Remark 3.2. *Apply Lemma 2.5 with $v = v_5$ and $U = \emptyset$, we have $R_{v_5,\emptyset} = \{\{x\}\}$ because any two elements in $R_{v_5,\emptyset}$ intersect. Similarly, $R_{v_5',\emptyset} = \{\{x\}\}$ and $R_{v_4,\emptyset} = R_{v_6,\emptyset} = \{\{x\}\}$.*

Proof. First, $e(V_2) = 0$. For, suppose there exists $xy \in E(G)$ with $x, y \in V_2$. Let $z \in N(y) - \{x\}$ and $F' = A(G - \{x, y, z\})$. Then $|F'| \geq \lceil (4(n-3)+3)/7 \rceil$. Clearly, $G[F' + \{x, y\}]$ is an induced forest in G; so $a(G) \geq |F'| + 2 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Next, we claim that for each $y \in V_2$, it is impossible that y has one neighbor of degree 3 and the other neighbor of degree at most 5. For otherwise, there exists a path xyz in G with $x \in V_3, y \in V_2, z \in V_{\leq 5}$. Let $N(x) - \{y\} = \{x_1, x_2\}$. Note that $\{x_1, x_2\} \subseteq N(z)$ since G is a quadrangulation. Then, $d(z) = 5$; otherwise, with $F' = A(G - \{x, y, z, x_1, x_2\})$, $G[F' + \{x, y, z\}]$ is an induced forest in G showing that $a(G) \geq |F'| + 3 \geq \lceil (4(n-5) + 3)/7 \rceil + 3 \geq \lceil (4n + 3)/7 \rceil$, a contradiction. So let $N(z) = \{x_1, y, x_2, z_2, z_1\}$ such that x_i and z_i are cofacial for $i = 1, 2$. If $|N(x_1) \cap N(z_1)| \leq 2$, then let $F' = A((G - \{x, y, z, x_2\})/x_1z_1)$ with w as the identification of x_1 and z_1; now $G[F' + \{x, y, z\}]$ (if $w \notin F'$) or $G[F' - w + \{x, y, x_1, z_1\}]$ (if $w \in F'$) is an induced forest in G showing that $a(G) \geq |F'| + 3 \geq \lceil (4(n-5)+3)/7 \rceil \geq \lceil (4n+3)/7 \rceil$, a contradiction. Thus, let $|N(x_1) \cap N(z_1)| \geq 3$. Then there exist $u \in N(x_1) \cap N(z_1) - \{z\}$ and a separation (G_1, G_2) in G such that $V(G_1 \cap G_2) = \{x_1, z_1, u\}$, $\{x, y, z, x_2, z_2\} \subseteq V(G_1)$, and $N(x_1) \cap N(z_1) - \{z\} \subseteq V(G_2)$. Let $F_1^{(1)} = A(G_1 - \{x_1, z_1, x, y, z, x_2\})$ and $F_2^{(1)} = A(G_2 - \{x_1, z_1\})$. Then $G[F_1^{(1)} \cup F_2^{(1)} + \{x, y, z\} - (\{u\} \cap (F_1^{(1)} \triangle F_2^{(1)}))]$ is an induced forest in G, which implies that

$$a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 2 \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil + \lceil (4(|G_2| - 2) + 3)/7 \rceil + 2.$$

Now let $F_1^{(2)} = A(G_1 - \{x_1, z_1, x, y, z, x_2, u\})$ and $F_2^{(2)} = A(G_2 - \{x_1, z_1, u\})$. Then $G[F_1^{(2)} \cup F_2^{(2)} + \{x, y, z\}]$ is an induced forest in G, showing that

$$a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + 3 \geq \lceil(4(|G_1| - 7) + 3)/7\rceil + \lceil(4(|G_2| - 3) + 3)/7\rceil + 3.$$

By Lemma 2.1, we have $a(G) \geq \lceil(4n + 3)/7\rceil$, a contradiction.

Thus, to complete the proof of Lemma 3.1, it suffices to show that for each $y \in V_2$, it is impossible that y has one neighbor of degree 4 and the other neighbor of degree at most 5. For otherwise, there exists a path xyz such that $x \in V_4, y \in V_2$ and $z \in V_{\leq 5}$. Thus, $z \in V_4 \cup V_5$ by the above claims. Let $N(x) = \{x_1, x_2, x_3, y\}$ and $N(z) = \{z_1, z_2, x_2, x_3, y\}$ if $z \in V_5$ or $N(z) = \{z_1, x_2, x_3, y\}$ if $z \in V_4$.

Case 1. $N(x_2) \cap N(x_3) = \{x, z\}$ and either $|N(z_1) \cap N(z_2)| \leq 2$ or $z \in V_4$.

Let $F' = A((G - \{x, y, z\})/\{x_2x_3, z_1z_2\})$ (when $z \in V_5$) and $F' = A((G - \{x, y, z, z_1\})/x_2x_3)$ (when $z \in V_4$). Let x' (respectively, z' when $z \in V_5$) denote the identification of x_2 and x_3 (respectively, z_1 and z_2). Let $z' = z_1$ if $z \in V_4$. By the choice of G, $|F'| \geq \lceil(4(n - 5) + 3)/7\rceil$. It is easy to see that one of the following is an induced forest in G: $G[F' + \{x, y, z\}]$ (if $x', z' \notin F'$), or $G[(F' - z') + \{x, y, z_1, z_2\}]$ (if $x' \notin F'$ and $z' \in F'$), or $G[(F' - x') + \{x_2, x_3, y, z\}]$ (if $x' \in F'$ and $z' \notin F'$), or $G[(F' - \{x', z'\}) + \{x_2, x_3, y, z_1, z_2\}]$ (if $x', z' \in F'$). Therefore, $a(G) \geq |F'| + 3 \geq \lceil(4n + 3)/7\rceil$, a contradiction.

Case 2. $|N(x_2) \cap N(x_3)| \geq 3$ and either $|N(z_1) \cap N(z_2)| \leq 2$ or $z \in V_4$.

Then there exist $w \in N(x_2) \cap N(x_3)$ and a separation (G_1, G_2) in G such that $V(G_1 \cap G_2) = \{w, x_2, x_3, x\}$, $\{y, z, z_1, z_2\} \subseteq V(G_1)$, and $N(x_2) \cap N(x_3) - \{z\} \subseteq V(G_2)$. Let $F_1^{(1)} = A((G_1 - \{w, x_2, x_3, x, y, z\})/z_1z_2)$ (when $z \in V_5$) or $F_1^{(1)} = A(G_1 - \{w, x_2, x_3, x, y, z, z_1\})$ (when $z \in V_4$), and let $F_2^{(2)} = A(G_2 - \{w, x_2, x_3, x\})$. Let z' denote the identification of z_1 and z_2. Then $G[F_1^{(1)} \cup F_2^{(1)} + \{x, y, z\}]$ (if $z \in V_4$ or if $z \in V_5$ and $z' \notin F_1^{(1)}$), or $G[F_1^{(1)} \cup F_2^{(1)} - z' + \{x, y, z_1, z_2\}]$ (if $z \in V_5$ and $z' \in F_1^{(1)}$) is an induced forest in G, showing that

$$a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 3 \geq \lceil(4(|G_1| - 7) + 3)/7\rceil + \lceil(4(|G_2| - 4) + 3)/7\rceil + 3.$$

Let $F_1^{(2)} = A((G_1 - \{x, x_2, x_3, x, y, z\})/z_1z_2)$ (when $z \in V_5$) with z' as the identification of z_1 and z_2, or $F_1^{(2)} = A(G_1 - \{x_2, x_3, x, y, z, z_1\})$ (when $z \in V_4$), and let $F_2^{(2)} = A(G_2 - \{x_2, x_3, x\})$. Then $G[F_1^{(2)} \cup F_2^{(2)} + \{x, y, z\} -$

$(\{w\} \cap (F_1^{(2)} \triangle F_2^{(2)}))]$ (if $z \in V_4$ or $z \in V_5$ and $z' \notin F_1^{(2)}$), or $G[F_1^{(2)} \cup F_2^{(2)} - z' + \{x, y, z_1, z_2\} - (\{w\} \cap (F_1^{(2)} \triangle F_2^{(2)}))]$ (if $z \in V_5$ and $z' \in F_1^{(2)}$) is an induced forest in G, giving

$$a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + 2 \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil + \lceil (4(|G_2| - 3) + 3)/7 \rceil + 2.$$

Hence, by Lemma 2.2(1) (with $k = 1$, $a = |G_1| - 6, a_1 = |G_2| - 3, c = 3, L = \emptyset$), $a(G) \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Case 3. $N(x_2) \cap N(x_3) = \{x, z\}$ and $|N(z_1) \cap N(z_2)| \geq 3$.

Then there exist $u \in N(z_1) \cap N(z_2)$ and a separation (G_1, G_2) in G such that $V(G_1 \cap G_2) = \{z_1, z_2, u\}$, $\{x, y, z, x_2, x_3\} \subseteq V(G_1)$, and $N(z_1) \cap N(z_2) - \{z\} \subseteq V(G_2)$. Let $F_1^{(1)} = A((G_1 - \{z_1, z_2, x, y, z\})/x_2x_3)$ with x' as the identification of x_2 and x_3, and $F_2^{(1)} = A(G_2 - \{z_1, z_2\})$. Then $G[F_1^{(1)} \cup F_2^{(1)} + \{x, y, z\} - (\{u\} \cap (F_1^{(2)} \triangle F_2^{(2)}))]$ (if $x' \notin F_1^{(1)}$) or $G[(F_1^{(1)} - x') \cup F_2^{(1)} + \{x_2, x_3, y, z\} - (\{u\} \cap (F_1^{(2)} \triangle F_2^{(2)}))]$ (if $x' \in F_1^{(1)}$) is an induced forest in G, which, by the choice of G, implies

$$a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 2 \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil + \lceil (4(|G_2| - 2) + 3)/7 \rceil + 2.$$

Let $F_1^{(2)} = A((G_1 - \{u, z_1, z_2, x, y, z\})/x_2x_3)$ with x' as the identification of x_2 and x_3, and $F_2^{(2)} = A(G_2 - \{u, z_1, z_2\})$. Then $G[F_1^{(2)} \cup F_2^{(2)} + \{x, y, z\}]$ (if $z' \notin F_1^{(2)}$) or $G[(F_1^{(2)} - x') \cup F_2^{(2)} + \{x_2, x_3, y, z\}]$ (if $z' \in F_1^{(2)}$) is an induced forest in G. So by the choice of G.

$$a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + 3 \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil + \lceil (4(|G_2| - 3) + 3)/7 \rceil + 3.$$

So by Lemma 2.2(1) (with $k = 1$, $a = |G_1| - 6, a_1 = |G_2| - 2, c = 3, L = \emptyset$), $a(G) \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Case 4. $|N(x_2) \cap N(x_3)| \geq 3$ and $|N(z_1) \cap N(z_2)| \geq 3$.

Then there exist $w \in N(x_2) \cap N(x_3) - \{x, z\}$, $u \in N(z_1) \cap N(z_2) - \{z\}$, and subgraphs G_1, G_2, G_3 of G such that G_2 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle wx_2xx_3w containing $N(x_2) \cap N(x_3) - \{z\}$, G_3 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle zz_1uz_2z containing $N(z_1) \cap N(z_2) - \{z\}$, and G_1 is obtained from G by removing $G_2 - \{w, x, x_2, x_3\}$ and $G_3 - \{u, z, z_1, z_2\}$.

Define $A_i = \{u\}$ for $i = 1, 3$, $A_i = \emptyset$ for $i = 2, 4$, and $\overline{A_i} = \{u\} - A_i$. Define $W_i = \{w\}$ for $i = 3, 4$, $W_i = \emptyset$ for $i = 1, 2$ and $\overline{W_i} = \{w\} -$

W_i. For $i \in [4]$, let $F_1^{(i)} = A(G_1 - \{x, y, z, x_2, x_3, z_1, z_2\} - A_i - W_i)$ and $F_2^{(i)} = A(G_2 - \{x_2, x_3, x\} - W_i)$ and $F_3^{(i)} = A(G_3 - \{z_1, z_2\} - A_i)$. Then $|F_1^{(i)}| \geq \lceil(4(|G_1| - 7 - |A_i| - |W_i|) + 3)/7\rceil$, $|F_2^{(i)}| \geq \lceil(4(|G_2| - 3 - |W_i|) + 3)/7\rceil$, and $|F_3^{(i)}| \geq \lceil(4(|G_3| - 2 - |A_i|) + 3)/7\rceil + 3$. Since $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{x, y, z\} - \{u, w\} \cap (F_1^{(i)} \triangle (F_2^{(i)} \cup F_3^{(i)}))]$ is an induced forest in G, $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 3 - (1 - |A_i|) - (1 - |W_i|)$. Let $(n_1, n_2, n_3) := (4(|G_1| - 7) + 3, 4(|G_2| - 3) + 3, 4(|G_3| - 2) + 3)$. So by Lemma 2.2(4) (with $a = |G_1| - 7, a_1 = |G_2| - 3, a_2 = |G_3| - 2, c = 3$),

$$(n_1, n_2, n_3) \equiv (1, 0, 0), (4, 0, 4), (4, 4, 0), (0, 4, 4) \mod 7.$$

Subcase 4.1. $(n_1, n_2, n_3) \equiv (1, 0, 0)$ (resp. $(4, 4, 0)$) $\mod 7$.

Let $W_5 = \overline{W_6} = \{w\}$ and $W_6 = \overline{W_5} = \emptyset$. Let $i = 5$ if $(n_1, n_2, n_3) \equiv (1, 0, 0) \mod 7$ and $i = 6$ if $(n_1, n_2, n_3) \equiv (4, 4, 0) \mod 7$. Let $F_1^{(i)} = A((G_1 - \{x, y, z, x_2, x_3\} - W_i)/z_1z_2)$ with z' as the identification of z_1 and z_2, $F_2^{(i)} = A(G_2 - \{x_2, x_3, x\} - W_i)$, and $F_3^{(i)} = A(G_3)$. By the choice of G, $|F_1^{(i)}| \geq \lceil(4(|G_1| - 6 - |W_i|) + 3)/7\rceil$, $|F_2^{(i)}| \geq \lceil(4(|G_2| - 3 - |W_i|) + 3)/7\rceil$, and $|F_3^{(i)}| \geq \lceil(4|G_3| + 3)/7\rceil$. Then $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{x, y, z\} - \{z_1, z_2, u, w\} \cap (F_1^{(i)} \triangle (F_2^{(i)} \cup F_3^{(i)})]$ (if $z' \notin F_1^{(i)}$) or $G[(F_1^{(i)} - z') \cup F_2^{(i)} \cup F_3^{(i)} + \{x, y, z_1, z_2\} - \{u, w, z_1, z_2\} \cap ((F_1^{(i)} \cup \{z_1, z_2\}) \triangle (F_2^{(i)} \cup F_3^{(i)})]$ (if $z' \in F_1^{(i)}$) is an induced forest in G, showing that $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 3 - 3 - |\overline{W_i}| \geq \lceil(4n + 3)/7\rceil$, a contradiction.

Subcase 4.2. $(n_1, n_2, n_3) \equiv (4, 0, 4) \mod 7$.

Let $F_1^{(7)} = A(G_1 - \{x, y, z, x_2, x_3, z_1, w\})$, $F_2^{(7)} = A(G_2 - \{x_2, x_3, x, w\})$, and $F_3^{(7)} = A(G_3 - \{z_1\})$. Then $|F_1^{(7)}| \geq \lceil(4(|G_1| - 7) + 3)/7\rceil$, $|F_2^{(7)}| \geq \lceil(4(|G_2| - 4) + 3)/7\rceil$, and $|F_3^{(7)}| \geq \lceil(4(|G_3| - 1) + 3)/7\rceil$. Clearly, $G[F_1^{(7)} \cup F_2^{(7)} \cup F_3^{(7)} + \{x, y, z\} - \{u, z_2\} \cap (F_1^{(7)} \triangle (F_2^{(7)} \cup F_3^{(7)})]$ is an induced forest in G, showing that $a(G) \geq |F_1^{(7)}| + |F_2^{(7)}| + |F_3^{(7)}| + 1 \geq \lceil(4n + 3)/7\rceil$, a contradiction.

Subcase 4.3. $(n_1, n_2, n_3) \equiv (0, 4, 4) \mod 7$.

Let $F_1^{(8)} = A(G_1 - \{y, z, x_2, x_3, z_1\} + xz_2)$, $F_2^{(8)} = A(G_2 - \{x_2, x_3\})$, and $F_3^{(8)} = A(G_3 - \{z_1\})$. Then $|F_1^{(8)}| \geq \lceil(4(|G_1| - 5) + 3)/7\rceil$, $|F_2^{(8)}| \geq \lceil(4(|G_2| - 2) + 3)/7\rceil$, and $|F_3^{(8)}| \geq \lceil(4(|G_3| - 1) + 3)/7\rceil$. Now $G[F_1^{(8)} \cup F_2^{(8)} \cup F_3^{(8)} + \{y, z\} - (\{u, w, x, z_2\} \cap (F_1^{(8)} \triangle (F_2^{(8)} \cup F_3^{(8)}))]$ is an induced forest in G, which implies that $a(G) \geq |F_1^{(8)}| + |F_2^{(8)}| + |F_3^{(8)}| - 2 \geq \lceil(4n + 3)/7\rceil$, a contradiction. This completes the proof of Lemma 3.1.

4. Structure around 3-vertices

In this section, we derive useful information about strutures around a 3-vertex.

Lemma 4.1. *Let* $x_1 \in V_3$ *and* $N(x_1) = \{x, y_1, z_1\}$, *with* $y_1, z_1 \in V_4$, $x_2 \in N(x) \cap N(y_1) - \{x_1\}$ *and* $xx_1y_1x_2x$ *be a facial cycle in* G. *Then* $z_1x_2 \notin E(G)$.

Proof. For, suppose $z_1x_2 \in E(G)$. Then G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{x_1, x_2, z_1\}$, $y_1 \in V(G_1)$, and $x \in V(G_2)$. For $i = 1, 2$, let $F_i^{(1)} = A(G_i - \{z_1, x_1, x_2\})$; so $|F_i^{(1)}| \geq \lceil (4(|G_i| - 3) + 3)/7 \rceil$. Now $G[F_1^{(1)} \cup F_2^{(1)} + x_1]$ is an induced forest in G, giving $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 1$.

Let $F_1^{(2)} = A(G_1 - \{z_1, x_1, x_2, y_1\})$ and $F_2^{(2)} = A(G_2 - \{z_1, x_1, x_2, x\})$. Then $|F_i^{(2)}| \geq \lceil (4(|G_i| - 4) + 3)/7 \rceil$ for $i = 1, 2$. If $N(z_1) \cap V(G_1) - \{x_1, x_2\} \neq \emptyset$ and $N(z_1) \cap V(G_2) - \{x_1, x_2\} \neq \emptyset$, then $G[F_1^{(2)} \cup F_2^{(2)} + \{x_1, z_1\}]$ is an induced forest in G, giving $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + 2$. Thus, by Lemma 2.1, $a(G) \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

If $N(z_1) \cap V(G_1) - \{x_1, x_2\} = \emptyset$, then since G is a quadrangulation, y_1, x_1, z_1, x_2 are incident to a common face. This is a contradiction since $|N(y_1)| = 4$. So $N(z_1) \cap V(G_2) - \{x_1, x_2\} = \emptyset$. Then since G is a quadrangulation, x, x_1, z_1, x_2 are incident to a common face. This implies that $|N(x)| = 2$. So $G[F_1^{(2)} \cup F_2^{(2)} + \{x_1, x\}]$ is an induced forest in G, giving $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + 2$. Thus, by Lemma 2.1, $a(G) \geq \lceil (4n + 3)/7 \rceil$, a contradiction. □

Lemma 4.2. $\Delta(G[V_{\leq 3}]) \leq 1$.

Proof. First, we claim $e(V_2) = 0$. For, suppose there exists $xy \in E(G)$ with $x, y \in V_2$. Let $z \in N(y) - \{x\}$ and $F' = A(G - \{x, y, z\})$. Then $|F'| \geq \lceil (4(n - 3) + 3)/7 \rceil$. Clearly, $G[F' + \{x, y\}]$ is an induced forest in G; so $a(G) \geq |F'| + 2 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Suppose $G[V_{\leq 3}]$ contains a path, say xyz. By the claim above and Lemma 2.5, we may assume that $|N(y)| = |N(z)| = 3$. Suppose $|N(x)| = 2$. Since every face of G has length 4, x and z have a common neighbor, say s. Let $N(x) = \{s, y\}$, $N(y) = \{y_1, x, z\}$ and $N(z) = \{z_1, s, y\}$. Let $F' = A(G - \{x, y, z, s, y_1\})$. Then by the choice of G, $|F'| \geq \lceil (4(n-5)+3)/7 \rceil$. Now $G[F' + \{x, y, z\}]$ is an induced forest in G and, hence, $a(G) \geq |F'| + 3 \geq \lceil (4n + 3)/7 \rceil$, a contradiction. So $|N(x)| = 3$.

Since every face of G has length 4, x and z have a common neighbor, say s. Let $N(x) = \{x_1, s, y\}$, $N(y) = \{y_1, x, z\}$ and $N(z) = \{z_1, s, y\}$. If $x_1 = z_1$, let $F' = A(G - \{x, y, z, s, x_1\})$. Then by the choice of G, $|F'| \geq$

$\lceil (4(n-5)+3)/7 \rceil$. Now $G[F' + \{x, y, z\}]$ is an induced forest in G and, hence, $a(G) \geq |F'| + 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction. So $x_1 \neq z_1$.

If $N(x_1) \cap N(z_1) = \{y_1\}$, let $F' = A((G - \{x, y, z, s\})/x_1 z_1)$ with x' as the identification of x_1 and z_1. Then $|F'| \geq \lceil (4(n-5)+3)/7 \rceil$. Now $G[F' + \{x, y, z\}]$ (if $x' \notin F'$) or $G[(F' - x') + \{x, z, x_1, z_1\}]$ (if $x' \in F'$) is an induced forest in G. So $a(G) \geq |F'| + 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

So $|N(x_1) \cap N(z_1)| \geq 2$. Then there exist $w \in N(x_1) \cap N(z_1) - \{y_1\}$ and a separation (G_1, G_2) in G such that $V(G_1 \cap G_2) = \{w, x_1, y_1, z_1\}$, $\{x, y, z, s\} \subseteq V(G_1)$, and $N(x_1) \cap N(z_1) \subseteq V(G_2)$. Let $W_1 = \overline{W_2} = \{w\}$ and $\overline{W_1} = W_2 = \emptyset$. For $i = 1, 2$, let $F_1^{(i)} = A(G_1 - \{s, x, y, z, x_1, y_1, z_1\} - W_i)$ and $F_2^{(i)} = A(G_2 - \{x_1, z_1\} - W_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 7 - |W_i|) + 3)/7 \rceil$ and $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |W_i|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} + \{x, y, z\} - (\{w\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ is an induced forest in G, giving $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + 3 - |\overline{W_i}|$. By Lemma 2.2(1) (with $k = 1$, $a = |G_1| - 7, a_1 = |G_2| - 2, L = \emptyset, c = 3$), $a(G) \geq \lceil (4n+3)/7 \rceil$, a contradiction. □

Lemma 4.3. *Let $x \in V_3$. If $y \in N(x)$ and $R_{y,\{x\}} \neq \emptyset$ then for any $z \in N(x) - \{y\}$, $R_{z,\{x\}} = \emptyset$.*

Proof. For otherwise, suppose $z \in N(x) - \{y\}$ and $R_{z,\{x\}} \neq \emptyset$. Let $R_1 \in R_{y,\{x\}}$ and $R_2 \in R_{z,\{x\}}$.

If $|R_1| = 1$ or $|R_2| = 1$, let $F' = A(((G - \{x, y, z\}) * R_1) * R_2)$. Then $|F'| \geq \lceil (4(n-5)+3)/7 \rceil$. Now $G[((F' + x) \cdot R_1) \cdot R_2]$ is an induced forest in G, showing $a(G) \geq |F'| + 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

So $|R_1| = |R_2| = 2$, let $R_1 = \{r_1, r_2\}$ and $yr_1y'r_2y$ bound a 4-face. Suppose $y' = z$. Let $F' = A(G - \{x, y, z, r_1, r_2\})$. Then $|F'| \geq \lceil (4(n-5)+3)/7 \rceil$. Now $G[F' + \{x, r_1, r_2\}]$ is an induced forest in G, showing $a(G) \geq |F'| + 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Now, we may assume $y' \neq z$. Suppose $R_1 \cap R_2 \neq \emptyset$. Without loss of generality, let $R_2 = \{r_2, r_3\}$. Since G is a quadrangulation, $zr_2y'r_3z$ bounds a 4-face. Let $F'' = A(G - \{x, y, z, r_1, r_2, r_3, y'\})$. Then $|F''| \geq \lceil (4(n-7)+3)/7 \rceil$. Now $G[F'' + \{x, r_1, r_2, r_3\}]$ is an induced forest in G, showing $a(G) \geq |F''| + 4 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Finally, we may assume $R_1 \cap R_2 = \emptyset$. Let $F''' = A((G - \{x, y, z, r_1, r_2, y'\}) * R_2)$. Then $|F'''| \geq \lceil (4(n-7)+3)/7 \rceil$. Now $G[(F''' + \{x, r_1, r_2\}) \cdot R_2]$ is an induced forest in G, showing $a(G) \geq |F'''| + 4 \geq \lceil (4n+3)/7 \rceil$, a contradiction. □

Lemma 4.4. *Let $x \in V_3$. If $y \in N(x) \cap V_{\leq 4}$ then for any $z \in N(x) - \{y\}$, $R_{z,\{x\}} = \emptyset$.*

Proof. Let $N(x) = \{u, y, z\}$, $y \in V_{\leq 4}$ and $R \in R_{z,\{x\}}$. Let $vyxzv$ be a facial cycle, $N(y) = \{y_1, x, v\}$ if $y \in V_3$ and $N(y) = \{y_1, y_2, x, v\}$ if $y \in V_4$. In the proof below, we assume $y \in V_4$ as for $y \in V_3$. We simply delete y_1 instead of identifying y_1 and y_2. Define $W_i = \{v\}$ for $i = 1, 3, 5, 8$ and $W_i = \emptyset$ if $i = 2, 4, 6, 7$, and let $\overline{W_i} = \{v\} - W_i$ for $i \in [8]$.

Suppose $R = \{y_2\}$. This implies that $zy_2 \in E(G)$ and $|N(y_2)| = 2$. Since G is a plane graph, $uv \notin E(G)$. Let $F = A(G - \{x, y, z, y_1, y_2\} + uv)$. By the choice of G, $|F| \geq \lceil (4(n-5)+3)/7 \rceil$. Then $G[F + \{x, y, y_2\}]$ is an induced forest in G. So $a(G) \geq |F| + 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction. So $R \neq \{y_2\}$. Similarly, $R \neq \{y_1\}$.

Case 1. $|N(y_1) \cap N(y_2)| \leq 2$ and $uv \notin E(G)$.

Let $F' = A((G - \{x, y, z\}) * R)/y_1y_2 + uv)$ with y' as the identification of y_1 and y_2. By the choice of G, $|F'| \geq \lceil (4(n-5)+3)/7 \rceil$. Then $G[(F' + \{x, y\}) \cdot R]$ (if $y' \notin F'$) or $G[(F' - \{y'\} + \{x, y_1, y_2\}) \cdot R]$ (if $y' \in F'$) is an induced forest in G. So $a(G) \geq |F'| + 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Case 2. $|N(y_1) \cap N(y_2)| \leq 2$ and $uv \in E(G)$.

Then G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{u, v, x\}$, $\{y, y_1, y_2\} \subseteq V(G_1)$, $z \in V(G_2)$. For $i = 1, 2$, let $F_1^{(i)} = A((G_1 - \{u, x, y\} - W_i)/y_1y_2))$ with y' as the identification of y_1 and y_2, and $F_2^{(i)} = A((G_2 - \{u, x, z\} - W_i) * R)$. Then $|F_j^{(i)}| \geq \lceil (4(|G_j| - 4 - |W_i|) + 3)/7 \rceil$ for $j = 1, 2$. Now $G[(F_1^{(i)} \cup F_2^{(i)} + \{x, y\}) \cdot R - (\{v\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ (if $y' \notin F_1^{(i)}$) or $G[((F_1^{(i)} - y') \cup F_2^{(i)} + \{x, y_1, y_2\}) \cdot R - (\{v\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ is an induced forest in G, showing that $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + 3 - |\overline{W_i}|$. By Lemma 2.2(1) (with $k = 1, a = |G_1| - 4, a_1 = |G_2| - 4, L = \emptyset, c = 3$), $a(G) \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Case 3. $|N(y_1) \cap N(y_2)| \geq 3$ and $uv \notin E(G)$.

There exist $w \in N(y_1) \cap N(y_2)$ and a separation (G_1, G_2) in G such that $V(G_1 \cap G_2) = \{y_1, y_2, w\}$, $\{x, y, z, u, v\} \subseteq V(G_1)$, and $N(y_1) \cap N(y_2) - \{y\} \subseteq V(G_2)$. Define $A_i = \{w\}$ if $i = 1, 3, 4$ and $A_i = \emptyset$ if $i = 2, 5, 6$, and let $\overline{A_i} = \{w\} - A_i$. For $i = 1, 2$, let $F_1^{(i)} = A((G_1 - \{x, y, z, y_1, y_2\} - A_i) * R + uv)$, and $F_2^{(i)} = A(G_2 - \{y_1, y_2\} - A_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 6 - |A_i|) + 3)/7 \rceil$, and $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |A_i|) + 3)/7 \rceil$. Now $G[(F_1^{(i)} \cup F_2^{(i)} + \{x, y\}) \cdot R - (\{w\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ is an induced forest in G, implying $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 3 - |\overline{A_i}|$. So by Lemma 2.2(1) (with $k = 1, a = |G_1| - 6, a_1 = |G_2| - 2, L = \emptyset, c = 3$), $a(G) \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Case 4. $|N(y_1) \cap N(y_2)| \geq 3$ and $uv \in E(G)$.

There exist $w \in N(y_1) \cap N(y_2)$ and subgraphs G_1, G_2, G_3 of G such that G_2 is the maximal subgraph of G contained in the closed region of the plane bounded by $uxzvu$ and containing R, G_3 is obtained by deleting y from the maximal subgraph of G contained in the closed region bounded by $y_1yy_2wy_1$ and containing $N(y_1) \cap N(y_2)$, and G_1 is obtained from G by removing $G_2 - \{u, v, x\}$ and $G_3 - \{w, y_1, y_2\}$. For $i = 3, 4, 5, 6$, let $F_1^{(i)} = A(G_1 - \{x, u, y, y_1, y_2\} - A_i - W_i)$, $F_2^{(i)} = A((G_2 - \{u, x, z\} - W_i) * R)$, and $F_3^{(i)} = A(G_3 - \{y_1, y_2\} - A_1)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 5 - |W_i| - |A_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2| - 4 - |W_i|) + 3)/7 \rceil$, and $|F_3^{(i)}| \geq \lceil (4(|G_3| - 2 - |A_i|) + 3)/7 \rceil$. Now $G[(F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{x, y\}) \cdot R - \{v, w\} \cap (F_1^{(i)} \triangle (F_2^{(i)} \cup F_3^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 3 - |\overline{W_i}| - |\overline{A_i}|$. Let $(n_1, n_2, n_3) := (4(|G_1| - 5) + 3, 4(|G_2| - 4) + 3, 4(|G_3| - 2) + 3)$. By Lemma 2.2(4) (with $a = |G_1| - 5, a_1 = |G_2| - 4, a_2 = |G_3| - 2$), $(n_1, n_2, n_3) \equiv (1, 0, 0), (4, 0, 4), (4, 4, 0), (0, 4, 4) \mod 7$.

Subcase 4.1. $(n_1, n_2, n_3) \equiv (1, 0, 0)$ (resp. $(4, 4, 0) \mod 7$).

For $i = 7$ (resp. $i = 8$), let $F_1^{(i)} = A(G_1 - \{u, x, y\} - W_i)$, $F_2^{(i)} = A(G_2 - \{u, x\} - W_i)$ and $F_3^{(i)} = A(G_3)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 3) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2) + 3)/7 \rceil$ and $|F_3^{(i)}| \geq \lceil (4|G_3| + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{x\} - \overline{W_i} - \{y_1, y_2, w\} \cap (F_1^{(i)} \triangle F_3^{(i)})]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 1 - 3 - |\overline{W_i}| \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Subcase 4.2. $(n_1, n_2, n_3) \equiv (4, 0, 4), (0, 4, 4) \mod 7$.

Let $F_1^{(9)} = A(G_1 - \{u, x, y, y_1, v\})$, $F_2^{(9)} = A(G_2 - \{u, x, v\})$ and $F_3^{(9)} = A(G_3 - \{y_1\})$. Then $|F_1^{(9)}| \geq \lceil (4(|G_1| - 5) + 3)/7 \rceil$, $|F_2^{(9)}| \geq \lceil (4(|G_2| - 3) + 3)/7 \rceil$, and $|F_3^{(9)}| \geq \lceil (4(|G_3| - 1) + 3)/7 \rceil$. Now $G[F_1^{(9)} \cup F_2^{(9)} \cup F_3^{(9)} + \{x, y\} - \{y_2, w\} \cap (F_1^{(9)} \triangle F_3^{(9)})]$ is an induced forest in G, showing $a(G) \geq |F_1^{(9)}| + |F_2^{(9)}| + |F_3^{(9)}| + 2 - 2 \geq \lceil (4n + 3)/7 \rceil$, a contradiction. □

Lemma 4.5. *For each $x \in V_3$, $N(x) \nsubseteq V_{\leq 4}$.*

Proof. Let $x \in V_3$ with $N(x) = \{w, y, z\} \subseteq V_{\leq 4}$. By Lemma 4.2, $|N(x) \cap V_{\leq 3}| \leq 1$; so let $N(z) = \{x, z_1, z_2, w_1\}$ and $N(w) = \{x, w_1, w_2, y_1\}$. Suppose $y \in V_2$. Let $N(y) = \{x, y_1\}$. Since G is a quadrangulation, we may assume $z_1 = y_1$. Let $F = A(G - \{x, y, z, w, y_1, w_1, z_2\})$. Then $|F| \geq \lceil (4(n-7)+3)/7 \rceil$. Therefore, $G[F + \{x, y, z, w\}]$ is an induced forest in G, showing that $a(G) \geq |F| + 4 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Now let $N(y) = \{x, y_1, z_1\}$ if $y \in V_3$ and $N(y) = \{x, y_1, y_2, z_1\}$ if $y \in V_4$. In the argument to follow, we treat the case $y \in V_4$, as the proof for $y \in V_3$ is the same by replacing identification of y_1 and y_2 with the deletion of y_1.

Case 1. $|N(y_1) \cap N(y_2)| \leq 2$, $|N(z_1) \cap N(z_2)| \leq 2$ and $|N(w_1) \cap N(w_2)| \leq 2$.

Let $F' = A(G - \{x, y, z, w\}/\{y_1y_2, z_1z_2, w_1w_2\})$ with y', z', w' as the identifications of y_1 and y_2, z_1 and z_2, and w_1 and w_2, respectively. Then $|F'| \geq \lceil (4(n-7)+3)/7 \rceil$. Let $F = F' + \{x, y, z, w\}$ if $w', y', z' \notin F'$, and otherwise, let F be obtained from $F' + \{x, y, z, w\}$ by deleting w, w' (respectively, y, y', z, z') adding $\{w_1, w_2\}$ (respectively, $\{y_1, y_2\}$, $\{z_1, z_2\}$) if $w' \in F'$ (respectively, $y' \in F'$, $z' \in F'$). Then $G[F]$ is an induced forest in G, showing that $a(G) \geq |F'| + 4 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Case 2. Exactly one of $|N(y_1) \cap N(y_2)|$, $|N(z_1) \cap N(z_2)|$, $|N(w_1) \cap N(w_2)|$ is greater than 2.

By symmetry, assume $|N(z_1) \cap N(z_2)| \geq 3$. Then there exist $z' \in N(z_1) \cap N(z_2)$ and a separation (G_1, G_2) in G such that $V(G_1 \cap G_2) = \{z_1, z_2, z'\}$, $\{x, y, z, w, y_1, y_2, w_1, w_2\} \subseteq V(G_1)$, $N(z_1) \cap N(z_2) - \{z\} \subseteq V(G_2)$. Define $A_i = \{z'\}$ for $i = 1, 5$ or $A_i = \emptyset$ for $i = 2, 6$, and let $\overline{A_i} = \{z'\} - A_i$. For $i = 1, 2$, let $F_1^{(i)} = A((G_1 - \{x, y, z, w, z_1, z_2\} - A_i)/\{y_1y_2, w_1w_2\})$ with y', w' as the identifications of y_1 and y_2, w_1 and w_2, respectively, and let $F_2^{(i)} = A(G_2 - \{z_1, z_2\} - A_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |A_i|) + 3)/7 \rceil$ and $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |A_i|) + 3)/7 \rceil$. Let $F^{(i)} = F_1^{(i)} \cup F_2^{(i)} + \{x, y, z, w\} - (\{z'\} \cap (F_1^{(i)} \triangle F_2^{(i)}))$ if $w', y' \notin F_1^{(i)}$, and otherwise, let $F^{(i)}$ be obtained from $F_1^{(i)} \cup F_2^{(i)} + \{x, y, z, w\} - (\{z'\} \cap (F_1^{(i)} \triangle F_2^{(i)}))$ by deleting $\{y, y'\}$ (respectively, $\{w, w'\}$) and adding $\{y_1, y_2\}$ (respectively, $\{w_1, w_2\}$) when $y' \in F_1^{(i)}$ (respectively, $w' \in F_1^{(i)}$). Then $G[F^{(i)}]$ is an induced forest in G, giving $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + 4 - |\overline{A_i}|$. By Lemma 2.2(2) (with $a = |G_1| - 8, a_1 = |G_2| - 2, c = 4$), $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (4, 0), (0, 4) \mod 7$.

Subcase 2.1. $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (4, 0) \mod 7$.

Let $F_1^{(3)} = A((G_1 - \{x, y, z, w\})/\{y_1y_2, w_1w_2, z_1z_2\})$ with y', w', z'' as the identification of y_1 and y_2, w_1 and w_2, and z_1 and z_2, respectively, and let $F_2^{(3)} = A(G_2)$. Then $|F_1^{(3)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$ and $|F_2^{(3)}| \geq \lceil (4|G_2| + 3)/7 \rceil$. Let $F^{(3)} = \overline{F_1}^{(3)} \cup F_2^{(3)} - (\{z', z_1, z_2\} \cap (\overline{F_1}^{(3)} \triangle F_2^{(3)}))$ where $\overline{F_1}^{(3)} = F_1{}^{(3)} + \{x, y, z, w\}$ if $w', y', z'' \notin F_1^{(3)}$; otherwise, let $\overline{F_1}^{(3)}$ be obtained from $F_1{}^{(3)} + \{x, y, z, w\}$ by deleting y, y' (respectviely, w, w', z, z'') and adding $\{y_1, y_2\}$ (respectively, $\{w_1, w_2\}$, $\{z_1, z_2\}$) when $y' \in F_1^{(3)}$ (respectively, $w' \in$

$F_1^{(3)}$, $z'' \in F_1^{(3)}$). Therefore, $G[F^{(3)}]$ is an induced forest in G, showing that $a(G) \geq |F_1^{(3)}| + |F_2^{(3)}| + 4 - 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 2.2. $(4(|G_1|-8)+3, 4(|G_2|-2)+3) \equiv (0,4) \mod 7$.

If $wz_2 \notin E(G)$, then let $F_1^{(4)} = A((G_1 - \{x,y,z,z_1,w_1\})/y_1y_2 + wz_2)$ with y' as the identification of y_1 and y_2, and $F_2^{(4)} = A(G_2 - \{z_1\})$. Then $|F_1^{(4)}| \geq \lceil (4(|G_1|-6)+3)/7 \rceil$ and $|F_2^{(4)}| \geq \lceil (4(|G_2|-1)+3)/7 \rceil$. Now $G[F_1^{(4)} \cup F_2^{(4)} + \{x,y,z\} - (\{z', z_2\} \cap (F_1^{(4)} \triangle F_2^{(4)}))]$ (if $y' \notin F_1^{(4)}$) or $G[(F_1^{(4)} - y') \cup F_2^{(4)} + \{x, y_1, y_2, z\} - (\{z', z_2\} \cap (F_1^{(4)} \triangle F_2^{(4)}))]$ (if $y' \in F_1^{(4)}$) is an induced forest in G, giving $a(G) \geq |F_1^{(4)}| + |F_2^{(4)}| + 3 - 2 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

So $wz_2 \in E(G)$. Then there exist subgraphs G_1', G_2', G_3' of G such that $G_2' = G_2$, G_3' is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $wxzz_2w$ and containing $N(w) \cap N(z) - \{x\}$, and G_1' is obtained from G by removing $G_2' - \{z_1, z_2, z'\}$ and $G_3' - \{w, z, z_2\}$. For $i = 5, 6$, let $F_1^{(i)} = A(G_1' - \{w, x, z, z_1, z_2\} - A_i)$, $F_2^{(i)} = A(G_2' - \{z_1, z_2\} - A_i)$, and $F_3^{(i)} = A(G_3' - \{w, z, z_2\})$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1'| - 5 - |A_i|)+3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2'| - 2 - |A_i|)+3)/7 \rceil$ and $|F_3^{(i)}| \geq \lceil (4(|G_3'| - 3)+3)/7 \rceil$. So $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{x, z\} - (\{z'\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ is an induced forest in G, giving $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 2 - |\overline{A_1}|$. By Lemma 2.2(1) (with $k = 1, L = \{1\}, a = |G_1'| - 5, a_1 = |G_2'| - 2, b_1 = |G_3'| - 3, c = 2$), $a(G) \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Thus, by symmetry, we have Case 3. At least two of $|N(y_1) \cap N(y_2)|$, $|N(z_1) \cap N(z_2)|$ and $|N(w_1) \cap N(w_2)|$ are greater than 2, and at least two of $|N(z_1) \cap N(y_2)|$, $|N(w_1) \cap N(z_2)|$ and $|N(w_2) \cap N(y_1)|$ are greater than 2.

First, suppose $|N(y_1) \cap N(y_2)| > 2$, $|N(y_1) \cap N(w_2)| > 2$, $|N(w_1) \cap N(w_2)| > 2$ and $|N(w_1) \cap N(z_2)| > 2$. Then there exist $y' \in N(y_1) \cap N(y_2) - \{y\}$, $w' \in N(y_1) \cap N(w_2) - \{w\}$, $w'' \in N(w_1) \cap N(w_2) - \{w\}$, $z' \in N(w_1) \cap N(z_2) - \{z\}$, and subgraphs G_1, G_2, G_3, G_4, G_5 of G such that G_2 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $yy_1y'y_2y$ and containing $N(y_1) \cap N(y_2) - \{y\}$, G_3 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $wy_1w'w_2w$ and containing $N(y_1) \cap N(w_2) - \{w\}$, G_4 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $ww_1w''w_2w$ and containing $N(w_1) \cap N(w_2) - \{w\}$, G_5 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $zw_1w''z_2z$ and containing $N(z_2) \cap N(w_1) - \{z\}$, and G_1 is obtained from G by removing $G_2 - \{y_1, y_2, y'\}$, $G_3 - \{y_1, w_2, w'\}$, $G_4 - \{w_1, w_2, w''\}$ and $G_5 - \{w_1, z_2, z'\}$. Let $A_1 \subseteq \{y'\}$, $B_1 \subseteq \{w'\}$, $C_1 \subseteq \{w''\}$, $D_1 \subseteq \{z'\}$. Let

$\overline{A_1} = \{y'\} - A_1$, $\overline{B_1} = \{w'\} - B_1$, $\overline{C_1} = \{w''\} - C_1$, $\overline{D_1} = \{z'\} - D_1$. For all choices of A_1, B_1, C_1, D_1, let $F_1^{(i)} = A(G_1 - \{w, y, x, z, z_1, z_2, y_1, y_2, w_1, w_2\} - A_1 - B_1 - C_1 - D_1)$, $F_2^{(i)} = A(G_2 - \{y_1, y_2\} - A_1)$, $F_3^{(i)} = A(G_3 - \{y_1, w_2\} - B_1)$, $F_4^{(i)} = A(G_4 - \{w_1, w_2\} - C_1)$, and $F_5^{(i)} = A(G_5 - \{w_1, z_2\} - D_1)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 10 - |A_1| - |B_1| - |C_1| - |D_1|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |A_1|) + 3)/7 \rceil$, $|F_3^{(i)}| \geq \lceil (4(|G_3| - 2 - |B_1|) + 3)/7 \rceil$, $|F_4^{(i)}| \geq \lceil (4(|G_4| - 2 - |C_1|) + 3)/7 \rceil$, and $|F_5^{(i)}| \geq \lceil (4(|G_5| - 2 - |D_1|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} \cup F_4^{(i)} \cup F_5^{(i)} + \{w, x, y, z\} - (\{y'\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{w'\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{w''\} \cap (F_1^{(i)} \triangle F_4^{(i)})) - (\{z'\} \cap (F_1^{(i)} \triangle F_5^{(i)}))]$ is an induced forest in G. Hence, $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + |F_4^{(i)}| + |F_5^{(i)}| + 4 - |\overline{A_1}| - |\overline{B_1}| - |\overline{C_1}| - |\overline{D_1}|$. By Lemma 2.2(1) (with $k = 4, a = |G_1| - 10, a_j = |G_{j+1}| - 2$ for $j = 1, 2, 3, 4$, $L = \emptyset, c = 4$), $a(G) \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Thus, by symmetry, we may assume that $|N(y_1) \cap N(y_2)| > 2$, $|N(y_1) \cap N(w_2)| > 2$, $|N(z_1) \cap N(z_2)| > 2$ and $|N(w_1) \cap N(z_2)| > 2$. Then there exist $y' \in N(y_1) \cap N(y_2) - \{y\}$, $w' \in N(y_1) \cap N(w_2) - \{w\}$, $z' \in N(z_1) \cap N(z_2) - \{w\}$, $z'' \in N(w_1) \cap N(z_2) - \{z\}$, and subgraphs G_1, G_2, G_3, G_4, G_5 of G such that G_2 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $yy_1y'y_2y$ and containing $N(y_1) \cap N(y_2) - \{y\}$, G_3 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $wy_1w'w_2w$ and containing $N(y_1) \cap N(w_2) - \{w\}$, G_4 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $zz_1z'z_2z$ and containing $N(z_1) \cap N(z_2) - \{z\}$, G_5 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $zw_1z''z_2z$ and containing $N(z_2) \cap N(w_1) - \{z\}$, and G_1 is obtained from G by removing $G_2 - \{y_1, y_2, y'\}$, $G_3 - \{y_1, w_2, w'\}$, $G_4 - \{z_1, z_2, z'\}$ and $G_5 - \{w_1, z_2, z''\}$. Let $A_1 \subseteq \{y'\}$, $B_1 \subseteq \{w'\}$, $C_1 \subseteq \{z'\}$, $D_1 \subseteq \{z''\}$. Let $\overline{A_1} = \{y'\} - A_1$, $\overline{B_1} = \{w'\} - B_1$, $\overline{C_1} = \{z'\} - C_1$, $\overline{D_1} = \{z''\} - D_1$. For all choices of A_1, B_1, C_1, D_1, let $F_1^{(i)} = A(G_1 - \{w, y, x, z, z_1, z_2, y_1, y_2, w_1, w_2\} - A_1 - B_1 - C_1 - D_1)$, $F_2^{(i)} = A(G_2 - \{y_1, y_2\} - A_1)$, $F_3^{(i)} = A(G_3 - \{y_1, w_2\} - B_1)$, $F_4^{(i)} = A(G_4 - \{z_1, z_2\} - C_1)$, and $F_5^{(i)} = A(G_5 - \{w_1, z_2\} - D_1)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 10 - |A_1| - |B_1| - |C_1| - |D_1|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |A_1|) + 3)/7 \rceil$, $|F_3^{(i)}| \geq \lceil (4(|G_3| - 2 - |B_1|) + 3)/7 \rceil$, $|F_4^{(i)}| \geq \lceil (4(|G_4| - 2 - |C_1|) + 3)/7 \rceil$, and $|F_5^{(i)}| \geq \lceil (4(|G_5| - 2 - |D_1|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} \cup F_4^{(i)} \cup F_5^{(i)} + \{w, x, y, z\} - (\{y'\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{w'\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{z'\} \cap (F_1^{(i)} \triangle F_4^{(i)})) - (\{z''\} \cap (F_1^{(i)} \triangle F_5^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + |F_4^{(i)}| + |F_5^{(i)}| + 4 - |\overline{A_1}| - |\overline{B_1}| -$

$|\overline{C_1}| - |\overline{D_1}|$. By Lemma 2.2(1) (with $k = 4, a = |G_1| - 10, a_j = |G_{j+1}| - 2$ for $j = 1, 2, 3, 4$, $L = \emptyset, c = 4$), $a(G) \geq \lceil (4n + 3)/7 \rceil$, a contradiction. □

By Lemmas 4.3, 4.4, 4.5, we have the following:

Corollary 4.6. *Let $x \in V_3$. Then there exists $v \in N(x) \cap V_{\geq 5}$ such that $R_{v,\{x\}} = \emptyset$.*

5. A forbidden configuration around a 3-vertex

We prove the following, which eliminates two configurations around a 3-vertex.

Lemma 5.1. *Let $x \in V_3$, $\{y, z\} \subseteq V_{\leq 4}$, $N(x) = \{w, y, z\}$. Suppose $xzvwx$ is a facial cycle and $w \in V_5$. Then $R_{v,\{w,z\}} = \emptyset$ and $v \notin V_{\leq 4}$.*

Proof. We may assume $\{y, z\} \subseteq V_4$ because the case when $y \in V_3$ or $z \in V_3$ is identical by replacing identifying neighbors of 4-vertex with deleting a neighbor of 3-vertex.

In the first part, we prove $R_{v,\{w,z\}} = \emptyset$. For, suppose $R \in R_{v,\{w,z\}}$. Let $N(y) = \{x, y_1, y_2, z_1\}$, $N(z) = \{x, v, z_1, z_2\}$ and $y_2 w \in E(G)$.

First, we claim that $wz_1 \notin E(G)$. For, suppose $wz_1 \in E(G)$. There exists a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{w, z, z_1\}$, $\{x, y, y_1, y_2\} \subseteq V(G_1)$, and $v \in V(G_2)$. Let $F_1^{(1)} = A(G_1 - \{z_1, z, w, x\})$, and $F_2^{(1)} = A((G_2 - \{z_1, z, w, v\}) * R)$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1| - 4) + 3)/7 \rceil$, and $|F_2^{(1)}| \geq \lceil (4(|G_2| - 5) + 3)/7 \rceil$. Now $G[(F_1^{(1)} \cup F_2^{(1)} + \{x, z\}) \cdot R]$ is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 3 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Secondly, we claim that $wz_2 \notin E(G)$. For otherwise, there exists a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{w, v, z_2\}$, $\{x, y, z, z_1, y_1, y_2\} \subseteq V(G_1)$, and $R \subseteq V(G_2)$. Let $F_1^{(2)} = A(G_1 - \{x, z, z_1, z_2, w, v\})$, and $F_2^{(2)} = A(G_2 - \{z_2, w\})$. Then $|F_1^{(2)}| \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil$, and $|F_2^{(2)}| \geq \lceil (4(|G_2| - 2) + 3)/7 \rceil$. Now $G[F_1^{(2)} \cup F_2^{(2)} + \{x, z\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + 2$. This implies $4(|G_2| - 2) + 3 \equiv 0, 5, 6 \mod 7$. If $|N(y_1) \cap N(y_2)| \leq 2$, let $F_1^{(3)} = A((G_1 - \{x, y, z, z_1, w, v\})/y_1 y_2)$ with y' as the identification of $\{y_1, y_2\}$, and $F_2^{(3)} = A((G_2 - \{w, v\}) * R)$. Then $|F_1^{(3)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$, and $|F_2^{(3)}| \geq \lceil (4(|G_2| - 3) + 3)/7 \rceil$. Now $F^{(3)} := G[(F_1^{(3)} \cup F_2^{(3)} + \{x, y, z\}) \cdot R - (\{z_2\} \cap (F_1^{(3)} \triangle F_2^{(3)}))]$ (if $y' \notin F_1^{(3)}$) or $G[((F_1^{(3)} - y') \cup F_2^{(3)} + \{x, y_1, y_2, z\}) \cdot R - (\{z_2\} \cap (F_1^{(3)} \triangle F_2^{(3)}))]$ (if $y' \in F_1^{(3)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(3)}| + |F_2^{(3)}| + 4 - 1$. By Lemma 2.2(1) (with $k = 1, a = |G_1| - 6, a_1 = |G_2| - 2, L = \emptyset, c = 3$),

$a(G) \geq \lceil (4n+3)/7 \rceil$, a contradiction. So $|N(y_1) \cap N(y_2)| > 2$. Then there exist $a_1 \in N(y_1) \cap N(y_2)$ and subgraphs G'_1, G'_2, G'_3 of G such that $G'_2 = G_2$, G'_3 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $yy_1a_1y_2y$ and containing $N(y_1) \cap N(y_2) - \{y\}$, and G_1 is obtained from G by removing $G'_2 - \{w, v, z_2\}$ and $G'_3 - \{a_1, y_2, y_1\}$. Let $A_4 = \emptyset$ and $A_5 = \{a_1\}$. For $i = 4, 5$, let $F_1^{(i)} = A(G'_1 - \{x, y, z, z_1, w, v, y_1, y_2, z_2\} - A_i)$, $F_2^{(i)} = A((G'_2 - \{w, v, z_2\}) * R)$, and $F_3^{(i)} = A(G'_3 - \{y_1, y_2\} - A_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G'_1| - 9 - |A_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G'_2| - 4) + 3)/7 \rceil$, and $|F_3^{(i)}| \geq \lceil (4(|G'_3| - 2 - |A_i|) + 3)/7 \rceil$. Now $G[(F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{x, y, z, w\}) \cdot R - (\{a_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 5 - (1 - |A_i|)$. By Lemma 2.2(1) (with $k = 1, a = |G'_1| - 8, a_1 = |G'_3| - 2, L = \{1\}, b_1 = |G'_2| - 4, c = 5$), $a(G) \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Case 1. $|N(y_1) \cap N(y_2)| \leq 2$ and $|N(z_1) \cap N(z_2)| \leq 2$.

Let $F' = A(((G - \{x, y, z, v\}) * R)/\{y_1y_2, z_1z_2\} + wz')$ with y' (respectively, z') as the identifications of $\{y_1, y_2\}$ (respectively, $\{z_1, z_2\}$). Then $|F'| \geq \lceil (4(n-7)+3)/7 \rceil$. Let $F := (F' + \{x, y, z\}) \cdot R$ if $y', z' \notin F'$, and otherwise F' obtained from $(F' + \{x, y, z\}) \cdot R$ by deleting $\{y, y'\}$ (respectively, $\{z', z\}$) and adding $\{y_2, y_1\}$ (respectively, $\{z_1, z_2\}$) when $y' \in F'$ (respectively, $z' \in F'$). Therefore, $G[F']$ is an induced forest in G, showing $a(G) \geq |F'| + 4 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Case 2. $|N(y_1) \cap N(y_2)| > 2$.

There exist $a_1 \in N(y_1) \cap N(y_2)$ and a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{y_1, y_2, a_1\}$, and $\{x, y, z, w, v\} \subseteq V(G_1)$, $N(y_1) \cap N(y_2) - \{y\} \subseteq V(G_2)$. Define $A_1 = \overline{A_2} = \{a_1\}$ and $A_2 = \overline{A_1} = \emptyset$. For $i = 1, 2$, let $F_1^{(i)} = A((G_1 - \{x, y, z, z_1, y_1, y_2, v\}) * R - A_i + wz_2)$, and $F_2^{(i)} = A(G_2 - \{y_1, y_2\} - A_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |A_i|) + 3)/7 \rceil$, and $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |A_i|) + 3)/7 \rceil$. Now $G[(F_1^{(i)} \cup F_2^{(i)} + \{x, y, z\}) \cdot R - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + 4 - (1 - |A_i|)$. By Lemma 2.2(2), $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (4, 0), (0, 4) \mod 7$.

Subcase 2.1. $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (4, 0) \mod 7$.

Let $F_1^{(5)} = A(((G_1 - \{x, y, z, z_1, v\}) * R)/y_1y_2 + wz_2)$ with y' as the identification of $\{y_1, y_2\}$, and $F_2^{(5)} = A(G_2)$. Then $|F_1^{(5)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$, and $|F_2^{(i)}| \geq \lceil (4|G_2| + 3)/7 \rceil$. Now $G[(F_1^{(5)} \cup F_2^{(5)} + \{x, y, z\}) \cdot R - (\{y_1, y_2, a_1\} \cap (F_1^{(5)} \triangle F_2^{(5)}))]$ (if $y' \notin F_1^{(5)}$) or $G[(F_1^{(5)} \cup F_2^{(5)} + \{x, y_1, y_2, z\} - \{y'\}) \cdot R - (\{a_1\} \cap (F_1^{(5)} \triangle F_2^{(5)})) - (\{y_1, y_2\} - F_2^{(5)})]$ (if $y' \in F_1^{(5)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(5)}| + |F_2^{(5)}| + 4 - 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 2.2. $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (0, 4) \mod 7$.

If $wy_1 \notin E(G)$ and $|N(v) \cap N(z_2)| \leq 2$, then let $F_1^{(6)} = A(G_1 - \{x, y, z, z_1, y_2\})/vz_2 + wy_1)$ with z' as the identification of $\{v, z_2\}$, and $F_2^{(6)} = A(G_2 - y_2)$. Then $|F_1^{(6)}| \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil$, and $|F_2^{(6)}| \geq \lceil (4(|G_2| - 1) + 3)/7 \rceil$. Let $F = F_1^{(6)} \cup F_2^{(6)} + \{x, y, z\} - (\{y_1, a_1\} \cap (F_1^{(6)} \triangle F_2^{(6)}))$. Now $G[F]$ (if $z' \notin F_1^{(6)}$) or $G[F - \{z, z'\} + \{v, z_2\}]$ (if $z' \in F_1^{(6)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(6)}| + |F_2^{(6)}| + 3 - 2 \geq \lceil (4n+3)/7 \rceil$, a contradiction. So we have $wy_1 \in E(G)$ or $|N(v) \cap N(z_2)| \geq 3$.

If $wy_1 \in E(G)$, then there exists a separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{y_1, y_2, w\}$, $\{x, y, z, v\} \subseteq V(G_1')$, and $N(y_1) \cap N(y_2) - \{y\} \subseteq V(G_2')$. Let $F_1^{(7)} = A((G_1 - \{w, x, y, z, y_1, y_2, z_1, v\}) * R)$ and $F_2^{(7)} = A(G_2 - \{y_1, w\})$. Then $|F_1^{(7)}| \geq \lceil (4(|G_1'| - 9) + 3)/7 \rceil$ and $|F_2^{(7)}| \geq \lceil (4(|G_2'| - 2) + 3)/7 \rceil$. Now $G[(F_1^{(7)} \cup F_2^{(7)} + \{x, y, z\}) \cdot R]$ is an induced forest in G, showing $a(G) \geq |F_1^{(7)}| + |F_2^{(7)}| + 4$. Let $F_1^{(8)} = A((G_1 - \{w, x, y, z, y_1, y_2, z_1, z_2, v\}) * R)$, and $F_2^{(8)} = A(G_2 - \{y_1, w, y_2\})$. Then $|F_1^{(8)}| \geq \lceil (4(|G_1'| - 10) + 3)/7 \rceil$, and $|F_2^{(8)}| \geq \lceil (4(|G_2'| - 3) + 3)/7 \rceil$. Now $F^{(8)} := G[(F_1^{(8)} \cup F_2^{(8)} + \{x, y, z, w\}) \cdot R]$ is an induced forest in G, showing $a(G) \geq |F_1^{(8)}| + |F_2^{(8)}| + 5$. By Lemma 2.2(1) (with $k = 1, a = |G_1'| - 9, a_1 = |G_2'| - 2, L = \emptyset, c = 4$), $a(G) \geq \lceil (4n+3)/7 \rceil$, a contradiction.

If $|N(v) \cap N(z_2)| > 2$, then there exist $c_1 \in N(v) \cap N(z_2)$ and subgraphs G_1'', G_2'', G_3'' of G such that $G_2'' = G_2$, G_3'' is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle zvc_1z_2 and containing $N(v) \cap N(z_2) - \{z\}$, and G_1'' is obtained from G by removing $G_2'' - \{y_1, y_2, a_1\}$ and $G_3'' - \{v, z_2, c_1\}$. By symmetry, assume $R \subseteq G_1''$. Define $C_8 = \overline{C_9} = \{c_1\}$ and $C_9 = \overline{C_8} = \emptyset$. For $i = 8, 9$, let $F_1^{(i)} = A((G_1'' - \{x, y, z, y_2, z_1, z_2, v\} - C_i) * R + wy_1)$, $F_2^{(i)} = A(G_2'' - y_2)$, and $F_3^{(i)} = A(G_3'' - \{c, z_2\} - C_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1''| - 8 - |C_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2''| - 1) + 3)/7 \rceil$, and $|F_3^{(i)}| \geq \lceil (4(|G_3''| - 2 - |C_i|) + 3)/7 \rceil$. Now $G[(F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{x, y, z\}) \cdot R - (\{y_1, a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{c_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - 2 - |\overline{C_1}|$. By Lemma 2.2(1) (with $k = 1, a = |G_1''| - 8, a_1 = |G_3''| - 2, L = \{1\}, b_1 = |G_2''| - 1, c = 2$), $a(G) \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Case 3. $|N(z_1) \cap N(z_2)| > 2$.

There exist $b_1 \in N(z_1) \cap N(z_2)$ and a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{z_1, z_2, b_1\}$, $\{x, y, z, w, v\} \subseteq V(G_1)$, and $N(z_1) \cap N(z_2) - \{z\} \subseteq V(G_2)$. Let $B_1 = \{b_1\}$ and $B_2 = \emptyset$. For $i = 1, 2$, let $F_1^{(i)} = A(((G_1 - \{x, y, z, z_1, z_2, v\} - B_i) * R)/y_1y_2)$ with y' as the identification of $\{y_1, y_2\}$, and $F_2^{(i)} = A(G_2 - \{z_1, z_2\} - B_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |B_i|) + 3)/7 \rceil$, and $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |B_i|) + 3)/7 \rceil$. Let $F = (F_1^{(i)} \cup F_2^{(i)} + \{x, y, z\}) \cdot R - (\{b_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))$. Now $G[F]$ (if $y' \notin F_1^{(i)}$) or $G[F - \{y, y'\} + \{y_1, y_2\}]$ (if $y' \in F_1^{(i)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + 4 - (1 - |B_i|)$. By Lemma 2.2(2), $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (0, 4), (4, 0) \mod 7$.

Subcase 3.1. $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (0, 4) \mod 7$.

Let $F_1^{(3)} = A(((G_1 - \{x, y, z, v, z_1\}) * R)/y_1y_2 + wz_2)$ with y' as the identification of $\{y_1, y_2\}$, and $F_2^{(3)} = A(G_2 - z_1)$. Then $|F_1^{(3)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$, and $|F_2^{(3)}| \geq \lceil (4(|G_2| - 1) + 3)/7 \rceil$. Let $F = (F_1^{(3)} \cup F_2^{(3)} + \{x, y, z\}) \cdot R - (\{z_2, b_1\} \cap (F_1^{(3)} \triangle F_2^{(3)}))$. Now $G[F]$ (if $y' \notin F_1^{(3)}$) or $G[F - \{y, y'\} + \{y_1, y_2\}]$ (if $y' \in F_1^{(3)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(3)}| + |F_2^{(3)}| + 4 - 2 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Subcase 3.2. $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (4, 0) \mod 7$.

Let $F_1^{(4)} = A(((G_1 - \{x, y, z, v\}) * R)/\{y_1y_2, z_1z_2\} + wz')$ with y' (respectively, z') as the identification of $\{y_1, y_2\}$ (respectively, $\{z_1, z_2\}$), and $F_2^{(9)} = A(G_2)$. Then $|F_1^{(4)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$, and $|F_2^{(4)}| \geq \lceil (4|G_2| + 3)/7 \rceil$. Now $F^{(4)} := (\overline{F_1}^{(4)} \cup F_2^{(4)}) \cdot R - (\{z_1, z_2, b_1\} \cap (\overline{F_1}^{(4)} \triangle F_2^{(4)}))$ where $\overline{F_1}^{(4)} = {F_1}^{(4)} + \{x, y, z\}$ if $y', z' \notin F_1^{(4)}$; or obtained from ${F_1}^{(4)} + \{x, y, z\}$ by deleting $\{z, z'\}$ ($\{y, y'\}$ respectively) and adding $\{z_1, z_2\}$ ($\{y_1, y_2\}$ respectively) when $z' \in {F_1}^{(4)}$ ($y' \in {F_1}^{(4)}$ respectively). Therefore, $G[F^{(4)}]$ is an induced forest of size $|F_1^{(4)}| + |F_2^{(4)}| + 4 - 3 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

We now prove $v \notin V_{\leq 4}$. By Lemma 3.1, $v \not\in N_2$. For otherwise, $v \in V_4$. The case $v \in V_3$ is identical by replacing identification of neighbors of v with deletion of a neighbor of v. Let $N(y) = \{x, y_1, y_2, z_1\}$ and $N(z) = \{x, v, z_1, z_2\}$ and $y_2w \in E(G)$. Let $N(v) = \{z, w, v_1, v_2\}$ and $v_1vzz_2v_1$ be a facial cycle.

Claim 1. $wz_1 \notin E(G)$.

For, suppose $wz_1 \in E(G)$. There exists a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{w, x, z_1\}$, $\{y, y_1, y_2\} \subseteq V(G_1)$, and $\{z, v\} \subseteq V(G_2)$. For $i = 1, 2$, let $F_i^{(1)} = A(G_i - \{z_1, w, x\})$. Then $|F_i^{(1)}| \geq \lceil (4(|G_i| - 3) + 3)/7 \rceil$. Now $G[F_1^{(1)} \cup F_2^{(1)} + x]$ is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 1$.

By Lemma 2.2(7) (with $k = 1, a_i = |G_i| - 3$ for $i = 1, 2$), $(4(|G_1| - 3) + 3, 4(|G_2| - 3) + 3) \equiv (0, 0), (0, 6), (6, 0) \mod 7$.

If $wz_2 \notin E(G)$ and $wy_1 \notin E(G)$, let $F_1^{(2)} = A(G_1 - \{z_1, x, y, y_2\} + wy_1)$, and $F_2^{(2)} = A(G_2 - \{z_1, z, x, v\} + wz_2)$. For $i = 1, 2$ $|F_i^{(2)}| \geq \lceil (4(|G_i| - 4) + 3)/7 \rceil$. Now $G[F_1^{(2)} \cup F_2^{(2)} + \{x, y, z\} - (\{w\} \cap (F_1^{(2)} \triangle F_2^{(2)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + 3 - 1 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

If $wz_2 \in E(G)$, let $F_1^{(3)} = A(G_1 - \{z_1, w, x, y, y_2\})$, and $F_2^{(3)} = A(G_2 - \{z_1, w, x, z, v, z_2\})$. Then $|F_1^{(3)}| \geq \lceil (4(|G_1| - 5) + 3)/7 \rceil$, and $|F_2^{(3)}| \geq \lceil (4(|G_2| - 6) + 3)/7 \rceil$. Now $G[F_1^{(3)} \cup F_2^{(3)} + \{x, y, z, w\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(3)}| + |F_2^{(3)}| + 4 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

So $wy_1 \in E(G)$. Let $F_1^{(4)} = A(G_1 - \{z_1, w, x, y, y_2, y_1\})$, and $F_2^{(4)} = A(G_2 - \{z_1, w, x, z, v\})$. Then $|F_1^{(4)}| \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil$, and $|F_2^{(4)}| \geq \lceil (4(|G_2| - 5) + 3)/7 \rceil$. Now $G[F_1^{(4)} \cup F_2^{(4)} + \{x, y, z, w\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(4)}| + |F_2^{(4)}| + 4 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Claim 2. $wz_2 \notin E(G)$. *(By symmetry,* $wy_1 \notin E(G)$.*)*

For, suppose $wz_2 \in E(G)$. There exists a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{w, z, z_2\}$, $\{x, y, y_1, y_2\} \subseteq V(G_1)$, and $v \in V(G_2)$. Let $F_1^{(1)} = A(G_1 - \{w, z, z_2, x, z_1\})$, and $F_2^{(1)} = A(G_2 - \{w, z, z_2\})$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1| - 5) + 3)/7 \rceil$, and $|F_2^{(1)}| \geq \lceil (4(|G_2| - 3) + 3)/7 \rceil$. Now $G[F_1^{(1)} \cup F_2^{(1)} + \{x, z\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 2$. By Lemma 2.2(8) (with $a = |G_1| - 5, a_1 = |G_2| - 3$), $(4(|G_1| - 5) + 3, 4(|G_2| - 3) + 3) \equiv (0, 0), (0, 6), (0, 5), (5, 0), (6, 6), (6, 0) \mod 7$.

If $wy_1 \notin E(G)$, let $w' \in N(w) - \{y_2, v, z_2, x, y_1\}$. Let $e = w'y$ if $w' \in G_1$ and otherwise $e = \emptyset$. Let $F_1^{(2)} = A(G_1 - \{w, z, z_2, x, y_2, z_1\} + e)$, and $F_2^{(2)} = A(G_2 - \{w, z, z_2, v\})$. Then $|F_1^{(2)}| \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil$, and $|F_2^{(2)}| \geq \lceil (4(|G_2| - 4) + 3)/7 \rceil$. Now $G[F_1^{(2)} \cup F_2^{(2)} + \{w, x, z\}]$ is an induced forest of size $|F_1^{(2)}| + |F_2^{(2)}| + 3 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

So $wy_1 \in E(G)$. By Lemma 3.1, $|N(y_2)| > 2$. There exist subgraphs G_1', G_2', G_3' of G such that $G_2' = G_2$, G_3' is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle wy_2yy_1w and containing $N(y_2)$, and G_1' is obtained from G by removing $G_2' - \{w, z, z_2\}$ and $G_3' - \{w, y_2, y, y_1\}$. Note $G_3' - \{y, y_1, y_2, w\} \neq \emptyset$ since $|N(y_2)| > 2$. Let $F_1^{(4)} = A(G_1' - \{z_1, w, x, y, z, y_1, y_2, z_2\})$, $F_2^{(4)} = A(G_2' - \{z_2, w, z, v\})$, and $F_3^{(3)} = A(G_3' - \{y_1, y_2, w, y\})$. Then $|F_1^{(4)}| \geq \lceil (4(|G_1'| - 8) + 3)/7 \rceil$, $|F_i^{(4)}| \geq \lceil (4(|G_i'| - 4) + 3)/7 \rceil$ for $i = 2, 3$. Note $|F_2^{(4)}| \geq \lceil (4(|G_2| - 4) + 3)/7 \rceil \geq$

$(4(|G_2|-4)+3)/7+4/7$. Now $G[F_1^{(4)} \cup F_2^{(4)} \cup F_3^{(4)} + \{x, y, z, w\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(4)}| + |F_2^{(4)}| + |F_3^{(4)}| + 4 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Note that we did not use the information on v in the above proof. So by symmetry, $wy_1 \notin E(G)$.

Claim 3. $v_2z_2 \notin E(G)$.

For, suppose $v_2z_2 \in E(G)$. There exists a separation (G_1, G_2) of G such that $V(G_1 \cap G_2) = \{v, v_2, z_2\}$, $\{x, y, y_1, y_2\} \subseteq V(G_1)$, and $v_1 \in V(G_2)$. Let $F_1^{(1)} = A(G_1 - \{v, v_2, z_2, z, x\} + wz_1)$, and $F_2^{(1)} = A(G_2 - \{z_2, v, v_2\})$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1|-5)+3)/7 \rceil$, and $|F_2^{(1)}| \geq \lceil (4(|G_2|-3)+3)/7 \rceil$. Now $G[F_1^{(1)} \cup F_2^{(1)} + \{v, z\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 2$. By Lemma 2.2(8) (with $a = |G_1| - 5, a_1 = |G_2| - 3$), $(4(|G_1| - 5) + 3, 4(|G_2| - 3) + 3) \equiv (0,0), (0,6), (0,5), (5,0), (6,6), (6,0) \mod 7$.

If $|N(z_1) \cap N(z_2)| \leq 2$, then let $F_1^{(2)} = A((G_1 - \{x, z, v, w, v_2\})/z_1z_2)$ with z' as the identification of $\{z_1, z_2\}$, and $F_2^{(2)} = A(G_2 - \{v, v_2\})$. Then $|F_1^{(2)}| \geq \lceil (4(|G_1|-6)+3)/7 \rceil$, and $|F_2^{(2)}| \geq \lceil (4(|G_2|-2)+3)/7 \rceil$. Now $G[F_1^{(2)} \cup F_2^{(2)} + \{x, v, z\} - \{z_2\}]$ (if $z' \notin F_1^{(2)}$) or $G[(F_1^{(2)} - z') \cup F_2^{(2)} + \{x, v, z_2, z_1\} - (\{z_2\} - F_2^{(2)})]$ (if $z' \in F_1^{(2)}$) is an induced forest of size $|F_1^{(2)}| + |F_2^{(2)}| + 3 - 1$, which implies $a(G) \geq \lceil (4n+3)/7 \rceil$, a contradiction.

So $|N(z_1) \cap N(z_2)| > 2$. Then there exist $a_1 \in N(z_1) \cap N(z_2)$ and subgraphs G'_1, G'_2, G'_3 such that $G'_2 = G_2$, G'_3 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $zz_1a_1z_2z$ and containing $N(z_1) \cap N(z_2) - \{z\}$, and G'_1 is obtained from G by removing $G'_2 - \{v_2, v, z_2\}$ and $G'_3 - \{z_2, a_1, z_1\}$. Let $A_3 = \{a_1\}$ and $A_4 = \emptyset$. For $i = 3, 4$, let $F_1^{(i)} = A(G'_1 - \{x, w, z, v, z_1, z_2, v_2\} - A_i)$, $F_2^{(i)} = A(G'_2 - \{z_2, v_2, v\})$, and $F_3^{(i)} = A(G'_3 - \{z_1, z_2\} - A_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G'_1| - 7 - |A_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G'_2| - 3) + 3)/7 \rceil$, and $|F_3^{(i)}| \geq \lceil (4(|G'_3| - 2 - |A_i|) + 3)/7 \rceil$. Now $F^{(i)} := G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{x, v, z\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ is an induced forest in G showing $a(G) \geq |F_1^{(3)}| + |F_2^{(3)}| + |F_3^{(3)}| + 3 - (1 - |A_i|)$. Let $(n_1, n_2, n_3) := (4(|G'_1| - 7) + 3, 4(|G'_2| - 3) + 3, 4(|G'_3| - 2) + 3)$. By Lemma 2.2(2), $(n_1, n_2, n_3) \equiv (0,0,4), (4,0,0) \mod 7$.

If $(n_1, n_2, n_3) \equiv (0,0,4) \mod 7$, let $F_1^{(4)} = A(G'_1 - \{x, y, z, v, z_1\} + wz_2)$, $F_2^{(4)} = A(G'_2 - v)$, and $F_3^{(4)} = A(G'_3 - z_1)$. Then $|F_1^{(4)}| \geq \lceil (4(|G'_1|-5)+3)/7 \rceil$, $|F_2^{(4)}| \geq \lceil (4(|G'_2|-1)+3)/7 \rceil$, and $|F_3^{(4)}| \geq \lceil (4(|G'_3|-1)+3)/7 \rceil$. Now $G[F_1^{(4)} \cup F_2^{(4)} \cup F_3^{(4)} + \{x, z\} - (\{a_1, z_2\} \cap (F_1^{(4)} \triangle F_3^{(4)})) - (\{v_2, z_2\} \cap (F_1^{(4)} \triangle F_2^{(4)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(4)}| + |F_2^{(4)}| + |F_3^{(4)}| + 2 - 4 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

If $(n_1, n_2, n_3) \equiv (4, 0, 0) \mod 7$, then by Lemma 4.1, $yv \notin E(G)$. Let $F_1^{(5)} = A(G_1' - \{w, x, z, z_1, z_2, a_1\} + yv)$, $F_2^{(5)} = A(G_2' - z_2)$, and $F_3^{(5)} = A(G_3' - \{z_1, z_2, a_1\})$. Then $|F_1^{(5)}| \geq \lceil (4(|G_1'| - 6) + 3)/7 \rceil$, $|F_2^{(5)}| \geq \lceil (4(|G_2'| - 1) + 3)/7 \rceil$, and $|F_3^{(5)}| \geq \lceil (4(|G_3'| - 3) + 3)/7 \rceil$. Now $G[F_1^{(5)} \cup F_2^{(5)} \cup F_3^{(5)} + \{x, z\} - (\{v, v_2\} \cap (F_1^{(5)} \triangle F_2^{(5)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(5)}| + |F_2^{(5)}| + |F_3^{(5)}| + 2 - 2 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Claim 4. $v_2 z_1 \notin E(G)$.

For, suppose $v_2 z_1 \in E(G)$. By Lemma 4.1, $y \notin \{v_1, v_2\}$. There exists a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{v, v_2, z, z_1\}$, $\{x, y, y_1, y_2\} \subseteq V(G_1)$, and $\{z_2, v_1\} \subseteq V(G_2)$. Let $F_1^{(1)} = A(G_1 - \{v, v_2, z, z_1, w, x\})$ and $F_2^{(1)} = A(G_2 - \{v, v_2, z, z_1, v_1\})$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil$, and $|F_2^{(1)}| \geq \lceil (4(|G_2| - 5) + 3)/7 \rceil$. Now $G[F_1^{(1)} \cup F_2^{(1)} + \{v, z, x\}]$ is an induced forest of size $|F_1^{(1)}| + |F_2^{(1)}| + 3$. Let $(n_1, n_2) := (4(|G_1| - 6) + 3, 4(|G_2| - 5) + 3)$. By Lemma 2.2(3), $(n_1, n_2) \equiv (0,0), (0,6), (0,5), (0,4), (4,0), (6,5), (5,6), (5,0), (6,6), (6,0) \mod 7$.

If $(n_1, n_2) \equiv (0,0), (0,6), (0,5), (0,4) \mod 7$, then for $i = 1, 2$, let $F_i^{(2)} = A(G_i - \{v, v_2, z, z_1\})$. Then $|F_i^{(2)}| \geq \lceil (4(|G_i| - 4) + 3)/7 \rceil$. Now $G[F_1^{(2)} \cup F_2^{(2)} + \{v\}]$ is an induced forest of size $|F_1^{(2)}| + |F_2^{(2)}| + 1 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

If $(n_1, n_2) \equiv (5,0), (6,0), (6,5), (5,6), (6,6) \mod 7$, then let $F_1^{(3)} = A(G_1 - \{x, y, z, v, z_1, v_2, y_2\} + wy_1)$ and $F_2^{(3)} = A(G_2 - \{v, v_2, z, z_1\})$ Then $|F_1^{(3)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$ and $|F_2^{(3)}| \geq \lceil (4(|G_2| - 4) + 3)/7 \rceil$. Now $G[F_1^{(3)} \cup F_2^{(3)} + \{x, y, z\}]$ is an induced forest of size $|F_1^{(3)}| + |F_2^{(3)}| + 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

So $(n_1, n_2) \equiv (4, 0) \mod 7$, then let $F_1^{(4)} = A(G_1 - \{z, z_1, v\} + xv_2)$ and $F_2^{(4)} = A(G_2 - \{z, z_1, v\})$ Then $|F_1^{(4)}| \geq \lceil (4(|G_1| - 3) + 3)/7 \rceil$ and $|F_2^{(4)}| \geq \lceil (4(|G_2| - 3) + 3)/7 \rceil$. Now $G[F_1^{(4)} \cup F_2^{(4)} + \{z\} - \{v_2\} \cap (F_1^{(4)} \triangle F_2^{(4)})]$ is an induced forest of size $|F_1^{(4)}| + |F_2^{(4)}| + 1 - 1 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Claim 5. $yz_2 \notin E(G)$, $y_1 z \notin E(G)$.

By symmetry, suppose that $y_1 = z_2$. By Lemma 3.1, $|N(z_1)| \geq 3$. Then there exists a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{z_1, y_1\}$, $\{x, y, z\} \subseteq V(G_1)$, and $N(z_1) - \{y, z\} \subseteq V(G_2)$. Let $F_1^{(1)} = A(G_1 - \{x, y, z, y_1, z_1\})$, and $F_2^{(1)} = A(G_2 - \{y_1, z_1\})$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1| -$

$5)+3)/7\rceil$, and $|F_2^{(1)}| \geq \lceil(4(|G_2|-2)+3)/7\rceil$. Now $G[F_1^{(1)} \cup F_2^{(1)} + \{y, z\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 2$. By Lemma 2.2(8) (with $a = |G_1| - 5, a_1 = |G_2| - 2, c = 2$), $(4(|G_1|-5)+3, 4(|G_2|-2)+3) \equiv (0,0), (0,6), (0,5), (5,0), (6,6), (6,0) \mod 7$. Let $F_1^{(2)} = A(G_1 - \{x, y, z, y_1, z_1, v\})$ and $F_2^{(2)} = A(G_2 - y_1)$. Then $|F_1^{(2)}| \geq \lceil(4(|G_1|-6)+3)/7\rceil$ and $|F_2^{(2)}| \geq \lceil(4(|G_2|-1)+3)/7\rceil$. Now $G[F_1^{(2)} \cup F_2^{(2)} + \{y, z\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + 2 \geq \lceil(4n+3)/7\rceil$, a contradiction.

Next, we distinguish several cases.

Case 1. $|N(z_1) \cap N(z_2)| \leq 2$, $|N(y_1) \cap N(y_2)| \leq 2$ and $|N(w) \cap N(v_2)| \leq 2$.

Let $F' = A((G - \{x, y, z, v\})/\{z_1z_2, y_1y_2, wv_2\} + v'z')$ with z' (respectively, y', v') as the identification of $\{z_1, z_2\}$ (respectively, $\{y_1, y_2\}, \{w, v_2\}$). Then $|F'| \geq \lceil(4(n-7)+3)/7\rceil$. Let $F := F' + \{x, y, v, z\}$ if $z', y', v' \notin F'$ and otherwise, let F be obtained by $F' + \{x, y, v, z\}$ by deleting $\{z, z'\}$ (respectively, $\{y, y'\}, \{v, v'\}$) and adding $\{z_1, z_2\}$ (respectively, $\{y_1, y_2\}, \{v_2, w\}$) when $z' \in F'$ (respectively, when $y' \in F', v' \in F'$). Therefore, $G[F]$ is an induced forest in G, showing $|F'| + 4 \geq \lceil(4n+3)/7\rceil$, a contradiction.

Case 2. $|N(z_1) \cap N(z_2)| > 2$, $|N(y_1) \cap N(y_2)| \leq 2$ and $|N(w) \cap N(v_2)| \leq 2$.

There exist $a_1 \in N(z_1) \cap N(z_2)$ and a separation (G_1, G_2) of G such that $V(G_1 \cap G_2) = \{z_1, z_2, a_1\}$, $\{x, y, y_1, y_2\} \subseteq V(G_1)$, and $N(z_1) \cap N(z_2) - \{z\} \subseteq V(G_2)$. Let $A_1 = \overline{A_2} = \{a_1\}$ and $A_2 = \overline{A_1} = \emptyset$. For $i = 1, 2$, let $F_1^{(i)} = A((G_1 - \{x, y, z, v, z_1, z_2\} - A_i)/\{y_1y_2, wv_2\})$ with y' (respectively, v') as the identification of $\{y_1, y_2\}$ (respectively, $\{w, v_2\}$), and $F_2^{(i)} = A(G_2 - \{z_1, z_2\} - A_i)$. Then $|F_1^{(i)}| \geq \lceil(4(|G_1| - 8 - |A_i|)+3)/7\rceil$, and $|F_2^{(i)}| \geq \lceil(4(|G_2| - 2 - |A_i|)+3)/7\rceil$. Let $F^{(i)} = F_1^{(i)} \cup F_2^{(i)} + \{x, y, v, z\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))$ if $y', v' \notin F'$ and otherwise, let $F^{(i)}$ be obtained by $F_1^{(i)} \cup F_2^{(i)} + \{x, y, v, z\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))$ by deleting $\{y, y'\}$ (respectively, $\{v, v'\}$) and adding $\{y_1, y_2\}$ (respectively, $\{v_2, w\}$) when $y' \in F'$ (respectively, $v' \in F'$). Therefore, $G[F^{(i)}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + 4 - |\overline{A_i}|$. By Lemma 2.2(2) (with $a = |G_1| - 8, a_1 = |G_2| - 2$), $(4(|G_1|-8)+3, 4(|G_2|-2)+3) \equiv (0,4), (4,0) \mod 7$.

Subcase 2.1. $(4(|G_1|-8)+3, 4(|G_2|-2)+3) \equiv (4,0) \mod 7$.

Let $F_1^{(1)} = A((G_1 - \{x, y, z, v\})/\{z_1z_2, y_1y_2, wv_2\} + v'z')$ with z' (respectively, y', v') as the identification of $\{z_1, z_2\}$ (respectively, $\{y_1, y_2\}, \{w, v_2\}$), and $F_2^{(1)} = A(G_2)$. Then $|F_1^{(1)}| \geq \lceil(4(|G_1|-7)+3)/7\rceil$, and $|F_2^{(1)}| \geq \lceil(4|G_2|+3)/7\rceil$. Let $F^{(1)} = \overline{F_1}^{(1)} \cup F_2^{(1)} - (\{z_1, z_2, a_1\} \cap (\overline{F_1}^{(1)} \triangle F_2^{(1)}))$, where

$\overline{F_1}^{(1)} = F_1{}^{(1)} + \{x, y, v, z\}$ if $z', y', v' \notin F'$ and otherwise, let $\overline{F_1}^{(1)}$ be obtained by $F_1{}^{(1)} + \{x, y, v, z\}$ by deleting $\{z, z'\}$ (respectively, $\{y, y'\}, \{v, v'\}$) and adding $\{z_1, z_2\}$ (respectively, $\{y_1, y_2\}, \{v_2, w\}$) when $z' \in F'$ (respectively, when $y' \in F', v' \in F'$). Therefore, $G[F^{(1)}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 4 - 3 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Subcase 2.2. $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (0, 4) \mod 7$.

Let $F_1^{(2)} = A((G_1 - \{x, y, z, v, z_1\})/y_1y_2 + wz_2)$ with y' as the identification of $\{y_1, y_2\}$, and $F_2^{(2)} = F(G_2 - z_1)$. Then $|F_1^{(2)}| \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil$, and $|F_2^{(2)}| \geq \lceil (4(|G_2| - 1) + 3)/7 \rceil$. Let $F = F_1^{(2)} \cup F_2^{(2)} + \{x, y, z\} - (\{z_2, a_1\} \cap (F_1^{(2)} \triangle F_2^{(2)}))$. Now $G[F]$ (if $y' \notin F_1^{(2)}$) or $G[F - \{y, y'\} + \{y_1, y_2\}]$ (if $y' \in F_1^{(2)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + 3 - 2 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Case 3. $|N(z_1) \cap N(z_2)| \leq 2$, $|N(y_1) \cap N(y_2)| \leq 2$ and $|N(w) \cap N(v_2)| > 2$.

There exist $c_1 \in N(z_1) \cap N(z_2)$ and a separation (G_1, G_2) of G such that $V(G_1 \cap G_2) = \{w, v_2, c_1\}$, $\{x, y, y_1, y_2\} \subseteq V(G_1)$, and $N(w) \cap N(v_2) - \{v\} \subseteq V(G_2)$. Let $C_1 = \overline{C_2} = \{c_1\}$ and $\overline{C_1} = C_2 = \emptyset$. For $i = 1, 2$, let $F_1^{(i)} = A((G_1 - \{x, y, z, v, w, v_2\} - C_i)/\{y_1y_2, z_1z_2\})$ with y' (respectively, z') as the identification of $\{y_1, y_2\}$ (respectively, $\{z_1, z_2\}$) and $F_2^{(i)} = A(G_2 - \{w, v_2\} - C_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |C_i|) + 3)/7 \rceil$ and $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |C_i|) + 3)/7 \rceil$. Let $F^{(i)} = F_1^{(i)} \cup F_2^{(i)} + \{x, y, v, z\} - (\{c_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))$ if $y', z' \notin F_1^{(i)}$ and otherwise, let $F^{(i)}$ be obtained by $F_1^{(i)} \cup F_2^{(i)} + \{x, y, v, z\} - (\{c_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))$ by deleting $\{y, y'\}$ (respectively, $\{z, z'\}$) and adding $\{y_1, y_2\}$ (respectively, $\{z_1, z_2\}$) when $y' \in F'$ (respectively, $z' \in F'$). Therefore, $G[F^{(i)}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + 4 - |\overline{C_i}|$. By Lemma 2.2(2) (with $a = |G_1| - 8, a_1 = |G_2| - 2$), $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (0, 4), (4, 0) \mod 7$.

Subcase 3.1. $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (4, 0) \mod 7$.

Let $F_1^{(1)} = A((G_1 - \{x, y, z, v\})/\{z_1z_2, y_1y_2, wv_2\} + v'z')$ with z' (respectively, y', v') as the identification of $\{z_1, z_2\}$ (respectively, $\{y_1, y_2\}, \{w, v_2\}$), and $F_2^{(1)} = A(G_2)$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$ and $|F_2^{(1)}| \geq \lceil (4|G_2| + 3)/7 \rceil$. Let $F^{(1)} = \overline{F_1}^{(1)} \cup F_2^{(1)} - (\{w, v_2, c_1\} \cap (\overline{F_1}^{(1)} \triangle F_2^{(1)}))$, where $\overline{F_1}^{(1)} = F_1{}^{(1)} + \{x, y, v, z\}$ if $z', y', v' \notin F'$ and otherwise, let $\overline{F_1}^{(1)}$ be obtained by $F_1{}^{(1)} + \{x, y, v, z\}$ by deleting $\{z, z'\}$ (respectively, $\{y, y'\}, \{v, v'\}$) and adding $\{z_1, z_2\}$ (respectively, $\{y_1, y_2\}, \{v_2, w\}$) when $z' \in F'$ (respectively, $y' \in F', v' \in F'$). Therefore, $G[F^{(1)}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 4 - 3 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Subcase 3.2. $(4(|G_1|-8)+3, 4(|G_2|-2)+3) \equiv (0,4) \mod 7$.

Let $F_1^{(2)} = A(G_1 - \{w,x,y,z\}/\{y_1y_2, z_1z_2\})$ with y' (respectively, z') as the identification of $\{y_1,y_2\}$ (respectively, $\{z_1,z_2\}$), and $F_2^{(2)} = A(G_2 - w)$. Then $|F_1^{(2)}| \geq \lceil (4(|G_1|-6)+3)/7 \rceil$, and $|F_2^{(2)}| \geq \lceil (4(|G_2|-1)+3)/7 \rceil$. Let $F^{(2)} := \overline{F_1}^{(2)} \cup F_2^{(2)} - (\{v_2, c_1\} \cap (\overline{F_1}^{(2)} \triangle F_2^{(2)}))$, where $\overline{F_1}^{(2)} = F_1{}^{(2)} + \{x,y,z\}$ if $y', z' \notin F_1^{(2)}$, and otherwise, $\overline{F_1}^{(2)}$ obtained from $\overline{F_1}^{(2)} = F_1{}^{(2)} + \{x,y,z\}$ by deleting y, y' (respectively, z, z') and adding $\{y_1,y_2\}$ (respectively, $\{z_1,z_2\}$) when $y' \in F_1^{(2)}$ (respectively, $z' \in F_1^{(2)}$). Therefore, $G[F^{(2)}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + 3 - 2 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Case 4. $|N(z_1) \cap N(z_2)| \leq 2$, $|N(y_1) \cap N(y_2)| > 2$ and $|N(w) \cap N(v_2)| \leq 2$.

There exist $b_1 \in N(y_1) \cap N(y_2)$ and a separation (G_1, G_2) of G such that $V(G_1 \cap G_2) = \{y_1, y_2, b_1\}$, $\{x, z, y_1, y_2\} \subseteq V(G_1)$, and $N(y_1) \cap N(y_2) - \{y\} \subseteq V(G_2)$. Let $B_1 = \overline{B_2} = \{b_1\}$, and let $B_2 = \overline{B_1} = \emptyset$. For $i = 1, 2$, let $F_1^{(i)} = A((G_1 - \{x,y,z,v,y_1,y_2\} - B_i)/\{wv_2, z_1z_2\} + v'z')$ with v' (respectively, z') as the identification of $\{w, v_2\}, \{z_1, z_2\}$ and $F_2^{(i)} = A(G_2 - \{y_1,y_2\} - B_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1|-8-|B_i|)+3)/7 \rceil$ and $|F_2^{(i)}| \geq \lceil (4(|G_2|-2-|B_i|)+3)/7 \rceil$. Let $F^{(i)} := G[\overline{F_1}^{(i)} \cup F_2^{(i)} - (\{b_1\} \cap (\overline{F_1}^{(i)} \triangle F_2^{(i)}))]$, where $\overline{F_1}^{(i)} := F_1{}^{(i)} + \{x,y,v,z\}$ if $v', z' \notin F_1^{(i)}$, and otherwise let $\overline{F_1}^{(i)}$ be obtained from $F_1{}^{(i)} + \{x,y,v,z\}$ by deleting $\{z, z'\}$ (respectively, $\{v, v'\}$) and adding $\{z_1, z_2\}$ (respectively, $\{v_2, w\}$) when $z' \in F_1{}^{(i)}$ (respectively, $y' \in F_1{}^{(i)}$). Therefore, $F^{(i)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + 4 - |\overline{B_i}|$. By Lemma 2.2(2) (with $a = |G_1| - 8, a_1 = |G_2| - 2$), $(4(|G_1|-8)+3, 4(|G_2|-2)+3) \equiv (4,0), (0,4)$ mod 7.

Subcase 4.1. $(4(|G_1|-8)+3, 4(|G_2|-2)+3) \equiv (4,0) \mod 7$.

Let $F_1^{(1)} = A((G_1 - \{x,y,z,v\})/\{z_1z_2, y_1y_2, wv_2\} + v'z')$ with z' (respectively, y', v') as the identification of $\{z_1, z_2\}$ (respectively, $\{y_1, y_2\}, \{w, v_2\}$), and $F_2^{(1)} = A(G_2)$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1|-7)+3)/7 \rceil$, and $|F_2^{(1)}| \geq \lceil (4|G_2|+3)/7 \rceil$. Let $F^{(1)} = \overline{F_1}^{(1)} \cup F_2^{(1)} - (\{y_1, y_2, b_1\} \cap (\overline{F_1}^{(1)} \triangle F_2^{(1)}))$, where $\overline{F_1}^{(1)} = F_1{}^{(1)} + \{x,y,v,z\}$ if $z', y', v' \notin F'$, and otherwise, $\overline{F_1}^{(1)}$ obtained from $F_1{}^{(1)} + \{x,y,v,z\}$ by deleting $\{z, z'\}$ (respectively, $\{y, y'\}, \{v, v'\}$) and adding $\{z_1, z_2\}$ (respectively, $\{y_1, y_2\}, \{v_2, w\}$) when $z' \in F'$ (respectively, $y' \in F', v' \in F'$). Therefore, $G[F^{(1)}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 4 - 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 4.2. $(4(|G_1|-8)+3, 4(|G_2|-2)+3) \equiv (0,4) \mod 7$.

By Claim 3, $v_2z_2 \notin E(G)$; so $|N(v) \cap N(z_2)| \leq 2$. Let $F_1^{(2)} = A((G_1 - \{z_1, x, y, z, y_2\})/vz_2 + wy_1)$ with z' as the identification of $\{v, z_2\}$, and $F_2^{(2)} = A(G_2 - y_2)$. Then $|F_1^{(2)}| \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil$, and $|F_2^{(2)}| \geq \lceil (4(|G_2| - 1) + 3)/7 \rceil$. Let $F = F_1^{(2)} \cup F_2^{(2)} + \{x, y, z\} - (\{y_1, b_1\} \cap (F_1^{(2)} \triangle F_2^{(2)}))$. Now $G[F]$ (if $z' \notin F_1^{(2)}$) or $G[F - \{z, z'\} + \{v, z_2\}]$ (if $z' \in F_1^{(2)}$) is an induced forest of size $|F_1^{(2)}| + |F_2^{(2)}| + 3 - 2 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Case 5. $|N(z_1) \cap N(z_2)| > 2$, $|N(y_1) \cap N(y_2)| > 2$.

There exist $a_1 \in N(z_1) \cap N(z_2), b_1 \in N(y_1) \cap N(y_2)$ and subgraphs G_1, G_2, G_3 of G such that G_2 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $zz_1a_1z_2z$ and containing $N(z_1) \cap N(z_2) - \{z\}$, G_3 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $yy_1b_1y_2y$ and containing $N(y_1) \cap N(y_2) - \{y\}$, and G_1 is obtained from G by removing $G_2 - \{z_1, z_2, a_1\}$ and $G_3 - \{y_1, b_1, y_2\}$. Let $A_i = \{a_1\}$ if $i = 1, 2$ and $A_i = \emptyset$ if $i = 3, 4$. Let $B_i = \{b_1\}$ if $i = 1, 3$ and $B_i = \emptyset$ if $i = 2, 4$. For $i \in [4]$, let $F_1^{(i)} = A((G_1 - \{x, y, z, v, y_1, y_2, z_1, z_2\} - A_i - B_i)/wv_2)$ with v' as the identification of $\{w, v_2\}$, $F_2^{(i)} = A(G_2 - \{z_1, z_2\} - A_i)$, and $F_3^{(i)} = A(G_3 - \{y_1, y_2\} - B_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 9 - |A_i| - |B_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |A_i|) + 3)/7 \rceil$, and $|F_3^{(i)}| \geq \lceil (4(|G_3| - 2 - |B_i|) + 3)/7 \rceil$. Let $F = F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{x, y, v, z\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))$. Now $G[F]$ (if $v' \notin F_1^{(i)}$) or $G[F - \{v', v\} + \{w, v_2\}]$ (if $v' \in F_1^{(i)}$) is an induced forest of size $|F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - (1 - |A_i|) - (1 - |B_i|)$. Let $(n_1, n_2, n_3) := (4(|G_1| - 9) + 3, 4(|G_2| - 2) + 3, 4(|G_3| - 2) + 3)$. By Lemma 2.2(5) (with $a = |G_1| - 9, a_1 = |G_2| - 2, a_2 = |G_3| - 2$), $(n_1, n_2, n_3) \equiv (0,0,0)$, $(1,0,0)$, $(4,0,3), (4,3,0), (3,0,4), (4,0,4), (3,4,0), (4,4,0), (1,6,0), (1,0,6), (0,3,4), (0,4,3), (0,4,4), (6,4,4), (4,4,6), (4,6,4) \mod 7$.

Subcase 5.1. $(n_1, n_2, n_3) \equiv (0,0,0), (1,0,0) \mod 7$.

If $|N(w) \cap N(v_2)| \leq 2$, let $F_1^{(1)} = A((G_1 - \{x, y, z, v\})/\{z_1z_2, y_1y_2, wv_2\} + v'z')$ with z' (respectively, y', v') as the identification of $\{z_1, z_2\}$ (respectively, $\{y_1, y_2\}, \{w, v_2\}$) and $F_2^{(1)} = A(G_2)$, and $F_3^{(1)} = A(G_3)$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$, $|F_2^{(1)}| \geq \lceil (4|G_2| + 3)/7 \rceil$, and $|F_3^{(1)}| \geq \lceil (4|G_3| + 3)/7 \rceil$. Let $F^{(1)} := G[\overline{F_1}^{(1)} \cup F_2^{(1)} \cup F_3^{(1)} - (\{z_1, z_2, a_1\} \cap (\overline{F_1}^{(1)} \triangle F_2^{(1)})) - (\{y_1, y_2, b_1\} \cap (\overline{F_1}^{(1)} \triangle F_3^{(1)}))]$ where $\overline{F_1}^{(1)} := F_1{}^{(1)} + \{x, y, v, z\}$ if $v', y', z' \notin F_1^{(1)}$ and otherwise $\overline{F_1}^{(1)}$ obtained from $F_1{}^{(1)} + \{x, y, v, z\}$ by deleting $\{z, z'\}$ (respectively, $\{v, v'\}, \{y, y'\}$) and adding $\{z_1, z_2\}$ (respectively, $\{v_2, w\}, \{y_1, y_2\}$)

when $z' \in {F_1}^{(1)}$ (respectively, $v' \in {F_1}^{(1)}$, $y' \in {F_1}^{(1)}$). Therefore, $F^{(1)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(1)}|+|F_2^{(1)}|+|F_3^{(1)}|+4-6 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

If $|N(w) \cap N(v_2)| > 2$, there exist $c_1 \in N(w) \cap N(v_2)$ and subgraphs G'_1, G'_2, G'_3, G'_4 of G such that $G'_2 = G_2, G'_3 = G_3$, G'_4 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle vwc_1v_2v and containing $N(w) \cap N(v_2) - \{v\}$, and G'_1 is obtained from G by removing $G_2 - \{z_1, z_2, a_1\}$, $G'_3 - \{y_1, b_1, y_2\}$ and $G'_4 - \{w, v_2, v\}$. Let $C_9 = \emptyset$ and $C_{10} = \{c_1\}$. For $i = 9, 10$, let $F_1^{(i)} = A((G'_1 - \{x, y, w, z, v, v_2\} - C_i)/\{y_1y_2, z_1z_2\})$ with y' (respectively, z') as identification of $\{y_1, y_2\}$ (respectively, $\{z_1, z_2\}$), $F_2^{(i)} = A(G'_2)$, $F_3^{(i)} = A(G'_3)$, and $F_4^{(i)} = A(G_4 - \{w, v_2\} - C_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G'_1| - 8 - |C_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4|G'_2| + 3)/7 \rceil$, $|F_3^{(i)}| \geq \lceil (4|G'_3| + 3)/7 \rceil$, and $|F_4^{(i)}| \geq \lceil (4(|G'_4| - 2 - |C_i|) + 3)/7 \rceil$. Let $F^{(i)} := G[\overline{F_1}^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} \cup F_4^{(i)} - (\{z_1, z_2, a_1\} \cap (\overline{F_1}^{(i)} \triangle F_2^{(i)})) - (\{y_1, y_2, b_1\} \cap (\overline{F_1}^{(i)} \triangle F_2^{(i)})) - (\{c_1\} \cap (\overline{F_1}^{(i)} \triangle F_4^{(i)}))]$ where $\overline{F_1}^{(i)} = {F_1}^{(i)} + \{x, y, z, v\}$ if $y', z' \notin F_1^{(i)}$ and $\overline{F_1}^{(i)}$ obtained from ${F_1}^{(i)} + \{x, y, z, v\}$ by deleting y, y' (respectively, $\{z, z'\}$) and adding $\{y_1, y_2\}$ (respectively, $\{z_1, z_2\}$) when $y' \in {F_1}^{(i)}$ (respectively, $z' \in {F_1}^{(i)}$). Therefore, $F^{(i)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}|+|F_2^{(i)}|+|F_3^{(i)}|+|F_4^{(i)}|+4-6-(1-|C_i|)$. Let $(n'_1, n'_2, n'_3, n'_4) := (4(|G'_1|-8)+3, 4(|G'_2|-2)+3, 4(|G'_3|-2)+3, 4(|G'_4|-2)+3)$. By Lemma 2.2(2), $(n'_1, n'_2, n'_3, n'_4) \equiv (4, 0, 0, 0), (0, 0, 0, 4) \mod 7$.

If $(n'_1, n'_2, n'_3, n'_4) \equiv (0, 0, 0, 4) \mod 7$, let $F_1^{(11)} = A((G'_1 - \{x, y, z, w\})/\{z_1z_2, y_1y_2\})$ with z' (respectively, y') as the identification of $\{z_1, z_2\}$ (respectively, $\{y_1, y_2\}$), $F_2^{(11)} = A(G'_2)$, $F_3^{(11)} = A(G'_3)$, and $F_4^{(11)} = A(G'_4 - w)$. Then $|F_1^{(11)}| \geq \lceil (4(|G'_1| - 6) + 3)/7 \rceil$, $|F_2^{(11)}| \geq \lceil (4|G'_2| + 3)/7 \rceil$, $|F_3^{(11)}| \geq \lceil (4|G'_3| + 3)/7 \rceil$ and $|F_4^{(11)}| \geq \lceil (4(|G'_4| - 1) + 3)/7 \rceil$. Let $F^{(11)} := G[\overline{F_1}^{(11)} \cup F_2^{(11)} \cup F_3^{(11)} \cup F_4^{(11)} - (\{z_1, z_2, a_1\} \cap (\overline{F_1}^{(11)} \triangle F_2^{(11)})) - (\{y_1, y_2, b_1\} \cap (\overline{F_1}^{(11)} \triangle F_3^{(11)})) - (\{v_2, c_1\} \cap (\overline{F_1}^{(11)} \triangle F_4^{(11)}))]$ where $\overline{F_1}^{(11)} = {F_1}^{(11)} + \{x, y, z\}$ when $z', y' \notin F_1^{(11)}$, and otherwise, let $\overline{F_1}^{(11)}$ be obtained from ${F_1}^{(11)}$ by deleting $\{z, z'\}$ (respectively, $\{y, y'\}$) and adding $\{z_1, z_2\}$ (respectively, $\{y_1, y_2\}$) when $z' \in {F_1}^{(11)}$ (respectively, $y' \in {F_1}^{(11)}$). Therefore, $F^{(11)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(11)}| + |F_2^{(11)}| + |F_3^{(11)}| + |F_4^{(11)}| + 3 - 8 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

If $(n'_1, n'_2, n'_3, n'_4) \equiv (4, 0, 0, 0) \mod 7$, let $F_1^{(12)} = A((G'_1 - \{x, y, z, v\})/\{z_1z_2, y_1y_2, wv_2\} + v'z')$ with z' (respectively, y', v') as the identification of $\{z_1, z_2\}$ (respectively, $\{y_1, y_2\}, \{w, v_2\}$), $F_2^{(12)} = A(G'_2)$, $F_3^{(12)} = A(G'_3)$, and

$F_4^{(12)} = A(G_4')$. Then $|F_1^{(12)}| \geq \lceil (4(|G_1'|-7)+3)/7 \rceil$, $|F_2^{(12)}| \geq \lceil (4|G_2'|+3)/7 \rceil$, $|F_3^{(12)}| \geq \lceil (4|G_3'|+3)/7 \rceil$ and $|F_4^{(12)}| \geq \lceil (4|G_4'|+3)/7 \rceil$. Let $F^{(12)} := G[\overline{F_1}^{(12)} \cup F_2^{(12)} \cup F_3^{(12)} \cup F_4^{(12)} - (\{z_1, z_2, a_1\} \cap (\overline{F_1}^{(12)} \triangle F_2^{(12)})) - (\{y_1, y_2, b_1\} \cap (\overline{F_1}^{(12)} \triangle F_3^{(12)})) - (\{w, v_2, c_1\} \cap (\overline{F_1}^{(12)} \triangle F_4^{(12)}))]$ where $\overline{F_1}^{(12)} = F_1{}^{(12)} + \{x, y, z, v\}$ when $z', y', v' \notin F_1^{(12)}$, and otherwise, let $\overline{F_1}^{(12)}$ be obtained from $F_1{}^{(12)}$ by deleting $\{z, z'\}$ (respectively, $\{y, y'\}, \{v, v'\}$) and adding $\{z_1, z_2\}$ (respectively, $\{y_1, y_2\}, \{w, v_2\}$) when $z' \in F_1{}^{(12)}$ (respectively, $y', v' \in F_1{}^{(12)}$). Therefore, $F^{(12)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(12)}| + |F_2^{(12)}| + |F_3^{(12)}| + |F_4^{(12)}| + 4 - 9 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 5.2. $(n_1, n_2, n_3) \equiv (3,0,4), (4,0,4), (3,4,0), (4,4,0), (0,4,4), (6,4,4), (4,4,6), (4,6,4) \mod 7$.

Let $F_1^{(2)} = A((G_1 - \{x, y, z, y_2, z_1\})/vz_2 + wy_1)$ with v' as the identification of $\{v, z_2\}$, $F_2^{(2)} = A(G_2 - z_1)$, and $F_3^{(2)} = A(G_3 - y_2)$. Then $|F_1^{(2)}| \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil$, $|F_2^{(2)}| \geq \lceil (4(|G_2| - 1) + 3)/7 \rceil$, and $|F_3^{(2)}| \geq \lceil (4(|G_3| - 1) + 3)/7 \rceil$. Now $G[F_1^{(2)} \cup F_2^{(2)} \cup F_3^{(2)} + \{x, y, z\} - (\{z_2, a_1\} \cap (F_1^{(2)} \triangle F_2^{(2)})) - (\{y_1, b_1\} \cap (F_1^{(2)} \triangle F_3^{(2)}))]$ (if $v' \notin F_1^{(2)}$) or $G[(F_1^{(2)} - v') \cup F_2^{(2)} \cup F_3^{(2)} + \{x, y, v, z_2\} - (\{z_2, a_1\} \cap ((F_1^{(2)} \cup \{z_2\}) \triangle F_2^{(2)})) - (\{y_1, b_1\} \cap (F_1^{(2)} \triangle F_3^{(2)}))]$ (if $v' \in F_1^{(2)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + |F_3^{(2)}| + 3 - 4 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 5.3. $(n_1, n_2, n_3) \equiv (0,4,3) \mod 7$.

Let $F_1^{(3)} = A(G_1 - \{x, y, z, v, y_1, y_2, z_1\} + wz_2)$, $F_2^{(3)} = A(G_2 - z_1)$, and $F_3^{(3)} = A(G_3 - \{y_1, y_2\})$. Then $|F_1^{(3)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$, $|F_2^{(3)}| \geq \lceil (4(|G_2| - 1) + 3)/7 \rceil$ and $|F_3^{(3)}| \geq \lceil (4(|G_3| - 2) + 3)/7 \rceil$. Now $G[F_1^{(3)} \cup F_2^{(3)} \cup F_3^{(3)} + \{x, y, z\} - (\{z_2, a_1\} \cap (F_1^{(3)} \triangle F_2^{(3)})) - (\{b_1\} \cap (F_1^{(3)} \triangle F_3^{(3)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(3)}| + |F_2^{(3)}| + |F_3^{(3)}| + 3 - 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 5.4. $(n_1, n_2, n_3) \equiv (0,3,4) \mod 7$.

Let $F_1^{(4)} = A(G_1 - \{x, y, z, v, y_2, z_1, z_2\} + wy_1)$, $F_2^{(4)} = A(G_2 - \{z_1, z_2\})$, and $F_3^{(4)} = A(G_3 - y_2)$. Then $|F_1^{(4)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$, $|F_2^{(4)}| \geq \lceil (4(|G_2| - 2) + 3)/7 \rceil$ and $|F_3^{(4)}| \geq \lceil (4(|G_3| - 1) + 3)/7 \rceil$. Now $G[F_1^{(4)} \cup F_2^{(4)} \cup F_3^{(4)} + \{x, y, z\} - (\{a_1\} \cap (F_1^{(4)} \triangle F_2^{(4)})) - (\{b_1, y_1\} \cap (F_1^{(4)} \triangle F_3^{(4)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(4)}| + |F_2^{(4)}| + |F_3^{(4)}| + 3 - 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 5.5. $(n_1, n_2, n_3) \equiv (4,3,0) \mod 7$ (respectively, $(1,6,0) \mod 7$).

If $|N(w) \cap N(v_2)| \leq 2$, let $A_5 = \emptyset$ and $A_6 = \{a_1\}$. For $i = 5$ (respectively, $i = 6$), let $F_1^{(i)} = A((G_1 - \{x, y, z, v, z_1, z_2\} - A_i)/\{y_1y_2, wv_2\})$ with y' (respectively, v') as the identification of $\{y_1, y_2\}$ (respectively, $\{w, v_2\}$), $F_2^{(i)} = A(G_2 - \{z_1, z_2\} - A_i)$ and $F_3^{(i)} = A(G_3)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |A_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |A_i|) + 3)/7 \rceil$ and $|F_3^{(i)}| \geq \lceil (4|G_3| + 3)/7 \rceil$. Let $F^{(i)} := G[\overline{F_1}^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} - (\{a_1\} \cap (\overline{F_1}^{(i)} \triangle F_2^{(i)})) - (\{y_1, y_2, b_1\} \cap (\overline{F_1}^{(i)} \triangle F_3^{(i)}))]$, where $\overline{F_1}^{(i)} := F_1{}^{(i)} + \{x, y, v, z\}$ if $v', y' \notin F_1^{(1)}$, and otherwise, $\overline{F_1}^{(1)}$ obtained from $F_1{}^{(i)} + \{x, y, v, z\}$ by deleting $\{y, y'\}$ (respectively, $\{v, v'\}$) and adding $\{y_1, y_2\}$ (respectively, $\{v_2, w\}$) when $y' \in F_1{}^{(i)}$ (respectively, $v' \in F_1{}^{(i)}$). Therefore, $F^{(i)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - 3 - (1 - |A_i|) \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

If $|N(w) \cap N(v_2)| > 2$, there exist $c_1 \in N(w) \cap N(v_2)$ and subgraphs G'_1, G'_2, G'_3, G'_4 of G such that $G'_2 = G_2, G'_3 = G_3$, G'_4 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle vwc_1v_2v and containing $N(w) \cap N(v_2) - \{v\}$, and G'_1 is obtained from G by removing $G_2 - \{z_1, z_2, a_1\}$, $G'_3 - \{y_1, b_1, y_2\}$ and $G'_4 - \{w, v_2, v\}$. Let $A_i = \{a_1\}$ if $i = 13, 14, 17, 19$ and $A_i = \emptyset$ if $i = 15, 16, 18, 20$. Let $C_i = \{c_1\}$ if $i = 13, 15$ and $C_i = \emptyset$ if $i = 14, 16$. For $i = 13, 14, 15, 16$, let $F_1^{(i)} = A((G'_1 - \{x, y, w, z, v, v_2, z_1, z_2\} - A_i - C_i)/y_1y_2)$ with y' as the identification of $\{y_1, y_2\}$, $F_2^{(i)} = A(G'_2 - \{z_1, z_2\} - A_i)$, $F_3^{(i)} = A(G'_3)$, and $F_4^{(i)} = A(G'_4 - \{w, v_2\} - C_i)$. Note $|F_1^{(i)}| \geq \lceil (4(|G'_1| - 9 - |A_i| - |C_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G'_2| - 2 - |A_i|) + 3)/7 \rceil$, $|F_3^{(i)}| \geq \lceil (4|G'_3| + 3)/7 \rceil$, and $|F_4^{(i)}| \geq \lceil (4(|G'_4| - 2 - |C_i|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} \cup F_4^{(i)} + \{x, y, z, v\} - (\{y_1, y_2, b_1\} \cap (F_1{}^{(i)} \triangle F_3^{(i)})) - (\{a_1\} \cap (F_1{}^{(i)} \triangle F_2^{(i)})) - (\{c_1\} \cap (F_1{}^{(i)} \triangle F_4^{(i)}))]$ (if $y' \notin F_1^{(i)}$) or $G[(F_1^{(i)} - y') \cup F_2^{(i)} \cup F_3^{(i)} \cup F_4^{(i)} + \{x, y_1, y_2, z, v\} - (\{y_1, y_2, b_1\} \cap ((F_1{}^{(i)} + \{y_1, y_2\}) \triangle F_3^{(i)})) - (\{a_1\} \cap (F_1{}^{(i)} \triangle F_2^{(i)})) - (\{c_1\} \cap (F_1{}^{(i)} \triangle F_4^{(i)}))]$ (if $y' \in F_1^{(i)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + |F_4^{(i)}| + 4 - 3 - (1 - |A_i|) - (1 - |C_i|)$. Let $(n'_1, n'_2, n'_3, n'_4) := (4(|G'_1| - 9) + 3, 4(|G'_2| - 2) + 3, 4(|G'_3| - 2) + 3, 4(|G'_4| - 2) + 3)$. By Lemma 2.2(2), $(n'_1, n'_2, n'_3, n'_4) \equiv (4, 3, 0, 0), (4, 6, 0, 0), (0, 3, 0, 4), (0, 6, 0, 4) \mod 7$.

If $(n'_1, n'_2, n'_3, n'_4) \equiv (4, 6, 0, 0) \mod 7$ (respectively, $(4, 3, 0, 0) \mod 7$), for $i = 17$ (respectively, $i = 18$), let $F_1^{(i)} = A((G'_1 - \{x, y, z, v, z_1, z_2\} - A_i)/\{y_1y_2, wv_2\})$ with y' (respectively, v') as the identification of $\{y_1, y_2\}$ (respectively $\{w, v_2\}$), $F_2^{(i)} = A(G'_2 - \{z_1, z_2\} - A_i)$, $F_3^{(i)} = A(G'_3)$, and $F_4^{(i)} = A(G'_4)$. Then $|F_1^{(i)}| \geq \lceil (4(|G'_1| - 8 - |A_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G'_2| -$

$2-|A_i|)+3)/7\rceil$, $|F_3^{(i)}| \geq \lceil(4|G_3'|+3)/7\rceil$, and $|F_4^{(i)}| \geq \lceil(4|G_4'|+3)/7\rceil$. Let $F^{(i)} := G[\overline{F_1}^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} \cup F_4^{(i)} - (\{y_1,y_2,b_1\} \cap (\overline{F_1}^{(i)} \triangle F_3^{(i)})) - (\{w,v_2,c_1\} \cap (\overline{F_1}^{(i)} \triangle F_3^{(i)})) - (\{a_1\} \cap (\overline{F_1}^{(i)} \triangle F_4^{(i)}))]$, where $\overline{F_1}^{(i)} = {F_1}^{(i)} + \{x,y,z,v\}$ if $y',v' \notin F_1^{(i)}$, and otherwise, $\overline{F_1}^{(i)}$ obtained from ${F_1}^{(i)} + \{x,y,z,v\}$ by deleting $\{y,y'\}$ (respectively, $\{v,v'\}$) and adding $\{y_1,y_2\}$ (respectively, $\{v_2,w\}$) when $y' \in {F_1}^{(i)}$ (respectively, $v' \in {F_1}^{(i)}$). Therefore, $F^{(i)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}|+|F_2^{(i)}|+|F_3^{(i)}|+|F_4^{(i)}|+4-6-(1-|A_i|) \geq \lceil(4n+3)/7\rceil$, a contradiction.

If $(n_1',n_2',n_3',n_4') \equiv (0,6,0,4) \mod 7$ (respectively, $(0,3,0,4) \mod 7$), for $i=19$ (respectively, $i=20$), let $F_1^{(i)} = A((G_1' - \{x,y,z,w,z_1,z_2\} - A_i)/y_1y_2)$ with y' as the identification of $\{y_1,y_2\}$, $F_2^{(i)} = A(G_2' - \{z_1,z_2\} - A_i)$, $F_3^{(i)} = A(G_3')$, and $F_4^{(i)} = A(G_4' - w)$. Then $|F_1^{(i)}| \geq \lceil(4(|G_1'|-7-|A_i|)+3)/7\rceil$, $|F_2^{(i)}| \geq \lceil(4(|G_2'|-2-|A_i|)+3)/7\rceil$, $|F_3^{(i)}| \geq \lceil(4|G_3'|+3)/7\rceil$, and $|F_4^{(i)}| \geq \lceil(4(|G_4'|-1)+3)/7\rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} \cup F_4^{(i)} + \{x,y,z\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{y_1,y_2,b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{v_2,c_1\} \cap (F_1^{(i)} \triangle F_4^{(i)}))]$ (if $y' \notin F_1^{(i)}$) or $G[(F_1^{(i)} - y') \cup F_2^{(i)} \cup F_3^{(i)} \cup F_4^{(i)} + \{x,y_1,y_2,z\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{y_1,y_2,b_1\} \cap ((F_1^{(i)} + \{y_1,y_2\}) \triangle F_3^{(i)})) - (\{v_2,c_1\} \cap (F_1^{(i)} \triangle F_4^{(i)}))]$ (if $y' \in F_1^{(i)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(i)}|+|F_2^{(i)}|+|F_3^{(i)}|+|F_4^{(i)}|+3-5-(1-|A_i|) \geq \lceil(4n+3)/7\rceil$, a contradiction.

Subcase 5.6. $(n_1,n_2,n_3) \equiv (4,0,3) \mod 7$ (respectively, $(1,0,6) \mod 7$).

If $|N(v_2) \cap N(w)| \leq 2$, let $B_7 = \emptyset$ and $B_8 = \{b_1\}$. For $i=7$ (respectively, $i=8$), let $F_1^{(i)} = A((G_1 - \{x,y,z,v,y_1,y_2\} - B_i)/\{z_1z_2, wv_2\} + z'v')$ with z' (respectively, v') as the identification of $\{z_1,z_2\}$ (respectively, $\{w,v_2\}$), $F_2^{(i)} = A(G_2)$, and $F_3^{(i)} = A(G_3 - \{y_1,y_2\} - B_1)$. Then $|F_1^{(i)}| \geq \lceil(4(|G_1|-8-|B_i|)+3)/7\rceil$, $|F_2^{(i)}| \geq \lceil(4|G_2|+3)/7\rceil$, and $|F_3^{(i)}| \geq \lceil(4(|G_3|-2-|B_1|)+3)/7\rceil$. Let $F^{(i)} := G[\overline{F_1}^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} - (\{b_1\} \cap (\overline{F_1}^{(i)} \triangle F_3^{(i)})) - (\{z_1,z_2,a_1\} \cap (\overline{F_1}^{(i)} \triangle F_2^{(i)}))]$, where $\overline{F_1}^{(i)} := {F_1}^{(i)} + \{x,y,v,z\}$ if $z',v' \notin F_1^{(1)}$, and otherwise, $\overline{F_1}^{(1)}$ obtained from ${F_1}^{(i)} + \{x,y,v,z\}$ by deleting $\{z,z'\}$ (respectively, $\{v,v'\}$) and adding $\{z_1,z_2\}$ (respectively, $\{v_2,w\}$) when $z' \in {F_1}^{(i)}$ (respectively, $v' \in {F_1}^{(i)}$). Therefore, $F^{(i)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}|+|F_2^{(i)}|+|F_3^{(i)}|+4-3-(1-|B_i|) \geq \lceil(4n+3)/7\rceil$, a contradiction.

If $|N(v_2) \cap N(w)| > 2$, there exist $c_1 \in N(w) \cap N(v_2)$ and subgraphs G_1', G_2', G_3', G_4' of G such that $G_2' = G_2, G_3' = G_3$, G_4' is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle vwc_1v_2v and containing $N(w) \cap N(v_2) - \{v\}$, and G_1' is obtained from G by removing $G_2 - \{z_1, z_2, a_1\}$, $G_3' - \{y_1, b_1, y_2\}$ and $G_4' - \{w, v_2, v\}$. Let $B_1 = \{b_1\}$ if $i = 21, 22, 25, 27$ and $\emptyset$ if $i = 23, 24, 26, 28$. Let $C_1 = \{c_1\}$ if $i = 21, 23$ and $\emptyset$ if $i = 22, 24$. For $i = 21, 22, 23, 24$, let $F_1^{(i)} = A((G_1' - \{x, y, w, z, v, v_2, y_1, y_2\} - B_i - C_i)/z_1z_2)$ with z' as the identification of $\{z_1, z_2\}$, $F_2^{(i)} = A(G_2')$, $F_3^{(i)} = A(G_3' - \{y_1, y_2\} - B_i)$, and $F_4^{(i)} = A(G_4' - \{w, v_2\} - C_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1'| - 9 - |A_i| - |C_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4|G_2'| + 3)/7 \rceil$, $|F_3^{(i)}| \geq \lceil (4(|G_3'| - 2 - |B_i|) + 3)/7 \rceil$ and $|F_4^{(i)}| \geq \lceil (4(|G_4'| - 2 - |C_i|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} \cup F_4^{(i)} + \{x, y, z, v\} - (\{z_1, z_2, a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{c_1\} \cap (F_1^{(i)} \triangle F_4^{(i)}))$ (if $z' \notin F_1^{(i)}$) or $G[(F_1^{(i)} - z') \cup F_2^{(i)} \cup F_3^{(i)} \cup F_4^{(i)} + \{x, y, z_1, z_2, v\} - (\{z_1, z_2, a_1\} \cap ((F_1^{(i)} + \{z_1, z_2\}) \triangle F_2^{(i)})) - (\{b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{c_1\} \cap (F_1^{(i)} \triangle F_4^{(i)}))$ (if $z' \in F_1^{(i)}$) is an induced forest of size $|F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + |F_4^{(i)}| + 4 - 3 - (1 - |B_i|) - (1 - |C_i|)$. Let $(n_1', n_2', n_3', n_4') := (4(|G_1'| - 2) + 3, 4(|G_2'| - 2) + 3, 4(|G_3'| - 2) + 3, 4(|G_4'| - 2) + 3)$. By Lemma 2.2(2), $(n_1', n_2', n_3', n_4') \equiv (4, 0, 3, 0), (4, 0, 6, 0), (0, 0, 3, 4), (0, 0, 6, 4) \mod 7$.

If $(n_1', n_2', n_3', n_4') \equiv (4, 0, 6, 0) \mod 7$ (respectively, $(4, 0, 3, 0) \mod 7$), for $i = 25$ (respectively, $i = 26$), let $F_1^{(i)} = A((G_1' - \{x, y, z, v, y_1, y_2\} - B_i)/\{z_1z_2, wv_2\} + z'v')$ with z' (respectively, v') as the identification of $\{z_1z_2\}$ (respectively $\{w, v_2\}$), $F_2^{(i)} = A(G_2')$, $F_3^{(i)} = A(G_3' - \{z_1, z_2\} - B_i)$, and $F_4^{(i)} = A(G_4')$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1'| - 8 - |B_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4|G_2'| + 3)/7 \rceil$, $|F_3^{(i)}| \geq \lceil (4(|G_3'| - 2 - |B_i|) + 3)/7 \rceil$ and $|F_4^{(i)}| \geq \lceil (4|G_4'| + 3)/7 \rceil$. Now $F^{(i)} := G[\overline{F_1}^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} \cup F_4^{(i)} - (\{z_1, z_2, a_1\} \cap (\overline{F_1}^{(i)} \triangle F_3^{(i)})) - (\{b_1\} \cap (\overline{F_1}^{(i)} \triangle F_3^{(i)})) - (\{w, v_2, c_1\} \cap (\overline{F_1}^{(i)} \triangle F_4^{(i)}))]$, where $\overline{F_1}^{(i)} = F_1^{(i)} + \{x, y, z, v\}$ if $z', v' \notin F_1^{(i)}$, and otherwise, let $\overline{F_1}^{(i)}$ be obtained from $F_1^{(i)} + \{x, y, z, v\}$ by deleting $\{z, z'\}$ (respectively, $\{v, v'\}$) and adding $\{z_1, z_2\}$ (respectively, $\{v_2, w\}$) when $z' \in F_1^{(i)}$ (respectively, $v' \in F_1^{(i)}$). Therefore, $F^{(i)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + |F_4^{(i)}| + 4 - 6 - (1 - |B_i|) \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

If $(n_1', n_2', n_3', n_4') \equiv (0, 0, 6, 4) \mod 7$ (respectively, $(0, 0, 3, 4) \mod 7$), for $i = 27, 28$, let $F_1^{(i)} = A(G_1' - \{x, y, z, w, y_1, y_2\} - B_i)/z_1z_2)$ with z' as the identification of $\{z_1, z_2\}$, $F_2^{(i)} = A(G_2')$, $F_3^{(i)} = A(G_3' - \{z_1, z_2\} - B_i)$, and $F_4^{(i)} = A(G_4' - w)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1'| - 7 - |B_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4|G_2'| + 3)/7 \rceil$, $|F_3^{(i)}| \geq \lceil (4(|G_3'| - 2 - |B_i|) + 3)/7 \rceil$ and $|F_4^{(i)}| \geq \lceil (4(|G_4'| - 1) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} \cup F_4^{(i)} + \{x, y, z\} - (\{z_1, z_2, a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{v_2, c_1\} \cap (F_1^{(i)} \triangle F_4^{(i)}))]$ (if $z' \notin F_1^{(i)}$) or $G[(F_1^{(i)} - z') \cup F_2^{(i)} \cup$

$F_3^{(i)} \cup F_4^{(i)} + \{x, y, z_1, z_2\} - (\{z_1, z_2, a_1\} \cap ((F_1^{(i)} + \{z_1, z_2\}) \triangle F_2^{(i)})) - (\{b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{v_2, c_1\} \cap (F_1^{(i)} \triangle F_4^{(i)}))]$ (if $z' \in F_1^{(i)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + |F_4^{(i)}| + 3 - 5 - (1 - |B_i|) \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Case 6. $|N(z_1) \cap N(z_2)| > 2$, $|N(y_1) \cap N(y_2)| \leq 2$ and $|N(w) \cap N(v_2)| > 2$.

There exist $a_1 \in N(z_1) \cap N(z_2)$, $c_1 \in N(w) \cap N(v_2)$ and subgraphs G_1, G_2, G_3 of G such that G_2 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $zz_1a_1z_2z$ and containing $N(z_1) \cap N(z_2) - \{z\}$, G_3 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle vwc_1v_2v and containing $N(w) \cap N(v_2) - \{v\}$, and G_1 is obtained from G by removing $G_2 - \{z_1, z_2, a_1\}$ and $G_3 - \{w, c_1, v_2\}$. Let $A_i = \{a_1\}$ if $i = 1, 2$ and $A_i = \emptyset$ if $i = 3, 4$. Let $C_i = \{c_1\}$ if $i = 1, 3$ and $C_i = \emptyset$ if $i = 2, 4$. For $i \in [4]$, let $F_1^{(i)} = A((G_1 - \{x, y, z, v, w, v_2, z_1, z_2\} - A_i - C_i)/y_1y_2)$ with y' as the identification of $\{y_1, y_2\}$, $F_2^{(i)} = A(G_2 - \{z_1, z_2\} - A_i)$, and $F_3^{(i)} = A(G_3 - \{w, v_2\} - C_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 9 - |A_i| - |C_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |A_i|) + 3)/7 \rceil$, and $|F_3^{(i)}| \geq \lceil (4(|G_3| - 2 - |C_i|) + 3)/7 \rceil$. Let $F = F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{x, y, v, z\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{c_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))$. Now $G[F]$ (if $y' \notin F_1^{(i)}$) or $G[F - \{y, y'\} + \{y_1, y_2\}]$ (if $y' \in F_1^{(i)}$) is an induced forest of size $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - (1 - |A_1|) - (1 - |C_1|)$. Let $(n_1, n_2, n_3) := (4(|G_1| - 9) + 3, 4(|G_2| - 2) + 3, 4(|G_3| - 2) + 3)$. By Lemma 2.2(5) (with $a = |G_1| - 9, a_1 = |G_2| - 2, a_2 = |G_3| - 2, c = 4$), $(n_1, n_2, n_3) \equiv (0,0,0), (1,0,0), (4,0,3), (4,3,0), (3,0,4), (4,0,4), (3,4,0), (4,4,0), (1,6,0), (1,0,6), (0,3,4), (0,4,3), (0,4,4), (6,4,4), (4,4,6), (4,6,4) \mod 7$.

Subcase 6.1. $|N(v_1) \cap N(v_2)| > 2$.

There exist $d_1 \in N(v_1) \cap N(v_2)$ and subgraphs G_1', G_3', G_4' such that $G_3' = G_3$, G_4' is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $vv_1d_1v_2v$ and containing $N(v_1) \cap N(v_2) - \{v\}$, and G_1' is obtained from G by removing $G_3' - \{w, c_1, v_2\}$ and $G_4' - \{v_1, v_2, v\}$. Let $C_i = \{c_1\}$ if $i = 1, 2$ and $C_i = \emptyset$ if $i = 3, 4$. Let $D_i = \{d_1\}$ if $i = 1, 3$ and $D_i = \emptyset$ if $i = 2, 4$. For $i \in [4]$, let $F_1^{(i)} = A(G_1' - \{x, z, z_1, v, w, v_1, v_2\} - C_i - D_i + yz_2)$, $F_2^{(i)} = A(G_3' - \{w, v_2\} - C_i)$, and $F_3^{(i)} = A(G_4' - \{v_1, v_2\} - D_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1'| - 7 - |C_i| - |D_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_3'| - 2 - |C_i|) + 3)/7 \rceil$, and $|F_3^{(i)}| \geq \lceil (4(|G_4'| - 2 - |D_i|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{x, z, v\} - (\{c_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{d_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 3 - (1 - |C_i|) - (1 - |D_i|)$.

Let $(n_1', n_2', n_3') := (4(|G_1'| - 7) + 3, 4(|G_3'| - 2) + 3, 4(|G_4'| - 2) + 3)$. By Lemma 2.2(4) (with $a = |G_1'| - 7, a_1 = |G_3'| - 2, a_2 = |G_4'| - 2, c = 3$), $(n_1', n_2', n_3') \equiv (1, 0, 0), (0, 4, 4), (4, 4, 0), (4, 0, 4) \mod 7$.

If $(n_1', n_2', n_3') \equiv (0, 4, 4) \mod 7$, let $F_1^{(5)} = A(G_1' - \{x, z, w, v, v_1\})$, $F_2^{(5)} = A(G_3' - w)$, and $F_3^{(5)} = A(G_4' - v_1)$. Then $|F_1^{(5)}| \geq \lceil (4(|G_1'| - 5) + 3)/7 \rceil$, $|F_2^{(5)}| \geq \lceil (4(|G_3'| - 1) + 3)/7 \rceil$, and $|F_3^{(5)}| \geq \lceil (4(|G_4'| - 1) + 3)/7 \rceil$. Now $G[F_1^{(5)} \cup F_2^{(5)} \cup F_3^{(5)} + \{x, v\} - (\{c_1, v_2\} \cap (F_1^{(5)} \triangle F_2^{(5)})) - (\{d_1, v_2\} \cap (F_1^{(5)} \triangle F_3^{(5)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(5)}| + |F_2^{(5)}| + |F_3^{(5)}| + 2 - 4 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

If $(n_1', n_2', n_3') \equiv (4, 4, 0) \mod 7$, let $F_1^{(6)} = A((G_1' - \{x, z, w, v, z_1\})/v_1v_2 + yz_2)$ with v' as the identification of $\{v_1, v_2\}$, $F_2^{(6)} = A(G_3' - w)$, and $F_3^{(6)} = A(G_4')$. Then $|F_1^{(6)}| \geq \lceil (4(|G_1'| - 6) + 3)/7 \rceil$, $|F_2^{(6)}| \geq \lceil (4(|G_3'| - 1) + 3)/7 \rceil$, and $|F_3^{(6)}| \geq \lceil (4|G_4'| + 3)/7 \rceil$. Now $G[F_1^{(6)} \cup F_2^{(6)} \cup F_3^{(6)} + \{x, v, z\} - (\{v_2, c_1\} \cap (F_1^{(6)} \triangle F_2^{(6)})) - (\{v_1, v_2, d_1\} \cap (F_1^{(6)} \triangle F_3^{(6)}))]$ (if $v' \notin F_1^{(6)}$) or $G[(F_1^{(6)} - v') \cup F_2^{(6)} \cup F_3^{(6)} + \{x, v_1, v_2, z\} - (\{v_2, c_1\} \cap ((F_1^{(6)} + v_2) \triangle F_2^{(6)})) - (\{v_1, v_2, d_1\} \cap ((F_1^{(6)} + \{v_1, v_2\}) \triangle F_3^{(6)}))]$ (if $v' \notin F_1^{(6)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(6)}| + |F_2^{(6)}| + |F_3^{(6)}| + 3 - 5 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

If $(n_1', n_2', n_3') \equiv (4, 0, 4) \mod 7$, let $F_1^{(7)} = A(G_1' - \{x, z, w, v, v_2, c_1\})$, $F_2^{(7)} = A(G_3' - \{w, v_2, c_1\})$, and $F_3^{(7)} = A(G_4' - \{v_2\})$. Then $|F_1^{(7)}| \geq \lceil (4(|G_1'| - 6) + 3)/7 \rceil$, $|F_2^{(7)}| \geq \lceil (4(|G_3'| - 3) + 3)/7 \rceil$, and $|F_3^{(7)}| \geq \lceil (4(|G_4'| - 1) + 3)/7 \rceil$. Now $G[F_1^{(7)} \cup F_2^{(7)} \cup F_3^{(7)} + \{x, v\} - (\{v_1, d_1\} \cap (F_1^{(7)} \triangle F_3^{(7)}))]$ is an induced forest in G, showing $|F_1^{(7)}| + |F_2^{(7)}| + |F_3^{(7)}| + 2 - 2 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

If $(n_1', n_2', n_3') \equiv (1, 0, 0) \mod 7$, then there exist $a_1 \in N(z_1) \cap N(z_2)$ and subgraphs $G_1'', G_2'', G_3'', G_4''$ of G such that $G_3'' = G_3', G_4'' = G_4'$, G_2'' is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $zz_1a_1z_2z$ and containing $N(v_1) \cap N(v_2) - \{v\}$, and G_1'' is obtained from G by removing $G_2'' - \{z_1, a_1, z_2\}$, $G_3'' - \{w, c_1, v_2\}$ and $G_4'' - \{v_1, v_2, v\}$. Let $A_8 = \{a_1\}$ and $A_9 = \emptyset$. For $i = 8, 9$, let $F_1^{(i)} = A((G_1'' - \{x, y, z, v, z_1, z_2\} - A_i)/\{y_1y_2, wv_2\})$ with y' (respectively, v') as the identification of $\{y_1, y_2\}$ (respectively, $\{w, v_2\}$), $F_2^{(i)} = A(G_2'' - \{z_1, z_2\} - A_i)$, $F_3^{(i)} = A(G_3'')$, and $F_4^{(i)} = A(G_4'')$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1''| - 8 - |A_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2''| - 2 - |A_i|) + 3)/7 \rceil$, $|F_3^{(i)}| \geq \lceil (4|G_3''| + 3)/7 \rceil$, and $|F_4^{(i)}| \geq \lceil (4|G_4''| + 3)/7 \rceil$. Let $F^{(i)} := G[\overline{F_1}^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} \cup F_4^{(i)} - (\{a_1\} \cap (\overline{F_1}^{(i)} \triangle F_2^{(i)})) - (\{w, v_2, c_1\} \cap (\overline{F_1}^{(i)} \triangle F_3^{(i)})) - (\{v_1, v_2, d_1\} \cap (\overline{F_1}^{(i)} \triangle F_4^{(i)}))]$, where $\overline{F_1}^{(i)} = F_1{}^{(i)} + \{x, y, v, z\}$

if $y', v' \notin F_1^{(i)}$, and otherwise, $\overline{F_1}^{(i)}$ obtained from ${F_1}^{(i)} + \{x, y, v, z\}$ by deleting $\{y, y'\}$ (respectively, $\{v, v'\}$) and adding $\{y_1, y_2\}$ (respectively, $\{w, v_2\}$) when $y' \in {F_1}^{(i)}$ (respectively, $v' \in {F_1}^{(i)}$). Therefore, $F^{(i)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - 6 - (1 - |A_i|)$. By Lemma 2.2(2), $(4(|G_1''|-8)+3, 4(|G_2''|-2)+3, 4(|G_3''|-2)+3, 4(|G_4''|-2)+3) \equiv (4,0,0,0), (0,4,0,0) \mod 7$.

If $(4(|G_1''| - 8) + 3, 4(|G_2''| - 2) + 3, 4(|G_3''| - 2) + 3, 4(|G_4''| - 2) + 3) \equiv (4,0,0,0) \mod 7$, let $F_1^{(10)} = A((G_1'' - \{x, y, z, v\})/\{z_1z_2, y_1y_2, wv_2\} + v'z')$ with z' (respectively, y', v') as the identification of $\{z_1, z_2\}$ (respectively, $\{y_1, y_2\}, \{w, v_2\}$) and $F_2^{(10)} = A(G_2'')$, $F_3^{(10)} = A(G_3'')$, and $F_4^{(10)} = A(G_4'')$. Then $|F_1^{(10)}| \geq \lceil (4(|G_1''| - 7) + 3)/7 \rceil$, $|F_2^{(10)}| \geq \lceil (4|G_2''| + 3)/7 \rceil$, $|F_3^{(10)}| \geq \lceil (4|G_3''| + 3)/7 \rceil$, and $|F_4^{(10)}| \geq \lceil (4|G_4''| + 3)/7 \rceil$. Let $F^{(10)} := G[\overline{F_1}^{(10)} \cup F_2^{(10)} \cup F_3^{(10)} - (\{z_1, z_2, a_1\} \cap (\overline{F_1}^{(10)} \triangle F_2^{(10)})) - (\{w, v_2, c_1\} \cap (\overline{F_1}^{(10)} \triangle F_3^{(10)})) - (\{v_1, v_2, d_1\} \cap (\overline{F_1}^{(10)} \triangle F_4^{(10)}))]$, where $\overline{F_1}^{(1)} := {F_1}^{(10)} + \{x, y, v, z\}$ if $v', y', z' \notin F_1^{(10)}$, and otherwise, $\overline{F_1}^{(10)}$ obtained from ${F_1}^{(10)} + \{x, y, v, z\}$ by deleting $\{z, z'\}$ (respectively, $\{v, v'\}, \{y, y'\}$) and adding $\{z_1, z_2\}$ (respectively, $\{v_2, w\}, \{y_1, y_2\}$) when $z' \in {F_1}^{(1)}$ (respectively, $v' \in {F_1}^{(10)}$, $y' \in {F_1}^{(10)}$). Therefore, $F^{(10)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(10)}| + |F_2^{(10)}| + |F_3^{(10)}| + |F_4^{(10)}| + 4 - 9 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

If $(4(|G_1''|-8)+3, 4(|G_2''|-2)+3, 4(|G_3''|-2)+3, 4(|G_4''|-2)+3) \equiv (0,4,0,0) \mod 7$, let $F_1^{(11)} = A(G_1'' - \{x, z, w, v, v_1, v_2, z_1, c_1, d_1\} + yz_2)$, $F_2^{(11)} = A(G_2'' - \{z_1\})$, $F_3^{(11)} = A(G_3'' - \{w, v_2, c_1\})$, and $F_4^{(11)} = A(G_4'' - \{v_1, v_2, d_1\})$. Then $|F_1^{(11)}| \geq \lceil (4(|G_1''| - 9) + 3)/7 \rceil$, $|F_2^{(11)}| \geq \lceil (4(|G_2''| - 1) + 3)/7 \rceil$, $|F_3^{(11)}| \geq \lceil (4(|G_3''|-3)+3)/7 \rceil$, and $|F_4^{(11)}| \geq \lceil (4(|G_4''|-3)+3)/7 \rceil$. Now $G[F_1^{(11)} \cup F_2^{(11)} \cup F_3^{(11)} \cup F_4^{(11)} + \{x, z, v\}] - (\{z_2, a_1\} \cap (F_1^{(11)} \triangle F_2^{(11)}))$. Therefore, $F^{(16)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(11)}| + |F_2^{(11)}| + |F_3^{(11)}| + |F_4^{(11)}| + 3 - 2 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Subcase 6.2. $|N(v_1) \cap N(v_2)| \leq 2$.

Subcase 6.2.1. $(n_1, n_2, n_3) \equiv (0,0,0), (1,0,0) \mod 7$.

Let $F_1^{(1)} = A((G_1 - \{x, y, z, v\})/\{z_1z_2, y_1y_2, wv_2\} + v'z')$ with z' (respectively, y', v') as the identification of $\{z_1, z_2\}$ (respectively, $\{y_1, y_2\}, \{w, v_2\}$), $F_2^{(1)} = A(G_2)$, and $F_3^{(1)} = A(G_3)$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$, $|F_2^{(1)}| \geq \lceil (4|G_2| + 3)/7 \rceil$, and $|F_3^{(1)}| \geq \lceil (4|G_3| + 3)/7 \rceil$. Let $F^{(1)} := G[\overline{F_1}^{(1)} \cup F_2^{(1)} \cup F_3^{(1)} - (\{z_1, z_2, a_1\} \cap (\overline{F_1}^{(1)} \triangle F_2^{(1)})) - (\{w, v_2, c_1\} \cap (\overline{F_1}^{(1)} \triangle F_3^{(1)}))]$, where $\overline{F_1}^{(1)} := {F_1}^{(1)} + \{x, y, v, z\}$ if $v', y', z' \notin F_1^{(1)}$, and otherwise, $\overline{F_1}^{(1)}$ obtained from ${F_1}^{(1)} + \{x, y, v, z\}$ by deleting $\{z, z'\}$ (respectively, $\{v, v'\}, \{y, y'\}$) and adding $\{z_1, z_2\}$ (respectively, $\{v_2, w\}, \{y_1, y_2\}$) when $z' \in {F_1}^{(1)}$ (respectively, $v' \in {F_1}^{(1)}$, $y' \in {F_1}^{(1)}$). Therefore, $F^{(1)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + |F_3^{(1)}| + 4 - 6 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 6.2.2. $(n_1, n_2, n_3) \equiv (3,0,4), (4,0,4), (3,4,0), (4,4,0), (0,4,4), (6,4,4), (4,4,6), (4,6,4) \mod 7$.

Let $F_1^{(2)} = A(G_1 - \{x, z, z_1, w, v\}/v_1v_2 + yz_2)$ with v' as the identification of $\{v_1, v_2\}$, $F_2^{(2)} = A(G_2 - \{z_1\})$, and $F_3^{(2)} = A(G_3 - \{w\})$. Then $|F_1^{(2)}| \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil$, $|F_2^{(2)}| \geq \lceil (4(|G_2| - 1) + 3)/7 \rceil$, and $|F_3^{(2)}| \geq \lceil (4(|G_3| - 1) + 3)/7 \rceil$. Now $G[F_1^{(2)} \cup F_2^{(2)} \cup F_3^{(2)} + \{x, v, z\} - (\{z_2, a_1\} \cap (F_1^{(2)} \triangle F_2^{(2)})) - (\{c_1, v_2\} \cap (F_1^{(2)} \triangle F_2^{(2)}))]$ (if $v' \notin F_1^{(2)}$) or $G[(F_1^{(2)} - v') \cup F_2^{(2)} \cup F_3^{(2)} + \{x, v, z\} - (\{z_2, a_1\} \cap (F_1^{(2)} \triangle F_2^{(2)})) - (\{c_1, v_2\} \cap ((F_1^{(2)} + v_2) \triangle F_2^{(2)}))]$ (if $v' \in F_1^{(2)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + |F_3^{(2)}| + 3 - 4 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 6.2.3. $(n_1, n_2, n_3) \equiv (0,4,3) \mod 7$.

Let $F_1^{(3)} = A(G_1 - \{x, z, v, w, z_1, v_1, v_2\} + yz_2)$, $F_2^{(3)} = A(G_2 - z_1)$, and $F_3^{(3)} = A(G_3 - \{w, v_2\})$. Then $|F_1^{(3)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$, $|F_2^{(3)}| \geq \lceil (4(|G_2| - 1) + 3)/7 \rceil$ and $|F_3^{(3)}| \geq \lceil (4(|G_3| - 2) + 3)/7 \rceil$. Then $G[F_1^{(3)} \cup F_2^{(3)} \cup F_3^{(3)} + \{x, v, z\} - (\{z_2, a_1\} \cap (F_1^{(3)} \triangle F_2^{(3)})) - (\{c_1\} \cap (F_1^{(3)} \triangle F_3^{(3)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(3)}| + |F_2^{(3)}| + |F_3^{(3)}| + 3 - 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 6.2.4. $(n_1, n_2, n_3) \equiv (0,3,4) \mod 7$.

Let $F_1^{(4)} = A((G_1 - \{x, y, z, w, z_1, z_2\})/y_1y_2)$ with y' as the identification of $\{y_1, y_2\}$, $F_2^{(4)} = A(G_2 - \{z_1, z_2\})$, and $F_3^{(4)} = A(G_3 - w)$. Then $|F_1^{(4)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$, $|F_2^{(4)}| \geq \lceil (4(|G_2| - 2) + 3)/7 \rceil$ and $|F_3^{(4)}| \geq \lceil (4(|G_3| - 1) + 3)/7 \rceil$. Let $F = F_1^{(4)} \cup F_2^{(4)} \cup F_3^{(4)} + \{x, y, z\} - (\{a_1\} \cap (F_1^{(4)} \triangle F_2^{(4)})) - (\{c_1, v_2\} \cap (F_1^{(4)} \triangle F_3^{(4)}))$. Now $G[F]$ (if $y' \notin F_1^{(4)}$) or $G[F - \{y', y\} + \{y_1, y_2\}]$ (if $y' \in F_1^{(4)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(4)}| + |F_2^{(4)}| + |F_3^{(4)}| + 3 - 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 6.2.5. $(n_1, n_2, n_3) \equiv (4,3,0) \mod 7$ (respectively $(1,6,0) \mod 7$).

Let $A_5 = \emptyset$ and $A_6 = \{a_1\}$. For $i = 5, 6$, let $F_1^{(i)} = A((G_1 - \{x, y, z, v, z_1, z_2\} - A_i)/\{y_1y_2, wv_2\})$ with y' (respectively, v') as the identification of $\{y_1, y_2\}$ (respectively, $\{w, v_2\}$), $F_2^{(i)} = A(G_2 - \{z_1, z_2\} - A_i)$, and $F_3^{(i)} =$

$A(G_3)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1|-8-|A_i|)+3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2|-2-|A_i|)+3)/7 \rceil$ and $|F_3^{(i)}| \geq \lceil (4|G_3|+3)/7 \rceil$. Let $F^{(i)} := G[\overline{F_1}^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} - (\{a_1\} \cap (\overline{F_1}^{(i)} \triangle F_2^{(i)})) - (\{w, v_2, c_1\} \cap (\overline{F_1}^{(i)} \triangle F_3^{(i)}))]$, where $\overline{F_1}^{(i)} = F_1{}^{(i)} + \{x, y, v, z\}$ if $y', v' \notin F_1^{(i)}$, and otherwise, $\overline{F_1}^{(i)}$ obtained from $F_1{}^{(i)}$ by deleting $\{y, y'\}$ (respectively, $\{v, v'\}$) and adding $\{y_1, y_2\}$ (respectively, $\{w, v_2\}$) when $y' \in F_1{}^{(i)}$ (respectively, $v' \in F_1{}^{(i)}$). Therefore, $F^{(i)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - 3 - (1 - |A_i|) \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 6.2.6. $(n_1, n_2, n_3) \equiv (4, 0, 3) \bmod 7$ (respectively, $(1, 0, 6) \bmod 7$).

Let $C_7 = \emptyset$ and $C_8 = \{c_1\}$. For $i = 7, 8$, let $F_1^{(i)} = A((G_1 - \{x, y, z, v, w, v_2\} - C_i)/\{z_1 z_2, y_1 y_2\})$ with z' (respectively, y') as the identification of $\{z_1, z_2\}$ (respectively, $\{y_1, y_2\}$), $F_2^{(i)} = A(G_2)$, and $F_3^{(i)} = A(G_3 - \{w, v_2\} - C_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |C_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4|G_2| + 3)/7 \rceil$ and $|F_3^{(i)}| \geq \lceil (4(|G_3| - 2 - |C_i|) + 3)/7 \rceil$. Let $F^{(i)} := G[\overline{F_1}^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} - (\{c_1\} \cap (\overline{F_1}^{(i)} \triangle F_3^{(i)})) - (\{z_1, z_2, a_1\} \cap (\overline{F_1}^{(i)} \triangle F_2^{(i)}))]$, where $\overline{F_1}^{(i)} = F_1{}^{(i)} + \{x, y, v, z\}$ if $y', z' \notin F_1^{(i)}$, and otherwise $\overline{F_1}^{(i)}$ obtained from $F_1{}^{(i)}$ by deleting $\{y, y'\}$ (respectively, $\{z, z'\}$) and adding $\{y_1, y_2\}$ (respectively, $\{z_1, z_2\}$) when $y' \in F_1{}^{(i)}$ (respectively, $z' \in F_1{}^{(i)}$). Therefore, $F^{(i)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - 3 - (1 - |C_i|) \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Case 7. $|N(z_1) \cap N(z_2)| \leq 2$, $|N(y_1) \cap N(y_2)| > 2$ and $|N(w) \cap N(v_2)| > 2$.

There exist $b_1 \in N(y_1) \cap N(y_2)$, $c_1 \in N(w) \cap N(v_2)$ and subgraphs G_1, G_2, G_3 of G such that G_2 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $yy_1b_1y_2y$ and containing $N(y_1) \cap N(y_2) - \{y\}$, G_3 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle vwc_1v_2v and containing $N(w) \cap N(v_2) - \{v\}$, and G_1 is obtained from G by removing $G_2 - \{y_1, y_2, b_1\}$ and $G_3 - \{w, c_1, v_2\}$. Let $B_i = \{b_1\}$ if $i = 1, 2$ and $B_i = \emptyset$ if $i = 3, 4$. Let $C_i = \{c_1\}$ if $i = 1, 3$ and $C_i = \emptyset$ if $i = 2, 4$. For $i \in [4]$, let $F_1^{(i)} = A((G_1 - \{x, y, z, v, y_1, y_2, w, v_2\} - B_i - C_i)/z_1 z_2)$ with z' as the identification of $\{z_1, z_2\}$, $F_2^{(i)} = A(G_2 - \{y_1, y_2\} - B_i)$, and $F_3^{(i)} = A(G_3 - \{w, v_2\} - C_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 9 - |B_i| - |C_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2|-2-|B_i|)+3)/7 \rceil$, and $|F_3^{(i)}| \geq \lceil (4(|G_3|-2-|C_i|)+3)/7 \rceil$. Let $F = F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{x, y, v, z\} - (\{b_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{c_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))$. Now $G[F]$ (if $z' \notin F_1^{(i)}$) or $G[F - \{z, z'\} + \{z_1, z_2\}]$ (if $z' \in F_1^{(i)}$) is an induced forest of size $|F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - (1 - |B_i|) - (1 - |C_i|)$. Let

$(n_1, n_2, n_3) := (4(|G_1|-9)+3, 4(|G_2|-2)+3, 4(|G_3|-2)+3)$. By Lemma 2.2(5) (with $a = |G_1| - 9, a_1 = |G_2| - 2, a_2 = |G_3| - 2, c = 4$), $(n_1, n_2, n_3) \equiv (0,0,0), (1,0,0), (4,0,3), (4,3,0), (3,0,4), (4,0,4), (3,4,0), (4,4,0), (1,6,0), (1,0,6), (0,3,4), (0,4,3), (0,4,4), (6,4,4), (4,4,6), (4,6,4) \mod 7$. Let $B_i = \emptyset$ if $i = 1,4,8$ and $B_i = \{b_1\}$ if $i = 3,5,9$.

Subcase 7.1. $(n_1, n_2, n_3) \equiv (0,0,0), (1,0,0) \mod 7$ (respectively $(4,4,0) \mod 7$).

For $i = 1$ (respectively, $i = 5$), let $F_1^{(i)} = A((G_1 - \{x,y,z,v\} - B_i)/\{z_1z_2, y_1y_2, wv_2\} + v'z')$ with z' (respectively, y', v') as the identification of $\{z_1, z_2\}$ (respectively, $\{y_1, y_2\}, \{w, v_2\}$) and $F_2^{(i)} = A(G_2 - B_i)$, and $F_3^{(i)} = A(G_3)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 7 - |B_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2| - |B_i|) + 3)/7 \rceil$, and $|F_3^{(i)}| \geq \lceil (4|G_3| + 3)/7 \rceil$. Let $F^{(i)} := G[\overline{F_1}^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} - (\{y_1, y_2, b_1\} \cap (\overline{F_1}^{(i)} \triangle F_2^{(i)})) - (\{w, v_2, c_1\} \cap (\overline{F_1}^{(i)} \triangle F_3^{(i)}))]$, where $\overline{F_1}^{(i)} := F_1{}^{(i)} + \{x, y, v, z\}$ if $v', y', z' \notin F_1^{(i)}$, and otherwise, $\overline{F_1}^{(i)}$ obtained from $F_1{}^{(i)} + \{x, y, v, z\}$ by deleting $\{z, z'\}$ (respectively, $\{v, v'\}, \{y, y'\}$) and adding $\{z_1, z_2\}$ (respectively, $\{v_2, w\}, \{y_1, y_2\}$) when $z' \in F_1{}^{(i)}$ (respectively, $v' \in F_1{}^{(i)}$, $y' \in F_1{}^{(i)}$). Therefore, $F^{(i)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - 5 - (1 - |B_i|) \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 7.2. $(n_1, n_2, n_3) \equiv (0,4,4), (6,4,4) \mod 7$.

Let $F_1^{(2)} = A(G_1 - \{x, y, w, z_1, y_2\} + y_1z)$, $F_2^{(2)} = A(G_2 - y_2)$, and $F_3^{(2)} = A(G_3 - w)$. Then $|F_1^{(2)}| \geq \lceil (4(|G_1| - 5) + 3)/7 \rceil$, $|F_2^{(2)}| \geq \lceil (4(|G_2| - 1) + 3)/7 \rceil$, and $|F_3^{(2)}| \geq \lceil (4(|G_3| - 1) + 3)/7 \rceil$. Now $G[F_1^{(2)} \cup F_2^{(2)} \cup F_3^{(2)} + \{x, y\} - (\{y_1, b_1\} \cap (F_1^{(2)} \triangle F_2^{(2)})) - (\{v_2, c_1\} \cap (F_1^{(2)} \triangle F_3^{(2)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + |F_3^{(2)}| + 2 - 4 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 7.3. $(n_1, n_2, n_3) \equiv (4,0,4), (4,6,4) \mod 7$ (respectively, $(0,3,4) \mod 7$).

For $i = 3$ (respectively, $i = 4$), let $F_1^{(i)} = A((G_1 - \{x, y, z, w, y_1, y_2\} - B_i)/z_1z_2)$ with z' as the identification of $\{z_1, z_2\}$, $F_2^{(i)} = A(G_2 - \{y_1, y_2\} - B_i)$, and $F_3^{(i)} = A(G_3 - w)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 7 - |B_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |B_i|) + 3)/7 \rceil$ and $|F_3^{(i)}| \geq \lceil (4(|G_3| - 1) + 3)/7 \rceil$. Let $F = F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{x, y, z\} - (\{v_2, c_1\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{b_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))$. Now $G[F]$ (if $z' \notin F_1^{(i)}$) or $G[F - \{z, z'\} + \{z_1, z_2\}]$ (if $z' \in F_1^{(i)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 3 - 2 - (1 - |B_i|) \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 7.4. $(n_1, n_2, n_3) \equiv (4,0,3) \mod 7$ (respectively, $(1,0,6) \mod 7$).

Let $C_6 = \emptyset$ and $C_7 = \{c_1\}$. For $i = 6,7$, let $F_1^{(i)} = A((G_1 - \{x,y,z,w,v,v_2\} - C_i)/\{y_1y_2, z_1z_2\})$ with y' (respectively, z') as the identification of $\{y_1, y_2\}$ (respectively, $\{z_1, z_2\}$), $F_2^{(i)} = A(G_2)$, and $F_3^{(i)} = A(G_3 - \{w, v_2\} - C_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |C_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4|G_2| + 3)/7 \rceil$, and $|F_3^{(i)}| \geq \lceil (4(|G_3| - 2 - |C_i|) + 3)/7 \rceil$. Let $F^{(i)} := G[\overline{F_1}^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} - (\{y_1, y_2, b_1\} \cap (\overline{F_1}^{(i)} \triangle F_2^{(i)})) - (\{c_1\} \cap (\overline{F_1}^{(i)} \triangle F_3^{(i)}))]$, where $\overline{F_1}^{(i)} := {F_1}^{(i)} + \{x,y,v,z\}$ if $y', z' \notin F_1^{(i)}$, and otherwise, $\overline{F_1}^{(i)}$ obtained from ${F_1}^{(i)} + \{x,y,v,z\}$ by deleting $\{z, z'\}$ (respectively, $\{y, y'\}$) and adding $\{z_1, z_2\}$ (respectively, $\{y_1, y_2\}$) when $z' \in {F_1}^{(i)}$ (respectively, $y' \in {F_1}^{(i)}$). Therefore, $F^{(i)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - 3 - (1 - |C_i|) \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 7.5. $(n_1, n_2, n_3) \equiv (4,3,0) \mod 7$ (respectively, $(1,6,0) \mod 7$).

For $i = 8$ (respectively, $i = 9$), let $F_1^{(i)} = A((G_1 - \{x,y,z,v,y_1,y_2\} - B_i)/\{wv_2, z_1z_2\} + v'z')$ with v' (respectively, z') as the identification of $\{w, v_2\}$ (respectively, $\{z_1, z_2\}$), $F_2^{(i)} = A(G_2 - \{y_1, y_2\} - B_i)$, and $F_3^{(i)} = A(G_3)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |B_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |B_i|) + 3)/7 \rceil$ and $|F_3^{(i)}| \geq \lceil (4|G_3| + 3)/7 \rceil$. Let $F^{(i)} := G[\overline{F_1}^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} - (\{b_1\} \cap (\overline{F_1}^{(i)} \triangle F_2^{(i)})) - (\{w, v_2, c_1\} \cap (\overline{F_1}^{(i)} \triangle F_3^{(i)}))]$, where $\overline{F_1}^{(i)} := {F_1}^{(i)} + \{x,y,v,z\}$ if $v', z' \notin F_1^{(i)}$, and otherwise, $\overline{F_1}^{(i)}$ obtained from ${F_1}^{(i)} + \{x,y,v,z\}$ by deleting $\{z, z'\}$ (respectively, $\{v, v'\}$) and adding $\{z_1, z_2\}$ (respectively, $\{w, v_2\}$) when $z' \in {F_1}^{(i)}$ (respectively, $v' \in {F_1}^{(i)}$). Therefore, $F^{(i)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - 3 - (1 - |B_i|) \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 7.6. $(n_1, n_2, n_3) \equiv (3,0,4) \mod 7$.

Let $F_1^{(10)} = A((G_1 - \{x,y,z,w\})/\{y_1y_2, z_1z_2\})$ with y' (respectively, z') as the identification of $\{y_1, y_2\}$ (respectively, $\{z_1, z_2\}$), $F_2^{(10)} = A(G_2)$, and $F_3^{(10)} = A(G_3 - w)$. Then $|F_1^{(10)}| \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil$, $|F_2^{(10)}| \geq \lceil (4|G_2| + 3)/7 \rceil$ and $|F_3^{(10)}| \geq \lceil (4(|G_3| - 1) + 3)/7 \rceil$. Let $F^{(10)} := G[\overline{F_1}^{(10)} \cup F_2^{(10)} \cup F_3^{(10)} - (\{y_1, y_2, b_1\} \cap (\overline{F_1}^{(10)} \triangle F_2^{(10)})) - (\{v_2, c_1\} \cap (\overline{F_1}^{(10)} \triangle F_3^{(10)}))]$, where $\overline{F_1}^{(10)} := {F_1}^{(10)} + \{x,y,z\}$ if $y', z' \notin F_1^{(10)}$, and otherwise, $\overline{F_1}^{(10)}$ obtained from ${F_1}^{(10)} + \{x,y,z\}$ by deleting $\{z, z'\}$ (respectively, $\{y, y'\}$) and adding $\{z_1, z_2\}$ (respectively, $\{y_1, y_2\}$) when $z' \in {F_1}^{(10)}$ (respectively, $y' \in {F_1}^{(10)}$). Therefore, $F^{(10)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(10)}| + |F_2^{(10)}| + |F_3^{(10)}| + 3 - 5 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 7.7. $(n_1, n_2, n_3) \equiv (4, 4, 6) \mod 7$.

By Claim 5, $wy_1 \notin E(G)$. Let $w' \in N(w) - \{v, c_1, x, y_2\}$. Let $F_1^{(11)} = A(G_1 - \{x, z, w, v, y_2, v_1, v_2, c_1\} + w'y)$, $F_2^{(11)} = A(G_2 - y_2)$, and $F_3^{(11)} = A(G_3 - \{w, v_2, c_1\})$. Then $|F_1^{(11)}| \geq \lceil(4(|G_1|-8)+3)/7\rceil$, $|F_2^{(11)}| \geq \lceil(4(|G_2|-1)+3)/7\rceil$, and $|F_3^{(11)}| \geq \lceil(4(|G_3|-3)+3)/7\rceil$. Now $G[F_1^{(11)} \cup F_2^{(11)} \cup F_3^{(11)} + \{x, w, v\} - (\{y_1, b_1\} \cap (F_1^{(11)} \triangle F_2^{(11)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(11)}| + |F_2^{(11)}| + |F_3^{(11)}| + 3 - 2 \geq \lceil(4n+3)/7\rceil$, a contradiction.

Subcase 7.8. $(n_1, n_2, n_3) \equiv (3, 4, 0) \mod 7$.

We claim that $|N(y_1) \cap N(z_1)| \leq 2$. Otherwise, there exist $d_1 \in N(y_1) \cap N(z_1)$ and subgraphs G'_1, G'_2, G'_3, G'_4 of G such that $G'_2 = G_2, G'_3 = G_3$, G'_4 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $zy_1d_1z_1z$ and containing $N(y_1) \cap N(z_1) - \{z\}$, and G'_1 is obtained from G by removing $G'_2 - \{y_1, y_2, b_1\}$, $G'_3 - \{w, c_1, v_2\}$ and $G'_4 - \{y_1, d_1, z_1\}$. Let $D_{14} = \{d_1\}$ and $D_{15} = \emptyset$. For $i = 14, 15$, let $F_1^{(i)} = A(G'_1 - \{x, y, z, w, v, y_1, y_2, v_2, c_1, z_1\} - D_i)$, $F_2^{(i)} = A(G'_2 - \{y_1, y_2\})$, $F_3^{(i)} = A(G'_3 - \{w, v_2, c_1\})$, and $F_4^{(i)} = A(G'_4 - \{y_1, z_1\} - D_i)$. Then $|F_1^{(i)}| \geq \lceil(4(|G'_1| - 10 - |D_i|) + 3)/7\rceil$, $|F_2^{(i)}| \geq \lceil(4(|G'_2| - 2) + 3)/7\rceil = \lceil(4(|G_2| - 2) + 3)/7\rceil = (4(|G_2| - 2) + 3)/7 + 3/7$, $|F_3^{(i)}| \geq \lceil(4(|G'_3| - 3) + 3)/7\rceil = \lceil(4(|G_3| - 3) + 3)/7\rceil = (4(|G'_3| - 3) + 3)/7 + 4/7$, and $|F_4^{(i)}| \geq \lceil(4(|G'_4| - 2 - |D_i|) + 3)/7\rceil$. Note $N(w) - \{y_1, x, v\} \subseteq V(G'_3)$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} \cup F_4^{(i)} + \{x, y, z, w\} - (\{b_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{d_1\} \cap (F_1^{(i)} \triangle F_4^{(i)}))]$ is an induced forest of size $|F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + |F_4^{(i)}| + 4 - 1 - (1 - |D_i|)$. By Lemma 2.2(1) (with $k = 1, a = |G'_1| - 10, a_1 = |G'_4| - 2, L = \{1, 2\}, b_1 = |G'_2| - 2, b_2 = |G'_3| - 3, c = 3$), $a(G) \geq \lceil(4n+3)/7\rceil$, a contradiction.

Let $F_1^{(12)} = A((G_1 - \{x, y, z, y_2, w, v, v_2, c_1\})/z_1y_1)$ with y' as the identification of $\{z_1, y_1\}$, $F_2^{(12)} = A(G_2 - y_2)$ and $F_3^{(12)} = A(G_3 - \{w, v_2, c_1\})$. Then $|F_1^{(12)}| \geq \lceil(4(|G_1| - 9) + 3)/7\rceil$, $|F_2^{(12)}| \geq \lceil(4(|G_2| - 1) + 3)/7\rceil$ and $|F_3^{(12)}| \geq \lceil(4(|G_3| - 3) + 3)/7\rceil$. Note $N(w) - \{y_2, x, v\} \subseteq V(G_3)$. Now $G[F_1^{(12)} \cup F_2^{(12)} \cup F_3^{(12)} + \{x, y, w, v\} - (\{y_1, b_1\} \cap (F_1^{(12)} \triangle F_2^{(12)}))]$ (if $y' \notin F_1^{(12)}$) or $G[(F_1^{(12)} - y') \cup F_2^{(12)} \cup F_3^{(12)} + \{x, y_1, z_1, w, v\} - (\{y_1, b_1\} \cap ((F_1^{(12)} + y_1) \triangle F_2^{(12)}))]$ (if $y' \in F_1^{(12)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(12)}| + |F_2^{(12)}| + |F_3^{(12)}| + 4 - 2 \geq \lceil(4n+3)/7\rceil$, a contradiction.

Subcase 7.9. $(n_1, n_2, n_3) \equiv (0, 4, 3) \mod 7$.

We claim that $|N(v_1) \cap N(v_2)| \leq 2$. Otherwise, there exist $e_1 \in N(v_1) \cap N(v_2)$ and subgraphs G'_1, G'_3, G'_4 such that $G'_3 = G_3$, G'_4 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $vv_1e_1v_2v$ and containing $N(v_1)\cap N(v_2)-\{v\}$, and G_1' is obtained from G by removing $G_3'-\{w,c_1,v_2\}$ and $G_4'-\{v_1,e_1,v_2\}$. Let $E_{16}=\{e_1\}$ and $E_{17}=\emptyset$. For $i=16,17$, let $F_1^{(i)}=A((G_1'-\{x,z,w,v,v_1,v_2\}-E_i)/z_1z_2)$ with z' as the identification of $\{z_1,z_2\}$, $F_3^{(i)}=A(G_3'-\{w,v_2\})$, and $F_4^{(i)}=A(G_4'-\{v_1,v_2\}-E_i)$. Then $|F_1^{(i)}|\geq\lceil(4(|G_1'|-7-|E_i|)+3)/7\rceil$, $|F_3^{(i)}|\geq\lceil(4(|G_3'|-2)+3)/7\rceil=\lceil(4(|G_3|-2)+3)/7\rceil=(4(|G_3|-2)+3)/7+4/7$, and $|F_4^{(i)}|\geq\lceil(4(|G_4'|-2-|E_i|)+3)/7\rceil$. Let $F=F_1^{(i)}\cup F_3^{(i)}\cup F_4^{(i)}+\{x,z,v\}-(\{c_1\}\cap(F_1^{(i)}\triangle F_3^{(i)}))-(\{e_1\}\cap(F_1^{(i)}\triangle F_4^{(i)}))$. Now $G[F]$ (if $z'\notin F_1^{(i)}$) or $G[F-\{z,z'\}+\{z_1,z_2\}]$ (if $z'\in F_1^{(i)}$) is an induced forest in G, showing $a(G)\geq|F_1^{(i)}|+|F_3^{(i)}|+|F_4^{(i)}|+3-1-(1-|E_i|)$. By Lemma 2.2(1) (with $k=1,a=|G_1'|-7,a_1=|G_4'|-2,L=\{1\},b_1=|G_3'|-2,c=2$), $a(G)\geq\lceil(4n+3)/7\rceil$, a contradiction.

Let $F_1^{(13)}=A((G_1-\{x,w,v,y_2,c_1\})/\{yz,v_1v_2\})$ with x' (respectively, v') as the identification of $\{y,z\}$ (respectively, $\{v_1,v_2\}$), $F_2^{(13)}=A(G_2-y_2)$, and $F_3^{(13)}=A(G_3-\{w,c_1\})$. Then $|F_1^{(13)}|\geq\lceil(4(|G_1|-7)+3)/7\rceil$, $|F_2^{(13)}|\geq\lceil(4(|G_2|-1)+3)/7\rceil$ and $|F_3^{(13)}|\geq\lceil(4(|G_3|-2)+3)/7\rceil$. Let $F^{(13)}:=G[\overline{F_1}^{(13)}\cup F_2^{(13)}\cup F_3^{(13)}-(\{y_1,b_1\}\cap(\overline{F_1}^{(13)}\triangle F_2^{(13)}))-(\{v_2\}\cap(\overline{F_1}^{(13)}\triangle F_3^{(13)}))]$, where $\overline{F_1}^{(13)}:={F_1}^{(13)}+\{x,w,v\}$ if $x',v'\notin F_1^{(13)}$, and $\overline{F_1}^{(13)}$ obtained from ${F_1}^{(13)}+\{x,w,v\}$ by deleting $\{x,x'\}$ (respectively, $\{v,v'\}$) and adding $\{y,z\}$ (respectively, $\{v_1,v_2\}$) when $x'\in{F_1}^{(13)}$ (respectively, $v'\in{F_1}^{(13)}$). Note $N(w)-\{y_2,x,v\}\subseteq V(G_3)$. Therefore, $F^{(13)}$ is an induced forest in G, showing $a(G)\geq|F_1^{(13)}|+|F_2^{(13)}|+|F_3^{(13)}|+3-3\geq\lceil(4n+3)/7\rceil$, a contradiction. □

6. Another forbidden configuration at a 3-vertex

In this section we prove that for any $x\in V_3$, $N(x)\cap V_i=\emptyset$ for some $i\in\{3,4,5\}$.

Lemma 6.1. *Let $x\in V_3$. Then $N(x)\cap V_3=\emptyset$, or $N(x)\cap V_4=\emptyset$, or $N(x)\cap V_5=\emptyset$.*

Proof. We begin the proof by assuming that $N(x)=\{w,y,z\}$ with $y\in V_3$, $z\in V_4$ and $w\in V_5$. Let $N(y)=\{y_1,x,z_1\}$, $N(z)=\{x,z_1,z_2,z_3\}$, and $N(w)=\{x,y_1,w_1,w_2,z_3\}$ where w_1 is co-facial with y_1.

Claim 1. $N(z_1)\cap N(z_3)=\{z\}$.

For, suppose $|N(z_1) \cap N(z_3)| \geq 2$. First, we claim that $N(z_1) \cap N(z_3) \cap N(w_2) = \emptyset$. Otherwise, there exist $a_1 \in N(z_1) \cap N(z_3)$ and subgraphs G_1, G_2, G_4 of G such that G_2 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $zz_1a_1z_3z$ and containing $N(z_1) \cap N(z_3) - \{z\}$, G_4 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $ww_2a_1z_3w$, and G_1 is obtained from G by removing $G_2 - \{z, z_1, a_1, z_3\}$ and $G_4 - \{w, w_2, a_1, z_3\}$. Let $F_1^{(1)} = A(G_1 - \{w, x, y, z, y_1, z_1, z_3, a_1, w_2\})$, $F_2^{(1)} = A(G_2 - \{z_1, z, z_3, a_1\})$, and $F_4^{(1)} = A(G_4 - \{w, z_3, a_1, w_2\})$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1| - 9) + 3)/7 \rceil$, $|F_2^{(1)}| \geq \lceil (4(|G_2| - 4) + 3)/7 \rceil$, and $|F_4^{(1)}| \geq \lceil (4(|G_4| - 4) + 3)/7 \rceil$. Now $G[F_1^{(1)} \cup F_2^{(1)} \cup F_4^{(1)} + \{w, x, y, z\}]$ is an induced forest in G, showing that $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + |F_4^{(1)}| + 4$. By Lemma 2.2(7) (with $k = 3, a_1 = |G_1| - 9, a_2 = |G_2| - 4, a_3 = |G_4| - 4, c = 4$), $a(G) \geq \lceil (4n+3)/7 \rceil$ unless $(4(|G_1|-9)+3, 4(|G_2|-4)+3), 4(|G_4|-4)+3) \equiv (0,0,0), (0,6,0), (0,0,6), (6,0,0) \mod 7$. In first three cases, let $F_1^{(2)} = A(G_1 - \{w, x, y, z, z_1, z_3, a_1\})$, $F_2^{(2)} = A(G_2 - \{z_1, z, z_3, a_1\})$, and $F_4^{(2)} = A(G_4 - \{w, z_3, a_1\})$. Then $|F_1^{(2)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$, $|F_2^{(2)}| \geq \lceil (4(|G_2| - 4) + 3)/7 \rceil$, and $|F_4^{(2)}| \geq \lceil (4(|G_4| - 3) + 3)/7 \rceil$. Now $G[F_1^{(2)} \cup F_2^{(2)} \cup F_4^{(2)} + \{x, y, z\} - \{w_2\} \cap (F_1^{(2)} \triangle F_4^{(2)})]$ is an induced forest in G, showing $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + |F_4^{(2)}| + 3 - 1 \geq \lceil (4n+3)/7 \rceil$, a contradiction. Now, assume $(4(|G_1|-9)+3, 4(|G_2|-4)+3), 4(|G_4|-4)+3) \equiv (6,0,0) \mod 7$. If $y_1a_1 \notin E(G)$, let $F_1^{(3)} = A(G_1 - \{w, x, y, z_1, z_3\} + y_1a_1)$, $F_2^{(3)} = A(G_2 - \{z_1, z_3\})$, and $F_4^{(3)} = A(G_4 - \{w, z_3\})$. Then $|F_1^{(3)}| \geq \lceil (4(|G_1| - 5) + 3)/7 \rceil$, $|F_2^{(3)}| \geq \lceil (4(|G_2|-2)+3)/7 \rceil$ and $|F_4^{(3)}| \geq \lceil (4(|G_4|-2)+3)/7 \rceil$. Now $G[F_1^{(3)} \cup F_2^{(3)} \cup F_4^{(3)} + \{x, y\} - (\{w_2, a_1\} \cap (F_1^{(3)} \triangle F_4^{(3)})) - (\{z, a_1\} \cap (F_1^{(3)} \triangle F_2^{(3)}))]$ an induced forest in G, showing $a(G) \geq |F_1^{(3)}| + |F_2^{(3)}| + |F_4^{(3)}| + 2 - 4 \geq \lceil (4n+3)/7 \rceil$, a contradiction. So $y_1a_1 \in E(G)$. Then there exist subgraphs G_1', G_2', G_4', G_5' of G such that $G_2' = G_2, G_4' = G_4$, G_5' is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $yy_1a_1z_1y$, and G_1 is obtained from G by removing $G_2' - \{z, z_1, a_1, z_3\}$, $G_4' - \{w, w_2, a_1, z_3\}$, and $G_5' - \{y, y_1, a_1, z_1\}$. Let $F_1^{(4)} = A(G_1' - \{w, x, y, z, y_1, z_1, z_3, w_2, a_1\})$, $F_2^{(4)} = A(G_2' - \{z_1, z, z_3, a_1\})$, $F_4^{(4)} = A(G_4' - \{w, z_3, a_1, w_2\})$ and $F_5^{(4)} = A(G_5' - \{y_1, y, z_1, a_1\})$. Then $|F_1^{(4)}| \geq \lceil (4(|G_1'|-9)+3)/7 \rceil$, $|F_2^{(4)}| \geq \lceil (4(|G_2'| - 2) + 3)/7 \rceil$, $|F_4^{(4)}| \geq \lceil (4(|G_4'| - 2) + 3)/7 \rceil$, and $|F_5^{(4)}| \geq \lceil (4(|G_5'| - 4) + 3)/7 \rceil$. Now $G[F_1^{(4)} \cup F_2^{(4)} \cup F_4^{(4)} \cup F_5^{(4)} + \{w, x, y, z\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(4)}| + |F_2^{(4)}| + |F_4^{(4)}| + |F_5^{(4)}| + 4 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Secondly, we claim that $N(z_1) \cap N(z_3) \cap N(y_1) = \emptyset$. For otherwise, there exist $a_1 \in N(z_1) \cap N(z_3)$ and subgraphs G_1, G_2, G_5 of G such that G_2 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $zz_1a_1z_3z$ and containing $N(z_1) \cap N(z_3) - \{z\}$, G_5 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $yy_1a_1z_1y$, and G_1 is obtained from G by removing $G_2 - \{z, z_1, a_1, z_3\}$ and $G_5 - \{y, y_1, a_1, z_1\}$. Let $F_1^{(1)} = A(G_1 - \{x, y, z, y_1, z_1, z_3, a_1\})$, $F_2^{(1)} = A(G_2 - \{z_1, z, z_3, a_1\})$ and $F_5^{(1)} = A(G_5 - \{y_1, y, z_1, a_1\})$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$, $|F_2^{(1)}| \geq \lceil (4(|G_2| - 4) + 3)/7 \rceil$ and $|F_5^{(1)}| \geq \lceil (4(|G_5| - 4) + 3)/7 \rceil$. Now $G[F_1^{(1)} \cup F_2^{(1)} \cup F_5^{(1)} + \{x, y, z\}]$ is an induced forest in G, showing that $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + |F_5^{(1)}| + 3$. By Lemma 2.2(6) (with $k = 3, a_1 = |G_1| - 7, a_2 = |G_2| - 4, a_3 = |G_5| - 4, c = 3$), $(4(|G_1| - 7) + 3, 4(|G_2| - 4) + 3), 4(|G_5| - 4) + 3) \equiv (0, 0, 0) \mod 7$. Let $F_1^{(2)} = A(G_1 - \{w, x, y, z, z_1\})$, $F_2^{(2)} = A(G_2 - \{z_1, z\})$, and $F_5^{(2)} = A(G_5 - \{y, z_1\})$. Then $|F_1^{(2)}| \geq \lceil (4(|G_1| - 5) + 3)/7 \rceil$, $|F_2^{(2)}| \geq \lceil (4(|G_2| - 2) + 3)/7 \rceil$, and $|F_5^{(2)}| \geq \lceil (4(|G_5| - 2) + 3)/7 \rceil$. Now $G[F_1^{(2)} \cup F_2^{(2)} \cup F_5^{(2)} + \{x, y\} - (\{z_3, a_1\} \cap (F_1^{(2)} \triangle F_2^{(2)})) - (\{y_1, a_1\} \cap (F_1^{(2)} \triangle F_5^{(2)}))]$ is an induced forest in G, showing that $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + |F_5^{(2)}| + 2 - 4 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Thirdly, we claim that $|N(y_1) \cap N(w_1)| \leq 2$. For otherwise, there exist $b_1 \in N(y_1) \cap N(w_1)$, $a_1 \in N(z_1) \cap N(z_3)$ and subgraphs G_1, G_2, G_3 of G such that G_2 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $zz_1a_1z_3z$ and containing $N(z_1) \cap N(z_3) - \{z\}$, G_3 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $wy_1b_1w_1w$ and containing $N(y_1) \cap N(w_1) - \{w\}$, and G_1 is obtained from G by removing $G_2 - \{z, z_1, a_1, z_3\}$ and $G_3 - \{y_1, b_1, w_1\}$. Let $B_1 = B_3 = \overline{B_2} = \overline{B_4} = \{b_1\}$ and $B_2 = \overline{B_1} = B_4 = \overline{B_3} = \emptyset$. For $i = 1, 2$, let $F_1^{(i)} = A(G_1 - \{w, x, y, z, y_1, z_1, z_3, a_1, w_1\} - B_i)$, $F_2^{(i)} = A(G_2 - \{z_1, z, z_3, a_1\})$, and $F_3^{(i)} = A(G_3 - \{y_1, w_1\} - B_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 9 - |B_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2| - 4) + 3)/7 \rceil$, and $|F_3^{(i)}| \geq \lceil (4(|G_3| - 2 - |B_i|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} \cup \{w, x, y, z\} - \{b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)})]$ is an induced forest in G, showing that $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - |\overline{B_i}|$. For $j = 3, 4$, let $F_1^{(j)} = A(G_1 - \{w, x, y, z, y_1, z_1, z_3, w_1\} - B_j + w_2a_1)$, $F_2^{(j)} = A(G_2 - \{z_1, z, z_3\})$, and $F_3^{(j)} = A(G_3 - \{y_1, w_1\} - B_1)$. Then $|F_1^{(j)}| \geq \lceil (4(|G_1| - 8 - |B_1|) + 3)/7 \rceil$, $|F_2^{(j)}| \geq \lceil (4(|G_2| - 3) + 3)/7 \rceil$, and $|F_3^{(j)}| \geq \lceil (4(|G_3| - 2 - |B_j|) + 3)/7 \rceil$. Now $G[F_1^{(j)} \cup F_2^{(j)} \cup F_3^{(j)} + \{w, x, y, z\} - (\{b_1\} \cap (F_1^{(j)} \triangle F_3^{(j)})) - (\{a_1\} \cap (F_1^{(j)} \triangle F_2^{(j)}))]$ is an induced forest in G, showing

$a(G) \geq |F_1^{(j)}| + |F_2^{(j)}| + |F_3^{(j)}| + 4 - 1 - |\overline{B_1}|$. By Lemma 2.2(1) (with $k = 1$), $a(G) \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Since $|N(z_1) \cap N(z_3)| \geq 2$, there exist $a_1 \in N(z_1) \cap N(z_3)$ and subgraphs G_1, G_2 of G such that G_2 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $zz_1a_1z_3z$ containing $N(z_1) \cap N(z_3) - \{z\}$, and G_1 is obtained from G by removing $G_2 - \{z, z_1, a_1, z_3\}$. Let $F_1^{(1)} = A(G_1 - \{w, x, y, z, z_1, z_3, a_1\}/w_1y_1)$ with w' as the identification of w_1 and y_1, and $F_2^{(1)} = A(G_2 - \{z_1, z, z_3, a_1\})$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1|-8)+3)/7 \rceil$, and $|F_2^{(1)}| \geq \lceil (4(|G_2|-4)+3)/7 \rceil$. Now $G[F_1^{(1)} \cup F_2^{(1)} + \{w, x, y, z\}]$ (if $w' \notin F_1^{(1)}$) or $G[(F_1^{(1)} - w') \cup F_2^{(1)} + \{w_1, y_1, x, y, z\}]$ (if $w' \in F_1^{(1)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 4$. By Lemma 2.2(6) (with $k = 2, a_1 = |G_1| - 8, a_2 = |G_2| - 4, c = 4$), $(4(|G_1| - 8) + 3, 4(|G_2| - 4) + 3)) \equiv (0, 0) \mod 7$. Let $F_1^{(2)} = A(G_1 - \{w, x, y, z, z_1, z_3\} + y_1a_1)$, and $F_2^{(2)} = A(G_2 - \{z_1, z, z_3\})$. Then $|F_1^{(2)}| \geq \lceil (4(|G_1|-6)+3)/7 \rceil$, and $|F_2^{(2)}| \geq \lceil (4(|G_2|-3)+3)/7 \rceil$. Now $G[F_1^{(2)} \cup F_2^{(2)} + \{x, y, z\} - (\{a_1\} \cap (F_1^{(2)} \triangle F_2^{(2)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + 3 - 1 \geq \lceil (4n+3)/7 \rceil$. This completes the proof of Claim 1.

Claim 2. $wz_2 \notin E(G)$.

Otherwise, $wz_2 \in E(G)$, there exists a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{w, x, z, z_2\}$, $y \in V(G_1)$, and $z_3 \in V(G_2)$. Let $F_1^{(1)} = A(G_1 - \{w, x, z, z_2, y, z_1\})$, and $F_2^{(1)} = A(G_2 - \{w, x, z, z_2\})$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil$, and $|F_2^{(1)}| \geq \lceil (4(|G_2| - 4) + 3)/7 \rceil$. Now $G[F_1^{(1)} \cup F_2^{(1)} \cup \{x, y, z\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction. This completes the proof of Claim 2.

Claim 3. $wz_1 \notin E(G)$.

Otherwise, $wz_1 \in E(G)$, there exists a separation (G_1, G_2) in G such that $V(G_1 \cap G_2) = \{w, x, y, z_1\}$, $y_1 \in V(G_1)$, and $z \in V(G_2)$. For $i = 1, 2$, let $F_i^{(1)} = A(G_i - \{w, x, y, z_1\})$; so $|F_i^{(1)}| \geq \lceil (4(|G_i| - 4) + 3)/7 \rceil$. Now $G[F_1^{(1)} \cup F_2^{(1)} + \{x, y\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 2 \geq \lceil (4n + 3)/7 \rceil$, a contradiction. This completes the proof of Claim 3.

We now distinguish several cases.

Case 1. $|N(y_1) \cap N(w_1)| \leq 2$, $|N(z_1) \cap N(z_2)| \leq 2$ and $|N(z_2) \cap N(z_3)| \leq 2$.

Let $F' = A(G - \{w, x, y, z\}/\{y_1w_1, z_1z_2z_3\})$ with w' (respectively, z') as identifications of $\{y_1, w_1\}$ (respectively, $\{z_1, z_2, z_3\}$). Then $|F'| \geq \lceil (4(n -$

$7)+3)/7\rceil$. Let $F = F' + \{w, x, y, z\}$ if $w', z' \notin F'$; $F = F' + \{w_1, y_1, x, y, z\}$ if $z' \notin F', w' \in F'$; $F = F' + \{x, y, z_1, z_2, z_3\} - \{z'\}$ if $w' \notin F', z' \in F'$; and $F = F' + \{w_1, y_1, x, z_1, z_2, z_3\} - \{w', z'\}$ if $w', z' \in F'$. Therefore, $G[F]$ is an induced forest in G, giving $a(G) \geq |F'| + 4 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Case 2. $|N(y_1) \cap N(w_1)| \geq 3$, $|N(z_1) \cap N(z_2)| \leq 2$ and $|N(z_2) \cap N(z_3)| \leq 2$.

There exist $b_1 \in N(y_1) \cap N(w_1)$ and a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{w_1, y_1, b_1\}$, $x \in V(G_1)$ and $N(y_1) \cap N(w_1) - \{w\} \subseteq V(G_2)$. Let $B_1 = \overline{B_2} = \{b_1\}$ and $B_2 = \overline{B_1} = \emptyset$. For $i = 1, 2$, let $F_1^{(i)} = A((G_1 - \{w, x, y, z, y_1, w_1\} - B_i)/\{z_1 z_2 z_3\})$ with z' as the identification of $\{z_1, z_2, z_3\}$, and $F_2^{(i)} = A(G_2 - \{y_1, w_1\} - B_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |B_i|) + 3)/7 \rceil$, and $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |B_i|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} + \{w, x, y, z\} - (\{b_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ (if $z' \notin F_1^{(i)}$) or $G[(F_1^{(i)} - z') \cup F_2^{(i)} + \{x, y, z_1, z_2, z_3\} - (\{b_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ (if $z' \in F_1^{(i)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + 4 - |\overline{B_i}|$. By Lemma 2.2(2) (with $a = |G_1| - 8, a_1 = |G_2| - 2, c = 4$), $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (4, 0), (0, 4) \mod 7$.

Subcase 2.1. $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (4, 0) \mod 7$.

Let $F_1^{(1)} = A((G_1 - \{w, x, y, z\})/\{y_1 w_1, z_1 z_2 z_3\})$ with w' (respectively z') as the identification of $\{y_1, w_1\}$ (respectively $\{z_1, z_2, z_3\}$), and $F_2^{(1)} = A(G_2)$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$, and $|F_2^{(1)}| \geq \lceil (4|G_2| + 3)/7 \rceil$. Let $F^{(1)} := \overline{F_1}^{(1)} \cup F_2^{(1)} - \{y_1, w_1, b_1\} \cap (\overline{F_1}^{(1)} \triangle F_2^{(1)})$ where $\overline{F_1}^{(1)} = F_1^{(1)} + \{w, x, y, z\}$ if $w', z' \notin F_1^{(1)}$; $\overline{F_1}^{(1)} = F_1^{(1)} + \{w_1, y_1, x, y, z\}$ if $z' \notin F_1^{(1)}, w' \in F_1^{(1)}$; $\overline{F_1}^{(1)} = F_1^{(1)} + \{x, y, z_1, z_2, z_3\} - \{z'\}$ if $w' \notin F_1^{(1)}, z' \in F_1^{(1)}$; and $\overline{F_1}^{(1)} = F_1^{(1)} + \{w_1, y_1, x, z_1, z_2, z_3\} - \{w', z'\}$ if $w', z' \in F_1^{(1)}$. Therefore, $G[F^{(1)}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 4 - 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 2.2. $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (0, 4) \mod 7$.

Let $F_1^{(2)} = A(G_1 - \{y_1, x, y, z\}/\{z_1 z_2 z_3\})$ with z' as the identification of $\{z_1, z_2, z_3\}$, and $F_2^{(2)} = A(G_2 - y_1)$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil$ and $|F_1^{(2)}| \geq \lceil (4(|G_2| - 1) + 3)/7 \rceil$. Now $G[F_1^{(2)} \cup F_2^{(2)} + \{x, y, z\} - (\{w_1, b_1\} \cap (F_1^{(2)} \triangle F_2^{(2)}))]$ (if $z' \notin F_1^{(2)}$) or $G[(F_1^{(2)} - z') \cup F_2^{(2)} + \{x, z_1, z_2, z_3\} - (\{w_1, b_1\} \cap (F_1^{(2)} \triangle F_2^{(2)}))]$ (if $z' \in F_1^{(2)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + 3 - 2 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Case 3. $|N(y_1) \cap N(w_1)| \leq 2$, $|N(z_1) \cap N(z_2)| > 2$, $|N(z_2) \cap N(z_3)| \leq 2$.

There exist $a_1 \in N(z_1) \cap N(z_2)$ and a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{z_1, z_2, a_1\}$, $x \in V(G_1)$, and $N(z_1) \cap N(z_2) - \{z\} \subseteq V(G_2)$.

Let $A_1 = \overline{A_2} = \{a_1\}$ and $A_2 = \overline{A_1} = \emptyset$. For $i = 1, 2$, let $F_1^{(i)} = A((G_1 - \{w, x, y, z, z_1, z_2, z_3\} - A_i)/y_1w_1)$ with w' as the identification of $\{y_1, w_1\}$, and $F_2^{(i)} = A(G_2 - \{z_1, z_2\} - A_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |A_i|) + 3)/7 \rceil$ and $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |A_i|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} + \{w, x, y, z\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ (if $w' \notin F_1^{(i)}$) or $G[(F_1^{(i)} - w') \cup F_2^{(i)} + \{w_1, y_1, x, y, z\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ (if $w' \in F_1^{(i)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + 4 - |\overline{A_i}|$. By Lemma 2.2(2) (with $a = |G_1| - 8, a_1 = |G_2| - 2, c = 4$), $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (4, 0), (0, 4) \mod 7$.

Subcase 3.1. $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (4, 0) \mod 7$.

Let $F_1^{(3)} = A((G_1 - \{w, x, y, z\})/\{y_1w_1, z_1z_2z_3\})$ with w' (respectively, z') as the identification of $\{y_1, w_1\}$ (respectively, $\{z_1, z_2, z_3\}$), and $F_2^{(3)} = A(G_2)$. Then $|F_1^{(3)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$, and $|F_2^{(3)}| \geq \lceil (4|G_2| + 3)/7 \rceil$. Let $F^{(3)} := \overline{F_1}^{(3)} \cup F_2^{(3)} - (\{z_1, z_2, a_1\} \cap (\overline{F_1}^{(3)} \triangle F_2^{(3)}))$ where $\overline{F_1}^{(3)} = F_1^{(3)} + \{w, x, y, z\}$ if $w', z' \notin F_1^{(3)}$; $\overline{F_1}^{(3)} = F_1^{(3)} + \{w_1, y_1, x, y, z\}$ if $z' \notin F_1^{(3)}, w' \in F_1^{(3)}$; $\overline{F_1}^{(3)} = F_1^{(3)} + \{x, y, z_1, z_2, z_3\} - \{z'\}$ if $w' \notin F_1^{(3)}, z' \in F_1^{(3)}$; and $\overline{F_1}^{(3)} = F_1^{(3)} + \{w_1, y_1, x, z_1, z_2, z_3\} - \{w', z'\}$ if $w', z' \in F_1^{(3)}$. Therefore, $G[F^{(3)}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(3)}| + |F_2^{(3)}| + 4 - 3 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Subcase 3.2. $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (0, 4) \mod 7$.

Let $F_1^{(4)} = A((G_1 - \{z_1, x, y, z, w\})/z_2z_3)$ with z' as the identification of $\{z_2, z_3\}$, and $F_2^{(4)} = A(G_2 - z_1)$. Then $|F_1^{(4)}| \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil$, and $|F_2^{(4)}| \geq \lceil (4(|G_2| - 1) + 3)/7 \rceil$. Now $G[F_1^{(4)} \cup F_2^{(4)} + \{x, y, z\} - (\{z_2, a_1\} \cap (F_1^{(4)} \triangle F_2^{(4)}))]$ (if $z' \notin F_1^{(2)}$) or $G[(F_1^{(4)} - z') \cup F_2^{(4)} + \{x, y, z_2, z_3\} - (\{z_2, a_1\} \cap ((F_1^{(4)} \cup \{z_2\}) \triangle F_2^{(4)}))]$ (if $z' \in F_1^{(2)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + 3 - 2 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Case 4. $|N(y_1) \cap N(w_1)| \leq 2$, $|N(z_1) \cap N(z_2)| \leq 2$, $|N(z_2) \cap N(z_3)| > 2$.

There exist $c_1 \in N(z_2) \cap N(z_3)$ and a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{z_2, z_3, c_1\}$, $x \in V(G_1)$, and $N(z_2) \cap N(z_3) - \{z\} \subseteq V(G_2)$. Let $C_1 = \overline{C_2} = \{c_1\}$ and $C_2 = \overline{C_1} = \emptyset$. For $i = 1, 2$, let $F_1^{(i)} = A((G_1 - \{w, x, y, z, z_1, z_2, z_3\} - C_i)/y_1w_1)$ with w' as the identification of $\{y_1, w_1\}$, and $F_2^{(i)} = A(G_2 - \{z_2, z_3\} - C_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |C_i|) + 3)/7 \rceil$, and $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |C_i|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} + \{w, x, y, z\} - (\{c_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ (if $w' \notin F_1^{(i)}$) or $G[(F_1^{(i)} - w') \cup F_2^{(i)} \cup \{w_1, y_1, x, y, z\} - (\{c_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ (if $w' \in F_1^{(i)}$) is an induced forest in G, showing $a(G) \geq$

$|F_1^{(i)}|+|F_2^{(i)}|+4-\overline{C_i}$. By Lemma 2.2(2), $(4(|G_1|-8)+3, 4(|G_2|-2)+3) \equiv (4,0), (0,4) \mod 7$.

Subcase 4.1. $(4(|G_1|-8)+3, 4(|G_2|-2)+3) \equiv (4,0) \mod 7$.

Let $F_1^{(3)} = A((G_1-\{w,x,y,z\})/\{y_1w_1, z_1z_2z_3\})$ with w' (respectively z') as the identification of $\{y_1,w_1\}$ (respectively $\{z_1,z_2,z_3\}$) and $F_2^{(3)} = A(G_2)$. Then $|F_1^{(3)}| \geq \lceil (4(|G_1|-7)+3)/7 \rceil$ and $|F_2^{(3)}| \geq \lceil (4|G_2|+3)/7 \rceil$. Let $F^{(3)} := \overline{F_1}^{(3)} \cup F_2^{(3)} - \{z_2,z_3,c_1\} \cap (\overline{F_1}^{(3)} \triangle F_2^{(3)})$ where $\overline{F_1}^{(3)} = F_1^{(3)} + \{w,x,y,z\}$ if $w',z' \notin F_1^{(3)}$; $\overline{F_1}^{(3)} = F_1^{(3)} + \{w_1,y_1,x,y,z\}$ if $z' \notin F_1^{(3)}, w' \in F_1^{(3)}$; $\overline{F_1}^{(3)} = F_1^{(3)} + \{x,y,z_1,z_2,z_3\} - \{z'\}$ if $w' \notin F_1^{(3)}, z' \in F_1^{(3)}$; and $\overline{F_1}^{(3)} = F_1^{(3)} + \{w_1,y_1,x,z_1,z_2,z_3\} - \{w',z'\}$ if $w',z' \in F_1^{(3)}$. Therefore, $G[F^{(3)}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(3)}|+|F_2^{(3)}|+4-3 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 4.2. $(4(|G_1|-8)+3, 4(|G_2|-2)+3) \equiv (0,4) \mod 7$.

Let $F_1^{(4)} = A((G_1-\{y_1,x,y,z,z_3\})/z_1z_2+wz')$ with z' as the identification of $\{z_1,z_2\}$, and $F_2^{(4)} = A(G_2-z_3)$. Then $|F_1^{(4)}| \geq \lceil (4(|G_1|-6)+3)/7 \rceil$, and $|F_1^{(4)}| \geq \lceil (4(|G_2|-1)+3)/7 \rceil$. Now $G[F_1^{(4)} \cup F_2^{(4)} + \{x,y,z\} - (\{z_2,c_1\} \cap (F_1^{(4)} \triangle F_2^{(4)}))]$ (if $z' \notin F_1^{(4)}$) or $G[(F_1^{(4)}-z') \cup F_2^{(4)} + \{x,y,z_1,z_2\} - (\{z_2,c_1\} \cap ((F_1^{(4)} \cup \{z_2\}) \triangle F_2^{(4)}))]$ (if $z' \in F_1^{(4)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(4)}|+|F_2^{(4)}|+3-2 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Case 5. $|N(z_1) \cap N(z_2)| > 2$, $|N(z_2) \cap N(z_3)| > 2$.

Subcase 5.1. $|N(y_1) \cap N(w_1)| \leq 2$.

There exist $a_1 \in N(z_1) \cap N(z_2)$, $c_1 \in N(z_2) \cap N(z_3)$ and subgraphs G_1, G_2, G_3 such that G_2 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $zz_1a_1z_2z$ and containing $N(z_1) \cap N(z_2) - \{z\}$, G_3 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $zz_3c_1z_2z$ and containing $N(z_3) \cap N(z_2) - \{z\}$, and G_1 is obtained from G by removing $G_2-\{z_1,a_1,z_2\}$ and $G_3-\{z_3,c_1,z_2\}$. Let $A_i = \{a_1\}$ if $i=1,2$ and $\emptyset$ if $i=3,4$ and $\overline{A_i} = \{a_1\}-A_i$. Let $C_i = \{c_1\}$ if $i=1,3$ and $\emptyset$ if $i=2,4$ and $\overline{C_i} = \{c_1\}-C_i$. For $i \in [4]$, let $F_1^{(i)} = A((G_1-\{w,x,y,z,z_1,z_2,z_3\}-A_i-C_i)/y_1w_1)$ with w' as the identification of $\{y_1,w_1\}$, $F_2^{(i)} = A(G_2-\{z_1,z_2\}-A_i)$, and $F_3^{(i)} = A(G_3-\{z_2,z_3\}-C_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1|-8-|A_i|-|C_i|)+3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2|-2-|A_i|)+3)/7 \rceil$, and $|F_3^{(i)}| \geq \lceil (4(|G_3|-2-|C_i|)+3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{w,x,y,z\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{c_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))]$

(if $w' \notin F_1^{(i)}$) or $G[(F_1^{(i)} - w') \cup F_2^{(i)} \cup F_3^{(i)} + \{w_1, y_1, x, y, z\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{c_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))]$ (if $w' \in F_1^{(i)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - |\overline{A_i}| - |\overline{C_i}|$. By Lemma 2.2(1) (with $k = 2, a = |G_1| - 8, a_1 = |G_2| - 2, a_2 = |G_3| - 2, L = \emptyset, c = 4$), $a(G) \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Subcase 5.2. $|N(y_1) \cap N(w_1)| \geq 3$.

There exist $a_1 \in N(z_1) \cap N(z_2)$, $b_1 \in N(y_1) \cap N(w_1)$, $c_1 \in N(z_2) \cap N(z_3)$ and subgraphs G_1, G_2, G_3, G_4 of G such that G_2 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $zz_1a_1z_2z$ and containing $N(z_1) \cap N(z_2) - \{z\}$, G_3 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $zz_3c_1z_2z$ and containing $N(z_3) \cap N(z_2) - \{z\}$, G_4 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $ww_1b_1y_1w$ and containing $N(y_1) \cap N(w_1) - \{w\}$, and G_1 is obtained from G by removing $G_2 - \{z_1, a_1, z_2\}$, $G_3 - \{z_3, c_1, z_2\}$ and $G_4 - \{w_1, b_1, y_1\}$. Let $A_i \subseteq \{a_1\}$ and $\overline{A_i} = \{a_1\} - A_i$. Let $B_i \subseteq \{b_1\}$ and $\overline{B_i} = \{b_1\} - B_i$. Let $C_i \subseteq \{c_1\}$ and $\overline{C_i} = \{c_1\} - C_i$. For each choice of A_i, B_i, C_i, let $F_1^{(i)} = A(G_1 - \{w, x, y, z, z_1, z_2, z_3, y_1, w_1\} - A_i - B_i - C_i)$, $F_2^{(i)} = A(G_2 - \{z_1, z_2\} - A_i)$, and $F_3^{(i)} = A(G_3 - \{z_2, z_3\} - C_i)$ and $F_4^{(i)} = A(G_4 - \{y_1, w_1\} - B_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 9 - |A_i| - |B_i| - |C_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |A_i|) + 3)/7 \rceil$, $|F_3^{(i)}| \geq \lceil (4(|G_3| - 2 - |C_i|) + 3)/7 \rceil$, and $|F_4^{(i)}| \geq \lceil (4(|G_4| - 2 - |B_i|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} \cup F_4^{(i)} + \{w, x, y, z\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{c_1\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{b_1\} \cap (F_1^{(i)} \triangle F_4^{(i)}))]$ is an induced forest in G, showing $|F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + |F_4^{(i)}| + 4 - |\overline{A_i}| - |\overline{B_i}| - |\overline{C_i}|$. By Lemma 2.2(1) (with $k = 3, a = |G_1| - 9, a_1 = |G_2| - 2, a_2 = |G_3| - 2, a_3 = |G_4| - 2, L = \emptyset, c = 4$), $a(G) \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Case 6. $|N(z_1) \cap N(z_2)| > 2$, $|N(z_2) \cap N(z_3)| \leq 2$ and $|N(y_1) \cap N(w_1)| > 2$.

There exist $a_1 \in N(z_1) \cap N(z_2), b_1 \in N(y_1) \cap N(w_1)$ and subgraphs G_1, G_2, G_3 of G such that G_2 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $zz_1a_1z_2z$ and containing $N(z_1) \cap N(z_2) - \{z\}$, G_3 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $ww_1b_1y_1w$ and containing $N(y_1) \cap N(w_1) - \{w\}$, and G_1 is obtained from G by removing $G_2 - \{z_1, a_1, z_2\}$ and $G_3 - \{w_1, b_1, y_1\}$. Let $A_i = \{a_1\}$ if $i = 1, 2, 5, 6, 9, 10$ and $A_i = \emptyset$ if $i = 3, 4, 7, 8, 11, 12$. Let $\overline{A_i} = \{a_1\} - A_i$. Let $B_i = \{b_1\}$ if $i = 1, 3, 5, 7$ and $B_i = \emptyset$ if $i = 2, 4, 6, 8$, and $\overline{B_i} = \{b_1\} - B_i$. For $i = 1, 2, 3, 4$, let $F_1^{(i)} = A(G_1 - \{w, x, y, z, z_1, z_2, z_3, y_1, w_1\} - A_i - B_i)$, $F_2^{(i)} = A(G_2 - \{z_1, z_2\} - A_i)$, and

$F_3^{(i)} = A(G_3 - \{y_1, w_1\} - B_i)$. Note $|F_1^{(i)}| \geq \lceil (4(|G_1| - 9 - |A_i| - |B_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |A_i|) + 3)/7 \rceil$, and $|F_3^{(i)}| \geq \lceil (4(|G_3| - 2 - |B_i|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{w, x, y, z\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - |\overline{A_i}| - |\overline{B_i}|$. By Lemma 2.2(5) (with $a = |G_1| - 9, a_1 = |G_2| - 2, a_2 = |G_3| - 2, c = 4$), $(n_1, n_2, n_3) := (4(|G_1| - 9) + 3, 4(|G_2| - 2) + 3, 4(|G_3| - 2) + 3) \equiv (0,0,0), (1,0,0), (4,0,3), (4,3,0), (3,0,4), (4,0,4), (3,4,0), (4,4,0), (1,6,0), (1,0,6), (0,3,4), (0,4,3), (0,4,4), (6,4,4), (4,4,6), (4,6,4) \mod 7$.

We claim that $4(|G_3| - 2) + 3 \not\equiv 4 \mod 7$. For, suppose that $4(|G_3| - 2) + 3 \equiv 4 \mod 7$. If $|N(w_2) \cap N(z_3)| \leq 2$, then for $i = 5, 7$, let $F_1^{(i)} = A((G_1 - \{w, x, y, z, z_1, z_2, y_1\} - A_i)/w_2 z_3)$ with w' as the identification of $\{w_2, z_3\}$, $F_2^{(i)} = A(G_2 - \{z_1, z_2\} - A_i)$, and $F_3^{(i)} = A(G_3 - \{y_1\})$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |A_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |A_i|) + 3)/7 \rceil$ and $|F_3^{(i)}| \geq \lceil (4(|G_3| - 1) + 3)/7 \rceil = (4(|G_3| - 1) + 3)/7 + 6/7$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{w, x, y, z\} - (\{w_1, b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ (if $w' \notin F_1^{(i)}$) or $G[(F_1^{(i)} - w') \cup F_2^{(i)} \cup F_3^{(i)} + \{w_2, z_3, x, y, z\} - (\{w_1, b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - 2 - (1 - |A_i|)$. By Lemma 2.2(1) (with $k = 1, a = |G_1| - 8, a_1 = |G_2| - 2, L = \{1\}, b_1 = |G_3| - 1, c = 2$), $a(G) \geq \lceil (4n + 3)/7 \rceil$, a contradiction. So $|N(w_2) \cap N(z_3)| > 2$. Then there exist $a_1 \in N(z_1) \cap N(z_2), b_1 \in N(y_1) \cap N(w_1), d_1 \in N(w_2) \cap N(z_3)$ and subgraphs G_1', G_2', G_3', G_4' of G such that $G_2' = G_2, G_3' = G_3$, G_4' is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $w w_2 d_1 z_3 w$ containing $N(w_2) \cap N(z_3) - \{w\}$, and G_1 is obtained from G by removing $G_2 - \{z_1, a_1, z_2\}$, $G_3 - \{w_1, b_1, y_1\}$ and $G_4 - \{w_2, d_1, z_3\}$. Let $D_i = \{d_1\}$ if $i = 9, 11$ and $D_i = \emptyset$ if $i = 10, 12$, and let $\overline{D_i} = \{d_1\} - D_i$. For $i = 9, 10, 11, 12$, let $F_1^{(i)} = A(G_1' - \{w, x, y, z, z_1, z_2, y_1, w_2, z_3\} - A_i - D_i)$, $F_2^{(i)} = A(G_2' - \{z_1, z_2\} - A_i)$, $F_3^{(i)} = A(G_3' - \{y_1\})$, and $F_4^{(i)} = A(G_4' - \{w_2, z_3\} - D_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1'| - 9 - |A_i| - |D_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2'| - 2 - |A_i|) + 3)/7 \rceil$, $|F_3^{(i)}| \geq \lceil (4(|G_3'| - 1) + 3)/7 \rceil = \lceil (4(|G_3| - 1) + 3)/7 \rceil = (4(|G_3| - 1) + 3)/7 + 6/7$, and $|F_4^{(i)}| \geq \lceil (4(|G_4'| - 2 - |D_i|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} \cup F_4^{(i)} + \{w, x, y, z\} - (\{w_1, b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{d_1\} \cap (F_1^{(i)} \triangle F_4^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + |F_4^{(i)}| + 4 - 2 - |\overline{A_i}| - |\overline{D_i}|$ By Lemma 2.2(1) (with $k = 2, a = |G_1'| - 9, a_1 = |G_2'| - 2, a_2 = |G_4'| - 2, L = \{1\}, b_1 = |G_3'| - 1, c = 2$) $a(G) \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Hence, $4(|G_3|-2)+3 \not\equiv 4 \mod 7$. Therefore, $(n_1,n_2,n_3) \equiv (0,0,0)$, $(1,0,0)$, $(4,0,3)$, $(4,3,0)$, $(3,4,0)$, $(4,4,0)$, $(1,6,0)$, $(1,0,6)$, $(0,4,3)$, $(4,4,6)$ mod 7.

Subcase 6.1. $(n_1,n_2,n_3) \equiv (0,0,0),(1,0,0) \mod 7$.

Let $F_1^{(1)} = A((G_1-\{w,x,y,z\})/\{y_1w_1, z_1z_2z_3\})$ with w' (respectively z') as the identifications of $\{y_1,w_1\}$ (respectively, $\{z_1,z_2,z_3\}$), $F_2^{(1)} = A(G_2)$, and $F_3^{(1)} = A(G_3)$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1|-7)+3)/7 \rceil$, $|F_2^{(1)}| \geq \lceil (4|G_2|+3)/7 \rceil$ and $|F_3^{(1)}| \geq \lceil (4|G_3|+3)/7 \rceil$. Let $F^{(1)} := \overline{F_1}^{(1)} \cup F_2^{(1)} \cup F_3^{(1)} - (\{z_1,z_2,a_1\} \cap (\overline{F_1}^{(1)} \triangle F_2^{(1)})) - (\{y_1,w_1,b_1\} \cap (\overline{F_1}^{(1)} \triangle F_3^{(1)}))$ where $\overline{F_1}^{(1)} = F_1^{(1)} + \{w,x,y,z\}$ if $w',z' \notin F_1^{(1)}$; $\overline{F_1}^{(1)} = F_1^{(1)} + \{w_1,y_1,x,y,z\}$ if $z' \notin F_1^{(1)}, w' \in F_1^{(1)}$; $\overline{F_1}^{(1)} = F_1^{(1)} + \{x,y,z_1,z_2,z_3\} - \{z'\}$ if $w' \notin F_1^{(1)}, z' \in F_1^{(1)}$; and $\overline{F_1}^{(1)} = F_1^{(1)} + \{w_1,y_1,x,z_1,z_2,z_3\} - \{w',z'\}$ if $w',z' \in F_1^{(1)}$. Therefore, $G[F^{(1)}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 4 - 6 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 6.2. $(n_1,n_2,n_3) \equiv (4,3,0),(4,4,0) \mod 7$ (respectively, $(1,6,0)$ mod 7).

Let $A_2 = \overline{A_3} = \emptyset$ and $A_3 = \overline{A_2} = \{a_1\}$. For $i=2$ (respectively, $i=3$), let $F_1^{(i)} = A((G_1 - \{w,x,y,z,z_1,z_2,z_3\} - A_i)/y_1w_1)$ with w' as the identification of $\{y_1,w_1\}$, $F_2^{(i)} = A(G_2 - \{z_1,z_2\} - A_i)$, and $F_3^{(i)} = A(G_3)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1|-8-|A_i|)+3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2|-2-|A_i|)+3)/7 \rceil$ and $|F_3^{(i)}| \geq \lceil (4|G_3|+3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{w,x,y,z\} - (\{y_1,w_1,b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ (if $w' \notin F_1^{(i)}$) or $G[(F_1^{(i)} - w') \cup F_2^{(i)} \cup F_3^{(i)} + \{w_1,y_1,x,y,z\} - (\{y_1,w_1,b_1\} \cap ((F_1^{(i)} \cup \{y_1,w_1\}) \triangle F_3^{(i)})) - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ (if $w' \in F_1^{(i)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - 3 - |\overline{A_i}| \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 6.3. $(n_1,n_2,n_3) \equiv (0,4,3) \mod 7$ (respectively, $(4,4,6) \mod 7$).

Let $B_4 = \overline{B_5} = \emptyset$ and $B_5 = \overline{B_4} = \{b_1\}$. For $i=4$ (respectively, $i=5$), let $F_1^{(i)} = A(G_1 - \{w,x,y,z_1,z_3,y_1,w_1\} - B_i + w_2z)$, $F_2^{(i)} = A(G_2 - \{z_1\})$, and $F_3^{(i)} = A(G_3 - \{y_1,w_1\} - B_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1|-7-|B_i|)+3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2|-1)+3)/7 \rceil$, and $|F_3^{(i)}| \geq \lceil (4(|G_3|-2-|B_i|)+3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{w,x,y\} - (\{z_2,a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 3 - 2 - |\overline{B_i}| \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 6.4. $(n_1,n_2,n_3) \equiv (4,0,3) \mod 7$ (respectively, $(1,0,6) \mod 7$).

Let $B_6 = \overline{B_7} = \emptyset$ and $B_7 = \overline{B_6} = \{b_1\}$. For $i = 6$ (resp. $i = 7$), let $F_1^{(i)} = A((G_1 - \{w, x, y, z, y_1, w_1\} - B_i)/\{z_1z_2z_3\})$, $F_2^{(i)} = A(G_2)$, and $F_3^{(i)} = A(G_3 - \{y_1, w_1\} - B_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |B_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4|G_2|+3)/7 \rceil$ and $|F_3^{(i)}| \geq \lceil (4(|G_3|-2-|B_i|)+3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{w, x, y, z\} - (\{z_1, z_2, a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))]$ (if $z' \notin F_1^{(i)}$) or $G[(F_1^{(i)} - z') \cup F_2^{(i)} \cup F_3^{(i)} + \{x, y, z_1, z_2, z_3\} - (\{z_1, z_2, a_1\} \cap ((F_1^{(i)} \cup \{z_1, z_2\}) \triangle F_2^{(i)})) - (\{b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))]$ (if $z' \in F_1^{(i)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(i)}|+|F_2^{(i)}|+|F_3^{(i)}|+4-3-|\overline{B_i}| \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 6.5. $(n_1, n_2, n_3) \equiv (3, 4, 0) \mod 7$.

Let $F_1^{(8)} = A(G_1 - \{w, x, y, z_1\} + zy_1)$, $F_2^{(8)} = A(G_2 - z_1)$ and $F_3^{(i)} = A(G_3)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 4) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2| - 1) + 3)/7 \rceil$ and $|F_3^{(i)}| \geq \lceil (4|G_3| + 3)/7 \rceil$. Now $G[F_1^{(8)} \cup F_2^{(8)} \cup F_3^{(8)} + \{x, y\} - (\{z_2, a_1\} \cap (F_1^{(8)} \triangle F_2^{(8)})) - (\{w_1, y_1, b_1\} \cap (F_1^{(8)} \triangle F_3^{(8)}))\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(8)}| + |F_2^{(8)}| + |F_3^{(8)}| + 2 - 5 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Case 7. $|N(z_1) \cap N(z_2)| \leq 2$, $|N(z_2) \cap N(z_3)| > 2$ and $|N(y_1) \cap N(w_1)| > 2$.

There exist $c_1 \in N(z_2) \cap N(z_3), b_1 \in N(y_1) \cap N(w_1)$ and subgraphs G_1, G_2, G_3 such that G_2 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $zz_2c_1z_3z$ and containing $N(z_2) \cap N(z_3) - \{z\}$, G_3 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $ww_1b_1y_1w$ and containing $N(y_1) \cap N(w_1) - \{w\}$, and G_1 is obtained from G by removing $G_2 - \{z_2, c_1, z_3\}$ and $G_3 - \{w_1, b_1, y_1\}$. Let $B_i = \{b_1\}$ if $i = 1, 2$ and $B_i = \emptyset$ if $i = 3, 4$ and $\overline{B_i} = \{b_1\} - B_i$. Let $C_i = \{c_1\}$ if $i = 1, 3$ and $C_i = \emptyset$ if $i = 2, 4$ and $\overline{C_i} = \{c_1\} - C_i$. For $i = 1, 2, 3, 4$, let $F_1^{(i)} = A(G_1 - \{w, x, y, z, z_1, z_2, z_3, y_1, w_1\} - B_i - C_i)$, $F_2^{(i)} = A(G_2 - \{z_2, z_3\} - C_i)$, and $F_3^{(i)} = A(G_3 - \{y_1, w_1\} - B_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 9 - |B_i| - |C_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |C_i|) + 3)/7 \rceil$ and $|F_3^{(i)}| \geq \lceil (4(|G_3| - 2 - |B_i|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{w, x, y, z\} - (\{c_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - |\overline{C_i}| - |\overline{B_i}|$. By Lemma 2.2(5) (with $a = |G_1| - 9, a_1 = |G_2| - 2, a_2 = |G_3| - 2, c = 4$), $(n_1, n_2, n_3) := (4(|G_1| - 9) + 3, 4(|G_2| - 2) + 3, 4(|G_3| - 2) + 3) \equiv (0, 0, 0)$, $(1, 0, 0)$, $(4, 0, 3)$, $(4, 3, 0)$, $(3, 0, 4)$, $(4, 0, 4)$, $(3, 4, 0)$, $(4, 4, 0)$, $(1, 6, 0)$, $(1, 0, 6)$, $(0, 3, 4)$, $(0, 4, 3)$, $(0, 4, 4)$, $(6, 4, 4)$, $(4, 4, 6)$, $(4, 6, 4) \mod 7$.

Subcase 7.1. $(n_1, n_2, n_3) \equiv (0, 0, 0), (1, 0, 0) \mod 7$.

Let $F_1^{(1)} = A((G_1 - \{w,x,y,z\})/\{y_1w_1, z_1z_2z_3\})$ with w' (respectively, z') as the identification of $\{y_1, w_1\}$ (respectively, $\{z_1, z_2, z_3\}$), $F_2^{(1)} = A(G_2)$, and $F_3^{(1)} = A(G_3)$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$, $|F_2^{(1)}| \geq \lceil (4|G_2| + 3)/7 \rceil$, and $|F_3^{(1)}| \geq \lceil (4|G_3| + 3)/7 \rceil$. Let $F^{(1)} := \overline{F_1}^{(1)} \cup F_2^{(1)} \cup F_3^{(1)} - \{z_3, z_2, c_1\} \cap (\overline{F_1}^{(1)} \triangle F_2^{(1)}) - \{y_1, w_1, b_1\} \cap (\overline{F_1}^{(1)} \triangle F_3^{(1)})$ where $\overline{F_1}^{(1)} = F_1^{(1)} + \{w, x, y, z\}$ if $w', z' \notin F_1^{(1)}$; $\overline{F_1}^{(1)} = F_1^{(1)} + \{w_1, y_1, x, y, z\}$ if $z' \notin F_1^{(1)}, w' \in F_1^{(1)}$; $\overline{F_1}^{(1)} = F_1^{(1)} + \{x, y, z_1, z_2, z_3\} - \{z'\}$ if $w' \notin F_1^{(1)}, z' \in F_1^{(1)}$; and $\overline{F_1}^{(1)} = F_1^{(1)} + \{w_1, y_1, x, z_1, z_2, z_3\} - \{w', z'\}$ if $w', z' \in F_1^{(1)}$. Therefore, $G[F^{(1)}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 4 - 6 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 7.2. $(n_1, n_2, n_3) \equiv (4, 4, 0), (3, 4, 0) \mod 7$.

Let $F_1^{(2)} = A((G_1 - \{w, x, y, z_1, z_3\})/y_1w_1 + \{w'z, w_2z\})$ with w' as the identification of $\{y_1, w_1\}$, $F_2^{(2)} = A(G_2 - \{z_3\})$, and $F_3^{(2)} = A(G_3)$. Then $|F_1^{(2)}| \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil$, $|F_2^{(2)}| \geq \lceil (4(|G_2| - 1) + 3)/7 \rceil$ and $|F_3^{(2)}| \geq \lceil (4|G_3| + 3)/7 \rceil$. Now $G[F_1^{(2)} \cup F_2^{(2)} \cup F_3^{(2)} + \{w, x, y\} - (\{z_2, c_1\} \cap (F_1^{(2)} \triangle F_2^{(2)})) - (\{w_1, y_1, b_1\} \cap (F_1^{(2)} \triangle F_3^{(2)}))]$ (if $w' \notin F_1^{(2)}$) or $G[(F_1^{(2)} - w') \cup F_2^{(2)} \cup F_3^{(2)} + \{w_1, y_1, x, y\} - (\{z_2, c_1\} \cap (F_1^{(2)} \triangle F_2^{(2)})) - (\{w_1, y_1, b_1\} \cap ((F_1^{(2)} \cup \{w_1, y_1\}) \triangle F_3^{(2)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + |F_3^{(2)}| + 3 - 5 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 7.3. $(n_1, n_2, n_3) \equiv (0, 4, 3), (0, 4, 4) \mod 7$ (respectively, $(4, 4, 6) \mod 7$).

Let $B_3 = \overline{B_4} = \emptyset$ and $B_4 = \overline{B_3} = \{b_1\}$. For $i = 3$ (respectively, $i = 4$), let $F_1^{(i)} = A(G_1 - \{w, x, y, z_1, z_3, y_1, w_1\} - B_i + w_2z)$, $F_2^{(i)} = A(G_2 - z_3)$, and $F_3^{(i)} = A(G_3 - \{y_1, w_1\} - B_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 7 - |B_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2| - 1) + 3)/7 \rceil$ and $|F_3^{(i)}| \geq \lceil (4(|G_3| - 2 - |B_i|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{w, x, y\} - (\{b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{z_2, c_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ is an induced forest in G, giving $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 3 - 2 - |\overline{B_i}| \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 7.4. $(n_1, n_2, n_3) \equiv (3, 0, 4), (4, 0, 4), (6, 4, 4), (4, 6, 4) \mod 7$.

Let $F_1^{(5)} = A(G_1 - \{x, y, z, z_1, z_3, y_1\} + z_2w)$, $F_2^{(5)} = A(G_2 - z_3)$, and $F_3^{(5)} = A(G_3 - \{y_1\})$. Then $|F_1^{(5)}| \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil$, $|F_2^{(5)}| \geq \lceil (4(|G_2| - 1) + 3)/7 \rceil$ and $|F_3^{(5)}| \geq \lceil (4(|G_3| - 1) + 3)/7 \rceil$. Now $G[F_1^{(5)} \cup F_2^{(5)} \cup F_3^{(5)} + \{z, x, y\} - (\{b_1, w_1\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{z_2, c_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(5)}| + |F_2^{(5)}| + |F_3^{(5)}| + 3 - 4 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 7.5. $(n_1, n_2, n_3) \equiv (4,3,0) \mod 7$ (respectively, $(1,6,0) \mod 7$). Let $C_6 = \overline{C_7} = \emptyset$ and $C_7 = \overline{C_6} = \{c_1\}$. For $i = 6$ (respectively, $i = 7$), let $F_1^{(i)} = A((G_1 - \{w,x,y,z,z_1,z_2,z_3\} - C_i)/y_1w_1)$ with w' as the identification of $\{y_1, w_1\}$, $F_2^{(i)} = A(G_2 - \{z_2, z_3\} - C_i)$, and $F_3^{(i)} = A(G_3)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |C_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |C_i|) + 3)/7 \rceil$ and $|F_3^{(i)}| \geq \lceil (4|G_3| + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{w,x,y,z\} - (\{y_1,b_1,w_1\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{c_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ (if $w' \notin F_1^{(i)}$) or $G[(F_1^{(i)} - w') \cup F_2^{(i)} \cup F_3^{(i)} + \{w_1, y_1, x, y, z\} - (\{y_1, b_1, w_1\} \cap ((F_1^{(i)} \cup \{w_1, y_1\}) \triangle F_3^{(i)})) - (\{c_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - 3 - |\overline{C_i}| \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 7.6. $(n_1, n_2, n_3) \equiv (4,0,3) \mod 7$ (respectively $(1,0,6) \mod 7$). Let $B_8 = \overline{B_9} = \emptyset$ and $B_9 = \overline{B_8} = \{b_1\}$. For $i = 8$ (respectively, $i = 9$), let $F_1^{(i)} = A((G_1 - \{w,x,y,z,y_1,w_1\} - B_i)/\{z_1z_2z_3\})$ with z' as the identification of $\{z_1, z_2, z_3\}$, $F_2^{(i)} = A(G_2)$, and $F_3^{(i)} = A(G_3 - \{y_1, w_1\} - B_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |B_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4|G_2| + 3)/7 \rceil$ and $|F_3^{(i)}| \geq \lceil (4(|G_3| - 2 - |B_i|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{w,x,y,z\} - (\{b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{z_2, z_3, c_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ (if $z' \notin F_1^{(i)}$) or $G[(F_1^{(i)} - z') \cup F_2^{(i)} \cup F_3^{(i)} + \{x, y, z_1, z_2, z_3\} - (\{b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{z_2, z_3, c_1\} \cap ((F_1^{(i)} \cup \{z_2, z_3\}) \triangle F_2^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - 3 - |\overline{B_i}| \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 7.7. $(n_1, n_2, n_3) \equiv (0,3,4) \mod 7$.

Let $F_1^{(10)} = A(G_1 - \{x,y,z,y_1,z_1,z_2,z_3\})$, $F_2^{(10)} = A(G_2 - \{z_2, z_3\})$, and $F_3^{(10)} = A(G_3 - \{y_1\})$. Then $|F_1^{(10)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$, $|F_2^{(10)}| \geq \lceil (4(|G_2| - 2) + 3)/7 \rceil$ and $|F_3^{(10)}| \geq \lceil (4(|G_3| - 1) + 3)/7 \rceil$. Now $F^{(10)} := G[F_1^{(10)} \cup F_2^{(10)} \cup F_3^{(10)} + \{z,x,y\} - (\{b_1, w_1\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{c_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(10)}| + |F_2^{(10)}| + |F_3^{(10)}| + 3 - 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction. □

7. Configurations around 5-vertices and 6-vertices

First, we define certain configurations around a 5-vertex or 6-vertex.

Definition 7.1. *Let x be a 5-vertex in G and x_1, x_2, x_3, x_4, x_5 be neighbors of x in cyclic order around x.*

(i) *x is of type **5-2-A** if $\{x_1, x_3\} \subseteq V_3, \{x_2, x_4, x_5\} \subseteq V_{\geq 4}$ such that if $N(x_1) = \{x_1', x_1'', x\}$ and $N(x_3) = \{x_3', x_3'', x\}$, then for $v \in \{x_1', x_1''\}$, either $v \in V_{\leq 4}$ or $R_{v,\{x_1\}} \neq \emptyset$; and for $u \in \{x_3', x_3''\}$, either $u \in V_{\leq 4}$ or $R_{u,\{x_3\}} \neq \emptyset$;*

(ii) *x is of type **5-2-B** if $\{x_1, x_3\} \subseteq V_3, \{x_2, x_4, x_5\} \subseteq V_{\geq 4}$ such that if $N(x_1) = \{x_1', x_1'', x\}$ and $N(x_3) = \{x_3', x_3'', x\}$, then for $v \in \{x_3', x_3''\}$, either $v \in V_{\leq 4}$ or $R_{v,\{x_3\}} \neq \emptyset$; and $x_1' \in V_{\geq 5}$ and $R_{x_1',\{x_1\}} = \emptyset$;*

(iii) *x is of type **5-2-C** if $\{x_1, x_3\} \subseteq V_3, \{x_2, x_4, x_5\} \subseteq V_{\geq 4}$ such that if $N(x_1) = \{x_1', x_1'', x\}$ and $N(x_3) = \{x_3', x_3'', x\}$, then $x_1' \in V_{\geq 5}, R_{x_1',\{x_1\}} = \emptyset$, $x_3' \in V_{\geq 5}$ and $R_{x_3',\{x_3\}} = \emptyset$;*

(iv) *x is of type **5-1-A** if $x_1 \in V_3$, $\{x_2, x_3, x_4, x_5\} \subseteq V_{\geq 4}$ such that if $N(x_1) = \{x_1', x_1'', x\}$, then for $v \in \{x_1', x_1''\}$, either $v \in V_{\leq 4}$ or $R_{v,\{x_1\}} \neq \emptyset$;*

(v) *x is of type **5-1-B** if $x_1 \in V_3$, $\{x_2, x_3, x_4, x_5\} \subseteq V_{\geq 4}$ such that if $N(x_1) = \{x_1', x_1'', x\}$, then $x_1' \in V_{\geq 5}$ and $R_{x_1',\{x_1\}} = \emptyset$;*

(vi) *x is of type **5-0** if $\{x_1, x_2, x_3, x_4, x_5\} \subseteq V_{\geq 4}$.*

Definition 7.2. *Let v be a 6-vertex in G and $v_1, v_2, v_3, v_4, v_5, v_6$ be neighbors of v in cyclic order around v.*

(i) *v is of type **6-3** if $\{v_1, v_3, v_5\} \subseteq V_3$ and $\{v_2, v_4, v_6\} \subseteq V_{\geq 4}$;*

(ii) *v is of type **6-2-A** if $\{v_1, v_3\} \subseteq V_3$ and $\{v_2, v_4, v_5, v_6\} \subseteq V_{\geq 4}$;*

(iii) *v is of type **6-2-B** if $\{v_1, v_4\} \subseteq V_3$ and $\{v_2, v_3, v_5, v_6\} \subseteq V_{\geq 4}$;*

(iv) *v is of type **6-1** if $\{v_1\} \subseteq V_3$ and $\{v_2, v_3, v_4, v_5, v_6\} \subseteq V_{\geq 4}$;*

(v) *v is of type **6-0** if $\{v_1, v_2, v_3, v_4, v_5, v_6\} \subseteq V_{\geq 4}$.*

Lemma 7.3. *The following configuration is impossible in G: x is a 5-vertex of type 5-2-B with neighbors x_1, y, x_3, z, x_2 in cyclic order around x, $\{y, z\} \subseteq V_3, x_1 \in V_4$, $N(z) = \{z_1, z_2\}$ with $\{z_1x_2, z_2x_3\} \subseteq E(G)$, $\{z_1, z_2\} \subseteq V_4$, and xx_2wx_1x forms a facial cycle where $w \in V_3$.*

Proof. Let $N(w) = \{x_1, w_1, x_2\}$, $N(z_1) = \{z, x_2, s_1, s_2\}$ and $N(z_2) = \{z, x_3, t,\ s_2\}$.

First, we claim that $|N(x_1) \cap N(y)| \leq 2$. For otherwise, suppose $N(x_1) \cap N(y) = \{x, p_1, p_2\}$. There exists a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{p_1, p_2\}$, $\{x, y, x_1\} \subseteq V(G_1)$, and $N(p_1) \cap N(p_2) - \{y, x_1\} \subseteq V(G_2)$. Let $F_1^{(1)} = A(G_1 - \{x, y, x_1, p_1, p_2\})$, and $F_2^{(1)} = A(G_2 - p_2)$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1| - 5) + 3)/7 \rceil$, and $|F_2^{(1)}| \geq \lceil (4(|G_2| - 1) + 3)/7 \rceil$. Now $G[F_1^{(1)} \cup F_2^{(1)} + \{x_1, y\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 2 \geq \lceil (4n + 3)/7 \rceil$, a contradiction. Thus, let $N(x_1) \cap N(y) = \{x, y_1\}$.

By Lemma 4.1, $z_2x_2 \notin E(G)$ and $z_1x_3 \notin E(G)$.

We also claim that $w_1z_2 \notin E(G)$. Otherwise, there exists a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{w_1, x_2, z, z_2\}$, $\{x, y, w, x_1\} \subseteq V(G_1)$, and $\{z_1, s_1, s_2\} \subseteq V(G_2)$. Let $F_1^{(3)} = A(G_1 - \{w_1, x_2, z, z_2, w, x, x_1, y, y_1\})$, and $F_2^{(3)} = A(G_2 - \{w_1, x_2, z, z_2, z_1, s_2\})$. Then $|F_1^{(3)}| \geq \lceil(4(|G_1| - 9) + 3)/7\rceil$, and $|F_2^{(3)}| \geq \lceil(4(|G_2| - 6) + 3)/7\rceil$. Now $G[F_1^{(3)} \cup F_2^{(3)} + \{z, z_1, z_2, w, x_1, y\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(3)}| + |F_2^{(3)}| + 6 \geq \lceil(4n+3)/7\rceil$, a contradiction.

We further claim that $s_1z_2 \notin E(G)$. Otherwise, there exists a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{s_1, z_1, z_2\}$, $\{x, y, w, x_1, z\} \subseteq V(G_1)$, and $s_2 \in V(G_2)$. Let $F_1^{(4)} = A(G_1 - \{s_1, z_1, z_2, z\})$, and $F_2^{(4)} = A(G_2 - \{s_1, z_1, z_2, s_2\})$. Then $|F_1^{(4)}| \geq \lceil(4(|G_1| - 4) + 3)/7\rceil$, and $|F_2^{(4)}| \geq \lceil(4(|G_2| - 4) + 3)/7\rceil$. Now $G[F_1^{(4)} \cup F_2^{(4)} + \{z_1, z_2\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(4)}| + |F_2^{(4)}| + 2$. By Lemma 2.2(8) (with $a = |G_1| - 4, a_1 = |G_2| - 4, c = 2$) $(4(|G_1| - 4) + 3, 4(|G_2| - 4) + 3) \equiv (0,0), (0,6), (0,5), (5,0), (6,6), (6,0) \mod 7$. Let $F_1^{(5)} = A(G_1 - \{s_1, z_1, z_2, x, z\})$, and $F_2^{(5)} = A(G_2 - \{s_1, z_1, z_2\})$. Then $|F_1^{(5)}| \geq \lceil(4(|G_1| - 5) + 3)/7\rceil$, and $|F_2^{(5)}| \geq \lceil(4(|G_2| - 3) + 3)/7\rceil$. Define $G[F_1^{(5)} \cup F_2^{(5)} + \{z, z_2\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(5)}| + |F_2^{(5)}| + 2 \geq \lceil(4n + 3)/7\rceil$, a contradiction.

Note that $s_1x \notin E(G)$. Otherwise, since G is simple, $s_1 \notin \{x_2, z\}$. $s_1 \notin \{x_1, y\}$ by Lemma 2.3 (G is a quadrangulation). $s_1 \neq x_3$ by second claim. Similarly, $tx \notin E(G)$.

We now distinguish several cases.

Case 1. $|N(w_1) \cap N(x_2)| \leq 2$ and $|N(s_1) \cap N(s_2)| \leq 2$.

Let $F' = A((G - \{w, x, z, z_1\})/\{x_1y, w_1x_2, s_1s_2\} + z_2u_2)$ with u_1 (respectively, u_2, u_3) as the identification of $\{x_1, y\}$ (respectively, $\{w_1, x_2\}, \{s_1, s_2\}$). Then $|F'| \geq \lceil(4(n-7)+3)/7\rceil$. Note $u_1 \in F'$ by Lemma 2.3 since $|N(u_1)| = 3$. Let $F = F' + \{x_1, y, z, z_1, w\} - \{u_1\}$ if $u_2, u_3 \notin F'$, and otherwise, F obtained from $F' + \{x_1, y, z, z_1, w\} - \{u_1\}$ by deleting $\{u_2, w\}$ (respectively, $\{u_3, z_1\}$) and adding $\{w_1, x_2\}$ (respectively, $\{s_1, s_2\}$) when $u_2 \in F'$ (respectively, $u_3 \in F'$). Therefore, $G[F]$ is an induced forest in G, showing $a(G) \geq |F'| + 4 \geq \lceil(4n + 3)/7\rceil$, a contradiction.

Case 2. $|N(w_1) \cap N(x_2)| \leq 2$ and $|N(s_1) \cap N(s_2)| > 2$.

There exist $a_1 \in N(s_1) \cap N(s_2)$ and a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{s_1, s_2, a_1\}$, $\{x, y, w, z, x_1, x_2, x_3\} \subseteq V(G_1)$, and $N(s_1) \cap N(s_2) - \{z_1\} \subseteq V(G_2)$. Let $A_1 = \{a_1\}$ and $A_2 = \emptyset$. For $i = 1, 2$, let

$F_1^{(i)} = A((G_1 - \{w, x, z, z_1, s_1, s_2\} - A_i)/\{x_1y, w_1x_2\} + u_2z_2)$ with u_1 (respectively, u_2) as the identification of $\{x_1, y\}$ (respectively, $\{w_1, x_2\}$), and $F_2^{(i)} = A(G_2 - \{s_1, s_2\} - A_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |A_i|) + 3)/7 \rceil$, and $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |A_i|) + 3)/7 \rceil$. Note $u_1 \in F_1^{(i)}$ by Lemma 2.3 since $|N(u_1)| = 3$. Let $F = (F_1^{(i)} - u_1) \cup F_2^{(i)} + \{x_1, y, z, z_1, w\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))$. Now $G[F]$ (if $u_2 \notin F_1^{(i)}$) or $G[F - \{u_2, w\} + \{w_1, x_2\}]$ (if $u_2 \in F_1^{(i)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + 4 - (1 - |A_i|)$. By Lemma 2.2(2) (with $a = |G_1| - 8, a_1 = |G_2| - 2, c = 4$), $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (4, 0), (0, 4) \mod 7$.

Subcase 2.1. $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (4, 0) \mod 7$.

Let $F_1^{(3)} = A((G_1 - \{w, x, z, z_1\})/\{x_1y, w_1x_2, s_1s_2\} + z_2u_2)$ with u_1 (respectively, u_2, u_3) as the identification of $\{x_1, y\}$ (respectively, $\{w_1, x_2\}$, $\{s_1, s_2\}$), and $F_2^{(3)} = A(G_2)$. Then $|F_1^{(3)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$ and $|F_2^{(3)}| \geq \lceil (4|G_2| + 3)/7 \rceil$. Note $u_1 \in F_1^{(4)}$ by Lemma 2.3 since $|N(u_1)| = 3$. Let $F^{(3)} := \overline{F_1}^{(3)} \cup F_2^{(3)} - (\{s_1, s_2, a_1\} \cap (\overline{F_1}^{(3)} \triangle F_2^{(3)}))$, where $\overline{F_1}^{(3)} = {F_1}^{(3)} + \{x_1, y, z, z_1, w\} - u_1$ if $u_2, u_3 \notin F_1^{(3)}$, and otherwise, $\overline{F_1}^{(3)}$ obtained from ${F_1}^{(3)} + \{x_1, y, z, z_1, w\} - u_1$ by deleting $\{u_2, w\}$ (respectively, $\{u_3, z_1\}$) and adding $\{w_1, x_2\}$ (respectively, $\{s_1, s_2\}$) when $u_2 \in {F_1}^{(3)}$ (respectively, $u_3 \in {F_1}^{(3)}$). Therefore, $G[F^{(3)}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(3)}| + |F_2^{(3)}| + 4 - 3 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Subcase 2.2. $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (0, 4) \mod 7$.

If $|N(x_3) \cap N(t)| \leq 2$, let $F_1^{(4)} = A((G_1 - \{s_2, x_2, z, z_1, z_2\})/x_3t + s_1x)$ with u as the identification of $\{x_3, t\}$, and $F_2^{(4)} = A(G_2 - \{s_2\})$. Then $|F_1^{(4)}| \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil$, and $|F_2^{(4)}| \geq \lceil (4(|G_2| - 1) + 3)/7 \rceil$. Let $F = F_1^{(4)} \cup F_2^{(4)} + \{z, z_1, z_2\} - (\{s_1, a_1\} \cap (F_1^{(4)} \triangle F_2^{(4)}))$. Now $G[F]$ (if $u \notin F_1^{(4)}$) or $G[F - \{u, z_2\} + \{x_3, t\}]$ (if $u \in F_1^{(4)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(4)}| + |F_2^{(4)}| + 3 - 2 \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

So, $|N(x_3) \cap N(t)| > 2$. There exist $b_1 \in N(x_3) \cap N(t)$ and subgraphs G_1', G_2', G_3' such that $G_2' = G_2$, G_3' is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $z_2x_3b_1tz_2$ and containing $N(x_3) \cap N(t) - \{z_2\}$, and G_1' is obtained from G by removing $G_2' - \{s_1, s_2, a_1\}$ and $G_3' - \{x_3, b_1, t\}$. Let $B_5 = \{b_1\}$ or $B_6 = \emptyset$. For $i = 5, 6$, let $F_1^{(i)} = A(G_1' - \{z, z_1, z_2, s_2, x_3, t, x_2\} - B_i + s_1x)$, $F_2^{(i)} = A(G_2' - \{s_2\})$, and $F_3^{(i)} = A(G_3' - \{x_3, t\} - B_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1'| - 7 - |B_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2'| - 1) + 3)/7 \rceil = \lceil (4(|G_2| - 1) + 3)/7 \rceil = (4(|G_2| - 1) + 3)/7 + 6/7$, and

$|F_3^{(i)}| \geq \lceil (4(|G_3'| - 2 - |B_i|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{z, z_1, z_2\} - (\{s_1, a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 3 - 2 - (1 - |B_i|) \geq \lceil (4n+3)/7 \rceil$. By Lemma 2.2(2) (with $a = |G_1'| - 7, a_1 = |G_3'| - 2, c = (4(|G_2| - 1) + 3)/7 + 6/7 + 1)$, $(4(|G_1'| - 7) + 3, 4(|G_2'| - 2) + 3, 4(|G_3'| - 2) + 3) \equiv (0, 4, 4), (4, 4, 0) \mod 7$.

If $(4(|G_1'| - 7) + 3, 4(|G_2'| - 2) + 3, 4(|G_3'| - 2) + 3) \equiv (4, 4, 0) \mod 7$, let $F_1^{(7)} = A((G_1' - \{z, z_1, z_2, s_2, x_2\})/x_3t + s_1x)$ with u as the identification of $\{x_3, t\}$, $F_2^{(7)} = A(G_2' - \{s_2\})$, and $F_3^{(7)} = A(G_3')$. Then $|F_1^{(7)}| \geq \lceil (4(|G_1'| - 6) + 3)/7 \rceil$, $|F_2^{(7)}| \geq \lceil (4(|G_2'| - 1) + 3)/7 \rceil$, and $|F_3(7)| \geq \lceil (4|G_3'| + 3)/7 \rceil$. Now $G[F_1^{(7)} \cup F_2^{(7)} \cup F_3^{(7)} + \{z, z_1, z_2\} - (\{s_1, a_1\} \cap (F_1^{(7)} \triangle F_2^{(7)})) - (\{t, b_1, x_3\} \cap (F_1^{(7)} \triangle F_3^{(7)}))]$ (if $u \notin F_1^{(7)}$) or $G[(F_1^{(7)} - u) \cup F_2^{(7)} \cup F_3^{(7)} + \{z, z_1, x_3, t\} - (\{s_1, a_1\} \cap (F_1^{(7)} \triangle F_2^{(7)})) - (\{t, b_1, x_3\} \cap ((F_1^{(7)} + \{x_3, t\}) \triangle F_3^{(7)}))]$ (if $u \in F_1^{(7)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(7)}| + |F_2^{(7)}| + |F_3^{(7)}| + 3 - 5 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

If $(4(|G_1'| - 7) + 3, 4(|G_2'| - 2) + 3, 4(|G_3'| - 2) + 3) \equiv (0, 4, 4) \mod 7$, let $F_1^{(8)} = A((G_1' - \{z, z_2, s_2, x_3\})/xz_1)$ with u as the identification of $\{x, z_1\}$, $F_2^{(8)} = A(G_2' - \{s_2\})$, and $F_3^{(8)} = A(G_3' - \{x_3\})$. Then $|F_1^{(8)}| \geq \lceil (4(|G_1'| - 5) + 3)/7 \rceil$, $|F_2^{(8)}| \geq \lceil (4(|G_2'| - 1) + 3)/7 \rceil$, and $|F_3^{(8)}| \geq \lceil (4(|G_3'| - 1) + 3)/7 \rceil$. Let $F = F_1^{(8)} \cup F_2^{(8)} \cup F_3^{(8)} + \{z, z_2\} - (\{s_1, a_1\} \cap (F_1^{(8)} \triangle F_2^{(8)})) - (\{t, b_1\} \cap (F_1^{(8)} \triangle F_3^{(8)}))$. Now $G[F]$ (if $u \notin F_1^{(8)}$) or $G[F - \{u, z\} + \{x, z_1\}]$ (if $u \in F_1^{(8)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(8)}| + |F_2^{(8)}| + |F_3^{(8)}| + 2 - 4 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Case 3. $|N(w_1) \cap N(x_2)| > 2$.

There exist $c_1 \in N(w_1) \cap N(x_2)$ and a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{w_1, x_2, c_1\}$, $\{x, y, w, z, x_1, x_3, z_1, z_2, s_1, s_2\} \subseteq V(G_1)$, and $N(w_1) \cap N(x_2) - \{w\} \subseteq V(G_2)$. By the fourth claim, $s_1z_2 \notin E(G)$. Let $C_1 = \{c_1\}$ and $C_2 = \emptyset$. For $i = 1, 2$, let $F_1^{(i)} = A((G_1 - \{w, x, w_1, x_2, z, z_1, s_2\} - C_i)/x_1y + s_1z_2)$ with u as the identification of $\{x_1, y\}$, and $F_2^{(i)} = A(G_2 - \{w_1, x_2\} - C_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |C_i|) + 3)/7 \rceil$, and $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |C_i|) + 3)/7 \rceil$. Note $u \in F_1^{(i)}$ by Lemma 2.3 since $|N(u)| = 3$. Now $G[(F_1^{(i)} - u) \cup F_2^{(i)} + \{w, x_1, y, z, z_1\} - (\{c_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + 4 - (1 - |C_i|)$. By Lemma 2.2(2) (with $a = |G_1| - 8, a_1 = |G_2| - 2, c = 4)$, $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (4, 0), (0, 4) \mod 7$.

Subcase 3.1. $(4(|G_1|-8)+3, 4(|G_2|-2)+3) \equiv (4,0) \mod 7$.

If $|N(s_1)\cap N(s_2)| \le 2$, then let $F_1^{(3)} = A((G_1-\{w,x,z,z_1\})/\{x_1y, w_1x_2, s_1s_2\}+z_2u_2)$ with u_1 (respectively, u_2, u_3) as the identification of $\{x_1,y\}$ (respectively, $\{w_1,x_2\},\{s_1,s_2\}$), and $F_2^{(3)} = A(G_2)$. Then $|F_1^{(3)}| \ge \lceil (4(|G_1|-7)+3)/7\rceil$, and $|F_2^{(3)}| \ge \lceil (4|G_2|+3)/7\rceil$. Note $u_1 \in F_1^{(10)}$ by Lemma 2.3 since $|N(u_1)| = 3$. Let $F^{(3)} := \overline{F_1}^{(3)} \cup F_2^{(3)} - (\{w_1,x_2,c_1\}\cap(\overline{F_1}^{(3)}\triangle F_2^{(3)}))$, where $\overline{F_1}^{(3)} = F_1{}^{(3)} + \{x_1,y,z,z_1,w\} - u_1$ if $u_2,u_3 \notin F_1^{(3)}$, and otherwise, $\overline{F_1}^{(3)}$ obtained from $F_1{}^{(3)} + \{x_1,y,z,z_1,w\} - u_1$ by deleting $\{u_2,w\}$ (respectively, $\{u_3,z_1\}$) and adding $\{w_1,x_2\}$ (respectively, $\{s_1,s_2\}$) when $u_2 \in F_1{}^{(3)}$ (respectively, $u_3 \in F_1{}^{(3)}$). Therefore, $G[F^{(3)}]$ is an induced forest in G, showing $a(G) \ge |F_1^{(3)}| + |F_2^{(3)}| + 4 - 3 \ge \lceil (4n+3)/7\rceil$, a contradiction.

So $|N(s_1)\cap N(s_2)| > 2$. There exist $a_1 \in N(s_1)\cap N(s_2)$ and subgraphs G_1', G_2', G_3' such that $G_2' = G_2$, G_3' is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $z_1s_1a_1s_2z_1$ and containing $N(s_1)\cap N(s_2) - \{z_1\}$, and G_1' is obtained from G by removing $G_2' - \{w_1,x_2,w\}$ and $G_3' - \{s_1,a_1,s_2\}$. Let $A_4 = \{a_1\}$ and $A_5 = \emptyset$. For $i = 4,5$, let $F_1^{(i)} = A((G_1' - \{w,x,z,z_1,s_1,s_2\} - A_i)/\{x_1y, w_1x_2\} + u_2z_2)$ with u_1 (respectively, u_2) as the identification of $\{x_1,y\}$, $F_2^{(i)} = A(G_2')$, and $F_3^{(i)} = A(G_3' - \{s_1,s_2\} - A_i)$. Then $|F_1^{(i)}| \ge \lceil (4(|G_1'|-8-|A_i|)+3)/7\rceil$, $|F_2^{(i)}| \ge \lceil (4|G_2'|+3)/7\rceil = \lceil (4|G_2|+3)/7\rceil = (4|G_2|+3)/7 + 6/7$, and $|F_3^{(i)}| \ge \lceil (4(|G_3'|-2-|A_i|)+3)/7\rceil$. Note $u_1 \in F_1^{(i)}$ by Lemma 2.3 since $|N(u_1)| = 3$. Now $G[(F_1^{(i)}-u_1)\cup F_2^{(i)}\cup F_3^{(i)}+\{x_1,y,z,z_1,w\}-(\{w_1,x_2,c_1\}\cap (F_1^{(i)}\triangle F_2^{(i)})) - (\{a_1\}\cap(F_1^{(i)}\triangle F_3^{(i)}))]$ (if $u_2 \notin F_1^{(i)}$) or $G[(F_1^{(i)} - \{u_1,u_2\}) \cup F_2^{(i)} \cup F_3^{(i)} + \{x_1,y,z,z_1,w_1,x_2\} - (\{w_1,x_2,c_1\}\cap((F_1^{(i)}\cup\{w_1,x_2\})\triangle F_2^{(i)})) - (\{a_1\}\cap(F_1^{(i)}\triangle F_3^{(i)}))]$ (if $u_2 \in F_1^{(i)}$) is an induced forest in G, showing $a(G) \ge |F_1^{(i)}| + |F_2^{(i)}| + 4 - 3 - (1-|A_i|)$. By Lemma 2.2(2) (with $a = |G_1'|-8, a_1 = |G_3'|-2, c = (4|G_2|+3)/7+6/7+1$), $(4(|G_1'|-8)+3, 4(|G_2'|-2)+3, 4(|G_3'|-2)+3) \equiv (4,0,0),(0,0,4) \mod 7$.

If $(4(|G_1'|-8)+3, 4(|G_2'|-2)+3, 4(|G_3'|-2)+3) \equiv (4,0,0) \mod 7$, let $F_1^{(6)} = A((G_1' - \{w,x,z,z_1\})/\{x_1y, w_1x_2, s_1s_2\} + z_2u_2)$ with u_1 (respectively, u_2, u_3) as the identification of $\{x_1,y\}$ (respectively, $\{w_1,x_2\},\{s_1,s_2\}$), $F_2^{(6)} = A(G_2')$, and $F_3^{(6)} = A(G_3')$. Then $|F_1^{(6)}| \ge \lceil (4(|G_1'|-7)+3)/7\rceil$, $|F_2^{(6)}| \ge \lceil (4|G_2'|+3)/7\rceil$, and $|F_3^{(6)}| \ge \lceil (4|G_3'|+3)/7\rceil$. Note $u_1 \in F_1^{(6)}$ by Lemma 2.3 since $|N(u_1)| = 3$. Let $F^{(6)} := \overline{F_1}^{(6)} \cup F_2^{(6)} \cup F_3^{(6)} - (\{w_1,x_2,c_1\}\cap (\overline{F_1}^{(6)}\triangle F_2^{(6)})) - (\{s_1,s_2,a_1\}\cap(\overline{F_1}^{(6)}\triangle F_3^{(6)}))$, where $\overline{F_1}^{(6)} = F_1{}^{(6)} + \{x_1,y,z,z_1,w\} - u_1$ if $u_2,u_3 \notin F_1^{(6)}$, and otherwise, $\overline{F_1}^{(6)}$ obtained from $F_1{}^{(6)} +$

$\{x_1, y, z, z_1, w\} - u_1$ by deleting $\{u_2, w\}$ (respectively, $\{u_3, z_1\}$) and adding $\{w_1, x_2\}$ (respectively, $\{s_1, s_2\}$) when $u_2 \in {F_1}^{(6)}$ (respectively, $u_3 \in {F_1}^{(6)}$). Therefore, $F^{(6)}$ is an induced forest in G, showing $|F_1^{(6)}| + |F_2^{(6)}| + |F_3^{(6)}| + 4 - 6 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

If $(4(|G'_1| - 8) + 3, 4(|G'_2| - 2) + 3, 4(|G'_3| - 2) + 3) \equiv (0, 0, 4) \mod 7$, let $F_1^{(7)} = A((G'_1 - \{w_1, x_2, w, x, z, z_1, c_1, s_2\} + s_1 z_2)/x_1 y)$ with u_1 as the identification of $\{x_1, y\}$, $F_2^{(7)} = A(G'_2 - \{w_1, x_2, c_1\})$, and $F_3^{(7)} = A(G'_3 - s_2)$. Then $|F_1^{(7)}| \geq \lceil (4(|G'_1| - 9) + 3)/7 \rceil$, $|F_2^{(7)}| \geq \lceil (4(|G'_2| - 3) + 3)/7 \rceil$, and $|F_3^{(7)}| \geq \lceil (4(|G'_3| - 1) + 3)/7 \rceil$. Note $u_1 \in F_1^{(7)}$ by Lemma 2.3 since $|N(u_1)| = 3$. Now $G[(F_1^{(7)} - u_1) \cup F_2^{(7)} \cup F_3^{(7)} + \{x_1, y, z, z_1, w\} - (\{s_1, a_1\} \cap (F_1^{(7)} \triangle F_2^{(7)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(7)}| + |F_2^{(7)}| + |F_3^{(7)}| + 4 - 2 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Subcase 3.2. $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (0, 4) \mod 7$.

If $|N(x_3) \cap N(t)| \leq 2$, let $F_1^{(8)} = A((G_1 - \{z, z_1, z_2, x_2, s_2\} + x s_1)/x_3 t)$ with u as the identification of $\{x_3, t\}$, and $F_2^{(8)} = A((G_2 - x_2)$. Then $|F_1^{(8)}| \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil$, and $|F_2^{(8)}| \geq \lceil (4(|G_2| - 1) + 3)/7 \rceil$. Let $F = F_1^{(8)} \cup F_2^{(8)} + \{z, z_1, z_2\} - (\{w_1, c_1\} \cap (F_1^{(8)} \triangle F_2^{(8)}))$. Now $G[F]$ (if $u \notin F_1^{(8)}$) or $G[F - \{u, z_2\} + \{x_3, t\}]$ (if $u \in F_1^{(8)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(8)}| + |F_2^{(8)}| + 3 - 2 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

So $|N(x_3) \cap N(t)| > 2$. There exist $b_1 \in N(x_3) \cap N(t)$ and subgraphs G'_1, G'_2, G'_3 such that $G'_2 = G_2$, G'_3 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $z_2 x_3 b_1 t z_2$ and containing $N(x_3) \cap N(t) - \{z_2\}$, and G'_1 is obtained from G by removing $G'_2 - \{w_1, x_2, c_1\}$ and $G'_3 - \{x_3, b_1, t\}$. Let $B_9 = \emptyset$ and $B_{10} = \{b_1\}$. For $i = 9, 10$, let $F_1^{(i)} = A(G'_1 - \{z, z_1, z_2, x_2, s_2, x_3, t\} - B_i + x s_1)$, $F_2^{(i)} = A(G'_2 - \{x_2\})$, and $F_3^{(i)} = A(G'_3 - \{x_3, t\} - B_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G'_1| - 7 - |B_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G'_2| - 1) + 3)/7 \rceil = \lceil (4(|G_2| - 1) + 3)/7 \rceil = (4(|G_2| - 1) + 3)/7 + 6/7$, and $|F_3^{(i)}| \geq \lceil (4(|G'_3| - 2 - |B_i|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{z, z_1, z_2\} - (\{c_1, w_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + 3 - 2 - (1 - |B_i|)$. By Lemma 2.2(2) (with $a = |G'_1| - 7, a_1 = |G'_3| - 2, c = (4(|G_2| - 1) + 3)/7 + 6/7 + 1$), $(4(|G'_1| - 7) + 3, 4(|G'_2| - 2) + 3, 4(|G'_3| - 2) + 3) \equiv (4, 4, 0), (0, 4, 4) \mod 7$.

If $(4(|G'_1| - 7) + 3, 4(|G'_2| - 2) + 3, 4(|G'_3| - 2) + 3) \equiv (4, 4, 0) \mod 7$, let $F_1^{(11)} = A((G'_1 - \{z, z_1, z_2, x_2, s_2\} + x s_1)/x_3 t)$ with u as the identification of $\{x_3, t\}$, $F_2^{(11)} = A(G'_2 - x_2)$, and $F_3^{(11)} = A(G'_3)$. Then $|F_1^{(11)}| \geq \lceil (4(|G'_1| - 6) + 3)/7 \rceil$, $|F_2^{(11)}| \geq \lceil (4(|G'_2| - 1) + 3)/7 \rceil$, and $|F_3^{(11)}| \geq \lceil (4|G'_3| + 3)/7 \rceil$. Now

$G[F_1^{(11)} \cup F_2^{(11)} \cup F_3^{(11)} + \{z, z_1, z_2\} - (\{w_1, c_1\} \cap (F_1^{(11)} \triangle F_2^{(11)})) - (\{x_3, t, b_1\} \cap (F_1^{(11)} \triangle F_3^{(11)}))]$ (if $u \notin F_1^{(11)}$) or $G[(F_1^{(11)} - u) \cup F_2^{(11)} \cup F_3^{(11)} + \{z, z_1, x_3, t\} - (\{w_1, c_1\} \cap (F_1^{(11)} \triangle F_2^{(11)})) - (\{x_3, t, b_1\} \cap ((F_1^{(11)} + \{x_3, t\}) \triangle F_3^{(11)}))]$ (if $u \in F_1^{(11)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(11)}| + |F_2^{(11)}| + |F_3^{(11)}| + 3 - 5 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

So $(4(|G'_1| - 7) + 3, 4(|G'_2| - 2) + 3, 4(|G'_3| - 2) + 3) \equiv (0, 4, 4) \mod 7$. If $|N(x_2) \cap N(s_1)| \leq 2$, let $F_1^{(12)} = A((G'_1 \cup G'_2 - \{z, z_1, z_2, s_2, x_3\})/x_2 s_1 + xt)$ with u as the identification of $\{x_2, s_1\}$, and $F_2^{(12)} = A(G'_3 - \{x_3\})$. Then $|F_1^{(12)}| \geq \lceil (4((n+3-|G'_3|) - 6) + 3)/7 \rceil$, and $|F_2^{(12)}| \geq \lceil (4(|G'_3| - 1) + 3)/7 \rceil$. Let $F = F_1^{(12)} \cup F_2^{(12)} + \{z, z_1, z_2\} - (\{b_1, t\} \cap (F_1^{(12)} \triangle F_2^{(12)}))$. Now $G[F]$ (if $u \notin F_1^{(12)}$) or $G[F - \{u, z_1\} + \{x_2, s_1\}]$ (if $u \in F_1^{(12)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(12)}| + |F_2^{(12)}| + 3 - 2 \geq \lceil (4n+3)/7 \rceil$, a contradiction. So $|N(x_2) \cap N(s_1)| > 2$. There exist $e_1 \in N(x_2) \cap N(s_1)$ and subgraphs $G''_1, G''_2, G''_3, G''_4$ such that $G''_2 = G'_2$, $G''_3 = G'_3$, G''_4 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $z_1 x_2 e_1 s_1 z_1$ and containing $N(s_1) \cap N(x_2) - \{z_1\}$, and G''_1 is obtained from G by removing $G''_2 - \{w_1, x_2, c_1\}$, $G''_3 - \{x_3, b_1, t\}$ and $G''_4 - \{s_1, x_2, e_1\}$. Let $E_{13} = \emptyset$ and $E_{14} = \{e_1\}$. For $i = 13, 14$, let $F_1^{(i)} = A(G''_1 - \{z, z_1, z_2, x_2, s_1, s_2, x_3\} - E_i + xt)$, $F_2^{(i)} = A(G''_2 - \{x_2\})$, $F_3^{(i)} = A(G''_3 - \{x_3\})$, and $F_4^{(i)} = A(G''_4 - \{s_1, x_2\} - E_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G''_1| - 7 - |E_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G''_2| - 1) + 3)/7 \rceil = \lceil (4(|G'_2| - 1) + 3)/7 \rceil = (4(|G'_2| - 1) + 3)/7 + 6/7$, $|F_3^{(i)}| \geq \lceil (4(|G''_3| - 1) + 3)/7 \rceil = \lceil (4(|G'_3| - 1) + 3)/7 \rceil = (4(|G'_3| - 1) + 3)/7 + 6/7$, and $|F_4^{(i)}| \geq \lceil (4(|G''_4| - 2 - |E_i|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{z, z_1, z_2\} - (\{w_1, c_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{b_1, t\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{e_1\} \cap (F_1^{(i)} \triangle F_4^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + |F_4^{(i)}| + 3 - 2 - (1 - |E_i|)$. By Lemma 2.2(1) (with $k = 1, a = |G''_1| - 7, a_1 = |G''_4| - 2, L = \{1, 2\}, b_1 = |G''_2| - 1, b_2 = |G''_3| - 1, c = 1$), $a(G) \geq \lceil (4n+3)/7 \rceil$, a contradiction. □

Lemma 7.4. *The following configuration is impossible in G: x is a 5-vertex of type 5-1-B in G with neighbors y_1, y_3, y_4, y_2, y_5 in cyclic order around x. $xy_1y'_3y_3x$, $xy_4y'_4y_2x$, $xy_1z_1y_5x$, $xy_2z_2y_5x$ are facial cycles. $\{y_1, y_2\} \subseteq V_4$, $\{z_1, z_2, y'_3, y_4\} \subseteq V_3$, $N(z_1) = \{y_1, y_5, w_1\}$ and $N(z_2) = \{y_2, y_5, w_2\}$.*

Proof. Let $N(y_1) = \{x, z_1, y'_3, y'_1\}$ and $N(y_2) = \{x, z_2, y'_4, y'_2\}$.

First, we claim that $w_1 x \notin E(G)$. Otherwise $w_1 x \in E(G)$. Since G is simple, $w_1 \notin \{y_1, y_5\}$. If $w_1 = y_2$ and $y_1 y'_4 \notin E(G)$, then let $F' = A(G - \{z_1, z_2, x, y_5, y_2\} + y_1 y'_4)$. Then $|F'| \geq \lceil (4(n-5)+3)/7 \rceil$. Now $G[F' + \{z_1, z_2, y_2\}]$ is an induced forest in G, showing $a(G) \geq |F'| + 3 \geq \lceil (4n +$

$3)/7\rceil$, a contradiction. If $w_1 = y_2$ and $y_1y_4' \in E(G)$, let $F' = A(G - \{z_1, z_2, x, y_5, y_2, y_1, y_4'\})$. Then $|F'| \geq \lceil (4(n-7)+3)/7\rceil$. Now $G[F' + \{z_1, z_2, y_2, y_1\}]$ is an induced forest in G, showing $a(G) \geq |F'| + 4 \geq \lceil (4n+3)/7\rceil$, a contradiction. If $w_1 = y_3$, then since G is plane, there exists a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{z_1, y_1, y_3\}$, $y_1' \in V(G_1)$, and $\{x, y_5\} \subseteq V(G_2)$. Let $F_1^{(1)} = A(G_1 - \{z_1, y_3\})$, and $F_2^{(1)} = A(G_2 - \{z_1, y_1, y_3, y_5\})$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1| - 2) + 3)/7\rceil$, and $|F_2^{(1)}| \geq \lceil (4(|G_2| - 4) + 3)/7\rceil$. Now $G[F_1^{(1)} \cup F_2^{(1)} + \{z_1\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 1$. By Lemma 2.2(7) (with $k = 2, a_1 = |G_1| - 2, a_2 = |G_2| - 4, c = 1$), $(4(|G_1|-2)+3, 4(|G_2|-4)+3) \equiv (0,6), (6,0), (0,0) \mod 7$. Let $F_i^{(2)} = A(G_i - \{z_1, y_1, y_3\})$ for $i = 1, 2$. Then $|F_i^{(2)}| \geq \lceil (4(|G_i|-3)+3)/7\rceil$. Now $G[F_1^{(2)} \cup F_2^{(2)} + \{z_1\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + 1 \geq \lceil (4n+3)/7\rceil$, a contradiction. If $w_1 = y_4$, then since G is plane, there exists a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{z_1, x, y_4\}$, $y_1 \in V(G_1)$, and $\{y_2, y_5\} \subseteq V(G_2)$. Let $F_1^{(3)} = A(G_1 - \{z_1, x, y_4, y_1\})$, and $F_2^{(3)} = A(G_2 - \{z_1, y_4\})$. Then $|F_1^{(3)}| \geq \lceil (4(|G_1| - 4) + 3)/7\rceil$, and $|F_2^{(3)}| \geq \lceil (4(|G_2| - 2) + 3)/7\rceil$. Now $G[F_1^{(3)} \cup F_2^{(3)} + \{z_1\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(3)}| + |F_2^{(3)}| + 1$. By Lemma 2.2(7) (with $k = 2, a_1 = |G_1| - 4, a_2 = |G_2| - 2, c = 1$), $(4(|G_1| - 4) + 3, 4(|G_2| - 2) + 3) \equiv (0,6), (6,0), (0,0) \mod 7$. Let $F_1^{(4)} = A(G_1 - \{z_1, x, y_4\})$, and $F_2^{(4)} = A(G_2 - \{z_1, x, y_4\})$. Then $|F_1^{(4)}| \geq \lceil (4(|G_1| - 3) + 3)/7\rceil$, and $|F_2^{(4)}| \geq \lceil (4(|G_2| - 3) + 3)/7\rceil$. Now $G[F_1^{(4)} \cup F_2^{(4)} + \{z_1\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + 1 \geq \lceil (4n+3)/7\rceil$, a contradiction. Similarly, $w_2x \notin E(G)$.

Secondly, we claim that $|N(y_1') \cap N(y_3')| \leq 2$. Otherwise, there exist $a_1 \in N(y_1') \cap N(y_3')$ and a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{y_1', y_3', a_1\}$, $\{x, y_1, y_2, y_3, y_4, y_5\} \subseteq V(G_1)$, and $N(y_1') \cap N(y_3') - \{y_1\} \subseteq V(G_2)$. Let $F_1^{(1)} = A(G_1 - \{y_1', y_3', a_1, y_1, z_1, y_5\} + w_1x)$, and $F_2^{(1)} = A(G_2 - \{y_1', y_3', a_1\})$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1| - 6) + 3)/7\rceil$, and $|F_2^{(1)}| \geq \lceil (4(|G_2| - 3) + 3)/7\rceil$. Now $G[F_1^{(1)} \cup F_2^{(1)} + \{z_1, y_1, y_3\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 3 \geq \lceil (4n+3)/7\rceil$, a contradiction.

Now we prove the lemma. If $|N(y_2') \cap N(y_4')| \leq 2$, let $F' = A((G - \{z_1, z_2, y_1, y_2, y_5\})/ \{y_1'y_3', y_2'y_4'\} + \{w_1x, w_2x\})$ with u_1 (respectively, u_2) as the identification of $\{y_1', y_3'\}$ (respectively, $\{y_2', y_4'\}$). Then $|F'| \geq \lceil (4(n-7) + 3)/7\rceil$. Let $F = F' + \{z_1, z_2, y_1, y_2\}$ if $u_1, u_2 \notin F'$, and otherwise F obtained from $F' + \{z_1, z_2, y_1, y_2\}$ by deleting $\{u_1, y_1\}$ (respectively, $\{u_2, y_2\}$) and adding $\{y_1', y_3'\}$ (respectively, $\{y_2', y_4'\}$) when $u_1 \in F'$ (respectively, $u_2 \in F'$). Therefore, $G[F]$ is an induced forest in G, showing $a(G) \geq |F'| + 4 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

So, $|N(y_2') \cap N(y_4')| > 2$. Let $B_1 = \{b_1\}$ and $B_2 = \emptyset$. There exist $b_1 \in N(y_2') \cap N(y_4')$ and a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{y_2', y_4', b_1\}$, $\{x, y_1, y_2, y_3, y_4, y_5\} \subseteq V(G_1)$, and $N(y_2') \cap N(y_4') - \{y_2\} \subseteq V(G_2)$. For $i = 1, 2$, let $F_1^{(i)} = A((G_1 - \{z_1, z_2, y_1, y_2, y_5, y_2', y_4'\} - B_i)/y_1'y_3' + \{w_1x, w_2x\})$ with u as the identification of $\{y_1', y_3'\}$, and $F_2^{(i)} = A(G_2 - \{y_2', y_4'\} - B_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |B_i|) + 3)/7 \rceil$, and $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |B_i|) + 3)/7 \rceil$. Let $F = F_1^{(i)} \cup F_2^{(i)} + \{z_1, z_2, y_1, y_2\} - (\{b_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))$. Now $G[F]$ ($u \notin F_1^{(i)}$) or $G[F - \{u, y_1\} + \{y_1', y_3'\}]$ ($u \in F_1^{(i)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + 4 - (1 - |B_i|)$. By Lemma 2.2(2) (with $a = |G_1| - 8, a_1 = |G_2| - 2, c = 4$), $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (4, 0), (0, 4) \mod 7$.

If $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (4, 0) \mod 7$, let $F_1^{(3)} = A((G - \{z_1, z_2, y_1, y_2, y_5\})/ \{y_1'y_3', y_2'y_4'\} + \{w_1x, w_2x\})$ with u_1 (respectively, u_2) as the identification of $\{y_1', y_3'\}$ (respectively, $\{y_2', y_4'\}$), and $F_2^{(3)} = A(G_2)$. Then $|F_1^{(3)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$, and $|F_2^{(3)}| \geq \lceil (4|G_2| + 3)/7 \rceil$. Let $F^{(3)} = \overline{F_1}^{(3)} \cup F_2^{(3)} - (\{b_1, y_4', y_2'\} \cap (\overline{F_1}^{(3)} \triangle F_2^{(3)}))$, where $\overline{F_1}^{(3)} = F_1{}^{(3)} + \{z_1, z_2, y_1, y_2\}$ if $u_1, u_2 \notin F'$, and let $\overline{F_1}^{(3)}$ obtained from $F_1{}^{(3)} + \{z_1, z_2, y_1, y_2\}$ by deleting $\{u_1, y_1\}$ (respectively, $\{u_2, y_2\}$) and adding $\{y_1', y_3'\}$ (respectively, $\{y_2', y_4'\}$) when $u_1 \in F'$ (respectively, $u_2 \in F'$). Therefore, $G[F^{(3)}]$ is an induced forest in G, showing $a(G) \geq |F_1{}^{(3)}| + |F_2{}^{(3)}| + 4 - 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

So $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (0, 4) \mod 7$. If $y_5y_2' \notin E(G)$, let $F_1^{(4)} = A(G_1 - \{x, y_4, y_4', y_2, z_2, w_2\} + y_5y_2')$, and $F_2^{(4)} = A(G_2 - \{y_4'\})$. Then $|F_1^{(4)}| \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil$, and $|F_2^{(4)}| \geq \lceil (4(|G_2| - 1) + 3)/7 \rceil$. Now $G[F_1^{(4)} \cup F_2^{(4)} + \{y_4, y_2, z_2\} - (\{b_1, y_2'\} \cap (F_1^{(4)} \triangle F_2^{(4)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(4)}| + |F_2^{(4)}| + 3 - 2 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

So, $y_5y_2' \in E(G)$, then there exists a separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{y_2', y_5, z_2\}$, $\{x, z_1, y_2, y_3, y_4\} \subseteq V(G_1')$, and $w_2 \in V(G_2')$. Let $F_1^{(5)} = A((G_1' - \{y_2', y_5, z_2, z_1, y_1\})/y_1'y_3' + w_1x)$ with u as the identification of $\{y_1', y_3'\}$ and $F_2^{(5)} = A(G_2' - \{y_2', y_5, z_2\})$. Then $|F_1^{(5)}| \geq \lceil (4(|G_1| - 6) + 3)/7 \rceil$, and $|F_2^{(5)}| \geq \lceil (4(|G_2| - 3) + 3)/7 \rceil$. Let $F = F_1^{(5)} \cup F_2^{(5)} + \{z_1, z_2, y_1\}$. Now $G[F]$ (if $u \notin F_1^{(5)}$) or $G[F - \{u, y_1\} + \{y_1', y_3'\}]$ (if $u \in F_1^{(5)}$) is an induced forest of size $a(G) \geq |F_1^{(5)}| + |F_2^{(5)}| + 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction. □

Lemma 7.5. *The following configuration is impossible in G: x is a 5^+-vertex in G with neighbors $x_1, x_2, x_3, ..., x_m$ in cyclic order around x. $\{x_1, x_k\} \subseteq$*

V_3, $N(x_1) = \{x, z_1, y_1\}$, $N(x_k) = \{x, z_2, y_2\}$, and $\{y_1x_2, y_2x_{k-1}\} \subseteq E(G)$. Moreover, for $v \in \{y_1, z_1\}$, either $v \subseteq V_{\leq 4}$ or $R_{v,\{x_1\}} \neq \emptyset$; and for $v \in \{y_2, z_2\}$, either $v \subseteq V_{\leq 4}$ or $R_{v,\{x_k\}} \neq \emptyset$.

Proof. By Lemmas 4.3, 4.4, we may assume that $\{y_1, z_1, y_2, z_2\} \subseteq V_4$. Let $N(z_1) = \{z_1', x_1, x_m, w_1\}$, $N(y_1) = \{x_1, w_1, x_2, y_1'\}$, $N(z_2) = \{z_2', x_k, x_{k+1}, w_2\}$, and $N(y_2) = \{x_k, w_2, x_{k-1}, y_2'\}$. By Lemma 4.1, $z_1x_2 \notin E(G)$, $y_1x_m \notin E(G)$, $z_2x_{k-1} \notin E(G)$, and $y_2x_{k+1} \notin E(G)$.

Claim 1. $z_1'x \notin E(G)$, $y_1'x \notin E(G)$, $z_2'x \notin E(G)$ and $y_2'x \notin E(G)$.

For, suppose $z_1'x \in E(G)$. Then there exists a separation (G_1, G_2) of G such that $V(G_1 \cap G_2) = \{x, z_1, z_1'\}$, $N(x) \cap N(z_1) - \{x_1\} \subseteq V(G_1)$ and $\{x_1, y_1\} \subseteq V(G_2)$. If $|N(w_1) \cap N(y_1')| \leq 2$, let $F_1 = A((G_1 - \{x, z_1, z_1', x_1, y_1\})/w_1y_1')$ with u as the identification of w_1 and y_1', and $F_2 = A(G_2 - \{x, z_1, z_1'\})$. Then $|F_1| \geq \lceil (4(|G_1|-6)+3)/7 \rceil$, and $|F_2| \geq \lceil (4(|G_2|-6)+3)/7 \rceil$. Let $F = F_1 \cup F_2 + \{z_1, x_1, y\}$. Now, $G[F]$ (if $u \notin F_1$) or $G[F - \{u, y_1\} + \{w_1, y_1'\}]$ (if $u \in F_1$) is an induced forest in G, showing $a(G) \geq |F_1| + |F_2| + 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

So $|N(w_1) \cap N(y_1')| > 2$. Then there exist $a_1 \in N(w_1) \cap N(y_1')$ and subgraphs G_1', G_2', G_3' of G such that $G_2' = G_2$, G_3' is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $yw_1a_1y_1'y$ and containing $N(y_1') \cap N(w_1) - \{y_1\}$, and G_1' is obtained from G by removing $G_2' - \{z_1', z_1, x\}$, and $G_3' - \{y_1', a_1, w_1\}$. Let $A_1 = \{a_1\}$ and $A_2 = \emptyset$. For $i = 1, 2$, let $F_1^{(i)} = A(G_1' - \{x, z_1, z_1', x_1, y_1, w_1, y_1'\} - A_i)$, $F_2^{(i)} = A(G_2' - \{x, z_1, z_1'\})$, and $F_3^{(i)} = A(G_3' - \{w_1, y_1'\} - A_i)$. Then, $|F_1^{(i)}| \geq \lceil (4(|G_1'| - 7 - |A_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2'| - 3) + 3)/7 \rceil$, and $|F_3^{(i)}| \geq \lceil (4(|G_3'| - 2 - |A_i|) + 3)/7 \rceil$. Now, $F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{z_1, x_1, y_1\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 3 - (1 - |A_i|)$. Let $(n_1, n_2, n_3) := (4(|G_1'|-7)+3, 4(|G_2'|-7)+3, 4(|G_3'|-7)+3)$. By Lemma 2.2(2), $(n_1, n_2, n_3) \equiv (4, 0, 0), (0, 0, 4) \mod 7$. If $(n_1, n_2, n_3) \equiv (4, 0, 0) \mod 7$, let $F_1^{(3)} = A((G_1' - \{x, z_1, z_1', x_1, y_1\})/w_1y_1')$ with u as the identification of w_1 and y_1', $F_2^{(3)} = A(G_2' - \{x, z_1, z_1'\})$, and $F_3^{(3)} = A(G_3')$. Then, $|F_1^{(3)}| \geq \lceil (4(|G_1'| - 6) + 3)/7 \rceil$, $|F_2^{(3)}| \geq \lceil (4(|G_2'| - 3) + 3)/7 \rceil$, and $|F_3^{(3)}| \geq \lceil (4|G_3'| + 3)/7 \rceil$. Now, $G[F_1^{(3)} \cup F_2^{(3)} \cup F_3^{(3)} + \{z_1, x_1, y_1\} - (\{w_1, y_1', a_1\} \cap (F_1^{(3)} \triangle F_3^{(3)}))]$ (if $u \notin F_1^{(3)}$) or $G[F_1^{(3)} \cup F_2^{(3)} \cup F_3^{(3)} + \{z_1, x_1, w_1, y_1'\} - (\{w_1, y_1', a_1\} \cap ((F_1^{(3)} \cup \{w_1, y_1'\}) \triangle F_3^{(3)}))]$ (if $u \in F_1^{(3)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(3)}| + |F_2^{(3)}| + |F_3^{(3)}| + 3 - 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction. If $(n_1, n_2, n_3) \equiv (0, 0, 4) \mod 7$, let $F_1^{(4)} = A(G_1' - \{x, z_1, z_1', x_1, w_1\})$, $F_2^{(4)} = A(G_2' - \{x, z_1, z_1'\})$, and $F_3^{(4)} = A(G'_3 - w_1)$. Then, $|F_1^{(4)}| \geq \lceil (4(|G'_1| - 5) + 3)/7 \rceil$, $|F_2^{(4)}| \geq \lceil (4(|G'_2| - 3) + 3)/7 \rceil$, and $|F_3^{(4)}| \geq \lceil (4(|G'_3| - 1) + 3)/7 \rceil$. Now, $G[F_1^{(4)} \cup F_2^{(4)} \cup F_3^{(4)} + \{z_1, x_1\} - (\{y'_1, a_1\} \cap (F_1^{(4)} \triangle F_3^{(4)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(4)}| + |F_2^{(4)}| + |F_3^{(4)}| + 2 - 2 \geq \lceil (4n+3)/7 \rceil$, a contradiction. By symmetry, we have $y'_1 x \notin E(G)$, $z'_2 x \notin E(G)$ and $y'_2 x \notin E(G)$.

Claim 2. If $|N(y'_1) \cap N(w_1)| > 2$, $|N(z'_1) \cap N(w_1)| > 2$, and there exist $a_1 \in N(y'_1) \cap N(w_1)$ and a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{w_1, y'_1, a_1\}$, $\{x_1, y_1\} \subseteq V(G_1)$, and $N(y'_1) \cap N(w_1) - \{y_1\} \subseteq V(G_2)$, then $4(|G_2| - 2) + 3 \not\equiv 4 \mod 7$.

For, suppose $4(|G_2| - 2) + 3 \equiv 4 \mod 7$. There exist $a_1 \in N(y'_1) \cap N(w_1), b_1 \in N(z'_1) \cap N(w_1)$ and subgraphs G'_1, G'_2, G'_3 such that $G'_2 = G_2$, G'_3 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $z_1 z'_1 b_1 w_1 z_1$ and containing $N(z'_1) \cap N(w_1) - \{z_1\}$, and G'_1 is obtained from G by removing $G'_2 - \{y'_1, a_1, w_1\}$, and $G'_3 - \{z'_1, b_1, w_1\}$. Let $B_1 = \{b_1\}$ and $B_2 = \emptyset$. If $|N(y'_1) \cap N(x_2)| \leq 2$, for $i = 1, 2$, let $F_1^{(i)} = A((G'_1 - \{x_1, y_1, z_1, w_1, x_m\} - B_i)/y'_1 x_2 + z'_1 x)$ with u as the identification of $\{y'_1, x_2\}$, $F_2^{(i)} = A(G'_2 - w_1)$, and $F_3^{(i)} = A(G'_3 - w_1 - B_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G'_1| - 6 - |B_i|) + 3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G'_2| - 1) + 3)/7 \rceil$, and $|F_3^{(i)}| \geq \lceil (4(|G'_3| - 1 - |B_i|) + 3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} \cup F_4^{(i)} + \{x_1, z_1, y_1\} - (\{y'_1, a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{z'_1, b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))]$ (if $u \notin F_1^{(i)}$) or $G[(F_1^{(i)} - u) \cup F_2^{(i)} \cup F_3^{(i)} \cup F_4^{(i)} + \{x_1, z_1, y'_1, x_2\} - (\{y'_1, a_1\} \cap ((F_1^{(i)} + y'_1) \triangle F_2^{(i)})) - (\{z'_1, b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))]$ (if $u \notin F_1^{(i)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + 3 - 3 - (1 - |B_i|)$. By Lemma 2.2(1) (with $k = 1$), $a(G) \geq \lceil (4n + 3)/7 \rceil$, a contradiction. Thus, $|N(y'_1) \cap x_2| > 2$. Similarly, $|N(z'_1) \cap N(x_m)| > 2$.

So $|N(y'_1) \cap N(x_2)| > 2$ and $|N(z'_1) \cap N(x_m)| > 2$. There exist $c_1 \in N(y'_1) \cap N(x_2), d_1 \in N(z'_1) \cap N(x_m)$ and subgraphs $G'''_1, G'''_2, G'''_3, G'''_4, G'''_5$ of G such that $G'''_2 = G'_2$, $G'''_3 = G'_3$, G'''_4 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $z_1 z'_1 d_1 x_m z_1$ and containing $N(z'_1) \cap N(x_m) - \{z_1\}$, G'''_5 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $y_1 y'_1 c_1 x_2 y_1$ and containing $N(y'_1) \cap N(x_2) - \{y_1\}$, and G''_1 is obtained from G by removing $G''_2 - \{y'_1, a_1, w_1\}$, $G''_3 - \{z'_1, b_1, w_1\}$, $G''_4 - \{z'_1, d_1, x_m\}$ and $G''_5 - \{y'_1, c_1, x_2\}$. Let $B_i \subseteq \{b_1\}$, $C_i \subseteq \{c_1\}$, and $D_i \subseteq \{d_1\}$. For each choice of B_i, C_i, D_i, let $F_1^{(i)} = A(G'''_1 - \{x_1, y_1, z_1, w_1, z'_1, y'_1, x_2, x_m\} - B_i - C_i - D_i)$, $F_2^{(i)} = A(G'''_2 - \{w_1, y'_1\})$, $F_3^{(i)} = A(G'''_3 - \{w_1, z'_1\} - B_i)$, $F_4^{(i)} = A(G'''_4 - \{x_m, z'_1\} - D_i)$, and $F_5^{(i)} = A(G'''_5 - \{x_2, y'_1\} - C_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G'''_1| - 8 - |B_i| - |C_i| - |D_i|) + 3)/7 \rceil$,

$|F_2^{(i)}| \geq \lceil (4(|G_2'''|-2)+3)/7 \rceil = \lceil (4(|G_2'|-2)+3)/7 \rceil = (4(|G_2'|-2)+3)/7 + 3/7$, $|F_3^{(i)}| \geq \lceil (4(|G_3'''|-2-|B_i|)+3)/7 \rceil$, $|F_4^{(i)}| \geq \lceil (4(|G_4'''|-2-|D_i|)+3)/7 \rceil$, and $|F_5^{(i)}| \geq \lceil (4(|G_5'''|-2-|C_i|)+3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} \cup F_4^{(i)} \cup F_5^{(i)} + \{x_1, z_1, y_1\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)})) - (\{c_1\} \cap (F_1^{(i)} \triangle F_4^{(i)})) - (\{d_1\} \cap (F_1^{(i)} \triangle F_5^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + |F_4^{(i)}| + |F_5^{(i)}| + 3 - 1 - (1-|B_i|) - (1-|C_i|) - (1-|D_i|)$. By Lemma 2.2(1) (with $k=3, a=|G_1'''|-8, a_2=|G_3'''|-2, a_3=|G_4'''|-2, a_4=|G_5'''|-2, L=\{1\}, b_1=|G_2'''|-2, c=2$), $a(G) \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Now we distinguish several cases.

Case 1. Either $|N(z_1') \cap N(w_1)| \leq 2$ or $|N(y_1') \cap N(w_1)| \leq 2$; and either $|N(z_2') \cap N(w_2)| \leq 2$ or $|N(y_2') \cap N(w_2)| \leq 2$.

We may assume that $|N(y_1') \cap N(w_1)| \leq 2$ and $|N(y_2') \cap N(w_2)| \leq 2$. Let $F' = A((G - \{x_1, x_k, y_1, y_2, x\})/\{y_1'w_1, y_2'w_2\} + \{z_1x_2, z_2x_{k-1}\})$ with u_1 (respectively, u_2) as the identification of $\{y_1', w_1\}$ (respectively, $\{y_2', w_2\}$). Then $|F'| \geq \lceil (4(n-7)+3)/7 \rceil$. Let $F = F' + \{x_1, x_k, y_1, y_2\}$ if $u_1, u_2 \notin F'$, and otherwise F obtained from $F' + \{x_1, x_k, y_1, y_2\}$ by deleting $\{y_1, u_1\}$ (respectively, $\{y_2, u_2\}$) and adding $\{y_1', w_1\}$ (respectively, $\{y_2', w_2\}$). Therefore, F is an induced forest in G, showing $a(G) \geq |F'| + 4 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Then, we have (both $|N(z_1') \cap N(w_1)| \geq 3$ and $|N(y_1') \cap N(w_1)| \geq 3$) or (both $|N(z_2') \cap N(w_2)| \geq 3$ and $|N(y_2') \cap N(w_2)| \geq 3$). Suppose $|N(z_1') \cap N(w_1)| \geq 3$ and $|N(y_1') \cap N(w_1)| \geq 3$.

Case 2. $|N(z_2') \cap N(w_2)| \leq 2$ or $|N(y_2') \cap N(w_2)| \leq 2$.

We may assume $|N(y_2') \cap N(w_2)| \leq 2$. There exist $a_1 \in N(y_1') \cap N(w_1)$ and a separation (G_1, G_2) of G such that $V(G_1 \cap G_2) = \{w_1, y_1', a_1\}$, $\{x, x_1, x_2, x_3, z_1, z_2\} \subseteq V(G_1)$, and $N(y_1') \cap N(w_1) - \{y_1\} \subseteq V(G_2)$. Let $A_1 = \emptyset$ and $A_2 = \{a_1\}$. For $i = 1, 2$, let $F_1^{(i)} = A((G_1 - \{x_1, x_k, y_1, y_2, x, y_1', w_1\} - A_i)/y_2'w_2 + z_2x_{k-1})$ with u as the identification of $\{y_2', w_2\}$, and $F_2^{(i)} = A(G_2 - \{y_1', w_1\} - A_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |A_i|) + 3)/7 \rceil$, and $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |A_i|) + 3)/7 \rceil$. Let $F = F_1^{(i)} \cup F_2^{(i)} + \{x_1, x_k, y_1, y_2\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))$. Now $G[F]$ (if $u \notin F_1^{(i)}$) or $G[F - \{u, y_2\} + \{y_2', w_2\}]$ (if $u \in F_1^{(i)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + 4 - (1 - |A_i|)$. By Lemma 2.2(2) (with $a = |G_1| - 8, a_2 = |G_2| - 2, c = 4$), $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (4, 0), (0, 4) \mod 7$. By *Claim 2*, we have $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (4, 0) \mod 7$. So assume it's the case. Let $F_1^{(3)} = A((G_1 - \{x_1, x_k, y_1, y_2, x\})/\{y_1'w_1, y_2'w_2\} + \{z_1x_2, z_2x_{k-1}\})$ with

u_1 (respectively, u_2) as the identification of $\{y_1', w_1\}$ (respectively, $\{y_2', w_2\}$), and $F_2^{(3)} = A(G_2)$. Then $|F_1^{(3)}| \geq \lceil (4(|G_1|-7)+3)/7 \rceil$, and $|F_2^{(3)}| \geq \lceil (4|G_2|+3)/7 \rceil$. Let $F^{(3)} = G[\overline{F_1}^{(3)} \cup F_2^{(3)} - (\{w_1, y_1', a_1\} \cap (\overline{F_1}^{(3)} \triangle F_2^{(3)}))]$, where $\overline{F_1}^{(3)} = F_1{}^{(3)} + \{x_1, x_k, y_1, y_2\}$ if $u_1, u_2 \notin F_1^{(3)}$, and otherwise, $\overline{F_1}^{(3)}$ obtained from $F_1{}^{(3)} + \{x_1, x_k, y_1, y_2\}$ by deleting $\{y_1, u_1\}$ (respectively, $\{y_2, u_2\}$) and adding $\{y_1', w_1\}$ (respectively, $\{y_2', w_2\}$). Therefore, $F^{(3)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(3)}| + |F_2^{(3)}| + 4 - 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

Case 3. $|N(z_2') \cap N(w_2)| \geq 3$ and $|N(y_2') \cap N(w_2)| \geq 3$.

There exist $a_1 \in N(y_1') \cap N(w_1)$, $c_1 \in N(y_2') \cap N(w_2)$, $b_1 \in N(z_1') \cap N(w_1)$, $d_1 \in N(z_2') \cap N(w_2)$ and subgraphs G_1, G_2, G_3, G_4, G_5 such that G_2 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $y_1 y_1' a_1 w_1 y_1$ and containing $N(y_1') \cap N(w_1) - \{y_1\}$, G_3 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $y_2 y_2' c_1 w_2 y_2$ and containing $N(y_2') \cap N(w_2) - \{y_2\}$, G_4 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $z_1 z_1' b_1 w_1 z_1$ and containing $N(z_1') \cap N(w_1) - \{z_1\}$, G_5 is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $z_2 z_2' d_1 w_2 z_2$ and containing $N(z_2') \cap N(w_2) - \{z_2\}$, and G_1 is obtained from G by removing $G_2 - \{y_1', a_1, w_1\}$, $G_3 - \{y_2', c_1, w_2\}$, $G_4 - \{z_1', b_1, w_1\}$ and $G_5 - \{z_2', d_1, w_2\}$. Let $A_i \subseteq \{a_1\}$, $B_i \subseteq \{b_1\}$, $C_i \subseteq \{c_1\}$ and $D_i \subseteq \{d_1\}$. Let $G_1' = G_1 \cup G_4 \cup G_5$. For each choice of A_i, C_i, let $F_1^{(i)} = A(G_1' - \{x_1, y_1, y_1', w_1, x, x_k, y_2', w_2, y_2\} - A_i - C_i + \{z_1 x_2, z_2 x_{k-1}\})$, $F_2^{(i)} = A(G_2 - \{w_1, y_1'\} - A_i)$, and $F_3^{(i)} = A(G_3 - \{w_2, y_2'\} - C_i)$. Then $|F_1^{(i)}| \geq \lceil (4(n+6-|G_2|-|G_3|-9-|A_i|-|C_i|)+3)/7 \rceil$, $|F_2^{(i)}| \geq \lceil (4(|G_2|-2-|A_i|)+3)/7 \rceil$, and $|F_3^{(i)}| \geq \lceil (4(|G_3|-2-|C_i|)+3)/7 \rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{x_1, y_1, x_k, y_2\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{c_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 4 - (1-|A_i|) - (1-|C_i|)$. By Lemma 2.2(5) (with $a = n+6-|G_2|-|G_3|-9, a_1 = |G_2|-2, a_2 = |G_3|-2, c = 4$), $4(|G_2|-2)+3 \equiv 0, 3, 4, 6 \mod 7$ and $4(|G_3|-2)+3 \equiv 0, 3, 4, 6 \mod 7$. By *Claim 2* and by symmetry, we have $4(|G_i|-2)+3 \equiv 0, 3, 6 \mod 7$ for $i = 2, 3, 4, 5$ and if $4(|G_2|-2)+3 \equiv 3, 6 \mod 7$ or $4(|G_4|-2)+3 \equiv 3, 6 \mod 7$, then $4(|G_3|-2)+3 \equiv 4(|G_5|-2)+3 \equiv 0 \mod 7$ and vice versa.

If $4(|G_2|-2)+3 \equiv 3, 6 \mod 7$, then $4(|G_3|-2)+3 \equiv 0 \mod 7$. Let $A_1 = \{a_1\}$ if $4(|G_2|-2)+3 \equiv 6 \mod 7$ and $A_1 = \emptyset$ otherwise. Let $F_1^{(1)} = A((G_1' - \{x_1, y_1, y_1', w_1, x, x_k, y_2\} - A_1)/y_2' w_2 + \{z_1 x_2, z_2 x_{k-1}\})$ with u as the identification of $\{y_2', w_2\}$, $F_2^{(1)} = A(G_2 - \{w_1, y_1'\} - A_1)$, and $F_3^{(1)} = A(G_3)$. Then $|F_1^{(1)}| \geq \lceil (4(n+6-|G_2|-|G_3|-8-|A_1|)+3)/7 \rceil$, $|F_2^{(1)}| \geq \lceil (4(|G_2|-$

$2-|A_1|)+3)/7\rceil$, and $|F_3^{(1)}| \geq \lceil (4|G_3|+3)/7\rceil$. Now $G[F_1^{(1)} \cup F_2^{(1)} \cup F_3^{(1)} + \{x_1, y_1, x_k, y_2\} - (\{a_1\} \cap (F_1^{(1)} \triangle F_2^{(1)})) - (\{w_2, y_2', c_1\} \cap (F_1^{(1)} \triangle F_3^{(1)}))]$ (if $u \notin F_1^{(1)}$) or $G[(F_1^{(1)} - u) \cup F_2^{(1)} \cup F_3^{(1)} + \{x_1, y_1, x_k, y_2', w_2\} - (\{a_1\} \cap (F_1^{(1)} \triangle F_2^{(1)})) - (\{w_2, y_2', c_1\} \cap ((F_1^{(1)} \cup \{y_2', w_2\}) \triangle F_3^{(1)}))]$ (if $u \in F_1^{(1)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + |F_3^{(1)}| + 4 - (1-|A_1|) - 3 \geq \lceil (4n+3)/7\rceil$, a contradiction.

So $4(|G_2|-2)+3 \equiv 4(|G_3|-2)+3 \equiv 0 \mod 7$ by symmetry. Let $F_1^{(2)} = A((G_1' - \{x_1, y_1, x, x_k, y_2\})/\{y_1'w_1, y_2'w_2\} + \{z_1x_2, z_2x_{k-1}\})$ with u_1,u_2 as the identification of $\{y_1', w_1\}$, $\{y_2', w_2\}$ respectively, and $F_2^{(2)} = A(G_2)$ and $F_3^{(2)} = A(G_3)$. Then $|F_1^{(2)}| \geq \lceil (4(|G_1|-7)+3)/7\rceil$, $|F_2^{(2)}| \geq \lceil (4|G_2|+3)/7\rceil$, and $|F_3^{(2)}| \geq \lceil (4|G_3|+3)/7\rceil$. Let $F^{(2)} = G[\overline{F_1}^{(2)} \cup F_2^{(2)} \cup F_3^{(2)} - (\{w_1, y_1', a_1\} \cap (\overline{F_1}^{(2)} \triangle F_2^{(2)})) - (\{w_2, y_2', c_1\} \cap (\overline{F_1}^{(2)} \triangle F_3^{(2)}))]$, where $\overline{F_1}^{(2)} = F_1{}^{(2)} + \{x_1, x_k, y_1, y_2\}$ if $u_1, u_2 \notin F_1^{(2)}$, and otherwise, $\overline{F_1}^{(2)}$ obtained from $F_1{}^{(3)} + \{x_1, x_k, y_1, y_2\}$ by deleting $\{y_1, u_1\}$ (respectively, $\{y_2, u_2\}$) and adding $\{y_1', w_1\}$ (respectively, $\{y_2', w_2\}$) when $u_1 \in F_1^{(2)}$ (respectively, $u_2 \in F_1^{(2)}$). Therefore, $F^{(2)}$ is an induced forest in G, showing $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + |F_3^{(2)}| + 4 - 6 \geq \lceil (4n+3)/7\rceil$, a contradiction. $\square$

Lemma 7.6. *The following configuration is impossible in G: v is a 5-vertex of type 5-2-C with neighbors v_1, v_2, v_3, v_4, v_5 in cyclic order, $\{v_1, v_3\} \subseteq V_3$, vv_4xv_5 is a facial cycle, v_4 is a 5-vertex of type 5-2-B with neighbors x, v, v_3', v_4', v_4'' in cyclic order, $v_5 \in V_4, \{x, v_4'\} \subseteq V_3$, $v_4v_3'yv_4'v$, $v_4v_4'zv_4''v$, $v_4v_4''x'xv$ are facial cycles and $x' \in V_4$.*

Proof. Let $N(v_5) = \{x, v, v_5', v_1'\}$ where $x'v_5' \in E(G)$ and $v_1v_1' \in E(G)$.

Case 1. $N(v_4'') \cap N(v_5') = \{x'\}$ and $|N(z) \cap N(y)| \leq 2$.

Let $F' = A((G - \{x', x, v_5, v_1', v_1, v_4, v, v_4', v_3', v_3\})/\{v_4''v_5', yz\})$ with v', y' as the identification of $\{v_4'', v_5'\}, \{y, z\}$ respectively. Then $|F'| \geq \lceil (4(n-12)+3)/7\rceil$. Let $F = F' + \{v_1, v_3, v_4, v_5, x, x', v_4'\}$ if $v', y' \notin F'$, and otherwise, F obtained from $F' + \{v_1, v_3, v_4, v_5, x, x', v_4'\}$ by deleting $\{y', v_4'\}$ (respectively, $\{v', x'\}$) and adding $\{y, z\}$ (respectively, $\{v_4'', v_5'\}$) when $y' \in F'$ (respectively, $v' \in F'$). Therefore, F is an induced forest in G, showing $a(G) \geq |F'| + 7 \geq \lceil (4n+3)/7\rceil$, a contradiction.

Case 2. $N(v_4'') \cap N(v_5') = \{x'\}$ and $|N(z) \cap N(y)| > 2$.

There exist $a_1 \in N(z) \cap N(y)$ and a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{y, z, a_1\}$, $\{v, v_1, v_2, v_3, v_4, v_5, x, x', v_4, v_4'', v_3'\} \subseteq V(G_1)$, and $N(z) \cap N(y) - \{v_4'\} \subseteq V(G_2)$. Let $A_1 = \emptyset$ and $A_2 = \{a_1\}$. For $i = 1, 2$, let $F_1^{(i)} = A((G_1 - \{x', x, v_5, v_1', v_1, v_4, v, v_3', v_3, y, z,\ v_4'\} - A_i)/v_4''v_5')$ with v' as the identification of $\{v_4'', v_5'\}$, and $F_2^{(i)} = A(G_2 - \{y, z\} - A_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 13 - |A_i|) + 3)/7 \rceil$, and $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |A_i|) + 3)/7 \rceil$. Let $F = F_1^{(i)} \cup F_2^{(i)} + \{v_1, v_3, v_4, v_5, x, x', v_4'\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))$. Now $G[F]$ (if $v' \notin F_1^{(i)}$) or $G[F - \{v', x'\} + \{v_4'', v_5'\}]$ (if $v' \in F_1^{(i)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + 7 - (1 - |A_i|)$. By Lemma 2.2(1) (with $k = 1, a = |G_1| - 13, a_1 = |G_2| - 2, L = \emptyset, c = 7$), $a(G) \geq \lceil (4n + 3)/7 \rceil$, a contradiction.

Case 3. $|N(v_4'') \cap N(v_5')| > 1$.

There exist $b_1 \in N(v_4'') \cap N(v_5')$ and a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{v_4'', v_5, x', b_1\}$, $\{v, v_1, v_2, v_3, v_4, v_5, x, x', v_4, v_4'', v_3'\} \subseteq V(G_1)$, and $N(v_4'') \cap N(v_5') - \{b_1\} \subseteq V(G_2)$. Let $F_1^{(1)} = A(G_1 - \{x', x, v_5, v_1', v_1, v_4, v, v_3', v_3, v_4'', v_5', b_1\})$, and $F_2^{(1)} = A(G_2 - \{x', v_4'', v_5', b_1\})$. Then $|F_1^{(1)}| \geq \lceil (4(|G_1| - 12) + 3)/7 \rceil$, and $|F_2^{(1)}| \geq \lceil (4(|G_2| - 4) + 3)/7 \rceil$. Then $G[F_1^{(1)} \cup F_2^{(1)} + \{v_1, v_3, v_4, v_5, x, x'\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(1)}| + |F_2^{(1)}| + 6$. If $v_4'b_1 \notin E(G)$, let $F_1^{(2)} = A(G_1 - \{x', x, v_5, v_1', v_1, v_4, v, v_3', v_3, v_4'', v_5'\} + v_4'b_1)$, and $F_2^{(2)} = A(G_2 - \{x', v_4'', v_5'\})$. Then $|F_1^{(2)}| \geq \lceil (4(|G_1| - 11) + 3)/7 \rceil$, and $|F_2^{(2)}| \geq \lceil (4(|G_2| - 3) + 3)/7 \rceil$. Now $G[F_1^{(2)} \cup F_2^{(2)} + \{v_1, v_3, v_4, v_5, x, x'\} - (\{b_1\} \cap (F_1^{(2)} \triangle F_2^{(2)}))]$ is an induced forest in G, showing $a(G) \geq |F_1^{(2)}| + |F_2^{(2)}| + 6 - 1$. By Lemma 2.2(1) (with $k = 1, a = |G_1| - 11, a_1 = |G_2| - 3, L = \emptyset, c = 6$), $a(G) \geq \lceil (4n + 3)/7 \rceil$, a contradiction. So $v_4'a_1 \in E(G)$. Let $F_1^{(3)} = A(G_1 - \{x', x, v_5, v_1', v_1, v_4, v, v_3', v_3, v_4'', v_5', b_1, v_4'\})$, and $F_2^{(3)} = A(G_2 - \{x', v_4'', v_5', b_1\})$. Then $|F_1^{(3)}| \geq \lceil (4(|G_1| - 13) + 3)/7 \rceil$, and $|F_2^{(3)}| \geq \lceil (4(|G_2| - 4) + 3)/7 \rceil$. Now $G[F_1^{(3)} \cup F_2^{(3)} + \{v_1, v_3, v_4, v_5, x, x', v_4'\}]$ is an induced forest in G, showing $a(G) \geq |F_1^{(3)}| + |F_2^{(3)}| + 7 \geq \lceil (4n + 3)/7 \rceil$, a contradiction. □

Lemma 7.7. *The following configuration is impossible in G: v is a 5-vertex of type 5-2-C with neighbors v_1, v_2, v_3, v_4, v_5 in cyclic order, $\{v_1, v_3\} \subseteq V_3$; vv_4xv_5v is a facial cycle. v_4 is a 5-vertex of type 5-1-A with neighbors x, v, v_3', v_4', v_4'' in cyclic order, $v_5 \in V_4, x \in V_3$; $v_4v_3'yv_4'v_4$, $v_4v_4'zv_4''v_4$, $v_4v_4''x'xv_4$ are facial cycles, $\{y, z\} \subseteq V_3$, and $x' \in V_{\leq 4}$.*

Proof. Let $xx'v_5'v_5$, $vv_5v_1'v_1$ bound 4-faces. Let $F' = A(G - \{v_5', v_1', x', v_5, x, v_1, v_4'', v_4, v, z, v_4',\ y, v_3', v_3\})$. Then $|F'| \geq \lceil (4(n - 14) + 3)/7 \rceil$. Now $G[F' + \{x', x, v_5, v_1, v_3, v_4, y, z\}]$ is an induced forest in G, showing $a(G) \geq |F'| + 8 \geq \lceil (4n + 3)/7 \rceil$, a contradiction. □

Lemma 7.8. *The following configuration is impossible in G: v is a 5-vertex of type 5-2-B with neighbors v_1, v_2, v_3, v_4, v_5 in cyclic order, $\{v_1, v_3\} \subseteq V_3$, $N(v_1) = \{v, v_1', v_1''\}$, $N(v_3) = \{v, v_3', v_3''\}$, $v_1'' \in V_{\geq 5}$ and $\{v_3', v_3''\} \subseteq V_4$; $vv_1v_1'v_2v$, $vv_2v_3'v_3v$, $vv_3v_3''v_4v$, $vv_5v_1''v_1v$ are facial cycles. v_2 is a 5-vertex of type 5-2-C with neighbors $v, v_1', v_2', v_2'', v_3'$ in cyclic order, $\{v_1', v_2''\} \subseteq V_3$.*

Proof. Let $t \in N(v_2'') \cap N(v_3')$ and $v_2v_2''tv_3'v_2$ bound a 4-face. Let $N(v_3'') = \{v_3, v_4, s_1, s_2\}$ and $v_3v_3's_2v_3''v_3$ bound a 4-face. Let $w \in N(v_1'') \cap N(v_1')$ and $v_1v_1''wv_1'v_1$ bound a 4-face.

By Lemma 4.1, $v_3'v_4 \notin E(G)$. We claim that $v_2v_1'' \notin E(G)$. Since G is simple, $v_1'' \notin \{v, v_1'\}$. Since $v_1'' \in V_{\geq 5}$, $v_1'' \notin \{v_3', v_2''\}$. If $v_2' = v_1''$, then since G is a quadrangulation, $v_2v_1''wv_1'v_2$ bound a 4-face and thus $N(w) = \{v_1'', v_1'\}$. But this contradicts Lemma 4.2.

If $|N(s_1) \cap N(s_2)| \leq 2$, then let $F' = A(G - \{v, v_1, v_1', v_3, v_3'', w\}/s_1s_2 + \{v_3'v_4, v_2v_1''\})$ with s' the identification of $\{s_1, s_2\}$. Then $|F'| \geq \lceil (4(n-7)+3)/7 \rceil$. Now $G[F' + \{v_3, v_3'', v_1, v_1'\}]$ (if $s' \notin F'$) or $G[F' - s' + \{v_3, s_1, s_2, v_1, v_1'\}]$ (if $s' \in F'$) is an induced forest in G, showing $a(G) \geq |F'| + 4 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

So $|N(s_1) \cap N(s_2)| > 2$. There exist $a_1 \in N(s_1) \cap N(s_2)$ and a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{s_1, s_2, a_1\}$, $\{v, v_1, v_2, v_3, v_4, v_5, v_1', v_1'', v_2', v_2'', t, v_3', v_3''\} \subseteq V(G_1)$, and $N(s_1) \cap N(s_2) - \{v_3''\} \subseteq V(G_2)$. Let $A_1 = \emptyset$ and $A_2 = \{a_1\}$. For $i = 1, 2$, let $F_1^{(i)} = A(G_1 - \{v, v_1, v_1', v_3, v_3'', w, s_1, s_2\} - A_i + \{v_3'v_4, v_2v_1''\})$ with s' the identification of $\{s_1, s_2\}$, and $F_2^{(i)} = A(G_2 - \{s_1, s_2\} - A_i)$. Then $|F_1^{(i)}| \geq \lceil (4(|G_1| - 8 - |A_i|) + 3)/7 \rceil$, and $|F_2^{(i)}| \geq \lceil (4(|G_2| - 2 - |A_i|) + 3)/7 \rceil$. Now $F_1^{(i)} \cup F_2^{(i)} + \{v_3, v_3'', v_1, v_1'\} - (\{a_1\} \cap (F_1^{(i)} \triangle F_2^{(i)}))$ is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + 4 - (1 - |A_i|)$. By Lemma 2.2(2) (with $a = |G_1| - 8, a_1 = |G_2| - 2, L = \emptyset, c = 4$), $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (0, 4), (4, 0) \mod 7$.

If $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (4, 0) \mod 7$, then let $F_1^{(3)} = A(G_1 - \{v, v_1, v_1', v_3, v_3'', w\}/s_1s_2 + \{v_3'v_4, v_2v_1''\})$ with s' the identification of $\{s_1, s_2\}$, and $F_2^{(3)} = A(G_2)$. Then $|F_1^{(3)}| \geq \lceil (4(|G_1| - 7) + 3)/7 \rceil$, and $|F_2^{(3)}| \geq \lceil (4|G_2| + 3)/7 \rceil$. Now $G[F_1^{(3)} \cup F_2^{(3)} + \{v_3, v_3'', v_1, v_1'\} - (\{a_1, s_1, s_2\} \cap (F_1^{(3)} \triangle F_2^{(3)}))]$ (if $s' \notin F_1^{(3)}$) or $G[F_1^{(3)} \cup F_2^{(3)} - s' + \{v_3, s_1, s_2, v_1, v_1'\} - (\{a_1, s_1, s_2\} \cap ((F_1^{(3)} + \{s_1, s_2\}) \triangle F_2^{(3)}))]$ (if $s' \in F_1^{(3)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(3)}| + |F_2^{(3)}| + 4 - 3 \geq \lceil (4n+3)/7 \rceil$, a contradiction.

So $(4(|G_1| - 8) + 3, 4(|G_2| - 2) + 3) \equiv (0, 4) \mod 7$. First, we claim that $vt \notin E(G)$. $t \notin \{v_2, v_3\}$ since G is simple. $t \neq v_4$ by Lemma 4.1. $t \neq v_1$ since $v_1'' \in V_{\geq 5}$ and $v_1' \in V_3$. Suppose $t = v_5$. Since G is a quadrangulation, $v_2''v_5 \in E(G)$. let $F_4 = A(G - \{v_3, v_3', v_3'', s_2, v_4, v, v_2, v_2'', v_1, v_1', v_5, w\})$. Then $|F_4| \geq$

$\lceil(4(n-12)+3)/7\rceil$. Now $G[F_4+\{v_3'', v_3, v_3', v, v_2'', v_1, v_1'\}]$ is an induced forest in G, showing $a(G) \geq |F_4|+7 \geq \lceil(4n+3)/7\rceil$, a contradiction. Secondly, suppose $|N(s_1)\cap N(v_4)| \leq 2$. Then let $F_1^{(5)} = A(G_1 - \{v_3'', s_2, v_3, v_3', v_2\}/s_1v_4 + \{vt\})$ with s' the identification of $\{s_1, v_4\}$, and $F_2^{(5)} = A(G_2 - s_2)$. Then $|F_1^{(5)}| \geq \lceil(4(|G_1|-6)+3)/7\rceil$, and $|F_2^{(5)}| \geq \lceil(4(|G_2|-1)+3)/7\rceil$. Now $G[F_1^{(5)} \cup F_2^{(5)} + \{v_3, v_3'', v_3'\} - (\{a_1, s_1\} \cap (F_1^{(5)} \triangle F_2^{(5)}))]$ (if $s' \notin F_1^{(5)}$) or $G[F_1^{(5)} \cup F_2^{(5)} - s' + \{s_1, v_4, v_3, v_3'\} - (\{a_1, s_1\} \cap ((F_1^{(5)} + s_1) \triangle F_2^{(5)}))]$ (if $s' \in F_1^{(5)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(5)}| + |F_2^{(5)}| + 3 - 2 \geq \lceil(4n+3)/7\rceil$, a contradiction. Now, $|N(s_1)\cap N(v_4)| > 2$. There exist $a_1 \in N(s_1)\cap N(s_2)$, $b_1 \in N(s_1)\cap N(v_4)$ and subgraphs G_1', G_2', G_3' such that $G_2' = G_2$, G_3' is the maximal subgraph of G contained in the closed region of the plane bounded by the cycle $v_3''s_1b_1v_4v_3''$ and containing $N(s_1)\cap N(v_4) - \{v_3''\}$, and G_1' is obtained from G by removing $G_2' - \{s_1, a_1, s_2\}$ and $G_3' - \{s_1, b_1, v_4\}$. Let $B_7 = \{b_1\}$ and $B_8 = \emptyset$. For $i = 7, 8$, let $F_1^{(i)} = A((G_1' - \{v_3, v_3'', s_1, v_4\} - B_i)/vv_3')$ with v' as the identification of $\{v, v_3'\}$, $F_2^{(i)} = A(G_2' - \{s_1\})$, and $F_3^{(i)} = A(G_3' - \{s_1, v_4\} - B_i)$. Then $|F_1^{(i)}| \geq \lceil(4(|G_1'|-5-|B_i|)+3)/7\rceil$, $|F_2^{(i)}| \geq \lceil(4(|G_2'|-1)+3)/7\rceil = \lceil(4(|G_2|-1)+3)/7\rceil = (4(|G_2|-1)+3)/7+6/7$, and $|F_3^{(i)}| \geq \lceil(4(|G_3'|-2-|B_i|)+3)/7\rceil$. Now $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} + \{v_3, v_3''\} - (\{a_1, s_2\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))]$ (if $v' \notin F_1^{(i)}$) or $G[F_1^{(i)} \cup F_2^{(i)} \cup F_3^{(i)} - v' + \{v, v_3', v_3''\} - (\{a_1, s_2\} \cap (F_1^{(i)} \triangle F_2^{(i)})) - (\{b_1\} \cap (F_1^{(i)} \triangle F_3^{(i)}))]$ (if $v' \in F_1^{(i)}$) is an induced forest in G, showing $a(G) \geq |F_1^{(i)}| + |F_2^{(i)}| + |F_3^{(i)}| + 2 - 2 - (1 - |B_i|)$. By Lemma 2.2(1) (with $k = 1, a = |G_1'| - 5, a_1 = |G_3'| - 2, L = \{1\}, b_1 = |G_2'| - 1, c = 0$), $a(G) \geq \lceil(4n+3)/7\rceil$, a contradiction. $\square$

8. Proof of Theorem 1.2

We define the discharging rules as follow: For each $v \in V(G)$, let $ch(v) := |N(v)| - 4$. Let $\mathcal{F}$ be the set of all the faces of graph G. For each $f \in \mathcal{F}$, let $ch(f) := |f| - 4$. Then, by Euler's Formula, the total charge of graph G is

$$\begin{aligned} \sum_{v\in V(G)} ch(v) + \sum_{f\in\mathcal{F}} ch(f) &= \sum_{v\in V(G)} (N(v)-4) + \sum_{f\in\mathcal{F}} (|f|-4) \\ &= 4|E(G)| - 4|V(G)| - 4|\mathcal{F}| \\ &= -8 \end{aligned}$$

Definition 8.1. *For $v \in V(G)$, suppose $|N(v)| \geq 5$. We redistribute the charges as follow:*

(i) *Suppose* $R_{v,U} \neq \emptyset$ *for some* $U \subseteq V(G)$*. If* $R_{v,U} = \{\{r\}\}$*, then* v *sends charge* $|N(v)|-4$ *to* r*; If* $R_{v,U} = \{\{r_1, r_2\}\}$*, then* v *sends charge* $(|N(v)|-4)/2$ *to both* r_1 *and* r_2*; If* $R_{v,U} = \{R_1, R_2\}$*, then* v *sends charge* $|N(v)|-4$ *to* $R_1 \cap R_2$*;*

(ii) *Suppose* $R_{v,\{u\}} = \emptyset$ *and* $vu \in E(G)$ *for some* $u \in V_3$*. Let* $N(u) = \{v, u_1, u_2\}$*. If for both* $w \in \{u_1, u_2\}$*, either* $w \in V_{\leq 4}$ *or* $R_{w,\{u\}} \neq \emptyset$*, then* v *sends charge* 1 *to* u*; If* $u_2 \in V_{\geq 5}$ *and* $R_{u_2,\{u\}} = \emptyset$*, then* v *sends charge* $1/2$ *to* u*;*

(iii) *Suppose* $R_{v,\{u\}} = \emptyset$ *and* $xwyvx$ *is a facial cycle such that* $\{x, y\} \subseteq N(v), x \in V_{\geq 5}$*,* $w \in V_3$ *and* $y \in V_4$*. If neither* v *nor* x *is of type 5-2-C, then* v *sends charge* $1/4$ *to* x*.*

We denote the new charge of v as $ch'(v)$. We remark that if v sends charge $1/4$ to x in both faces bounded by xw_1y_1vx and xw_2y_2vx by Definition 8.1 (iii), then v sends charge $1/2$ to x.

We show that for $v \in V(G)$, $ch'(v) \geq 0$. If $|N(v)| = 2$, then by Lemma 3.1 and Definition 8.1 (i), v either receives at least 1 from $\{v_5, v_5'\} \subseteq V_{\geq 5} \cap N(v)$ where $R_{v_5,\{v\}} = R_{v_5',\{v\}} = \emptyset$ or at least 2 from $v_6 \in V_{\geq 6} \cap N(v)$ where $R_{v_6,\{v\}} = \emptyset$. Hence, $ch'(v) \geq ch(v) + 2 = 0$. Suppose $|N(v)| = 3$ with $N(v) = \{v_1, v_2, v_3\}$. If $R_{v_3,\{v\}} \neq \emptyset$, then by Lemmas 4.3, 4.4, $\{v_1, v_2\} \subseteq V_{\geq 5}$, $R_{v_1,\{v\}} = R_{v_2,\{v\}} = \emptyset$; thus, by Definition 8.1(ii), v receives $1/2$ from each of v_1 and v_2, and $ch'(v) = ch(v) + 1/2 + 1/2 = 0$. Now, assume $R_{v_i,\{v\}} = \emptyset$ for $i = 1, 2, 3$. By Corollary 4.6, there exists $v_1 \in N(v) \cap V_{\geq 5}$. By Definition 8.1(ii), v receives at least 1 from $N(v)$ and thus $ch'(v) \geq ch(v) + 1 = 0$. If $|N(v)| = 4$, then v does not receive or send charge to other vertices. Therefore, $ch'(v) = ch(v) = 0$. If $|N(v)| \geq 5$ and $R_{v,\emptyset} \neq \emptyset$, then by Definition 8.1(i) and Lemma 2.5, v sends $|N(v)| - 4$ to $R_{v,\emptyset}$ only. Therefore, $ch'(v) = ch(v) - (|N(v)| - 4) = 0$.

Next, assume $|N(v)| \geq 5$ and $R_{v,\emptyset} = \emptyset$. We distinguish the cases by Definition 7.1:

- v is of type 5-2-A. By Lemma 7.5, v does not exist in G, a contradiction;
- v is of type 5-2-B. Let $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$ in order. Let $\{v_1, v_3\} \subseteq V_3$, $\{v_2, v_4, v_5\} \subseteq V_{\geq 4}$, $N(v_1) = \{v_1', v_1'', v\}$, $N(v_3) = \{v_3', v_3'', v\}$, $u \in V_{\leq 4}$ or $R_{u,\{v_3\}} \neq \emptyset$ for $u \in \{v_3', v_3''\}$ and $v_1' \in V_{\geq 5}$, $R_{v_1',\{v_1\}} = \emptyset$. By Definition 8.1(ii), v sends $1/2$ to v_1, 1 to v_3. By Lemma 4.3, 4.4, 6.1, $R_{u,\{v_3\}} = \emptyset$ and $u \in V_4$ for $u \in \{v_3', v_3''\}$. By Lemma 5.1, $\{v_2, v_4\} \subseteq V_{\geq 5}$ and $R_{v_2,\{v\}} = R_{v_4,\{v\}} = \emptyset$. By Lemma 7.6, v_4 is not of type 5-2-C. By Lemma 7.8, v_2 is not of type 5-2-C. Hence, v receives $1/4$ from each of v_2 and v_4 by Definition 8.1(iii). In addition, by Lemma 7.3, v does not send charge to v_4. So $ch'(v) = ch(v) - 1 - 1/2 + 1/4 + 1/4 = 0$;

- v is of type 5-2-C. Let $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$ in order. Let $\{v_1, v_3\} \subseteq V_3$, $\{v_2, v_4, v_5\} \subseteq V_{\geq 4}$, $N(v_1) = \{v_1', v_1'', v\}$, $N(v_3) = \{v_3', v_3'', v\}$, $v_1' \in V_{\geq 5}$, $R_{v_1', \{v_1\}} = \emptyset$, $v_3' \in V_{\geq 5}$, and $R_{v_3', \{v_3\}} = \emptyset$. By Definition 8.1(ii)(iii), v sends $1/2$ to both v_1 and v_3. So $ch'(v) = ch(v) - 1/2 - 1/2 = 0$;
- v is of type 5-1-A. Let $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$ in order. Let $v_1 \in V_3, \{v_2, v_3, v_4, v_5\} \subseteq V_{\geq 4}, N(v_1) = \{v_1', v_1'', v\}$, and $u \in V_{\leq 4}$ or $R_{u, \{v_1\}} \neq \emptyset$ for $u \in \{v_1', v_1''\}$. Let $vv_2v_2'v_3v$, $vv_3v_3'v_4v$, $vv_4v_4'v_5v$ be facial cycles. By Lemmas 4.3, 4.4, 6.1, $R_{u, \{v_1\}} = \emptyset$ and $u \in V_4$ for $u \in \{v_1', v_1''\}$. By Lemma 5.1, $\{v_2, v_5\} \subseteq V_{\geq 5}$ and $R_{v_2, \{v\}} = R_{v_5, \{v\}} = \emptyset$. If v_2, v_5 are not of type 5-2-C, then by Definition 8.1(iii) v receives $1/4$ from each of v_2 and v_5. By Definition 8.1(ii)(iii), v sends 1 to v_1 and $1/4$ to at most two of $\{v_2, v_3, v_4, v_5\}$. So $ch'(v) \geq ch(v) - 1 - 1/4 \times 2 + 1/4 + 1/4 = 0$; If both v_2 and v_5 are of type 5-2-C, then $\{v_2', v_4'\} \subseteq V_3$ and by Lemma 7.7 $v_3' \notin V_3$. By Definition 8.1(ii)(iii), v sends 1 to v_1 and no charge to $\{v_2, v_3, v_4, v_5\}$. So $ch'(v) \geq ch(v) - 1 = 0$; If exactly one of v_2 and v_5 is of type 5-2-C, say v_2, then by Lemma 7.7, $|\{v_3', v_4'\} \cap V_3| \leq 1$. By Definition 8.1(ii)(iii), v sends 1 to v_1 and $1/4$ to at most one of $\{v_3, v_4, v_5\}$ and v receives $1/4$ from v_5. So $ch'(v) \geq ch(v) + 1/4 - 1 - 1/4 = 0$;
- v is of type 5-1-B. Let $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$ in order. Let $v_1 \in V_3, \{v_2, v_3, v_4, v_5\} \subseteq V_{\geq 4}, N(v_1) = \{v_1', v_1'', v\}$, $v_1' \in V_{\geq 5}$, and $R_{v_1', \{v_1\}} = \emptyset$. By Lemma 7.4 and Definition 8.1(ii)(iii), v sends $1/2$ to v_1 and $1/4$ to at most two of $\{v_2, v_3, v_4, v_5\}$. So $ch'(v) \geq ch(v) - 1/2 - 1/4 \times 2 = 0$;
- v is of type 5-0. By Definition 8.1(iii), $ch'(v) \geq ch(v) - 1/4 \times 4 = 0$.

Suppose $|N(v)| = 6$ and $R_{v, \emptyset} = \emptyset$. We distinguish the cases by Definition 7.2:

- v is of type 6-3. By Lemma 7.5 and Definition 8.1(ii), v sends 1 to at most one of $\{v_1, v_3, v_5\}$. By Definition 8.1(iii), v sends no charge to $\{v_2, v_4, v_6\}$. So $ch'(v) \geq ch(v) - 1 - 1/2 \times 2 = 0$;
- v is of type 6-2-A. By Lemma 7.5 and Definition 8.1(ii), v sends 1 to at most one of $\{v_1, v_3\}$. By Definition 8.1(iii), v sends no charge to v_2 and $1/4$ to at most two of $\{v_4, v_5, v_6\}$. So $ch'(v) \geq ch(v) - 1 - 1/2 - 2 \times 1/4 = 0$;
- v is of type 6-2-B. Let $N(v_1) = \{v_1', v_1'', v\}$ and $N(v_3) = \{v_3', v_3'', v\}$. By Lemma 7.5 and Definition 8.1(ii), v sends 1 to at most one of $\{v_1, v_4\}$. By Definition 8.1(iii), v sends $1/4$ to at most one of $\{v_2, v_3\}$ and to at most one of $\{v_5, v_6\}$. So $ch'(v) \geq ch(v) - 1 - 1/2 - 2 \times 1/4 = 0$;
- v is of type 6-1. By Definition 8.1(ii), v sends at most 1 to v_1. By Definition 8.1(iii), v sends $1/4$ to at most four of $\{v_2, v_3, v_4, v_5, v_6\}$. So $ch'(v) \geq ch(v) - 1 - 4 \times 1/4 = 0$;

- v is of type 6-0. By Definition 8.1(iii), $ch'(v) \geq ch(v) - 6 \times 1/4 = 1/2$.

Suppose $|N(v)| = 7$ and $R_{v,\emptyset} = \emptyset$. Let $N(v) := \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ in order. If $|N(v) \cap V_3| = 3$, then we may assume that they are v_1, v_3, v_5. By Lemma 7.5 and Definition 8.1(ii), v sends 1 to at most one of $N(v) \cap V_3$. So $ch'(v) \geq ch(v) - 1 - 2 \times 1/2 - 1/4 > 0$. If $|N(v) \cap V_3| \leq 2$, then by Definition 8.1(iii), v sends $1/4$ to at most three of $N(v)$. So, $ch'(v) \geq ch(v) - 1 - 1 - 3 \times 1/4 > 0$. Suppose $|N(v)| \geq 8$. We observe that if we amortize the redistribution of charge to all the faces which v is incident with, then v sends at most $1/2$ in each face. So $ch'(v) \geq ch(v) - 1/2 \times |N(v)| = |N(v)|/2 - 4 \geq 0$.

Therefore, $ch'(v) \geq 0$ for $v \in V(G)$. Since G is a quadrangulation by Lemma 2.3, $ch'(f) := ch(f) = 0$. Then, the total charge after redistribution is $\sum_{v \in V(G)} ch'(v) + \sum_{f \in \mathcal{F}} ch'(f) \geq 0$, which contradicts Euler's Formula.

To conclude, the minimum counterexample G does not exist. This completes the proof of Theorem 1.2.

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