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Random subgraphs make identification affordable

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An identifying code of a graph is a dominating set which uniquely determines all the vertices by their neighborhood within the code. Whereas graphs with large minimum degree have small domination number, this is not the case for the identifying code number (the size of a smallest identifying code), which indeed is not even a monotone parameter with respect to graph inclusion.

We show that for every large enough Δ, every graph G on n vertices with maximum degree Δ and minimum degree $\delta \geq c \log \Delta$, for some constant $c > 0$, contains a large spanning subgraph which admits an identifying code with size $O\left(\frac{n \log \Delta}{\delta}\right)$. In particular, if $\delta = \Theta(n)$, then G has a dense spanning subgraph with identifying code $O(\log n)$, namely, of asymptotically optimal size. The subgraph we build is created using a probabilistic approach, and we use an interplay of various random methods to analyze it. Moreover we show that the result is essentially best possible, both in terms of the number of deleted edges and the size of the identifying code.

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1. Introduction

Consider any graph parameter that is not monotone with respect to graph inclusion. Given a graph G, a natural problem in this context is to study the minimum value of this parameter over all spanning subgraphs of G. In particular, how many edge deletions are sufficient in order to obtain from G a graph with near-optimal value of the parameter? Herein, we use random methods to study this question with respect to the identifying code number of a graph, a well-studied non-monotone parameter. An identifying code of graph G is a set C of vertices which is a dominating set, and such that the closed neighborhood within C of each vertex v uniquely determines v.

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Identifying codes were introduced in 1998 in [14] and have been studied extensively in the literature since then (see [17] for an on-line bibliography and [4, 13, 15, 16, 22] for some of its applications).

Let G be a simple, undirected and finite graph. The *open neighborhood* of a vertex v in G is the set of vertices in $V(G)$ that are adjacent to it, and will be denoted $N_G(v)$. The *closed neighborhood* of a vertex v in G is defined as $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex $u \in V(G)$, is defined as $d(v) = |N_G(v)|$. Similarly, we define, for a set $S \subseteq V(G)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = \bigcup_{v \in S} N_G[v]$. If two distinct vertices u, v are such that $N[u] = N[v]$, they are called *twins*. A graph is called *twin-free* if there is no pair of twins. The symmetric difference between two sets A and B is denoted by $A \oplus B$. We use $\log(x)$ to denote the natural logarithm of x.

Given a graph G and a subset C of vertices of G, C is called a *dominating set* if each vertex of $V(G) \setminus C$ has at least one neighbor in C. The set C is called a *separating set* of G if for each pair u, v of vertices of G, $N[u] \cap C \neq N[v] \cap C$ (equivalently, $(N[u] \oplus N[v]) \cap C \neq \emptyset$). If $x \in N[u]$, we say that x *dominates* u. If $x \in N[u] \oplus N[v]$, we say that x *separates* u, v.

Definition 1.1. A subset of vertices of a graph G which is both a dominating set and a separating set is called an *identifying code* of G.

Observe that the notion of separating set is very close to the one of identifying code. The only difference is that in a separating set there is at most one vertex which is not dominated, while in an identifying code, every vertex is dominated. In particular, this implies that the size of a minimum separating set and the size of a minimum identifying code differ at most by one. Since all the results of this paper are asymptotic in the order of the graph, any result stated for identifying codes can be directly translated to separating sets.

The following observation gives an equivalent condition for a set to be an identifying code, and follows from the fact that for two vertices u, v at distance at least 3 from each other, $N[u] \oplus N[v] = N[u] \cup N[v]$.

Observation 1.2. *For a graph G and a set $C \subseteq V(G)$, if C is dominating and $N[u] \cap C \neq N[v] \cap C$ for each pair of vertices u, v at distance at most two from each other, then C is an identifying code of the graph.*

The minimum size of a dominating set of graph G, its *domination number*, is denoted by $\gamma(G)$. Similarly, the minimum size of an identifying code of G, $\gamma^{\text{ID}}(G)$, is the *identifying code number* of G. It is known [6, 12, 14] that for every twin-free graph G on n vertices having at least one edge, we have the following tight bounds:

$$\log_2(n+1) \leq \gamma^{\mathrm{ID}}(G) \leq n-1 \ .$$

In view of the above lower bound, we say that an identifying code C of G is *asymptotically optimal* if

$$|C| = O(\log n) \ .$$

The problem we address in this paper is to deal with graphs that have a large identifying code number, or are not even identifiable. Our approach will consist in slightly modifying such a graph in order to decrease its identifying code number and obtain an asymptotically optimal identifying code, unless its domination number prevents us from doing so.

One of the reasons for a graph to have a large identifying code number is that it has a large domination number (this one being a monotone parameter under edge deletion). For instance, we need roughly $n/3$ vertices to dominate all the vertices in a path of order n. When this is the case, we cannot expect to decrease much the size of a minimum identifying code by deleting edges from G, as the deletion of edges cannot decrease the domination number.

However, there are many graphs with small domination number where the identifying code number is very large [9, 10]. For instance, consider the complete graph minus a perfect matching: whilst the domination number is 2, it can be checked that the identifying code number is $n-1$. Typically, this phenomenon appears in graphs having a specific, "rigid", structure. Supporting this intuition, Frieze, Martin, Moncel, Ruszinkó and Smyth [11] have shown that the random graph $G(n,p)$ with $p \in (0,1)$, admits an asymptotically optimal identifying code. In particular, they prove in [11] that with probability $1-o(1)$, we have

$$\gamma^{\mathrm{ID}}(G(n,p)) = (1+o(1))\frac{2\log n}{\log(1/q)} \ ,$$

where $q = p^2 + (1-p)^2$. This suggests that the lack of structure in dense graphs implies the existence of a small identifying code.

We will use standard asymptotic notation for classes of functions that depend on n, the order of the graph.

Our results and structure of the paper. In Section 2, we prove our main result by selecting at random a small set of edges that can be deleted to "add some randomness" to the graph,

Theorem 1.3. *For every large enough Δ and every graph G on n vertices with maximum degree Δ and minimum degree $\delta \geq 66 \log \Delta$, there exists a subset of edges $F \subset E(G)$ of size*

$$|F| \leq 83 n \log \Delta \ ,$$

such that

$$\gamma^{ID}(G \setminus F) \leq 134 \frac{n \log \Delta}{\delta} \ .$$

Observe that when $\delta = \Theta(n)$, the upper bound in Theorem 1.3 has the same order of magnitude as the value obtained in [11] for dense random graphs.

We then show in Section 3 that our result is asymptotically best possible in terms of both the number of deleted edges and of the size of the final identifying code for any graph with $\Delta = \mathrm{Poly}(\delta)$. For smaller values of the minimum degree, we prove that our result is almost optimal. We also show that the two assumptions, large enough Δ and $\delta \geq c \log \Delta$, for some constant $c > 0$, are necessary.

We present some consequences of our result in Section 4. When considering the case of adding edges to the graph, we get analogous (symmetric) results, showing that every graph is a large spanning subgraph of some graph that admits a small identifying code. This result also turns out to be tight. We also describe an application to the closely related topic of *watching systems.*

The paper concludes with some final remarks and open problems.

Our methods. The proofs of our results are based on defining a suitable random subgraph of G: we first randomly choose a code C, and then we randomly delete edges among the edges containing vertices of C. We then analyze the construction by applying concentration inequalities and the Lovász Local Lemma.

A similar approach has been used in the literature when considering *random subgraphs of a graph*: for every graph G, consider the graph G_p to be the subgraph of G obtained by keeping *each* edge from $E(G)$ independently with probability p. Our random subgraph model is adapted to the analysis of identifying codes, and can be seen as a weighted version of G_p.

2. Main theorem

In this section, we prove Theorem 1.3. The proof is structured in the following steps:

1. We select a set C at random, where each vertex is selected independently with probability p. Let A_C be the event that C is small enough. The probability p is chosen to ensure that A_C holds with high probability. From C, we construct a spanning subgraph $G(C,\pi)$ of G, to be defined below, for some suitable function π.
2. We use the Lovász Local Lemma (Lemma 2.4) and Lemma 2.3 to bound from below the probability that the following events (whose conjunction we call A_{LL}) hold jointly: (i) in $G(C,\pi)$, each pair of vertices that are at distance at most 2 from each other are separated by C; and (ii) for each such pair and each member of this pair in G, its degree within C in G is close to its expected value $d(v)p$. We show that with nonzero probability, A_C and A_{LL} hold jointly.
3. We find a dominating set D of G with $|D| = O(|C|)$; by Observation 1.2, if A_{LL} holds, then $C \cup D$ is an identifying code.
4. Finally, we show that, conditional on A_C and A_{LL}, the expected number of deleted edges is as small as desired.

2.1. Important tools and lemmas

In this section we include some lemmas we will use for the proof of the main theorem.

We will repeatedly use the following version of the well-known Chernoff's inequalities for the sum of independent bounded random variables (see e.g. [2, Corollary A.1.14]):

Lemma 2.1. *Let $X_1, \dots, X_N$ be independent Bernoulli random variable with probability p_i and define $X = \sum_{i=1}^N X_i$. Then, for all $\varepsilon > 0$,*

$$\Pr(|X - \mathbb{E}(X)| \geq \varepsilon \mathbb{E}(X)) < 2e^{-c_\varepsilon \mathbb{E}(X)} ,$$

where

$$c_\varepsilon = \min\left\{(1+\varepsilon)\log(1+\varepsilon) - \varepsilon, \frac{\varepsilon^2}{2}\right\} .$$

In what follows, for each any of vertices $B \subseteq V(G)$ and any $v \in V(G)$, we let $N_G^B(v) = N_G(v) \cap B$ be the set of neighbors of v in B. Analogously, $N_G^B[v] = N_G[v] \cap B$. We denote by $d_B(v) = |N_G^B(v)|$, the degree of v within set B.

Definition 2.2. Given a graph G and $B \subseteq V(G)$, a function $\pi : V(G) \to [0,1]$ is said to be (G,B)–*bounded* if for each pair u, v of vertices with $d_B(u) \geq d_B(v)$, we have $\pi(u) \leq \pi(v)$. Given a (G,B)–bounded function π, we define the random spanning subgraph $G(B,\pi)$ of G as follows:

- $G(B,\pi)$ contains all edges of the subgraph $G[V(G)\setminus B]$ induced by $V(G)\setminus B$, and
- each edge uv incident with B is independently chosen to be in $G(B,\pi)$ with probability $1-p_{uv}$, where

$$p_{uv} = \frac{1}{4}\left(\pi(u)+\pi(v)\right) .$$

Observe that, since $\pi(u) \le 1$ for each vertex $u \in V(G)$, we have $p_{uv} \le 1/2$.

The next lemma gives an exponential upper bound on the probability that two vertices of $G(B,\pi)$ are not separated by B. This lemma is a crucial one in our main proof.

Lemma 2.3. *Let G be a graph, $B \subseteq V(G)$, and π a (G,B)–bounded function. In the random subgraph $G(B,\pi)$, for every pair u,v of distinct vertices with $d_B(u) \ge d_B(v)$, we have*

$$\Pr\left(N^B_{G(B,\pi)}[u] = N^B_{G(B,\pi)}[v]\right) \le e^{-\frac{3\pi(u)}{16}d_B(u)} .$$

Proof. Consider the following partition of $S = N^B_G[u] \cup N^B_G[v]$ into three parts: S_1, the vertices of B dominating u but not v; S_2, the vertices of B dominating v but not u; and S_3, the vertices of B dominating both u and v.

Let D be the random variable which gives the size of the symmetric difference of $N^B_{G(B,\pi)}[u]$ and $N^B_{G(B,\pi)}[v]$. The statement of the lemma is equivalent to $\Pr(D=0) < e^{-\frac{3\pi(u)}{16}d_B(u)}$.

The random variable $D = |N^B_{G(B,\pi)}[u] \oplus N^B_{G(B,\pi)}[v]|$ can be written as the sum of independent Bernoulli variables

$$D = \sum_{w\in S} D_w ,$$

where $D_w = 1$ if and only if w dominates precisely one of the two vertices u or v in $G(B,\pi)$. Therefore, for any $w \notin \{u,v\}$,

$$\Pr(D_w = 1) = \begin{cases} 1-p_{uw} & w\in S_1\\ 1-p_{vw} & w \in S_2\\ p_{uw}(1-p_{vw}) + p_{vw}(1-p_{uw}) & w\in S_3\end{cases}$$

Since we want to bound from above the probability that $D=0$, we can always assume that $u,v \notin N^B_{G(B,\pi)}[u] \oplus N^B_{G(B,\pi)}[v]$. Recall that $d_B(u) \ge$

$d_B(v)$. By the definition of a (G, B)–bounded function, we have that $p_{uw} \leq p_{vw}$ for each $w \in S_3$. Since $x(1-x)$ has a unique maximum at $x = 1/2$ and $p_{uw} \leq 1/2$, we also have:

$$g(u) := \frac{\pi(u)}{4}\left(1 - \frac{\pi(u)}{4}\right) \leq p_{uw}(1 - p_{uw}) \ , \tag{1}$$

for each $w \in S_1 \cup S_3$.

For $w \in S$, denote by q_w the expected value of the Bernoulli random variable D_w. Then, using the definition of $g(u)$ in (1) we get

$$\begin{aligned}
\mathbb{E}(D) &\geq \sum_{w \in N_G^B(u)} q_w \\
&= \sum_{w \in S_1} q_w + \sum_{w \in S_3} q_w \\
&= \sum_{w \in S_1} (1 - p_{uw}) + \sum_{w \in S_3} (p_{uw}(1 - p_{vw}) + p_{vw}(1 - p_{uw})) \\
&\geq \sum_{w \in S_1} p_{uw}(1 - p_{uw}) + \sum_{w \in S_3} p_{uw}(1 - p_{uw}) \\
&\geq g(u) d_B(u) \\
&= \frac{\pi(u)}{4}\left(1 - \frac{\pi(u)}{4}\right) d_B(u) \\
&\geq \frac{3\pi(u)}{16} d_B(u) \ . \qquad (2)
\end{aligned}$$

Finally, we conclude the proof of the lemma

$$\Pr(D = 0) = \prod_{w \in S} (1 - q_w) \leq e^{-\sum_{w \in S} q_w} = e^{-\mathbb{E}(D)} \leq e^{-\frac{3\pi(u)}{16} d_B(u)} \ . \qquad \square$$

In the proof of our main result, we will use the following version of the Lovász Local Lemma, which can be found in e.g. [2, Corollary 5.1.2] (the lower bound on $\Pr(\bigcap_{i=1}^{M} \overline{E_i})$ can be derived from the general Lovász Local Lemma, see [2, Lemma 5.1.1], by setting $x_i = e \cdot p_{LL}$).

Lemma 2.4 (Symmetric Local Lemma)**.** *Let $\mathcal{E} = \{E_1, \ldots, E_M\}$ be a set of (typically "bad") events such that each E_i is mutually independent of $\mathcal{E} \setminus (\mathcal{D}_i \cup \{E_i\})$ for some $\mathcal{D}_i \subseteq \mathcal{E}$. Let $d_{LL} = |\mathcal{D}_i|$, and suppose that there exists a real $0 < p_{LL} < 1$ such that, for each $1 \leq i \leq M$,*

- $\Pr(E_i) \leq p_{LL}$, *and*

- $e \cdot p_{LL} \cdot (d_{LL} + 1) \leq 1$.

Then $\Pr(\bigcap_{i=1}^{M} \overline{E_i}) \geq (1 - e \cdot p_{LL})^M > 0$.

2.2. Proof of the main result

We are now ready to prove the main theorem.

Proof of Theorem 1.3. We will follow the four steps described in the beginning of Section 2.

Step 1. Constructing C and $G(C, \pi)$

Let $C \subseteq V(G)$ be a subset of vertices, where each vertex v in G is chosen to be in C independently with probability $p = \frac{66 \log \Delta}{\delta}$. Observe that $p \leq 1$ since $\delta \geq 66 \log \Delta$.

Consider the random variable $|C|$ and recall that $\mathbb{E}(|C|) = np$. Define A_C to be the event that

$$(A_C) \qquad |C| \leq 2np = \frac{132 n \log \Delta}{\delta}.$$

Since the choices of the elements in C are independent, by setting $\varepsilon = 1$ in Lemma 2.1, we have $c_\varepsilon > 1/3$ and

$$\Pr(\overline{A_C}) < e^{-\frac{22 n \log \Delta}{\delta}}. \tag{3}$$

We let $\pi(u) = \min\left(\frac{66 \log \Delta}{d_C(u)}, 1\right)$. Observe that π is (G, C)–bounded. We construct $G(C, \pi)$ as the random spanning subgraph of G given in Definition 2.2, where each edge uv incident to a vertex of C is deleted with probability p_{uv}.

Step 2. Applying the Lovász Local Lemma

Let u, v be a pair of vertices at distance at most 2 in G. We define the following events:

- A_{uv} is the event that there exists a vertex $w \in \{u, v\}$ such that the degree of w within C is deviating from its expected value $d(w)p$ by half, i.e. $|d_C(w) - d(w)p| \geq \frac{d(w)p}{2}$;
- B_{uv} is the event that $N^C_{G(C,\pi)}[u] = N^C_{G(C,\pi)}[v]$;
- E_{uv} is the event that A_{uv} or B_{uv} occurs;
- A_{LL} is the event that no event E_{uv} occurs.

In order to apply the Lovász Local Lemma, we wish to upper bound the probability of E_{uv}. We have:

$$\Pr(E_{uv}) \leq \Pr(A_{uv}) + \Pr(B_{uv})$$

$$= \Pr(A_{uv}) + \Pr(B_{uv}|A_{uv}) \cdot \Pr(A_{uv}) + \Pr(B_{uv}|\overline{A_{uv}}) \cdot \Pr(\overline{A_{uv}}) \ .$$

Let us upper bound $\Pr(A_{uv})$. We use Lemma 2.1 with $\varepsilon = 1/2$. Observe that $c_\varepsilon > \frac{1}{10}$, and thus

$$\begin{aligned}
\Pr(A_{uv}) &< \Pr\left(|d_C(u) - d(u)p| \geq \frac{d(u)p}{2}\right) + \Pr\left(|d_C(v) - d(v)p| \geq \frac{d(v)p}{2}\right) \\
&\leq 2e^{-\frac{1}{10}d(u)p} + 2e^{-\frac{1}{10}d(v)p} \\
&= 2e^{-\frac{66d(u)\log\Delta}{10\delta}} + 2e^{-\frac{66d(v)\log\Delta}{10\delta}} \\
&\leq 4e^{-\frac{33\log\Delta}{5}} \\
&\leq 4\Delta^{-\frac{33}{5}} \ .
\end{aligned}$$

Next, we give an upper bound for $\Pr(B_{uv}|\overline{A_{uv}})$. For such a purpose, we apply Lemma 2.3 with $B = C$ and $\pi(u) = \min(66\log\Delta/d_C(u), 1)$. Observe that π is (G, C)–bounded. Since A_{uv} does not hold, we know that $d_C(u)$ and $d_C(v)$ are large enough, i.e. for $w \in \{u, v\}$, $d_C(w) \geq \frac{d(w)p}{2} \geq \frac{\delta p}{2} = 33\log\Delta$; thus $\pi(w)d_C(w) = \min(66\log\Delta, d_C(w)) \geq 33\log\Delta$. We have:

$$\Pr(B_{uv}|\overline{A_{uv}}) \leq e^{-\frac{3\cdot 33\log\Delta}{16}} \leq \Delta^{-\frac{99}{16}} \ . \tag{4}$$

The probability that the event E_{uv} holds is

$$\begin{aligned}
\Pr(E_{uv}) &\leq \Pr(A_{uv}) + \Pr(B_{uv}|A_{uv}) \cdot \Pr(A_{uv}) + \Pr(B_{uv}|\overline{A_{uv}}) \cdot \Pr(\overline{A_{uv}}) \\
&\leq 4\Delta^{-\frac{33}{5}} + 1 \cdot 4\Delta^{-\frac{33}{5}} + \Delta^{-\frac{99}{16}} \cdot 1 \\
&\leq 2\Delta^{-\frac{99}{16}} = p_{LL} \ ,
\end{aligned}$$

where we used that Δ is large enough. We now note that each event E_{uv} is mutually independent of all but at most $2\Delta^6$ events $E_{u'v'}$. Indeed, E_{uv} depends on the random variables determining the existence of the edges incident to u and v. This is given by probabilities p_{uw} and p_{vw} that depend on $d_C(w)$, where w is at distance at most one from either u or v. Thus, E_{uv} depends only on the vertices at distance at most two from either u or v belonging to C. In other words, E_{uv} and $E_{u'v'}$ are mutually independent unless there exist a vertex w at distance at most two from both pairs; namely, $d(\{u, v\}, \{u', v'\}) \leq 4$. Hence, there are at most $2\Delta^4$ choices for the vertex among $\{u', v'\}$ that is closest from $\{u, v\}$ (say u'), and at most Δ^2 additional choices for v', since $d(u', v') \leq 2$.

Therefore, we can apply Lemma 2.4 if

$$e \cdot 2\Delta^{-\frac{99}{16}} \cdot (2\Delta^6 + 1) \leq 1 \ ,$$

which holds since Δ is large enough.

Now, by Lemma 2.4 and since there are at most $\frac{n\Delta^2}{2}$ events E_{uv} (one for each pair of vertices at distance at most 2 from each other) and $p_{LL} = 2\Delta^{-\frac{99}{16}}$,

$$\text{(5)} \qquad \Pr(A_{LL}) \geq (1 - e \cdot p_{LL})^M \geq e^{-2e \cdot p_{LL} M} \geq e^{-2en\Delta^{2-\frac{99}{16}}} \ ,$$

where we have used $(1 - x) = e^{-x(1-O(x))} \geq e^{-2x}$, if $x = o(1)$.

Step 3. Revealing the identifying code

Let us lower bound the probability that both A_C and A_{LL} hold, by using the inequalities (3) and (5):

$$\begin{aligned} \Pr(A_C \cap A_{LL}) &\geq \Pr(A_{LL}) - \Pr(\overline{A_C}) \\ &\geq e^{-2en\Delta^{2-\frac{99}{16}}} - e^{-\frac{22n\log\Delta}{\delta}} \ , \end{aligned}$$

which is strictly positive if

$$\frac{22\log\Delta}{\delta} > 2e\Delta^{2-\frac{99}{16}} \ .$$

The latter inequality holds since $\delta \leq \Delta$ and Δ is large enough.

Hence, there exists a set C of size $132\frac{n\log\Delta}{\delta}$ such that all vertices at distance 2 from each other are separated by C and such that the degree in C of each vertex does not deviate much from its degree in $G(C,\pi)$.

In order to build an identifying code, we must also make sure that all vertices are dominated. It is well-known that for every graph G, $\gamma(G) \leq (1+o(1))\frac{n\log\delta}{\delta}$ (see e.g. [2, Theorem 1.2.2]). Hence, we select a dominating set D of G with size $(1+o(1))\frac{n\log\delta}{\delta}$. Then, by Observation 1.2, $C \cup D$ is an identifying code of size at most

$$(132 + 1 + o(1))\frac{n\log\Delta}{\delta} \leq 134\frac{n\log\Delta}{\delta}.$$

Step 4. Estimating the number of deleted edges

Let $Y = |E(G) \setminus E(G(C,\pi))|$ be the number of edges we have deleted from G to obtain $G(C,\pi)$. Recall that each edge $uv \in E(G)$ is deleted independently from G with probability

$$p_{uv} = \frac{1}{4}\left(\pi(u) + \pi(v)\right) \ ,$$

if one of its endpoints is in C.

Since $\Pr(A_C \cap A_{LL}) > 0$, there is a small identifying code of G obtained by deleting at most $\mathbb{E}(Y|A_C \cap A_{LL})$ edges. We next give an upper bound for $\mathbb{E}(Y|A_C \cap A_{LL})$. If both A_C and A_{LL} hold, then

$$p_{uv} \leq \frac{1}{4}\left(\frac{66\log\Delta}{d_C(u)} + \frac{66\log\Delta}{d_C(v)}\right) .$$

The expected number of deleted edges is

$$\mathbb{E}(Y|A_C \cap A_{LL}) = \sum_{\substack{uv\in E(G)\\ (\{u,v\}\cap C)\neq\emptyset}} p_{uv} .$$

Observe that in order to estimate this quantity, we can split the two additive terms in each p_{uv} and group them depending whether u belongs to C or not: for every $u \notin C$, we sum all the terms $\frac{66\log\Delta}{4d_C(u)}$ for all $v \in C$ being neighbors of u; for every $u \in C$, we sum all the terms $\frac{66\log\Delta}{4d_C(u)}$ for all $v \in V(G)$ being neighbors of u.

$$\begin{aligned}
\mathbb{E}(Y|A_C \cap A_{LL}) &\leq \frac{1}{4}\left(\sum_{u\notin C}\sum_{v\in N_G^C(u)}\frac{66\log\Delta}{d_C(u)} + \sum_{u\in C}\sum_{v\in N_G(u)}\frac{66\log\Delta}{d_C(u)}\right)\\
&\leq \frac{1}{4}\left(\sum_{u\notin C} d_C(u)\frac{66\log\Delta}{d_C(u)} + \sum_{u\in C} d(u)\frac{66\log\Delta}{d_C(u)}\right)\\
&\leq \frac{1}{4}\left(|V(G)\setminus C|\cdot 66\log\Delta + \sum_{u\in C} 2\frac{66\log\Delta}{p}\right)\\
&\leq \frac{1}{4}\left(n\cdot 66\log\Delta + 2|C|\delta\right)\\
&\leq \frac{66n\log\Delta + 264n\log\Delta}{4}\\
&\leq 83n\log\Delta ,
\end{aligned}$$

where we used the fact (implied by A_{LL}) that for every vertex v, $\frac{d(v)p}{2} \leq d_C(v)$ at the second line, and that A_C implies $|C| \leq 132\frac{n\log\Delta}{\delta}$ at the fifth line.

Summarizing, we have shown the existence of a small identifying code in a spanning subgraph of G obtained by deleting at most $83n\log\Delta$ edges from G, which completes the proof. □

3. Asymptotic optimality of Theorem 1.3

In this section, we discuss the optimality of Theorem 1.3, first with respect to the size of the constructed code and the number of deleted edges, and then with respect to the hypothesis that Δ is a sufficiently large constant and that $\delta \geq 66 \log \Delta$.

3.1. On the size of the code and the number of deleted edges

Charon, Honkala, Hudry and Lobstein showed that deleting an edge from G can decrease by at most 2 the identifying code number of a graph [8]. That is, for every graph G and any edge $uv \in E(G)$,

$$\gamma^{\text{ID}}(G) \leq \gamma^{\text{ID}}(G \setminus uv) + 2 \ .$$

This directly implies that for every graph with linear identifying code number, one needs to delete a subset F of at least $\Omega(n)$ edges, to get a graph with $\gamma^{\text{ID}}(G \setminus F) = o(n)$.

We will show that, indeed, one needs to delete at least $\Omega(n \log n)$ edges from the complete graph to get a graph with an asymptotically optimal identifying code. Using this, we will derive a family of graphs with arbitrary minimum degree δ, that asymptotically attains the bound of Theorem 1.3, both in number of edges and size of the minimum code, when $\Delta = \text{Poly}(\delta)$.

First of all, we prove that every graph with an asymptotically optimal identifying code cannot contain too few edges.

Lemma 3.1. *For every $M' \geq 0$, there exists a constant $c_0 > 0$ such that every graph G with $\gamma^{ID}(G) \leq M' \log n$ contains at least $c_0 n \log n$ edges.*

Proof. Set α_0 as the smallest positive root of

$$f(\alpha) = \alpha \log\left(\frac{M' + \alpha}{\alpha} e\right) - 1/2 \ . \tag{6}$$

Note that $f(\alpha)$ is well-defined since we have that $\lim_{\alpha \to 0} f(\alpha) = -1/2$ and that $f(1) = \log(M' + 1) + 1/2 > 0$.

Suppose by contradiction that there exists a graph G containing less than $c_0 n \log n$ edges, with $c_0 = \alpha_0/4$, that admits an identifying code C of size at most $M' \log n$. Let U be the subset of vertices of degree at least $\alpha_0 \log n$. Notice that

$$|U| \leq \frac{2|E(G)|}{\alpha_0 \log n} \leq \frac{2c_0}{\alpha_0} n = \frac{n}{2} \ .$$

Since $|C| \leq M' \log n$ and every $v \in V(G) \setminus U$ has degree smaller than $\alpha_0 \log n$, the number of possible nonempty sets $N_G[v] \cap C$ for some vertex v with a small degree, is smaller than

$$\begin{aligned}\sum_{i=1}^{\alpha_0 \log n} \binom{|C|}{i} &\leq \binom{M' \log n + \alpha_0 \log n}{\alpha_0 \log n} \\ &\leq \left(\frac{(M' + \alpha_0)e}{\alpha_0}\right)^{\alpha_0 \log n} \\ &= n^{\alpha_0 \log\left(\frac{M'+\alpha_0}{\alpha_0} e\right)} \\ &= \sqrt{n}\ ,\end{aligned}$$

where we have used that $\binom{a}{b} \leq \left(\frac{ae}{b}\right)^b$ for the second inequality and the fact that α_0 is a root of (6) for the last inequality.

Since $|V(G) \setminus U| \geq n/2$ there must be at least two vertices $v_1, v_2 \in V(G) \setminus U$ such that $N_G[v_1] \cap C = N_G[v_2] \cap C$, and thus C cannot be an identifying code, a contradiction. □

The following lemma relates the identifying code number of a graph G to the one of its complement $\overline{G}$.

Lemma 3.2. *Let G be a twin-free graph. If $\overline{G}$ is twin-free, then*

$$\gamma^{ID}(\overline{G}) \leq 2\gamma^{ID}(G) + 1\ .$$

Proof. Let C_0 be a minimum identifying code of G. We will show that there exists a set C_1 of size at most $\gamma^{\mathrm{ID}}(G)$ and a special vertex v, such that $C = C_0 \cup C_1 \cup \{v\}$ is an identifying code of $\overline{G}$.

For the sake of simplicity, we define the following relation. Two vertices $u, v \in V(G)$ are in relation with each other if and only if $N_G(u) \cap C_0 = N_G(v) \cap C_0$ (notice that here we are considering the open neighborhoods) and u is not adjacent to v (i.e. considering C_0 in G, u, v are separated by one of u, v). This will be denoted as $u \equiv_G v$. It can be checked that this is an equivalence relation.

Claim. Every pair of distinct vertices $u \not\equiv_G v$ is separated by C_0 in $\overline{G}$.

Proof. By the definition of $\equiv_G$, either $N_G(u) \cap C_0 \neq N_G(v) \cap C_0$ or $u \sim v$.

If $N_G(u) \cap C_0 \neq N_G(v) \cap C_0$, there exists $w \in C_0$ (and $w \notin \{u, v\}$) such that $w \in N_G(u) \oplus N_G(v)$. Then, $w \in N_{\overline{G}}(u) \oplus N_{\overline{G}}(v)$, hence w still separates u, v in $\overline{G}$.

If $N_G(u) \cap C_0 = N_G(v) \cap C_0$, then $u \sim v$. If at least one of them belongs to C_0, then this vertex separates u, v in $\overline{G}$. Otherwise, $u, v \notin C_0$ and we have $N_G(u) \cap C_0 = N_G[u] \cap C_0$ and $N_G[v] \cap C_0 = N_G(v) \cap C_0$. Hence $N_G[u] \cap C_0 = N_G[v] \cap C_0$. But then C_0 does not separate u, v in G, a contradiction. □

In particular, the above claim implies that every vertex in an equivalence class of size one is separated by C_0 from all other vertices in $\overline{G}$.

Claim. Every equivalence class contains at most one element that is not in C_0.

Proof. Suppose that $u \equiv_G v$ and that $u, v \notin C_0$. Then, $N_G[u] \cap C_0 = N_G(u) \cap C_0$ and $N_G[v] \cap C_0 = N_G(v) \cap C_0$. Using that they are equivalent, we have that $N_G[u] \cap C_0 = N_G[v] \cap C_0$. Since C_0 is an identifying code of G, we must have $u = v$. □

Claim. Let U be an equivalence class of $\equiv_G$. Then all the pairs in U can be separated in $\overline{G}$ by using $|U| - 1$ vertices.

Proof. We will prove by induction on $s \geq 2$ that every pair in every subset of $U' \subseteq U$ with cardinality s can be separated in $\overline{G}$ by using $s - 1$ vertices. Let $u_1, u_2, \ldots, u_s$ denote the vertices of U'.

For $s = 2$ it holds: since $\overline{G}$ is twin-free, we can select $w \in N_{\overline{G}}[u_1] \oplus N_{\overline{G}}[u_2]$, and w separates u and v in $\overline{G}$.

For each $s > 2$, consider the vertices $u_1, u_2 \in U'$ and let $w \in N_{\overline{G}}[u_1] \oplus N_{\overline{G}}[u_2]$. Since U' forms a clique in $\overline{G}$, $w \notin U'$. Then w splits the set U' into U_1, the set of vertices of U' adjacent to w in $\overline{G}$, and U_2, the set of vertices in U' non-adjacent to w in $\overline{G}$. Let $|U_1| = s_1$ and $|U_2| = s_2$. We may assume $u_1 \in U_1$ and $u_2 \in U_2$. Hence, $s_1, s_2 \geq 1$. Clearly $s_1 + s_2 = s$, which implies $s_1, s_2 < s$. Now, the pairs of vertices of U' with one vertex from U_1 and one vertex from U_2 are separated by w. Since U_1 and U_2 are subsets of U of size at most $s - 1$, by induction, the pairs of vertices in U_1 can be separated using $s_1 - 1$ vertices and the ones in U_2 using $s_2 - 1$. Thus we need at most $(s_1 - 1) + (s_2 - 1) + 1 = s - 1$ vertices to separate all the pairs of vertices in U. □

From the first claim we deduce that only the pairs belonging to the same equivalence class need to be separated. From the second one, we know that in each equivalence class there is at most one vertex that does not belong to C_0. Thus, if there are t different equivalence classes, the number of vertices in them is at most $|C_0| + t$. Finally, it follows from the last claim, that at most $(|C_0| + t) - t = |C_0|$ vertices suffice to separate all the pairs of vertices in the same equivalent class. Thus, there exists a set C_1 of size at most $|C_0|$ vertices that separates all the pairs in $\overline{G}$ that are not separated by C_0.

Eventually, there might be a unique vertex v such that $N_{\overline{G}}[v] \cap (C_0 \cup C_1) = \emptyset$ (if there were two such vertices, they would not be separated by $C_0 \cup C_1$, a contradiction). Hence, $C = C_0 \cup C_1 \cup \{v\}$ is an identifying code of $\overline{G}$ of size at most $2|C_0| + 1 = 2\gamma^{\mathrm{ID}}(G) + 1$. □

Proposition 3.3. *For every $M \geq 0$, there exists a constant $c > 0$ such that, if $\gamma^{ID}(K_n \setminus F) \leq M \log n$ for some set of edges $F \subset E(K_n)$, then $|F| \geq cn \log n$.*

Proof. Set $M' = 3M$ and let $c = c_0$ be the constant given by Lemma 3.1 for this M'. Suppose that there exists a set F of edges, $|F| < cn \log n$ such that $G = K_n \setminus F$ satisfies $\gamma^{\mathrm{ID}}(G) \leq M \log n$. By Lemma 3.2, the graph $\overline{G}$ admits an identifying code of size at most $2M \log n + 1 \leq M' \log n$. By Lemma 3.1, we get a contradiction. □

Using the former proposition, for every δ we can provide an example of a graph with minimum degree δ for which the result of Theorem 1.3 is asymptotically tight when assuming that $\Delta = \mathrm{Poly}(\delta)$.

Fix $\delta > 0$ and consider the graph H_δ consisting of a disjoint union of cliques of order $\delta+1$. We may assume that $\delta+1$ divides n for the sake of simplicity. Denote by $H_\delta^{(1)}, \ldots, H_\delta^{(s)}$, $s = \frac{n}{\delta+1}$, the cliques composing H_δ. Since $H_\delta^{(i)}$ is a connected component, an asymptotically optimal identifying code for H_δ must also be asymptotically optimal for each $H_\delta^{(i)}$. By Proposition 3.3, we must delete at least $\Omega(\delta \log \delta)$ edges from $H_\delta^{(i)}$ to get an identifying code of size $O(\log \delta)$. Thus, one must delete at least $\Omega(s\delta \log \delta) = \Omega(n \log \delta)$ edges from H_δ to get an optimal identifying code.

Corollary 3.4. *For every large enough δ and every $M \geq 0$, there exists a constant $c > 0$ such that for every set of edges $F \subset E(H_\delta)$ satisfying $\gamma^{ID}(H_\delta \setminus F) \leq M \frac{n \log \delta}{\delta}$, we have $|F| \geq cn \log \delta$.*

We remark that a connected counterexample can also be constructed from H_δ by connecting its cliques using few edges, without affecting the above result.

Corollary 3.4 implies that Theorem 1.3 is asymptotically tight when $\Delta = \mathrm{Poly}(\delta)$, since in that case $\log \Delta = O(\log \delta)$. However, when δ is subpolynomial with respect to Δ, we do not know if Theorem 1.3 is asymptotically tight.

3.2. On the hypothesis on Δ and δ

We conclude this section by discussing the necessity of the hypothesis that Δ should be sufficiently large and that $\delta \geq 66 \log \Delta$ in Theorem 1.3.

First note that, if Δ is bounded by a constant, we need at least $\frac{n}{\Delta+1} = \Theta(n)$ vertices to dominate G. Thus, no code of size smaller than $\Theta(n)$ can be obtained by deleting edges of the graph.

On the other hand, the condition $\delta \geq c \log \Delta$ for some constant $c > 0$ is also necessary to prove that there exists a subgraph with small identifying code number (in the spirit of Theorem 1.3). This can be deduced from the following proposition that also implies $c > (2 \log 2)^{-1}$.

Proposition 3.5. *For arbitrarily large values of Δ, there exists a graph G with maximum degree Δ and minimum degree $\delta = \frac{\log_2 \Delta}{2}$ such that, for every spanning subgraph $H \subseteq G$,*

$$\gamma^{ID}(H) = (1 - o(1))n \; .$$

Proof. Consider the bipartite complete graph $G = K_{r,s}$ where $s = 2^{2r}$. Denote by V_1 the stable set of size r and by V_2 the stable set of size s. Observe that $\delta = r = \frac{\log_2 s}{2} = \frac{\log_2 \Delta}{2}$.

For every given twin-free spanning subgraph $H \subseteq G$, let $C \subseteq V(G)$ be an identifying code of H. Let us show that most of the vertices in V_2 must be in C. Let $S \subseteq V_2$ be the subset of vertices in V_2 that are not in the code. Thus, for every $u \in S$, $N_C[u] = N_C(u)$. Observe that $N_C(u) \subseteq V_1$, and hence, there are at most 2^r possible candidates for such $N_C(u)$. Since C is dominating and separating all the pairs in S, all the subsets $N_C(u)$ must be non empty and different, which implies, $|S| < 2^r$. Hence, we have

$$|C| \geq |V_2 \setminus S| \geq 2^{2r} - 2^r = (1 - o(1))2^{2r} = (1 - o(1))n \; . \qquad \square$$

4. Some consequences of Theorem 1.3

In the previous sections, we have studied how much the identifying code number can decrease when we delete few edges from the original graph. The next corollary deals with the symmetric question of how much the addition of edges can help to decrease this parameter.

The question of how much can a parameter decrease when deleting or adding edges has been widely studied in the literature for some monotone parameters. However, if the parameter is monotone, only one of either deleting or adding edges can help to decrease it. One of the interesting facts of the identifying code number, which is nonmonotone, is that one can have similar results for both procedures.

Corollary 4.1. *For every graph G on n vertices, with minimum degree δ such that $n-\delta$ is large enough, and maximum degree Δ such that $n-\Delta \geq 66\log(n-\delta)$, there exists a set of edges F with $F \cap E(G) = \emptyset$ of size*

$$|F| = O\left(n\log(n-\delta)\right) ,$$

such that

$$\gamma^{ID}(G \cup F) = O\left(\frac{n\log n}{n-\Delta}\right) .$$

This can be proven by using Lemma 3.2 and by applying Theorem 1.3 to the graph $\overline{G}$. Similar arguments than the ones shown before, prove that this corollary is also asymptotically tight.

Theorem 1.3 also has a direct application to *watching systems* [3]. In a watching system, we can place on each vertex v a set of *watchers*. To each watcher w placed on v, we assign a nonempty subset $Z(w) \subseteq N[v]$, its *watching zone*. We now ask each vertex to belong to a unique and nonempty set of watching zones; the minimum number of watchers that need to be placed on the vertices of G to obtain a watching system is the *watching number* $w(G)$ of G.

In [3], the authors provide a tight upper bound on the watching number of graphs with given maximum degree, $w(G) \leq \gamma(G)\lceil\log_2(\Delta+2)\rceil$. Since the domination number of a graph with minimum degree δ satisfies $\gamma(G) \leq (1+o(1))\frac{n\log\delta}{\delta}$, this implies

$$w(G) \leq \gamma(G)\lceil\log\Delta + 2\rceil = \Omega\left(\frac{n\log^2\delta}{\delta}\right) . \tag{7}$$

Nonetheless, observe that

$$w(G) \leq \min\{\gamma^{\mathrm{ID}}(H),\ \text{where } H \text{ is a spanning subgraph of } G\} .$$

Thus, Theorem 1.3 implies that

Corollary 4.2. *For every large enough Δ and every graph G on n vertices with maximum degree Δ and minimum degree $\delta \geq 66\log\Delta$, we have:*

$$w(G) \leq 134\frac{n\log\Delta}{\delta} .$$

This bound asymptotically improves (7) when the maximum degree is $\Delta = \mathrm{Poly}(\delta)$.

5. Concluding remarks and open questions

1. The kind of results we provide in this paper can be connected to the notion of resilience (see e.g. [21]). Given a graph property $\mathcal{P}$, the *global resilience* of G with respect to $\mathcal{P}$ is the minimum number of edges one has to delete to obtain a graph not satisfying $\mathcal{P}$.

Our result can be interpreted in terms of the resilience of the following (non-monotone) property $\mathcal{P}$: "G has a large identifying code number in terms of its degree parameters, δ and Δ". For every large enough Δ and every graph G on n vertices with maximum degree Δ and minimum degree $\delta \geq 66 \log \Delta$, Theorem 1.3 can be stated as follows: the resilience of G with respect to $\mathcal{P}$ is $O(n \log \Delta)$. Moreover, Corollary 3.4 shows that there are graphs that attain this value.

2. In Theorem 1.3, we show the existence of a small identifying code for a large spanning subgraph of G. However, our proof is not constructive and, besides, the probability that such pair exists is exponentially small, due to the use of the Lovász Local Lemma. However, the proof of the theorem can probably be adapted to explicitly find them by using the algorithmic version of the Lovász Local Lemma proposed by Moser and Tardos [18].

3. Note that a notion similar to identifying codes, *locating-dominating sets*, has also been extensively studied in the literature (see e.g. [17] for many references). In particular, it follows that every identifying code is a locating-dominating set, hence Theorem 1.3 also holds for this notion. In fact, the proof of Corollary 3.4 can be adapted for this case too.

4. Another notion on which we can apply our results is the *metric dimension* of a graph. A *resolving set* of a connected graph G is a set R of vertices such that for every pair u, v of vertices of G, there is a vertex $x \in R$ such that the distance between x and u is different from the distance between x and v. The metric dimension of G is the smallest size of a resolving set of G. This parameter has been extensively studied since its introduction in the mid-1970's (see for example [5, 19, 20] and the references therein). The metric dimension is also a non-monotone parameter with respect to graph inclusion and therefore it is natural to study the same problem for it. It is not difficult to see that any identifying code is a resolving set, therefore, Theorem 1.3 also holds if we change the identifying code number for the metric dimension. However, in this case, we do not expect Theorem 1.3 to be tight. Note that the metric dimension may be much smaller than $\Theta(\log n)$, for example any path has metric dimension 1 [20], and the hypercube with

$n = 2^d$ vertices has metric dimension $\Theta\left(\frac{\log n}{\log\log n}\right)$ [19]. The metric dimension of an Erdős-Rényi random graph, has been recently studied in [7]. It would be interesting to show how much can the metric dimension decrease, by randomly perturbing some edges of the graph.

5. As further research, it would be very interesting to close the gap between the result in Theorem 1.3 and the lower bound given by the example in Corollary 3.4. Motivated by this example, we ask the following question:

Question 5.1. *Is it true that for every graph G with minimum degree δ, there exists a subset of edges $F \subset E(G)$ of size*

$$|F| = O\left(n \log \delta\right) ,$$

such that

$$\gamma^{ID}(G \setminus F) = O\left(\frac{n \log \delta}{\delta}\right) ?$$

It seems to us that the techniques used in this paper will not provide an answer to the previous question. The main obstacle is the use of the Lovász Local Lemma, which forces us to take into account the role of the maximum degree of G.

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