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A generalization of the r-Whitney numbers of the second kind

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In this paper, we consider a (p, q)-generalization of the r-Whitney numbers of the second kind and of the associated r-Dowling polynomials. We obtain generalizations of some earlier results for these numbers, including recurrence and generating function formulas, that reduce to them when $p = q = 1$. Furthermore, some of our results appear to be new in the case $p = q = 1$ and thus yield additional formulas for the r-Whitney numbers. As a consequence, some new identities are obtained for the q-Stirling and r-Whitney numbers. In addition, the log-concavity of our generalized Whitney numbers is shown for certain values of the parameters p and q. Finally, we introduce (p, q)-Whitney matrices of the second kind and study some of their properties.

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1. Introduction

Given integers $r \geq 0$ and $m \geq 1$, let $W(n, k) = W(n, k; r, m)$ denote the connection constants in the polynomial identities

$$(1.1)\qquad (mx + r)^n = \sum_{k=0}^{n} W(n, k)m^k(x)_k, \qquad n \geq 0,$$

where $(x)_k = x(x - 1)\cdots(x - k + 1)$ if $k \geq 1$, with $(x)_0 = 1$. The $W(n, k)$ are known as *r-Whitney numbers of the second kind* and have recently been studied [7]. Equivalently, the $W(n, k)$ are determined by the recurrence

$$(1.2)\qquad W(n, k) = W(n - 1, k - 1) + (r + mk)W(n - 1, k), \qquad n, k \geq 1,$$

with the initial values

$$W(n, 0) = r^n \text{ and } W(0, k) = \delta_{k,0}$$

for all $n, k \geq 0$. Note that $W(n, k; 0, 1) = S(n, k)$ and $W(n, k; 1, 1) = S(n + 1, k + 1)$, where $S(n, k)$ is the classical Stirling number of the second kind. In [7], a combinatorial interpretation for $W(n, k)$ is given in terms of a kind of finite geometric lattice known as the Dowling lattice [8]. See also [1, 18], where various properties are proven for the numbers $W(n, k; 1, m)$.

The *r-Dowling polynomials*, which we will denote here by $D(n; x) = D(n; r, m, x)$, were defined as

$$D(n; x) = \sum_{k=0}^{n} W(n, k)x^k, \qquad n \geq 0, \tag{1.3}$$

in [7], where some algebraic properties were found using Riordan matrices. See also [2, 18], where various properties were determined in the case $r = 1$. When $m = 1$, note that the $W(n, k)$ and $D(n; x)$ reduce to the r-Stirling numbers and r-Bell polynomials, respectively, see [3, 14].

In this paper, we consider (p, q)-generalizations of the r-Whitney numbers and the r-Dowling polynomials, which we denote by $W_{p,q}(n, k)$ and $D_{p,q}(n; x)$, obtained by considering a certain pair of statistics on a class of colored set partitions enumerated by $W(n, k)$. We thereby obtain, via combinatorial arguments, polynomial generalizations of several identities given in [7] and [18], which were found by various algebraic methods. Thus, our results not only provide extensions that reduce to these prior identities when $p = q = 1$, but one also obtains combinatorial proofs of these identities. Furthermore, our combinatorial model allows one to derive formulas involving $W_{p,q}(n, k)$ and $D_{p,q}(n; x)$ that are apparently new in the case $p = q = 1$ (see, for example, Theorems 3.7 and 3.8 below). In one instance, additionally taking $r = m = 1$ yields a recurrence formula for the Stirling numbers of the second kind which seems to be new (see Corollary 3.9). Moreover, the $W_{p,q}(n, k)$ are seen to reduce to the q-Stirling numbers of Carlitz (see, e.g., [21]) when $m = p = 1$ and $r = 0$. See also [10] (which was submitted and first appeared while the current paper was under review) for a related generalization of the r-Whitney numbers studied from an algebraic standpoint.

The paper is divided as follows. In Section 2, we define $W_{p,q}(n, k)$ and $D_{p,q}(n; x)$ and determine the basic recurrence for $D_{p,q}(n, k)$. In the third section, we obtain, by combinatorial arguments, recurrence formulas for $D_{p,q}(n, k)$ and $D_{p,q}(n; x)$ involving different combinations of the variables n, k, r, and m, as well as explicit expressions in terms of the q-Stirling numbers. In addition, we provide a combinatorial proof of a prior formula from [17] that relates r-Bell polynomials to Stirling numbers of the first kind which was obtained using linear algebra techniques. In Section 4, we prove, bijectively, certain formulas for $D_{p,q}(n,k)$ and $D_{p,q}(n;x)$ in the case when $r=1$ by defining appropriate sign-changing involutions. In Section 5, we find some related generating function formulas and obtain as a consequence identities involving the q-Stirling and r-Whitney numbers. We also prove the strict log-concavity (and hence unimodality) of $W_{p,q}(n,k)$ for certain values of the parameters p and q. In the final section, we introduce the (p,q)-Whitney matrix of the second kind and give some factorizations involving this matrix.

We will make use of the following notation. If m and n are positive integers, then let $[m,n]=\{m,m+1,\ldots,n\}$ if $m\leq n$, with $[m,n]=\emptyset$ if $m>n$. We will denote the special case $[1,n]$ by $[n]$. Given a positive integer k and an indeterminate q, let $[k]_q=1+q+\cdots+q^{k-1}$, with $[0]_q=0$. Let $[k]_q!=\prod_{i=1}^{k}[i]_q$, with $[0]_q!=1$, denote the q-factorial. Define the q-binomial coefficient by $\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!}$, if $0\leq k\leq n$, and putting 0 otherwise. Throughout, empty sums will assume the value 0, and empty products the value 1, with $0^0=1$.

2. Preliminaries

In this section, we describe a combinatorial interpretation for the numbers $W_{p,q}(n,k)$. Before doing so, let us recall some terminology and make a few definitions. A *partition* of $[m]$ is a collection of pairwise disjoint subsets, called *blocks*, whose union is $[m]$. The cardinality of the set of partitions of $[m]$ having exactly k blocks is given by $S(m,k)$, the Stirling number of the second kind, with $B(m)=\sum_{k=0}^{m}S(m,k)$, the m-th Bell number, enumerating all partitions of $[m]$.

We now consider the following restricted subset of partitions of a given size.

Definition 2.1. *Given $0\leq r\leq m$, by an r-partition of $[m]$, we will mean one in which the elements of $[r]$ belong to distinct blocks. If $n,k,r\geq 0$, then let $\Pi_r(n,k)$ denote the set of all r-partitions of $[n+r]$ having $k+r$ blocks and $\Pi_r(n)=\cup_{k=0}^{n}\Pi_r(n,k)$.*

Note that when $r=0$, an r-partition of $[m]$ is the same as an ordinary partition. The cardinalities of $\Pi_r(n,k)$ and $\Pi_r(n)$ are given, respectively, by the r-Stirling number of the second kind and r-Bell number (see, e.g., [3] and [12]), which we will denote here by $S(n,k;r)$ and $B(n;r)$.

Within a member of $\Pi_r(n,k)$, we will refer to the blocks containing an element of $[r]$ as *special* and the remaining blocks comprised exclusively of elements of $[r+1,r+n]$ as *non-special*. (The members of $[r]$ themselves will also at times be described as *special*.) Furthermore, we will refer to an element within a member of $\Pi_r(n,k)$ that is the smallest within its block as *minimal*, and to all other elements as *non-minimal*.

We now allow for certain elements within an r-partition to be colored.

Definition 2.2. *Given an integer $m \geq 1$, let $\Pi_{r,m}(n,k)$ denote the set of r-partitions of $[n+r]$ having $k+r$ blocks wherein within each non-special block, every non-minimal element is assigned one of m colors, and let $\Pi_{r,m}(n) = \cup_{k=0}^{n}\Pi_{r,m}(n,k)$.*

Note that $|\Pi_{r,m}(n,k)| = W(n,k;r,m)$ for all relevant values of the parameters, upon making a comparison of recurrences and initial values. This interpretation of $W(n,k;r,m)$ is seen to be equivalent to the one given by Mihoubi and Rahmani [15] in terms of their partial r-Bell polynomials. Alternatively, one may also take m to be an indeterminate marking the total number of non-minimal elements in non-special blocks, or equivalently, by subtraction, the number of non-minimal elements in special blocks. Thus, for example, when $r = 1$ and $|\Pi_1(n,k)| = S(n+1,k+1)$, the number $m^{n-k}W(n,k;1,1/m)$ may be viewed as the distribution polynomial for the statistic recording the number of elements in the first block of an ordinary partition of $[n+1]$. For a combinatorial interpretation of $W(n,k;r,m)$ in terms of Dowling lattices, see [7, Section 2].

We now describe a pair of statistics on the set $\Pi_{r,m}(n,k)$, which will extend and refine a statistic originally considered on set partitions by Carlitz [6] and later studied by Wagner [22].

Definition 2.3. *Suppose $\pi \in \Pi_{r,m}(n,k)$ is represented as*

$$\pi = A_1/A_2/\cdots/A_r/B_1/B_2/\cdots/B_k,$$

where A_i denotes the special block containing the element i for $i \in [r]$ and non-special blocks are denoted by B_j, with $\min(B_1) < \min(B_2) < \cdots < \min(B_r)$. Define the statistics w_1 and w_2 on $\Pi_{r,m}(n,k)$ by letting

$$w_1(\pi) = \sum_{i=1}^{r}(i-1)(|A_i|-1)$$

and

$$w_2(\pi) = \sum_{i=1}^{k}(i-1)(|B_i|-1).$$

We now define (p,q)-generalizations of the r-Whitney numbers and of the r-Dowling polynomials.

Definition 2.4. *Define $W_{p,q}(n,k) = W_{p,q}(n,k;r,m)$ as the joint distribution polynomial for the w_1 and w_2 statistics on the set $\Pi_{r,m}(n,k)$, that is,*

$$W_{p,q}(n,k) = \sum_{\pi\in\Pi_{r,m}(n,k)} p^{w_1(\pi)}q^{w_2(\pi)}, \qquad n,k \geq 0,$$

where p and q are indeterminates. Define $D_{p,q}(n;x) = D_{p,q}(n;r,m,x)$ by setting

$$D_{p,q}(n;x) = \sum_{k=0}^{n} W_{p,q}(n,k)x^k, \qquad n \geq 0.$$

Note that $W_{p,q}(n,k)$ reduces to $W(n,k)$ and $D_{p,q}(n;x)$ to $D(n;x)$ when $p = q = 1$, by definition. The $W_{p,q}(n,k)$ satisfy the following two-term recurrence.

Proposition 2.5. *The array $W_{p,q}(n,k)$ for $n \geq k \geq 0$ is determined by the recurrence*

$$W_{p,q}(n,k) = W_{p,q}(n-1,k-1)+([r]_p+m[k]_q)W_{p,q}(n-1,k), \quad n,k \geq 1, \tag{2.1}$$

and the initial conditions $W_{p,q}(n,0) = [r]_p^n$ and $W_{p,q}(0,k) = \delta_{k,0}$ for $n,k \geq 0$.

Proof. The initial condition $W_{p,q}(0,k) = \delta_{k,0}$ follows from the definitions. Note that within a member of $\Pi_{r,m}(n,0)$, each element of $[r+1,r+n]$ belongs to a special block and thus contributes a factor of $1+p+\cdots+p^{r-1} = [r]_p$ towards the total w_1-weight, whence $W_{p,q}(n,0) = [r]_p^n$. To show (2.1), first note that the weight of all members of $\Pi = \Pi_{r,m}(n,k)$ in which $n+r$ belongs to its own block is $W_{p,q}(n-1,k-1)$, since in this case neither the w_1 nor the w_2 statistic values are changed by the addition of this element. On the other hand, if $n+r$ belongs to the i-th left-most special block within a member of Π, then the weight of all such members is $p^{i-1}W_{p,q}(n-1,k)$ for $1 \leq i \leq r$. Thus, the total weight of all members of Π in which $n+r$ belongs to a special block is $[r]_pW_{p,q}(n-1,k)$. Similarly, members of Π in which $n+r$ belongs to a non-special block contribute weight $m[k]_qW_{p,q}(n,k)$, since in this case the element $n+r$ is also assigned one of m colors. Combining the three previous cases yields (2.1). □

We will denote the $m = p = 1, r = 0$ case of $W_{p,q}(n,k;r,m)$ by $S_q(n,k)$ since it coincides with the q-Stirling number of Carlitz [5].

3. Generalized r-Whitney identities

In this section, we prove several identities of the generalized r-Whitney numbers by combinatorial arguments. We note that the $p = q = 1$ cases of the identities in the next four theorems (with the exception of the second identity in Theorem 3.3 and the first in Theorem 3.5, which we were unable to find in the literature) occur in [7], where they were proven algebraically using Riordan matrices. Our first result expresses $W_{p,q}(n,k)$ and $D_{p,q}(n;x)$ in terms of the generalized Stirling polynomial $S_q(n,k)$.

Theorem 3.1. *If $n, k \geq 0$, then*

$$W_{p,q}(n,k) = \sum_{j=k}^{n} m^{j-k} \binom{n}{j} [r]_p^{n-j} S_q(j,k) \tag{3.1}$$

and

$$D_{p,q}(n;x) = \sum_{j=0}^{n} \binom{n}{j} [r]_p^{n-j} \sum_{i=0}^{j} m^{j-i} x^i S_q(j,i). \tag{3.2}$$

Proof. To show (3.1), consider the number, $n-j$, of elements of $[r+1, r+n]$ within a member of Π that belong to special blocks. Note that there are $\binom{n}{j}[r]_p^{n-j}$ ways to choose and arrange these elements and $m^{j-k}S_q(j,k)$ ways in which to arrange the remaining j elements of $[r+1, r+n]$ in special blocks. The factor m^{j-k} accounts for the $j-k$ non-minimal elements within these blocks that are each to be colored in one of m ways. Similar reasoning applies to (3.2) except now the remaining j elements of $[r+1, r+n]$ can occupy any number i of blocks. □

Let $B(n;r,x) = \sum_{k=0}^{n} S(n,k;r)x^k$ denote the n-th r-Bell polynomial (see, e.g., [14]) and $B(n;x) = \sum_{k=0}^{n} S(n,k)x^k$ the n-th Bell polynomial. Taking $m = p = q = 1$ in (3.2) yields the relation

$$B(n;r,x) = \sum_{j=0}^{n} r^{n-j} \binom{n}{j} B(j;x), \qquad n, r \geq 0.$$

Since it is known (see [17]) that

$$B(n;r,x) = \sum_{i=0}^{r} x^{-r} s(r,i) B(n+i;x),$$

where $s(r,i)$ denotes the Stirling number of the first kind, we have the following relation.

Corollary 3.2. *If* $n,r \geq 0$, *then*

$$x^r B(n;r,x) = \sum_{i=0}^{n} r^{n-i} x^r \binom{n}{i} B(i;x) = \sum_{i=0}^{r} s(r,i) B(n+i;x). \tag{3.3}$$

Identity (3.3) was also obtained in [18]. A direct bijective proof of (3.3) is given at the end of this section.

The first identity in the next result provides a recurrence for the quantity $W_{p,q}(n,k;r,m)$ in terms of $W_{p,q}(i,k;s,m)$, where $i \leq n$ and $s \leq r$.

Theorem 3.3. *If* $n,k \geq 0$ *and* $r \geq s \geq 0$, *then*

$$W_{p,q}(n,k;r,m) = \sum_{i=k}^{n} p^{s(n-i)} \binom{n}{i} [r-s]_p^{n-i} W_{p,q}(i,k;s,m) \tag{3.4}$$

and

$$W_{p,q}(n,k;r,m) = \sum_{i=k}^{n} \sum_{j=0}^{i-k} p^{(r-s)j} m^{i-j-k} \binom{n}{i} \binom{i}{j} [r-s]_p^{n-i} [s]_p^j S_q(i-j,k). \tag{3.5}$$

Proof. To show (3.4), we count the members of Π according to the number, $n-i$, of elements of $[r+1,r+n]$ contained in the final $r-s$ special blocks. These elements may be selected in $\binom{n}{i}$ ways and arranged in $p^{s(n-i)}[r-s]_p^{n-i}$ ways, as each element contributes a factor of $p^s+p^{s+1}+\cdots+p^{r-1} = p^s[r-s]_p$ towards the w_1-weight. The remaining i members of $[r+1,r+n]$, together with the members of $[s]$, then have weight $W_{p,q}(i,k;s,m)$. Summing over i gives (3.4).

To show (3.5), we consider instead the number, $n-i$, of additional elements found within the *first* $r-s$ special blocks. These elements may be chosen and positioned in $\binom{n}{i}[r-s]_p^{n-i}$ ways. Once this has been done, we then select j of the remaining i members of $[r+1,r+n]$ to go in the final s special blocks, which can be done in $p^{(r-s)j}\binom{i}{j}[s]_p^j$ ways. The remaining $i-j$ members of $[r+1,r+n]$ are arranged in k (non-special) blocks and thus have weight $m^{i-j-k}S_q(i-j,k)$. Summing over i and j gives (3.5). □

Taking $s = r - 1$ in (3.4) gives the recurrence

$$W_{p,q}(n,k;r,m) = \sum_{i=k}^{n} p^{(r-1)(n-i)} \binom{n}{i} W_{p,q}(i,k;r-1,m), \qquad r \geq 1. \tag{3.6}$$

Our next formula expresses $W_{p,q}(n,k;r,m)$ in terms of $W_{p,q}(j,k;r,1)$ where $j \leq n$.

Theorem 3.4. *If $n, k \geq 1$, then*

$$\begin{aligned} &W_{p,q}(n,k;r,m) \\ &= \sum_{j=k}^{n} (-1)^{n-j} m^{j-k} ((m-1)[r]_p)^{n-j} \binom{n}{j} W_{p,q}(j,k;r,1). \end{aligned} \tag{3.7}$$

Proof. Let $\Pi^* = \Pi^*_{r,m}(n,k)$ denote a variant of the set Π which differs from it in the following two ways: (i) non-minimal elements in special blocks are also colored using one of m colors, and (ii) some subset (possibly empty) of these elements are circled and those that are circled may be colored using only the first $m-1$ colors. Suppose $\lambda \in \Pi^*$ has exactly $n-j$ circled elements among its special blocks. Define the (signed) weight of λ to be $(-1)^{n-j} p^{w_1(\lambda)} q^{w_2(\lambda)}$, where the w_1 and w_2 statistic values are computed as they would be ordinarily for a member of $\Pi_{r,m}(n,k)$. To compute the weight of all such λ, we describe a way in which they may be formed as follows. Note first that $j \geq k$ since members of Π^* are to contain k non-special blocks. Since the circled elements of λ belong to $[r+1, r+n]$, by definition, there are $\binom{n}{j}$ choices for these elements and $(m-1)^{n-j}$ ways in which to color them. The remaining j elements of $[r+1, r+n]$, together with the elements of $[r]$ (which are considered special), then contribute weight $m^{j-k} W_{p,q}(j,k;r,1)$, since all non-minimal elements except those that are circled are to be colored in one of m ways (note that there are $j + r - (k + r) = j - k$ non-minimal elements). Finally, we add the $n-j$ circled elements to the special blocks of this arrangement, which can be done in $[r]_p^{n-j}$ ways. Thus, the weight of the members of Π^* containing exactly $n-j$ circled elements is $(m-1)^{n-j}\binom{n}{j} \cdot m^{j-k} W_{p,q}(j,k;r,1) \cdot [r]_p^{n-j}$. Then the right-hand side of (3.7) gives the sum of the weights of all members of Π^*.

To complete the proof, we define a sign-changing, weight-preserving involution of $\Pi^* - \mathcal{B}$, where $\mathcal{B}$ is a subset of Π^* having weight $W_{p,q}(n,k;r,m)$ all of whose members have positive sign. Let $\mathcal{B}$ denote the set of arrangements in Π^* containing no circled elements and in which non-minimal elements in special blocks are colored using the m-th color only. Note that members of $\mathcal{B}$ have positive sign and are synonymous with members of Π. Given a member of $\Pi^* - \mathcal{B}$, denote by y the smallest element of $[r+1, r+n]$ that is a non-minimal element belonging to a special block and colored using one of the first $m-1$ colors. Either circle or uncircle the element y, keeping its color the same. This operation defines the desired involution of $\Pi^* - \mathcal{B}$ and completes the proof. □

We have the following additional formulas for $W_{p,q}(n,k)$ and $D_{p,q}(n;x)$ in terms of generalized Stirling numbers.

Theorem 3.5. *If* $n, k \geq 1$, *then*

$$\begin{aligned} W_{p,q}(n+1,k) &= [r]_p W_{p,q}(n,k) \\ &\quad + \sum_{i=k-1}^{n} \sum_{j=k-1}^{i} m^{n-i+j-k+1} q^{j-k+1} \binom{n}{i} \binom{i}{j} [r]_p^{i-j} S_q(j,k-1) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} D_{p,q}(n+1;x) &= [r]_p D_{p,q}(n;x) \\ &\quad + \sum_{i=0}^{n} \sum_{j=0}^{i} \sum_{\ell=0}^{j} m^{n-i+j-\ell} q^{j-\ell} x^{\ell+1} \binom{n}{i} \binom{i}{j} [r]_p^{i-j} S_q(j,\ell). \end{aligned} \tag{3.9}$$

Proof. To show (3.8), consider whether or not the element $r+1$ belongs to a special block within a member of $\Pi_{r,m}(n+1,k)$. If it does, then there are $[r]_p W_{p,q}(n,k)$ possibilities. Otherwise, suppose $r+1$ occupies a non-special block with exactly $n-i$ members of $I = [r+2, r+n+1]$, where $k-1 \leq i \leq n$. Then there are $m^{n-i}\binom{n}{n-i}$ possibilities concerning the selection and coloring of these elements. Suppose further that exactly j of the i remaining elements of I belong to non-special blocks, where $k-1 \leq j \leq i$. Then there are $\binom{i}{j}$ ways to choose these elements and $(mq)^{j-k+1} S_q(j,k-1)$ ways to arrange (and color) them. The factor q^{j-k+1} arises since each non-minimal element in an arrangement contributes one more than it would ordinarily towards the w_2 statistic value (due to the presence of the block containing $r+1$ preceding it). The remaining $i-j$ elements of I are then placed in special blocks, which contributes a factor of $[r]_p^{i-j}$ towards the overall weight. Considering all possible i and j gives (3.8). We reason similarly to show (3.9) except now the j elements of I not going in a special block or in the block containing $r+1$ can occupy any number ℓ of non-special blocks. □

Letting $p = q = 1$ in (3.9), and applying the $p = q = 1$ case of (3.2) to the inner two sums, gives the following recurrence for the r-Dowling polynomials.

Corollary 3.6. *If $n \geq 0$, then*

$$D(n+1;x) = rD(n;x) + x\sum_{i=0}^{n} m^{n-i}\binom{n}{i}D(i;x). \tag{3.10}$$

Note that (3.10) occurs as [7, Theorem 5.1]. Our next result is inspired by the Bell number formula of Spivey [19].

Theorem 3.7. *If $a, b, k \geq 0$, then*

$$W_{p,q}(a+b,k) = \sum_{i=0}^{a}\sum_{j=0}^{b}(mq^i)^{i+j-k}\binom{b}{j}([r]_p + m[i]_q)^{b-j}W_{p,q}(a,i)S_q(j,k-i) \tag{3.11}$$

and

$$D_{p,q}(a+b;x) = \sum_{i=0}^{a}\sum_{j=0}^{b}\sum_{\ell=0}^{j}(mq^i)^{j-\ell}x^{i+\ell}\binom{b}{j}([r]_p + m[i]_q)^{b-j}W_{p,q}(a,i)S_q(j,\ell). \tag{3.12}$$

Proof. If $a = 0$ or $b = 0$, then both identities are seen to hold, so assume $a, b \geq 1$. To show (3.11), consider the number, i, of non-special blocks that contain at least one element of $[r+1, r+a]$ within a member of $\Pi_{r,m}(a+b,k)$. Then there are $W_{p,q}(a,i)$ ways in which to arrange the elements of $[r+1, r+a]$, together with the elements of $[r]$. Suppose further that there are exactly $b-j$ elements of $J = [r+a+1, r+a+b]$ lying in blocks containing at least one member of $[r+a]$. Then there are $\binom{b}{j}([r]_p + m[i]_q)^{b-j}$ possibilities concerning the weight of these elements. The remaining j elements of J are arranged in $k-i$ non-special blocks in $(mq^i)^{i+j-k}S_q(j,k-i)$ ways. Note that the factor $q^{i(i+j-k)}$ arises due to the presence of i non-special blocks preceding those that contain exclusively members of J (which causes each of the $i+j-k$ non-minimal elements within these blocks to contribute i more than they would ordinarily towards the value of w_2). Summing over all possible i and j then gives (3.11). Identity (3.12) follows similarly except that there are now $x^iW_{p,q}(a,i)$ possibilities for arranging the elements of

$[a+r]$ in their blocks and $\sum_{\ell=0}^{j}(mq^i)^{j-\ell}x^\ell S_q(j,\ell)$ possibilities concerning the positions of the remaining j elements of J. □

There is the following further recurrence for $W_{p,q}(n,k)$.

Theorem 3.8. *If $n, k \geq 0$ and $r \geq 1$, then*

$$\begin{aligned} &W_{p,q}(n,k;r,m) - W_{p,q}(n,k;r-1,m) \\ &= p^{r-1}\sum_{i=0}^{n-1}\sum_{j=0}^{k}\sum_{\ell=k-j}^{n-i-1}(mq^j)^{j+\ell-k}\binom{n-i-1}{\ell}\cdot \\ &\qquad \cdot([r]_p+m[j]_q)^{n-i-\ell-1}W_{p,q}(i,j;r-1,m)S_q(\ell,k-j). \end{aligned} \tag{3.13}$$

Proof. We may assume that $k < n$ in (3.13), for otherwise the equality is trivial. Note that the left-hand side of (3.13) gives the total weight of all members of Π in which the block containing r contains at least one other element, by subtraction. Let us denote this subset of Π by Π', and we will show that the right-hand side of (3.13) also gives the weight of Π'. To do so, suppose that $\pi \in \Pi'$ and that the second smallest element in the block of π containing r is $r+i+1$, where $0 \leq i \leq n-1$. Suppose further that exactly j non-special blocks of π contain at least one element of $[r+1, r+i]$, where $0 \leq j \leq \min\{i,k\}$. Then there are $p^{r-1}W_{p,q}(i,j;r-1,m)$ possibilities concerning the placement of the members of $[r+i+1]$ since no member of $[r+1,r+i]$ can go in the block containing r, with the factor of p accounting for the placement of the element $r+i+1$.

Now suppose that there are exactly ℓ elements comprising the remaining $k-j$ non-special blocks of π (i.e., those that do *not* contain some member of $[r+1, r+i]$). Then there are $\binom{n-i-1}{\ell}$ choices for these elements (since they must belong to $[r+i+2, r+n]$) and their contribution towards the weight is $(mq^j)^{j+\ell-k}S_q(\ell,k-j)$, where the factor of q arises for the same reason as previously seen and where the factor of m accounts for the coloring of the non-minimal elements. Furthermore, there are $([r]_p+m[j]_q)^{n-i-\ell-1}$ possibilities concerning the placement (and coloring) of the remaining $n-i-\ell-1$ elements of $[r+i+2, r+n]$ since each may go, independently, in either a special block or in one of the first j non-special blocks. Considering all possible i, j, and ℓ gives the weight of all members of Π' and completes the proof. □

Taking all parameters to be unity in (3.13), and recalling $S(n+1,k+1) - S(n,k) = (k+1)S(n,k+1)$, yields the following Stirling number recurrence formula, which seems to be new.

Corollary 3.9. *If* $n, k \geq 1$, *then*

$$S(n,k) = \frac{1}{k}\sum_{i=0}^{n-1}\sum_{j=0}^{k-1}\sum_{\ell=k-j-1}^{n-i-1}\binom{n-i-1}{\ell}(j+1)^{n-i-\ell-1}S(i,j)S(\ell,k-j-1).$$

We conclude this section with a bijective proof of identity (3.3).

Combinatorial proof of (3.3). The first equality in (3.3) follows from specializing the argument given above for (3.2). To show the second equality, first recall that $s(r,i) = (-1)^{r-i}c(r,i)$, where $c(r,i)$ denotes the number of permutations of $[r]$ having exactly i cycles, the set of which we will denote by $\mathcal{C}(r,i)$. Assume that within each cycle the smallest element is written first. Given $0 \leq i \leq r$, let $\mathcal{C}_i$ denote the set of all ordered pairs (γ, δ) such that $\gamma \in \mathcal{C}(r,i)$ and δ is a partition of size $n+i$ of the elements of $[n]$, together with the cycles of γ, considered as elements. Define the weight of $(\gamma,\delta) \in \mathcal{C}_i$ as $(-1)^{r-i}x^{\nu(\delta)}$, where $\nu(\delta)$ denotes the number of blocks of δ. Then the third quantity in (3.3) above gives the sum of the weights of all members of $\mathcal{C} = \cup_{j=0}^{r}\mathcal{C}_i$.

To complete the proof, it is enough to define a sign-changing, weight-preserving involution of $\mathcal{C}$ off of a subset $\mathcal{C}^*$ of $\mathcal{C}$ having weight $x^r B(n;r,x)$. Let $\mathcal{C}$ consist of those members $(\gamma,\delta) \in \mathcal{C}$ such that γ is the identity permutation of $[r]$, with the cycles of γ belonging to distinct blocks of δ. Note that members of $\mathcal{C}$ all have positive sign and are synonymous with the members of $\Pi_r(n)$ since the r (singleton) cycles of γ function as special elements within a partition of size $n+r$, which implies the weight of $\mathcal{C}^*$ is as given.

Next observe that if $(\alpha,\beta) \in \mathcal{C} - \mathcal{C}^*$, then at least one block of β has two or more elements of $[r]$ contained within all of the cycles in the block combined. Identify the smallest element of $[r]$ found within a cycle contained within such a block of β. Let u denote this element, B be the block of β to which its cycle belongs, and v denote the second smallest element of $[r]$ contained within a cycle in B. If u and v belong to the same cycle in B, say as $(u\cdots v\cdots)$, then we split this cycle into two cycles $(u\cdots),(v\cdots)$, and vice-versa, if the elements u and v belong to distinct cycles in B. This operation is seen to define the desired involution of $\mathcal{C}-\mathcal{C}^*$, which completes the proof. □

4. Identities in the case $r = 1$

Let $W_q(n,k;m) = W_{1,q}(n,k;1,m)$. Then we have the following identities.

Proposition 4.1. *If $n, k, m \geq 1$, then*

$$W_q(n,k;m+1) = \frac{1}{(m+1)^k m^{n-k}} \sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j} (m+1)^j W_q(j,k;m), \tag{4.1}$$

with $W_q(n,k;1) = \sum_{j=k}^{n} \binom{n}{j} S_q(j,k)$. If $D_q(n;m,x) = \sum_{k=0}^{n} W_q(n,k;m)x^k$, then

$$D_q(n;m+1,x) = \frac{1}{m^n} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (m+1)^j D_q\left(j;m,\frac{mx}{m+1}\right), \tag{4.2}$$

with $D_q(n;1,x) = \sum_{j=0}^{n} \sum_{i=0}^{j} \binom{n}{j} S_q(j,i)x^i$.

Proof. The formulas for $W_q(n,k;1)$ and $D_q(n;1,x)$ follow from the definitions, upon considering the number, $n-j$, of additional elements belonging to the block containing 1. To show (4.1), given $n \geq k \geq 1$, let $\widetilde{\Pi}_{1,m}(n,k)$ denote the set obtained from $\Pi_{1,m}(n,k)$ by marking all elements other than 1 in one of $m+1$ ways; note that $\widetilde{\Pi}_{1,m}(n,k)$ has weight $(m+1)^n W_q(n,k;m)$. If $k \leq j \leq n$, then let $\mathcal{A}_j$ denote the set of all ordered pairs (α,β), where α is a subset of $[2,n+1]$ of size $n-j$ and β is a member of $\widetilde{\Pi}_{1,m}(j,k)$ whose elements belong to the set $[n+1]-\alpha$. Let $\mathcal{A} = \cup_{j=k}^{n} \mathcal{A}_j$ and define the (signed) weight of $(\alpha,\beta) \in \mathcal{A}$ by $(-1)^{|\alpha|} q^{w_2(\beta)}$. Then the sum of the weights of all members of $\mathcal{A}$ is given by $\sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j} (m+1)^j W_q(j,k;m)$.

To complete the proof of (4.1), it suffices to define a sign-changing involution of $\mathcal{A}$ off of a set having weight $(m+1)^k m^{n-k} W_q(n,k;m+1)$. Let $\mathcal{A}^*$ denote the subset of $\mathcal{A}_n$ where, in the members of which, all elements of $[2,n+1]$ within the block containing 1 are marked in one of the first m ways (but not in the $(m+1)$-st way). Let $(\alpha,\beta) \in \mathcal{A}^*$; note that $\alpha = \emptyset$. To show that $\mathcal{A}^*$ has weight $(m+1)^k m^{n-k} W_q(n,k;m+1)$, first observe that each non-minimal element within a block of β not containing 1 is colored in one of m ways and marked in one of $m+1$ ways, while each element of $[2,n+1]$ in the block of β containing 1 is marked in one of m ways. Regarding the $m+1$ types of markings in the previous case as colors, it follows that there are $m^{n-k} W_q(n,k;m+1)$ possibilities in which the minimal elements within each block of β are unmarked to start with (the factor of m^{n-k} accounts for the ways in which to mark the non-minimal elements in the block containing 1 and to color the non-minimal elements in the remaining blocks). Since each of the minimal elements in the non-special blocks are also to be marked in one of $m+1$ ways, the formula for the weight of $\mathcal{A}^*$ follows. Now suppose $(\alpha,\beta)\in\mathcal{A}-\mathcal{A}^*$ and let x denote the smallest member of $[2,n+1]$ such that (i) $x\in\alpha$, or (ii) x belongs to the block of β containing 1 and is marked in the $(m+1)$-st way. If (i) occurs, then move x to the block of β containing 1 and mark it in the $(m+1)$-st way, and vice-versa, if (ii) occurs. This operation is seen to define a sign-changing involution of $\mathcal{A}-\mathcal{A}^*$, as desired.

Identity (4.2) will follow from suitably modifying the proof of (4.1) as follows. Let $\mathcal{A}_j$ now consist of the ordered pairs (α,β), where α is as before and β is a "marked" member of $\Pi_{1,m}(j)$ (on the remaining elements of $[n+1]$) wherein the non-minimal elements are each marked in one of $m+1$ ways and each minimal element other than 1 is marked in m ways. Define the weight of (α,β) as $(-1)^{|\alpha|}q^{w_2(\beta)}x^{\nu(\beta)-1}$, where $\nu(\beta)$ denotes the number of blocks of β. Then the right-hand side of (4.2) (multiplied by m^n) is seen to give the weight of all members of $\mathcal{A}=\cup_{j=k}^{n}\mathcal{A}_j$. Applying the same involution as before to $\mathcal{A}$, one sees that the set of survivors consists of those ordered pairs (α,β) where $\alpha=\emptyset$ and all non-minimal elements within the block of β containing 1 are marked in one of the first m ways. Reasoning as before shows that the sum of the weights of all such ordered pairs is given by $m^nD_q(n;m+1,x)$, which completes the proof. □

We remark that identities (4.1) and (4.2) in the case $q=1$ occur as [18, Theorems 1 and 2], where generating function proofs were provided. We also have the following further identity involving $D_q(n;m,x)$, the $q=1$ case of which occurs as [18, Corollary 1].

Proposition 4.2. *If $n,m\geq 0$, then*

$$\sum_{k=0}^{n} m^{n-k}x^kS_q(n,k)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}D_q(k;m,x). \tag{4.3}$$

Proof. Given $0\leq k\leq n$, let $\mathcal{D}_k$ denote the set of ordered pairs (α,β), where α is a subset of $[2,n+1]$ of cardinality $n-k$ and $\beta\in\Pi_{1,m}(k)$ whose elements belong to the set $[n+1]-\alpha$. Let $\mathcal{D}=\cup_{k=0}^{n}\mathcal{D}_k$ and define the weight of $(\alpha,\beta)\in\mathcal{D}$ by $(-1)^{|\alpha|}q^{w_2(\beta)}x^{\nu(\beta)-1}$. Then the right-hand side of (4.3) gives the sum of the weights of all members of $\mathcal{D}$. Given $(\alpha,\beta)\in\mathcal{D}$, we identify the smallest $\ell\in[2,n+1]$, if it exists, such that either (i) ℓ belongs to α, or (ii) ℓ belongs to the block of β that contains 1. Define an involution of $\mathcal{D}$ by replacing option (i) with (ii), and vice-versa. Note that this operation reverses the sign since the cardinality of α changes by one. It is not defined for those members $(\alpha,\beta)\in\mathcal{D}$ such that α is empty and the block of β containing 1 is a singleton. Note that the sum of the weights of all such members of $\mathcal{D}$ is given by the left-hand side of (4.3), upon considering the number k of non-special blocks, which completes the proof. □

5. Generating function formulas

In this section, we provide generating function formulas for the array $W_{p,q}(n,k)$ and some related identities. Let

$$W_{p,q}^{(k)}(t) = \sum_{n\geq k} W_{p,q}(n,k)t^n \text{ and } W_{p,q}(t;x) = \sum_{n\geq 0} D_{p,q}(n;x)t^n.$$

Theorem 5.1. *Let $k \geq 0$. We have*

$$W_{p,q}^{(k)}(t) = \frac{t^k}{\prod_{j=0}^{k}(1-t(m[j]_q+[r]_p))}, \tag{5.1}$$

and

$$W_{p,q}(t;x) = \begin{cases} \sum_{j\geq 0} \frac{m^j t^j B_j(x/m)}{(1-t[r]_p)^{j+1}}, & \text{if } q=1; \\ \sum_{j\geq 0} \frac{(-mt)^j}{(1-q)^j \prod_{i=0}^{j}(1-(xq^i+[r]_p)t-\frac{mt}{1-q})}, & \text{if } q\neq 1, \end{cases} \tag{5.2}$$

where $B_j(z)$ denotes the j-th Bell polynomial.

Proof. Multiplying both sides of (2.1) by t^n and summing over $n \geq k$ gives

$$W_{p,q}^{(k)}(t) = \frac{t}{1-t(m[k]_q+[r]_p)} W_{p,q}^{(k-1)}(t), \qquad k \geq 1,$$

with $W_{p,q}^{(0)}(t) = \frac{1}{1-t[r]_p}$. Iterating this last recurrence, and noting the initial condition, gives (5.1). For the $q=1$ case of (5.2), first note that by (5.1), we have

$$\begin{aligned} W_{p,1}^{(k)}(t) &= \frac{1}{m^k(1-t[r]_p)} \cdot \frac{y^k}{(1-y)(1-2y)\cdots(1-ky)} \\ &= \frac{1}{m^k(1-t[r]_p)} \sum_{n\geq k} S(n,k)y^n, \end{aligned}$$

where $y = \frac{mt}{1-t[r]_p}$. This implies

$$W_{p,1}(t;x) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} W_{p,1}(n,k)x^k\right)t^n = \sum_{k\geq 0} x^k W_{p,1}^{(k)}(t)$$

$$= \sum_{k\geq 0} \frac{x^k}{m^k(1-t[r]_p)} \sum_{n\geq k} S(n,k)y^n = \sum_{k\geq 0} x^k \sum_{n\geq k} \frac{S(n,k)m^{n-k}t^n}{(1-t[r]_p)^{n+1}}$$

$$= \sum_{n\geq 0} \frac{m^n t^n}{(1-t[r]_p)^{n+1}} \sum_{k=0}^{n} S(n,k)(x/m)^k$$

$$= \sum_{n\geq 0} \frac{m^n t^n B_n(x/m)}{(1-t[r]_p)^{n+1}}.$$

If $q \neq 1$, then multiplying both sides of (2.1) by x^k, summing over $0 \leq k \leq n$, and writing $[k]_q = \frac{1-q^k}{1-q}$ yields the recurrence

$$D_{p,q}(n;x) = \left(x + [r]_p + \frac{m}{1-q}\right) D_{p,q}(n-1;x) - \frac{m}{1-q} D_{p,q}(n-1;qx), \tag{5.3}$$

with $D_{p,q}(0;x) = 1$. Multiplying both sides of (5.3) by t^n, and summing over $n \geq 1$, gives

$$W_{p,q}(t;x) = \frac{1}{1-(x+[r]_p)t - \frac{mt}{1-q}} - \frac{mt/(1-q)}{1-(x+[r]_p)t - \frac{mt}{1-q}} W_{p,q}(t;qx).$$

Iteration of this last equation completes the proof. □

We note that the $p = q = 1$ case of formula (5.1) occurs in [7, Section 2]. Reasoning as in the prior proof, and using the generating function formula (see, e.g., [21])

$$\sum_{n\geq k} S_q(n,k)z^n = \frac{z^k}{(1-z[1]_q)(1-z[2]_q)\cdots(1-z[k]_q)},$$

shows further that

$$W_{p,q}^{(k)}(t) = \frac{1}{m^k(1-t[r]_p)} \sum_{j\geq 0} S_q(j,k)y^j = \sum_{j\geq 0} S_q(j,k) \frac{m^{j-k}t^j}{(1-t[r]_p)^{j+1}}$$

$$= \sum_{i\geq 0} \sum_{j\geq 0} m^{j-k} \binom{i+j}{i} [r]_p^i S_q(j,k) t^{i+j}.$$

Comparing coefficients of t^n yields

$$W_{p,q}(n,k) = \sum_{j=0}^{n} m^{j-k} \binom{n}{n-j} [r]_p^{n-j} S_q(j,k), \qquad n,k \geq 0,$$

which was shown earlier by a combinatorial argument. We have the following (p,q)-identity.

Corollary 5.2. *If* $n,k\geq 0$, *then*

$$\sum_{j=0}^{n} m^{j-k}\binom{n}{j}[r]_p^{n-j}S_q(j,k)$$

$$= \frac{1}{m^k[k]_q!}\sum_{j=0}^{k}(-1)^{k-j}q^{-\binom{j}{2}-j(k-j)}\begin{bmatrix}k\\j\end{bmatrix}_q(m[j]_q+[r]_p)^n. \tag{5.4}$$

Proof. Suppose that the array $\{u(n,k)\}_{n,k\geq 0}$ is defined by the recurrence

$$u(n,k)=u(n-1,k-1)+(a_{n-1}+b_k)u(n-1,k), \qquad n,k\geq 1,$$

subject to the boundary conditions $u(n,0)=\prod_{i=0}^{n-1}(a_i+b_0)$ and $u(0,k)=\delta_{k,0}$ for all $n,k\geq 0$, where $(a_i)_{i\geq 0}$ and $(b_i)_{i\geq 0}$ are given sequences with the b_i distinct. By [11, Theorem 1.1], we have the formula

$$u(n,k)=\sum_{j=0}^{k}\left(\frac{\prod_{i=0}^{n-1}(a_i+b_j)}{\prod_{\substack{i=0\\ i\neq j}}^{k}(b_j-b_i)}\right), \qquad n,k\geq 0. \tag{5.5}$$

Taking $a_i=0$ and $b_i=[r]_p+m[i]_q$ for all i in (5.5), noting $[j]_q-[i]_q=q^i[j-i]_q$ if $j>i$, and simplifying yields the formula

$$W_{p,q}(n,k)=\frac{1}{m^k[k]_q!}\sum_{j=0}^{k}(-1)^{k-j}q^{-\binom{j}{2}-j(k-j)}\begin{bmatrix}k\\j\end{bmatrix}_q(m[j]_q+[r]_p)^n, \tag{5.6}$$

which implies (5.2). □

Note that when $m=q=1$ and $r=0$, equation (5.2) reduces to the well-known formula

$$S(n,k)=\frac{1}{k!}\sum_{j=0}^{n}(-1)^{k-j}\binom{k}{j}j^n;$$

see, e.g., [20, p. 34].

One can also give an explicit formula for the exponential generating function of $W_{p,q}(n,k)$ in the case when $q=1$. By the $q=1$ case of (5.2), we have

$$\sum_{k\geq 0}\left(\sum_{n\geq 0}W_{p,1}(n,k)\frac{t^n}{n!}\right)x^k=\sum_{k\geq 0}\left(\frac{1}{m^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}e^{(mj+[r]_p)t}\right)x^k$$
$$=e^{t[r]_p}\sum_{k\geq 0}\frac{(e^{mt}-1)^k}{m^k k!}x^k=e^{(e^{mt}-1)\frac{x}{m}+t[r]_p}.$$

Corollary 5.3. *If $n\geq 0$, then*

$$\sum_{k=0}^{n}W_{p,1}(n,k)m^k=\sum_{i=0}^{n}m^i\binom{n}{i}[r]_p^{n-i}B(i). \tag{5.7}$$

Proof. By the fact that $e^{e^x-1}=\sum_{n\geq 0}B(n)\frac{x^n}{n!}$ (see [20, p. 34]), we obtain

$$\sum_{k\geq 0}\left(\sum_{n\geq 0}W_{p,1}(n,k)\frac{t^n}{n!}\right)m^k=e^{(e^{mt}-1)+t[r]_p}=\sum_{i\geq 0}B(i)\frac{m^i t^i}{i!}\sum_{j\geq 0}\frac{t^j[r]_p^j}{j!},$$

and comparing coefficients of t^n gives (5.7). □

We conclude this section by considering the log-concavity of the array $W_{p,q}(n,k)$ for various values of its parameters. Recall that a sequence $(a_n)_{n\geq 0}$ is said to be *log-concave* if $a_n^2\geq a_{n-1}a_{n+1}$ for all $n\geq 1$, and *strictly log-concave* if the inequality is strict. A sequence $(f_n(x))_{n\geq 0}$ of polynomials having real coefficients is said to be *x-log-concave* if $f_n(x)^2-f_{n-1}(x)f_{n+1}(x)$ has all non-negative coefficients for each $n\geq 1$. Given an array $a(n,k)$, $0\leq k\leq n$, let $A_r(n;x)=\sum_{k=r}^{n}a(n,k)x^k$. Then $a(n,k)$ is said to be *LC-positive*, see [23], if the sequence $(A_r(n;x))_{n\geq r}$ is x-log-concave for each $r\geq 0$.

Lemma 5.4. *The array $a(n,k):=q^{-\binom{k}{2}-k(n-k)}\begin{bmatrix}n\\k\end{bmatrix}_q$ is LC-positive for all $q\geq 1$.*

Proof. Dividing both sides of the q-binomial coefficient recurrence

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\begin{bmatrix}n-1\\k-1\end{bmatrix}_q+q^k\begin{bmatrix}n-1\\k\end{bmatrix}_q$$

by $q^{\binom{k}{2}+k(n-k)}$, and simplifying, implies

$$a(n,k)=q^{-(n-1)}a(n-1,k-1)+a(n-1,k)$$

for $n \geq 1, k \geq 0$. Thus, we have

$$\begin{aligned} A_r(n;x) &= \sum_{k=r}^{n} a(n,k)x^k = \sum_{k=r}^{n}(q^{-(n-1)}a(n-1,k-1)+a(n-1,k))x^k \\ &= (1+q^{-(n-1)}x)A_r(n-1;x)+q^{-(n-1)}a(n-1,r-1)x^r, \quad n \geq 1, \end{aligned}$$

which gives

$$\begin{aligned} &A_r(n;x)^2 - A_r(n-1;x)A_r(n+1;x) \\ &= A_r(n;x)((1+q^{-(n-1)}x)A_r(n-1;x)+q^{-(n-1)}a(n-1,r-1)x^r) \\ &\quad - A_r(n-1;x)((1+q^{-n}x)A_r(n;x)+q^{-n}a(n,r-1)x^r) \\ &= \frac{(q-1)x}{q^n}A_r(n;x)A_r(n-1;x) \\ &\quad + \frac{x^r}{q^n}(qa(n-1,r-1)A_r(n;x)-a(n,r-1)A_r(n-1;x)). \end{aligned} \tag{5.8}$$

Note that the first quantity on the right-hand side of (5.8) has non-negative x-coefficients since $q \geq 1$. Furthermore, observe that $qa(n-1,r-1)a(n,k) \geq a(n,r-1)a(n-1,k)$ for all $n,k \geq r$ and $q \geq 1$ since it reduces to the obvious inequality $[n-r+1]_q \geq q^{k-r}[n-k]_q$. Thus, the x-coefficients of the polynomial corresponding to the second term in (5.8) are also non-negative, which implies the result. □

Theorem 5.5. *If $n \geq 1$, then $W_{p,q}(n,k)$ for $0 \leq k \leq n$ is strictly log-concave (and hence unimodal) for all $p \geq 0$ and $1 \leq q \leq 1+\frac{m}{[r]_p}$ if $r \geq 1$ and for all $q \geq 1$ if $r = 0$.*

Proof. We prove only the first statement, as the second will follow by slightly modifying our proof. First note that if a sequence $(b_n)_{n\geq 0}$ of positive real numbers is log-concave, then $\frac{b_n}{m^n[n]_q!}$ is strictly log-concave. Thus, by (5.6), to show that $W_{p,q}(n,k)$ is strictly log-concave, it suffices to show that the sequence $b_k = \sum_{j=0}^{k}(-1)^j a(k,j)(m[j]_q+[r]_p)^n$ is log-concave for each fixed $n \geq 1$, where $a(k,j)$ is as defined in the previous lemma. Since the array $a(k,j)$ is LC-positive for $q \geq 1$, in order to establish the log-concavity of the sequence b_k, it is enough to show that the sequence $c_j = (-1)^j(m[j]_q+[r]_p)^n$ for $j \geq 0$ is log-concave, by [23, Theorem 2.3]. To do so, note that $c_j^2 \geq c_{j-1}c_{j+1}$ if and only if

$$(m[j]_q+[r]_p)^2 - (m[j-1]_q+[r]_p)(m[j+1]_q+[r]_p)$$

$$= m^2([j]_q^2 - [j-1]_q[j+1]_q) \\ + m[r]_p(2[j]_q - [j-1]_q - [j+1]_q) \geq 0.$$

Upon substituting $[j-1]_q = [j]_q - q^{j-1}$ and $[j+1]_q = [j]_q + q^j$, and simplifying, the last inequality is equivalent to

$$m^2(q^{j-1}(1-q)[j]_q + q^{2j-1}) + mq^{j-1}(1-q)[r]_p \geq 0,$$

i.e., $mq^{j-1}(m + (1-q)[r]_p) \geq 0$, which is clearly true for $1 \leq q \leq 1 + \frac{m}{[r]_p}$. Thus, the sequence c_j is log-concave, which completes the proof. □

We conjecture that $W_{p,q}(n,k)$ is strictly log-concave for all $p \geq 0$ and $q \geq 1$, but do not have a complete proof.

6. The (p,q)-Whitney matrix of the second kind

In this section, we introduce the (p,q)-Whitney matrix of the second kind. Then we find some factorizations of this matrix in analogy with the results of Mező and Ramírez [13]. For the results of this section, we will need the following.

Theorem 6.1. *If $n \geq 0$, then*

$$(mx + [r]_p)^n = \sum_{k=0}^{n} W_{p,q}(n,k) m^k [x]_q^{\underline{k}}, \tag{6.1}$$

where $[x]_q^{\underline{k}} = x(x-[1]_q)(x-[2]_q)\cdots(x-[k-1]_q)$ and $[x]_q^{\underline{0}} = 1$.

Proof. We proceed by induction on n. The equality clearly holds for $n = 0, 1$. Now assume that the claim holds for n, and let us prove it for $n+1$:

$$\begin{aligned}
&(mx + [r]_p)^{n+1} \\
&= (mx + [r]_p) \sum_{k=0}^{n} W_{p,q}(n,k) m^k [x]_q^{\underline{k}} \\
&= x \sum_{k=0}^{n} W_{p,q}(n,k) m^{k+1} [x]_q^{\underline{k}} + [r]_p \sum_{k=0}^{n} W_{p,q}(n,k) m^k [x]_q^{\underline{k}} \\
&= (x - [k]_q) \sum_{k=0}^{n} m^{k+1} W_{p,q}(n,k) [x]_q^{\underline{k}} + (m[k]_q + [r]_p) \sum_{k=0}^{n} m^k W_{p,q}(n,k) [x]_q^{\underline{k}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} m^{k+1} W_{p,q}(n,k)[x]_q^{\underline{k+1}} + (m[k]_q + [r]_p) \sum_{k=0}^{n} m^k W_{p,q}(n,k)[x]_q^{\underline{k}} \\
&= \sum_{k=0}^{n+1} m^k W_{p,q}(n,k-1)[x]_q^{\underline{k}} + (m[k]_q + [r]_p) \sum_{k=0}^{n+1} m^k W_{p,q}(n,k)[x]_q^{\underline{k}} \\
&= \sum_{k=0}^{n+1} \left(W_{p,q}(n,k-1) + (m[k]_q + [r]_p) W_{p,q}(n,k)\right) m^k [x]_q^{\underline{k}} \\
&= \sum_{k=0}^{n+1} W_{p,q}(n+1,k) m^k [x]_q^{\underline{k}}. \qquad \square
\end{aligned}
$$

Definition 6.2. *The (p,q)-Whitney matrix of the second kind is the $n \times n$ matrix defined by*

$$\mathcal{W}_{p,q}(n) := \mathcal{W}_{p,q}^{(m,r)}(n) := [W_{p,q}(i,j;r,m)]_{0 \le i,j \le n-1}\,.$$

For example,

$$\mathcal{W}_{p,q}(4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ [r]_p & 1 & 0 & 0 \\ [r]_p^2 & m + 2[r]_p & 1 & 0 \\ [r]_p^3 & m^2 + 3m[r]_p + 3[r]_p^2 & (2+q)m + 3[r]_p & 1 \end{bmatrix}.$$

In particular, if $p = q = 1$, we obtain the r-Whitney matrix of the second kind [13]. If $m = p = 1$ and $r = 0$, we obtain the q-Stirling matrix of the second kind $S_{q,n} := [S_q(i,j)]_{0 \le i,j \le n-1}$; see, e.g., [9, 16].

We need to introduce some auxiliary matrices. The generalized $n \times n$ Pascal matrix $P_n[x]$ is defined as follows (see [4]):

$$P_n[x] := \left[x^{i-j} \binom{i}{j} \right]_{0 \le i,j \le n-1}.$$

If $x = 1$, we obtain the Pascal matrix P_n of order n.

From Theorem 3.1, we have the following factorization for any positive integer n:

$$\mathcal{W}_{p,q}(n) = P_n[[r]_p] S_{q,n}[m], \tag{6.2}$$

where $S_{q,n}[x] := [x^{i-j} S_q(i,j)]_{0 \le i,j \le n-1}$.

For example,

$$\mathcal{W}_{p,q}(4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ [r]_p & 1 & 0 & 0 \\ [r]_p^2 & 2[r]_p & 1 & 0 \\ [r]_p^3 & 3[r]_p^2 & 3[r]_p & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & m & 1 & 0 \\ 0 & m^2 & (q+2)m & 1 \end{bmatrix}$$
$$= P_4\left[[r]_p\right] S_{q,4}\left[m\right].$$

Moreover, from relation (3.4), we obtain that if $0 \le s \le r$, then

$$\mathcal{W}_{p,q}^{(m,r)}(n) = P_n[p^s[r-s]_p]\mathcal{W}_{p,q}^{(m,s)}(n).$$

We need the following factorization of the generalized Pascal matrix given by Zhang (see [24, Theorem 1]):

$$P_n[x] = G_n[x]G_{n-1}[x]\cdots G_1[x], \qquad n \ge 1, \tag{6.3}$$

where $G_k[x]$ is the $n \times n$ matrix defined as

$$G_k[x] = I_{n-k} \oplus S_k[x], \qquad 1 \le k \le n-1,$$

and $G_n[x] = S_n[x]$, where

$$S_n[x] := \left[x^{i-j}\right]_{0 \le j \le i \le n-1}$$

and $\oplus$ denotes the matrix direct sum.

Having these preliminaries, we obtain the following factorization of $\mathcal{W}_{p,q}(n)$.

Proposition 6.3. *For all $n \ge 2$, we have*

(6.4)
$$\mathcal{W}_{p,q}(n) = G_n[[r]_p]G_{n-1}[[r]_p]\cdots G_1[[r]_p]\overline{P}_{n-1}[m]\overline{P}_{n-2}[mq]\cdots\overline{P}_1[mq^{n-2}].$$

Here

$$\overline{P}_k[x] = I_{n-k} \oplus P_k[x].$$

Proof. By (6.2), we have

$$\mathcal{W}_{p,q}(n) = P_n[[r]_p]S_{q,n}[m].$$

The matrix $P_n[[r]_p]$ can be factorized by means of (6.3), while the matrix $S_{q,n}[m]$ can be factorized by a result of Oruç et al. [16] as

$$S_{q,n}[m] = \overline{P}_{n-1}[m]\overline{P}_{n-2}[mq]\cdots\overline{P}_1[mq^{n-2}].$$

Therefore, equation (6.4) follows. □

Now we introduce a generalized Vandermonde matrix and find some factorizations of it by using the (p,q)-Whitney matrix. Note first that relation (6.1) can be expressed as

$$(6.5) \qquad (mx+[r]_p)^n = \sum_{k=0}^{n} k!m^k W_{p,q}(n,k)\begin{vmatrix} x \\ k \end{vmatrix}_q,$$

where

$$\begin{vmatrix} x \\ k \end{vmatrix}_q = \begin{cases} \frac{[x]_q^{\underline{k}}}{k!}, & \text{if } k \geq 1; \\ 1, & \text{if } k = 0; \\ 0, & \text{if } k < 0, \end{cases}$$

for any real number x and integer k. Hence, equation (6.5) can be rewritten as follows:

$$(6.6) \qquad \mathbf{v}_p(x) = \widetilde{\mathcal{W}}_{p,q}(n)\mathbf{c}_{q,n}(x), \qquad n \geq 1,$$

where

$$\mathbf{v}_p(x) = \left[1, mx+[r]_p, (mx+[r]_p)^2, \ldots, (mx+[r]_p)^{n-1}\right]^T,$$
$$\mathbf{c}_{q,n}(x) = \left[\begin{vmatrix} x \\ 0 \end{vmatrix}_q, m\begin{vmatrix} x \\ 1 \end{vmatrix}_q, \ldots, m^{n-1}\begin{vmatrix} x \\ n-1 \end{vmatrix}_q\right]^T,$$

and $\widetilde{\mathcal{W}}_{p,q}(n) := \mathcal{W}_{p,q}(n)\cdot\mathrm{diag}(0!, 1!, \ldots, (n-1)!)$. For example, if $n = 4$, then

$$\begin{bmatrix} 1 \\ mx+[r]_p \\ (mx+[r]_p)^2 \\ (mx+[r]_p)^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ [r]_p & 1 & 0 & 0 \\ [r]_p^2 & m+2[r]_p & 2 & 0 \\ [r]_p^3 & m^2+3m[r]_p+3[r]_p^2 & 2(2+q)m+3[r]_p & 6 \end{bmatrix} \begin{bmatrix} \left|\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right|_q \\ m\left|\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right|_q \\ m^2\left|\begin{smallmatrix} x \\ 2 \end{smallmatrix}\right|_q \\ m^3\left|\begin{smallmatrix} x \\ 3 \end{smallmatrix}\right|_q \end{bmatrix}.$$

Let $V_{p,n}^{m,r}[x]$ be an $n \times n$ p-generalized Vandermonde matrix defined by

$$V_{p,n}^{(m,r)}[x] := V_{p,n}^{(m,r)}(mx+[r]_p, mx+m+[r]_p, \ldots, mx+(n-1)m+[r]_p)$$

$$= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ mx+[r]_p & mx+m+[r]_p & \cdots & mx+(n-1)m+[r]_p \\ (mx+[r]_p)^2 & (mx+m+[r]_p)^2 & \cdots & (mx+(n-1)m+[r]_p)^2 \\ \vdots & & \cdots & \vdots \\ (mx+[r]_p)^{n-1} & (mx+m+[r]_p)^{n-1} & \cdots & (mx+(n-1)m+[r]_p)^{n-1} \end{bmatrix}.$$

The matrix equation (6.6) implies the following factorization of $V_{p,n}^{(m,r)}[x]$:

$$V_{p,n}^{(m,r)}[x] = \widetilde{\mathcal{W}}_{p,q}(n) C_{q,n}^m[x], \tag{6.7}$$

where $C_{q,n}^m[x] := \left(x^i \left|{x+j \atop i}\right|_q\right)_{0\le i,j\le n-1}$.

So, for example, if $x=1$ and $n=4$, then

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ m+[r]_p & 2m+[r]_p & 3m+[r]_p & 4m+[r]_p \\ (m+[r]_p)^2 & (2m+[r]_p)^2 & (3m+[r]_p)^2 & (4m+[r]_p)^2 \\ (m+[r]_p)^3 & (2m+[r]_p)^3 & (3m+[r]_p)^3 & (4m+[r]_p)^3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ [r]_p & 1 & 0 & 0 \\ [r]_p^2 & m+2[r]_p & 2 & 0 \\ [r]_p^3 & m^2+3m[r]_p+3[r]_p^2 & 2(2+q)m+3[r]_p & 6 \end{bmatrix}$$
$$\times \begin{bmatrix} 1 & 1 & 1 & 1 \\ m\left|{x \atop 1}\right|_q & m\left|{x+1 \atop 1}\right|_q & m\left|{x+2 \atop 1}\right|_q & m\left|{x+3 \atop 1}\right|_q \\ m^2\left|{x \atop 2}\right|_q & m^2\left|{x+1 \atop 2}\right|_q & m^2\left|{x+2 \atop 2}\right|_q & m^2\left|{x+3 \atop 2}\right|_q \\ m^3\left|{x \atop 3}\right|_q & m^3\left|{x+1 \atop 3}\right|_q & m^3\left|{x+2 \atop 3}\right|_q & m^3\left|{x+3 \atop 3}\right|_q \end{bmatrix}.$$

Define the $n \times n$ matrix $\mathcal{L}_{q,n}[x]$ by $(\mathcal{L}_{q,n}[x])_{1\le i,j\le n} = x^{j-1}[i]_q^{\underline{j-1}}$. We find a factorization of $V_{p,n}^{(m,r)}[x]$ in terms of the (p,q)-Whitney and $\mathcal{L}_{q,n}[x]$ matrices.

Lemma 6.4. *The p-Vandermonde matrix $V_{p,n}^{(m,r)}[1]$ can be factorized as*

$$V_{p,n}^{(m,r)}[1] = \mathcal{W}_{p,q}(n)\mathcal{L}_{q,n}[m]^T.$$

Proof. From (6.1), we have

$$(\mathcal{W}_{p,q}(n)\mathcal{L}_{q,n}[m]^T)_{ij} = \sum_{k=0}^{i-1} W_{p,q}(i-1,k) m^k [j]_q^{\underline{k}} = (jm+[r]_p)^{i-1}$$
$$= (m+(j-1)m+[r]_p)^{i-1} = (V_{p,n}^{(m,r)}[1])_{ij}. \qquad \square$$

Theorem 6.5. *The p-Vandermonde matrix $V_{p,n}^{(m,r)}[x]$ can be factorized for any real number x as $V_{p,n}^{(m,r)}[x] = P_n[m(x-1)]\mathcal{W}_{p,q}(n)\mathcal{L}_{q,n}[m]^T$.*

Proof. From Lemma 6.4, we have

$$
\begin{aligned}
&(P_n[m(x-1)]\mathcal{W}_{p,q}(n)\mathcal{L}_{q,n}[m]^T)_{ij} = (P_n[m(x-1)]V_{p,n}^{(m,r)}[1])_{ij} \\
&= \sum_{k=0}^{i-1} \binom{i-1}{k} (m(x-1))^{i-1-k}(mj+[r]_p)^k \\
&= (m(x-1)+mj+[r]_p)^{i-1} = (mx+m(j-1)+[r]_p)^{i-1} = (V_{p,n}^{(m,r)}[x])_{ij}.
\end{aligned}
$$

□

For example, if $n=4$, then

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
mx+[r]_p & mx+m+[r]_p & mx+2m+[r]_p & mx+3m+[r]_p \\
(mx+[r]_p)^2 & (mx+m+[r]_p)^2 & (mx+3m+[r]_p)^2 & (mx+3m+[r]_p)^2 \\
(mx+[r]_p)^3 & (mx+m+[r]_p)^3 & (mx+3m+[r]_p)^3 & (mx+3m+[r]_p)^3
\end{bmatrix}
$$

$$
= \begin{bmatrix}
1 & 0 & 0 & 0 \\
m(x-1) & 1 & 0 & 0 \\
m^2(x-1)^2 & 2m(x-1) & 1 & 0 \\
m^3(x-1)^3 & 3m^2(x-1)^2 & 3m(x-1) & 1
\end{bmatrix}
$$

$$
\times \begin{bmatrix}
1 & 0 & 0 & 0 \\
[r]_p & 1 & 0 & 0 \\
[r]_p^2 & m+2[r]_p & 1 & 0 \\
[r]_p^3 & m^2+3m[r]_p+3[r]_p^2 & (2+q)m+3[r]_p & 1
\end{bmatrix}
$$

$$
\times \begin{bmatrix}
1 & 1 & 1 & 1 \\
m & 2m & 3m & 4m \\
0 & 2m^2 & 6m^2 & 12m^2 \\
0 & 2m^3(1-q) & 6m^3(2-q) & 12m^3(3-q)
\end{bmatrix}.
$$

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