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A note on the k^{th} tensor product of the defining representation

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Let D be the defining representation of the symmetric group S_n. We prove an identity which decomposes the tensor product of D with itself k times into irreducible components using a sign reversing involution.

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1. Introduction

Let S_n be the symmetric group on n letters and let D be the defining representation of S_n; that is, D is the representation which sends permutations to permutation matrices. We provide a sign reversing involution which gives a combinatorial algorithm to decompose the tensor product of D with itself k times into irreducible components. Our proof relies only on the combinatorics of permutations, set partitions, integer partitions, and rim hook tableaux. In general, decomposing the tensor product of representations into irreducible components is difficult, and so it is remarkable that this can be done relatively easily.

Let δ be the character of D and let δ^k be the character of $D \otimes \cdots \otimes D$, the representation of S_n found by taking the tensor product of D with itself k times. If $\mathrm{fxd}(\sigma)$ denotes the number of fixed points in $\sigma \in S_n$, then $\delta(\sigma) = \mathrm{fxd}(\sigma)$ since δ is the trace of a permutation matrix. Results which can express δ^k as the sum of irreducible characters of S_n have been found by Goupil and Chauve [1], but our formulation is simpler, our approach cleaner and more direct, and our combinatorial proof is noteworthy.

More explicitly, Goupil and Chauve show $\left\langle (\chi^{(n-1,1)})^k, \chi^\lambda \right\rangle$ is equal to

$$(1) \qquad f^{\overline{\lambda}} \sum_{m_1=0}^{|\overline{\lambda}|} \left(\binom{k}{m_1} \sum_{m_2=|\overline{\lambda}|-m_1}^{\lfloor (k-m_1)/2 \rfloor} \binom{m_2}{|\overline{\lambda}|-m_1} p_2(k-m_1, m_2) \right)$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on characters, χ^λ is the character of the irreducible representation of S_n corresponding to the integer partition λ, $\overline{\lambda}$ is the integer partition with largest part removed, $f^{\overline{\lambda}}$ is the number of standard tableaux of shape $\overline{\lambda}$, and $p_2(k-m_1, m_2)$ is the number of set partitions of $k - m_1$ into exactly m_2 sets of size at least 2. This identity holds only when $n \geq k + \lambda_2$ where λ_2 is the second largest part in λ.

The defining representation is reducible; specifically, $\delta = 1 + \chi^{(n-1,1)}$ where 1 represents the trivial representation. So, in order to find $\langle \delta^k, \chi^\lambda \rangle$, we can expand $(1 + \chi^{(n-1,1)})^k$ with the binomial theorem and use (1) on each term of the form $(\chi^{(n-1,1)})^i$. Doing this, we find $\langle \delta^k, \chi^\lambda \rangle$ is equal to

$$\sum_{i=0}^{k} \binom{k}{i} f^{\overline{\lambda}} \sum_{m_1=0}^{|\overline{\lambda}|} \left(\binom{i}{m_1} \sum_{m_2=|\overline{\lambda}|-m_1}^{\lfloor (i-m_1)/2 \rfloor} \binom{m_2}{|\overline{\lambda}| - m_1} p_2(i - m_1, m_2) \right),$$

provided $\lambda \neq (n)$ and $n \geq k + \lambda_2$.

In contrast, our identity is that $\langle \delta^k, \chi^\lambda \rangle$ is equal to

$$\sum_{L(\lambda) \subseteq \mu \subseteq \lambda} f^\mu \, S(k, |\mu|)$$

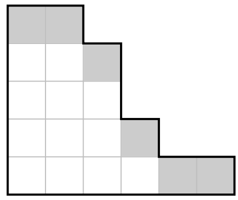
where $L(\lambda)$ is the integer partition found by removing the bottom row of the Young diagram of λ and $S(k, |\mu|)$ is the number of set partitions of k into $|\mu|$ parts (the Stirling number of the second kind).

We end this introduction by establishing some standard notation. If A and B are matrix representations of S_n, then the inner product of their characters χ^A and χ^B is

$$\langle \chi^A, \chi^B \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^A(\sigma) \chi^B(\sigma).$$

If B is irreducible, then $\langle \chi^A, \chi^B \rangle$ is the number of times B appears in A.

Our convention of drawing Young diagrams for the integer partition λ is drawing the largest part on the bottom row. If μ and λ are integer partitions such that the Young diagram for μ fits inside the Young diagram for λ, then we write $\mu \subseteq \lambda$. Let $L(\lambda)$ be the partition created by removing the bottom row from the Young diagram of λ. In the example illustrated below, $L(6, 4, 3, 3, 2) = (4, 3, 3, 2)$:

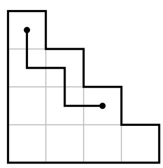


We define the ledge of λ to be the cells in the Young diagram of λ but not in $L(\lambda)$. The cells in the ledge of $(6, 4, 3, 3, 2)$ are shaded in the above example.

A rim hook ζ in a partition λ is a sequence of consecutive blocks along the north east border of the Young diagram of λ such that ζ does not contain any 2×2 block of cells and the removal of ζ from λ leaves a Young diagram for another partition. Let $\lambda \setminus \zeta$ denote the Young diagram created by removing ζ from λ. The length of a rim hook ζ is the number of cells in ζ, and the sign of ζ is defined by

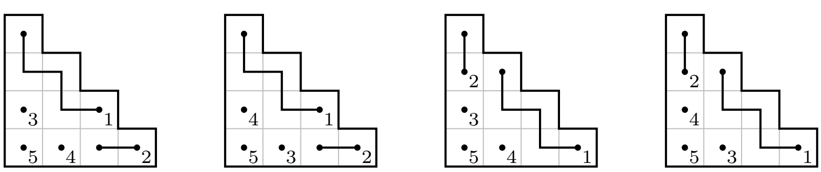
$$\text{sign}(\zeta) = (-1)^{(\text{the number of rows spanned by } \zeta)-1}.$$

For example, one rim hook of length 5 is shown inside the Young diagram for $(4, 3, 2, 1)$ below:



Let λ be an integer partition of n and let $\mu = (\mu_1, \ldots, \mu_\ell)$ be a list of nonnegative integers which sum to n (a weak composition of n). A rim hook tableaux of shape λ and type μ is created by placing a rim hook ζ_1 in λ of length μ_1 (so that ζ_1 is on the north east boundary of λ), placing a rim hook ζ_2 in $\lambda \setminus \zeta_1$ of length μ_2, placing a rim hook ζ_3 in $\lambda \setminus \zeta_1 \setminus \zeta_2$ of length μ_3, and so on. As an example, here are all possible rim hook tableau of shape $(4, 3, 2, 1)$ and type $(5, 2, 1, 1, 1)$:

(2)



The numbers below each rim hook indicate the order in which the rim hooks were placed into the diagram. The rim hook of length 5 was placed first along the north east edge of the diagram, followed by the rim hooks of length 2, 1, 1 and 1.

The sign of a rim hook tableaux T, denoted $\mathrm{sign}(T)$, is the product of the signs of the rim hooks in T. In the example displayed above, the first two rim hook tableaux have sign -1 while the last two rim hook tableau have sign $+1$.

Theorem 1 (The Murnaghan-Nakayama Rule)**.** If λ is a partition of n, $\sigma \in S_n$ is a permutation with cycle type μ, and χ^λ is the character of the irreducible representation of S_n indexed by λ, then

$$\chi^\lambda(\sigma) = \sum_{\substack{T \text{ is a rim hook tableau} \\ \text{of shape } \lambda \text{ and type } \mu}} \mathrm{sign}(T).$$

We define $f^\lambda = \chi^\lambda(\varepsilon)$. This number may be calculated using with either the Murnaghan-Nakayama rule or the well known hook length formula. More details about these topics can be found in [2].

2. The combinatorics of δ^k

Theorem 2. If k is a nonnegative integer, then

$$\left\langle \delta^k, \chi^\lambda \right\rangle = \sum_{L(\lambda) \subseteq \mu \subseteq \lambda} f^\mu \, S(k, |\mu|).$$

Using the definition of the inner product, this may be expressed as

$$\sum_{\sigma \in S_n} \mathrm{fxd}(\sigma)^k \chi^\lambda(\sigma) = n! \sum_{L(\lambda) \subseteq \mu \subseteq \lambda} f^\mu \, S(k, |\mu|). \tag{3}$$

It is this formulation of the identity which we prove using a sign reversing involution.

Proof. Build a set of combinatorial objects by following these steps:

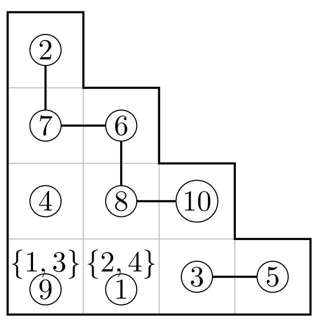
1. Select a permutation $\sigma \in S_n$ with at least one fixed point. Write σ in cyclic notation.
2. For each $i \in \{1, \ldots, k\}$, choose a fixed point j in σ and write i in a set above j. The cycles with a set of integers over them (which must be cycles of length one) will be called covered. Cycles without a set of integers over them will be called uncovered.

3. Write the cycles of σ such that each cycle begins with its least element. Order the cycles in σ by writing the uncovered cycles first in increasing order according to smallest elements. Then write the covered cycles in increasing order according to their single element.
4. Select a rim hook tableau of shape λ and type $\mu = (\mu_1, \ldots, \mu_\ell)$, where μ_i is the length of the i^{th} cycle in σ when reading from left to right.
5. Starting from the most north west cell in a rim hook of length μ_i, write down the elements in the i^{th} cycle of σ into the successive cells of the rim hook. If the cycle is covered, write down the set above the cycle inside of the rim hook tableau as well. Erase the labels $1, \ldots, \ell$ on the ℓ rim hooks—they are no longer needed to keep track of the order in which rim hooks are inserted into the tableau because that order can be deduced from the smallest element in each cycle.

For example, consider the case of $\lambda = (4, 3, 2, 1)$ and $k = 4$. We might select $\sigma = (1)(2\ 7\ 6\ 8\ 10)(3\ 5)(4)(9)$. We may then choose to place the integers 2 and 3 in a set over (1) and place the integers 1 and 4 in a set over (9). Following step 3, we reorder the cycles of σ to find

$$(2\ 7\ 6\ 8\ 10) \qquad (3\ 5) \qquad (4) \qquad \overset{\{2,4\}}{(1)} \quad \overset{\{1,3\}}{(9)}$$

Step 4 asks us to select a rim hook tableau of shape $(4, 3, 2, 1)$ and type $(5, 2, 1, 1, 1)$. We might choose the rim hook tableau written first in equation (2). Finally, inserting the cycles and sets into the rim hook tableau as described in step 5, we arrive at



Let $\mathcal{T}$ be the set of objects created in this manner. Given $T \in \mathcal{T}$, define $\text{sign}(T)$ to be the sign of the underlying rim hook tableau. It follows that

$$\sum_{\sigma \in S_n} \text{fxd}(\sigma)^k \chi^\lambda(\sigma) = \sum_{T \in \mathcal{T}} \text{sign}(T).$$

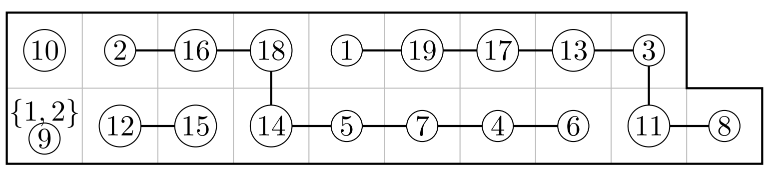
Define a cell c in $T \in \mathcal{T}$ to be a downstep if c is in the same rim hook as the cell immediately north of c. The cells containing 7 and 8 are both downsteps in the object displayed above.

We now describe a sign reversing involution on $\mathcal{T}$. Take $T \in \mathcal{T}$. Reading left to right, locate the first uncovered cell c in the bottom row of T that is either a downstep or is one cell south of the end of a rim hook. If no such cell c exists on the bottom row of T, locate c by proceeding inductively on the second row of T. If no such c exists in any row of T then we leave T fixed. Otherwise, define a sign reversing involution by considering the two possible cases for c separately.

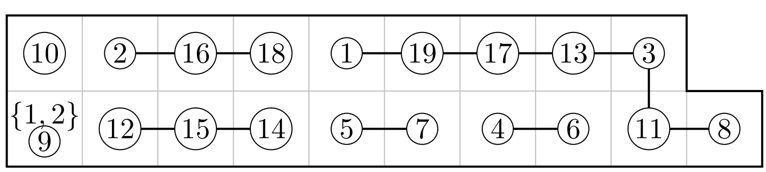
Case 1: The cell c is a downstep. In this case,

1. Erase the line connecting c and the cell above c.
2. Suppose there is an uncovered rim hook ζ which ends one cell west of c. If the integer in c is larger than the smallest integer in ζ, then connect c with ζ.
3. Let ξ be the rim hook which now contains c. Read the integers in ξ from right to left, looking for the first i in ξ which is smaller than every integer in ξ on its left. Break ξ into two rim hooks by erasing the line connecting i and the cell to the left of i. Iterate this procedure with the remaining portion of ξ. This step ensures that each rim hook begins with its smallest integer.

As an example, consider this object:



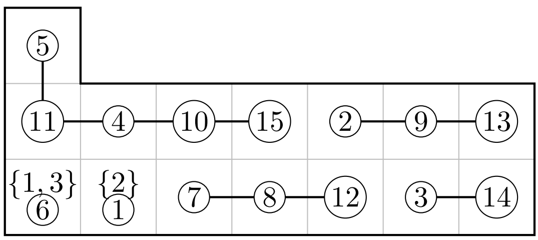
The cell c is the cell containing 14. Following the three steps, we first erase the line connecting 18 and 14 and then connect the 14 to the 15. The rim hook ξ, which now contains 14, also contains the integers $12, 15, 14, 5, 7, 4,$ and 6. We need to break ξ up into smaller pieces so that each rim hook begins with its smallest integer. After doing this, we find:



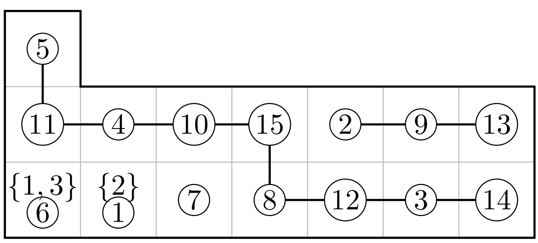
Case 2: The cell c is one cell south of the end of a rim hook. In this case,

1. If c is in the same rim hook as the cell to its west, erase the line connecting these two cells.
2. Connect c with the cell above c.
3. Let ζ be the rim hook which now contains c. Suppose there is a rim hook ξ which begins one cell east of the end of ζ. If the smallest integer in ζ is smaller than the smallest integer in ξ, connect ζ and ξ. Iterate this process until either there is no rim hook ξ that begins one cell east of the rim hook containing c or the smallest integer in ζ is larger than the smallest integer in ξ.

As a first example of Case 2, it can be verified that these three steps change the second object displayed in Case 1 into the first object displayed in Case 1. As a second example, consider this object:



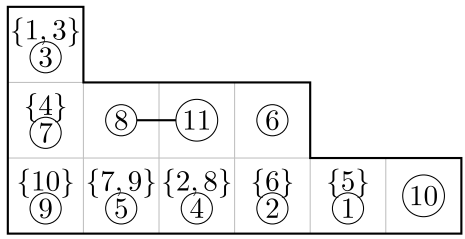
The cell c is the cell containing 8. We first erase the line connecting 7 and 8, and then connect the 15 and 8. The rim hook containing 8, ζ, begins with 5. The rim hook which begins one cell east of the end of ζ, ξ, begins with 3. Since $3 < 5$, we connect ζ and ξ. After doing this, there is no rim hook which begins one cell east of the rim hook containing 8. We end up with this:



Let T' be the image of $T \in \mathfrak{T}$ after applying either Case 1 or Case 2. The cell c is the first downstep in or cell below the end of a rim hook in T if and only if c is the first downstep in or cell below the end of a rim hook in T'. By construction, the operations described in Case 1 and Case 2 are inverses. Furthermore, this involution is sign reversing because Case

1 removes exactly one downstep and Case 2 adds exactly one downstep, thereby changing the sign by -1.

The fixed objects under this sign reversing involution are those $T \in \mathcal{T}$ for which no row of T contains a downstep and no row of T contains an uncovered cell below the end of a rim hook. Here is an example of one such fixed object:



These fixed objects have the following property that all uncovered cycles occur in the ledge of λ.

The sign reversing involution pairs each $-$ object with a $+$ object, leaving us with the fixed objects to count. To complete the proof, we will show that the number of such fixed objects is equal to the right hand side of (3). Count the number of fixed objects in this way:

1. Select a partition $\mu = (\mu_1, \ldots, \mu_\ell)$ to be the partition which contains the covered cycles in a fixed object T. The partition μ must satisfy $L(\lambda) \subseteq \mu \subseteq \lambda$.
2. Select which integers out of $\{1, \ldots, n\}$ are to be covered. There are $\binom{n}{|\mu|}$ ways to do this.
3. Select the subsets to cover the integers selected in step 2. The number of ways to do this is the number of ways to select a set partition of k into $|\mu|$ parts, namely $S(k, |\mu|)$.
4. Use the subsets selected in step 3 to cover the integers in step 2. There are $|\mu|!$ ways to do this.
5. Place the covered integers into the Young diagram of $\mu \subseteq \lambda$. There are f^μ choices here.
6. Select which of the remaining integers after step 2 will go into each of the rows in the uncovered portions of the ledge of λ. The number of ways to do this is the multinomial coefficient $\binom{n-|\mu|}{\lambda_1-\mu_1,\ldots,\lambda_\ell-\mu_\ell}$.
7. For each uncovered row in the ledge of λ, select a permutation with the choice of integers made in step 6. Write this permutation in cyclic notation, and place it into the row in the ledge as specified by the elements in $\mathcal{T}$. The number of ways to do this is $(\lambda_1 - \mu_1)! \cdots (\lambda_\ell - \mu_\ell)!$.

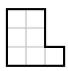
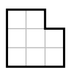
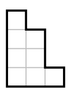
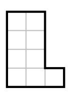
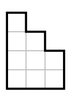
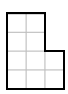
Therefore, the number of fixed points is

$$\sum_{L(\lambda)\subseteq\mu\subseteq\lambda} \binom{n}{|\mu|} S(k,|\mu|)|\mu|! f^{\mu} \binom{n-|\mu|}{\lambda_1-\mu_1,\ldots,\lambda_\ell-\mu_\ell} (\lambda_1-\mu_1)!\cdots(\lambda_\ell-\mu_\ell)!$$

Simplifying the above expression gives $n! \sum_{L(\lambda)\subseteq\mu\subseteq\lambda} f^{\mu} S(k,|\mu|)$, as desired.

□

As an example of the utility of Theorem 2, we find $\langle \delta^9, \chi^{(3,3,2,2)} \rangle$. For each integer partition μ which satisfies $L(\lambda) \subseteq \mu \subseteq \lambda$, we can calculate f^{μ} using the hook length formula and we can find $S(9,|\mu|)$ using the identity $S(k,i) = \frac{1}{i!}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}j^k$, say:

μ	f^{μ}	$S(9,\|\mu\|)$	Product
	21	462	9702
	42	36	1512
	70	36	2520
	84	1	84
	168	1	168
	252	0	0
		Total:	13986

Therefore $\langle \delta^9, \chi^{(3,3,2,2)} \rangle = 13986$, meaning that there are 13986 copies of the irreducible representation of S_{10} corresponding to the integer partition $(3,3,2,2)$ in the matrix representation of degree 10^9 found by taking the tensor product of the defining representation of S_{10} with itself 9 times.

Corollary 1. The inner product $\langle \delta^k, \chi^{\lambda} \rangle$ is 0 if and only if $k < |L(\lambda)|$.

Proof. If $\langle \delta^k, \chi^\lambda \rangle = 0$, then every term in the sum in Theorem 2 must be 0. In particular, $S(k, |\mu|) = 0$ when $\mu = L(\lambda)$. This can only happen if $L(\lambda)$ has more than k cells.

Conversely, any μ which satisfies $L(\lambda) \subseteq \mu$ must have at least as many cells as $L(\lambda)$. If the number of cells here is larger than k, then every $S(k, |\mu|)$ term in the sum found in Theorem 2 is equal to 0 since there are no set partitions of k with more than k parts. □

References

[1] A. Goupil, C. Chauve, Combinatorial operators for Kronecker powers of representations of S_n, Sém. Lothar. Combin. 54 (2005/07) Art. B54j, 13 pp. (electronic). MR2264927

[2] B. E. Sagan, The symmetric group, representations, combinatorial algorithms, and symmetric functions, 2nd Edition, Vol. 203 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2001. MR1824028

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