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On the longest common pattern contained in two or more random permutations

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We provide upper and lower bounds for the expected length $\mathbb{E}(L_{n,m})$ of the longest common pattern contained in m random permutations of length n. We also address the tightness of the concentration of $L_{n,m}$ around $\mathbb{E}(L_{n,m})$.

Keywords and phrases: Longest common pattern, pattern containment, random permutations.

1. Introduction

The goal of this paper is to investigate a statistic $L_{n,m}$ which can be thought of as a measure of similarity among a collection of permutations. To define this statistic, we must first define some more primitive concepts. Given two lists of real numbers $a(1), a(2), \dots, a(n)$ and $b(1), b(2), \dots, b(n)$, we say the lists are *order-isomorphic* if, for all $1 \le i < j \le n$, $a(i) < a(j)$ if and only if $b(i) < b(j)$. For example, the sequences $3, 1, 4, 2$ and $7, 3, 9, 6$ are order isomorphic. Given permutations $\sigma \in S_k$ and $\pi \in S_n$, we say that π *contains the pattern* σ if there exist indices $i_1 < i_2 < \cdots < i_k$ for which the subsequence $\pi(i_1), \pi(i_2), \dots, \pi(i_k)$ is order isomorphic to σ.

Given m permutations $\pi_1, \dots, \pi_m \in S_n$, a *common pattern* is a permutation σ which is a pattern contained in all of the π_i, and a *longest common pattern* is a common pattern of maximal length. Define $L_{n,m}$ to be the length of a longest common pattern (LCP) contained in m uniformly randomly chosen permutations of length n. Our main results show that $\mathbb{E}(L_{n,m}) = \Theta(n^{\frac{m}{2m-1}})$ as $n \to \infty$, and we give asymptotic bounds for $\mathbb{E}(L_{n,m})$.

The topics contained in this paper are strongly related to two classical and well-studied problems, namely those of the longest common subsequence (LCS) N_n between two random strings [17] and the longest monotone subsequence (LMS) of a random permutation [2]. Here is a summary of key results in these areas:

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First, consider the LCS problem. Given two independent, identically distributed binary strings $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$, subadditivity arguments yield that

$$\frac{\mathbb{E}(N_n)}{n} \to c$$

for some constant $c \in (0, 1)$. The value of c is not known to date (see [17]), and the best currently known bounds can be found in [14], namely $0.7880 \leq c \leq 0.8296$. The situation where the variables take values from an alphabet $\{0, 1, \ldots, d-1\}$ of size d is similarly in an incompletely understood state, though techniques such as Azuma's and Talagrand's inequalities [3] have been used to provide estimates of the width of concentration intervals of the LCS around its mean for all alphabet sizes. The work of Kiwi, Loebl, and Matoušek [12] is of particular relevance to this paper. They consider the case of large alphabet sizes and verify that the limiting constant c_d in the alphabet d LCS problem does indeed satisfy

$$\lim_{d \to \infty} c_d \sqrt{d} = 2,$$

as conjectured by Sankoff and Manville.

Good general references for the LMS problem are [17], [4], and [2], which describe the classical Erdős-Szekeres theorem (every permutation of $[n^2+1]$ contains a monotone sequence of length $n+1$); the work of Logan-Shepp [13] and Vershik-Kerov [18] (namely that the longest monotone subsequence of a random permutation on $[n]$ is asymptotic to $2\sqrt{n}$); the concentration results (Kim [11], Frieze [8]) that reveal that the standard deviation of the size of the LMS is of order $\Theta(n^{\frac{1}{6}})$; and the landmark work of Baik et al. [4] that exhibits the limiting law of a normalized version of the LMS.

Other forms of LMS problems have been considered in [16] and [1], and algorithmic results on the LCP problem that we study in the subsequent sections may be found in [6] and [7].

2. Upper bound

Theorem 1. $\mathbb{E}(L_{n,m}) \leq \lceil en^{\frac{m}{2m-1}} \rceil$.

Proof. First, we provide an upper bound on $\mathbb{P}(L_{n,m} \geq k)$, when $k > en^{\frac{m}{2m-1}}$. Let $S_1, \ldots, S_m$ be subsets of $[n]$, each of size k. These define m subsequences of $\pi_1, \ldots, \pi_n$, by considering the entries of π_i whose indices are in S_i. Since the π_i are chosen independently at random, the orderings of the subsequences will also be independent, and as each subsequence has $k!$ possible equally likely orderings, the probability that the subsequences will all be order isomorphic is $1/(k!)^{m-1}$. Furthermore, $L_{n,m} \geq k$ if and only if these subsequences are order isomorphic for at least one of the $\binom{n}{k}^m$ choices for the list $S_1, \ldots, S_m$, so

$$\mathbb{P}(L_{n,m} \geq k) \leq \binom{n}{k}^m \frac{1}{k!^{m-1}} \leq \left(\frac{n^k}{k!}\right)^m \frac{1}{k!^{m-1}} = \frac{n^{mk}}{k!^{2m-1}},$$

Using the bound $k! > \sqrt{2\pi k}\left(\frac{k}{e}\right)^k$, this implies

$$\mathbb{P}(L_{n,m} \geq k) \leq \frac{1}{(2\pi k)^{(m-\frac{1}{2})}} \left(\frac{e^{2m-1}n^m}{k^{2m-1}}\right)^k,$$

and thus when $k > en^{\frac{m}{2m-1}}$, we have $\mathbb{P}(L_{n,m} \geq k) \leq (2\pi k)^{-(m-\frac{1}{2})}$.

We can write $\mathbb{E}(L_{n,m})$ as

$$\begin{aligned}\mathbb{E}(L_{n,m}) = \sum_{k=1}^{n} \mathbb{P}(L_{n,m} \geq k) &= \sum_{k=1}^{\lfloor en^{\frac{m}{2m-1}} \rfloor} \mathbb{P}(L_{n,m} \geq k) + \sum_{k=\lceil en^{\frac{m}{2m-1}} \rceil}^{n} \mathbb{P}(L_{n,m} \geq k) \\ &\leq \lfloor en^{\frac{m}{2m-1}} \rfloor + \sum_{k=\lceil en^{\frac{m}{2m-1}} \rceil}^{n} \frac{1}{(2\pi k)^{(m-\frac{1}{2})}} \\ &\leq \lfloor en^{\frac{m}{2m-1}} \rfloor + \frac{1}{(2\pi)^{(m-\frac{1}{2})}} \sum_{k=1}^{\infty} \frac{1}{k^{(m-\frac{1}{2})}}\end{aligned}$$

The second term on the last line is known to have a sum less than 1 for $m \geq 2$, so it follows that $\mathbb{E}(L_{n,m}) \leq \lceil en^{\frac{m}{2m-1}} \rceil$. □

3. Lower bound

The purpose of this section is to prove the following asymptotic lower bound for $\mathbb{E}(L_{n,m})$.

Theorem 2. $\liminf_{n\to\infty} \mathbb{E}(L_{n,m})/n^{\frac{m}{2m-1}} \geq \frac{1}{2}$.

Proof. We first give a method to generate the m random permutations which will allow us to identify common patterns more easily. Let I be the interval $[0, 1]$. By choosing n points uniformly randomly in the unit square I^2, we can specify a permutation $\pi \in S_n$ uniformly at random as follows. Consider the point with the i^{th} smallest x coordinate. Assign $\pi(i) = j$ if that point has the j^{th} smallest y coordinate. For our proof, let $X_{i,j}$, for $1 \le i \le m$ and $1 \le j \le n$, be i.i.d. points distributed uniformly on I^2, and let π_i for $1 \le i \le m$, be the permutation specified in the above fashion by the points $X_{i,1}, \ldots, X_{i,n}$.

Furthermore, let $r = \lfloor n^{\frac{1}{2m-1}} \rfloor$. We can partition I^2 into a $r^m \times r^m$ array of r^{2m} equally sized square boxes. Call a box *full* if, for each $1 \le i \le m$, it contains at least one point from the set $\{X_{i,j} : 1 \le j \le n\}$. In other words, it contains a point used to define each of the m permutations. Furthermore, define a *scattering* to be a set of full boxes, no two on the same row or column. Scatterings are related to common patterns among $\pi_1, \ldots, \pi_m$ as follows: if there is a scattering of size k, there will be a common pattern among $\pi_1, \ldots, \pi_m$ of length k. This can be seen by examining the m subsequences defined by the points in these full boxes. In this proof, we find a probabilistic lower bound for the number of full boxes, and use this to find a lower bound for the expected size of the largest scattering.

We need one last tool. Let ρ be an ordering of the r^{2m} boxes chosen uniformly at random. Ordering the boxes randomly (as opposed to some arbitrary, deterministic ordering) will simplify parts of the proof later, which will consider the boxes in the order defined by ρ. Finally, let F_i be the event that $\rho(i)$ is full, and let F be the total number of full boxes.

Lemma 3. *For all $\varepsilon > 0$, $\mathbb{P}(F < (1-\varepsilon)r^m) \to 0$ as $n \to \infty$.*

Proof. Since $n \ge r^{2m-1}$, we have

$$\begin{aligned}
\mathbb{P}(F_i) &= \left(1 - \left(\frac{r^{2m}-1}{r^{2m}}\right)^n\right)^m \\
&\ge \left(1 - \left(1 - \frac{1}{r^{2m}}\right)^{r^{2m-1}}\right)^m \\
&\ge \left(1 - e^{-1/r}\right)^m \\
&\ge \left(\frac{1}{r} - \frac{1}{2r^2}\right)^m \\
&\ge \frac{1}{r^m}\left(1 - \frac{m}{2r}\right).
\end{aligned}$$

Then, by linearity of expectation, we have $\mathbb{E}(F) \ge \left(1 - \frac{m}{2r}\right) r^m$. Also, the inequalities $e^{-x} \ge 1 - x \ge e^{-x/(1-x)}$ show that

$$\mathbb{P}(F_i) = \left(1 - \left(\frac{r^{2m}-1}{r^{2m}}\right)^n\right)^m$$

$$\leq \left(1 - e^{-n/(r^{2m}-1)}\right)^m$$
$$\leq \left(\frac{n}{r^{2m}-1}\right)^m$$
$$\leq \frac{1}{r^m}(1 + o(1)).$$

We now give a bound for $\mathrm{Var}(F)$. Notice that the indicator variables 1_{F_i} for F_i are pairwise negatively correlated; given that F_i has occurred, it is less likely that F_j will occur (since there will be strictly fewer points that can land in the j^{th} box). Thus,

$$\mathrm{Var}(F) < \sum_{i=1}^{r^{2m}} \mathrm{Var}(1_{F_i}) < \sum_{i=1}^{r^{2m}} \mathbb{P}(F_i) < r^m(1 + o(1)).$$

Then, for any $\varepsilon > 0$, we have that

$$\begin{aligned}\mathbb{P}(F < (1-\varepsilon)r^m) &= \mathbb{P}\left(F < \left(1 - \frac{m}{2r}\right)r^m - \left(\varepsilon - \frac{m}{2r}\right)r^m\right)\\ &\leq \mathbb{P}\left(F < \mathbb{E}(F) - \left(\varepsilon - \frac{m}{2r}\right)r^m\right)\\ &\leq \mathbb{P}\left(|F - \mathbb{E}(F)| > \left(\varepsilon - \frac{m}{2r}\right)r^m\right).\end{aligned}$$

If we choose n sufficiently large so $\frac{m}{2r} < \varepsilon$, then by Chebychev's inequality, we have

$$\mathbb{P}(F < (1-\varepsilon)r^m) \leq \frac{\mathrm{Var}(F)}{\left(\varepsilon - \frac{m}{2r}\right)^2 r^{2m}} \leq \frac{1 + o(1)}{\left(\varepsilon - \frac{m}{2r}\right)^2 r^m} \to 0$$

as $n \to \infty$. □

Given that there are F full boxes, index them with the numbers 1 through F in the same order as ρ, and let B_k refer to the k^{th} full box. The fact that ρ was a random ordering ensures that, given $\{B_1, \ldots, B_{k-1}\}$, B_k is distributed uniformly among the $r^{2m} - k + 1$ locations not occupied by $\{B_1, \ldots, B_{k-1}\}$. Define the sequence of random variables $\{S_k\}_{k=0}^F$, where $S_0 = 0$ and S_k is the size of the largest scattering which is a subset of $\{B_1, \ldots, B_k\}$. Then $S_1 = 1$, and S_{k+1} is equal to either S_k or $S_k + 1$.

Let $\varepsilon > 0$ be given. Throughout the rest of this proof, we will use the expression S_x to mean $S_{\lfloor x \rfloor}$. The next lemma formalizes the previous observation that given a size k scattering, there will be a common pattern of length k.

Lemma 4. *For large enough n, $\mathbb{E}(L_{n,m}) \geq \mathbb{E}(S_{(1-\varepsilon)r^m})(1-o(1))$.*

Proof. Let $r^m = R$. By conditioning $L_{n,m}$ on the event $F > (1-\varepsilon)R$, we have

$$\mathbb{E}(L_{n,m}) \geq \mathbb{E}(L_{n,m}|F > (1-\varepsilon)R) \cdot \mathbb{P}(F > (1-\varepsilon)R).$$

Given $F > (1-\varepsilon)R$, the variable $S_{(1-\varepsilon)R}$ is well defined. Suppose that $S_{(1-\varepsilon)R} = k$, so that there exists a scattering of size k. The centers of these k boxes define a permutation $\sigma \in S_k$, as described in the beginning of this section. For any $i \in 1, \ldots, m$, since the boxes in the scattering are full, there will be a subsequence $\pi_i(j_1), \ldots, \pi_i(j_k)$, where the points corresponding to each entry will be in different boxes in the scattering. This implies the subsequence is order isomorphic to σ. Thus, σ is a common pattern among $\pi_1, \ldots, \pi_m$ of length k, implying $L_{n,m} \geq S_{(1-\varepsilon)R}$. Combining this with the proof of Lemma 3, which guarantees $P(F > (1-\varepsilon)R) \geq 1 - \frac{C}{r^m}$ for some constant C and large n, we get that

$$\mathbb{E}(L_{n,m}) \geq \mathbb{E}(L_{n,m}|F > (1-\varepsilon)R) \cdot \mathbb{P}(F > (1-\varepsilon)R) \geq \mathbb{E}(S_{(1-\varepsilon)R}) \left(1 - \frac{C}{r^m}\right),$$

as asserted. □

For the rest of the proof, we will assume $F > (1-\varepsilon)R$, so that $S_{(1-\varepsilon)R}$ is well defined. The next lemma provides a lower bound for $\mathbb{E}(S_k)$ in terms of another sequence.

Lemma 5. *For $R = r^m$, define the sequence $\{y_k\}_{k=0}^{R}$, where $y_0 = 0$, and*

$$y_{k+1} = y_k + \frac{1}{R}(1-y_k)^2 \tag{1}$$

Then, for all $0 \leq k \leq (1-\varepsilon)R$,

$$\frac{\mathbb{E}(S_k)}{R} \geq y_k.$$

Proof. Suppose that $S_k = s$ and let T be a scattering of size s (there may be other scatterings the same size as T). Notice that B_{k+1} can be appended to T to make a larger scattering if it is in one of the $(R-s)^2$ locations not sharing a row or column with any box in T, in which case there will exist a scattering of size $s+1$. This occurs with probability $\frac{(R-s)^2}{R^2-k}$, so that

$$\mathbb{E}(S_{k+1}) \quad = \quad \mathbb{E}(S_k) + \mathbb{E}(S_{k+1} - S_k)$$

$$\begin{aligned} &= \mathbb{E}(S_k) + \mathbb{P}(S_{k+1} - S_k = 1) \\ &\geq \mathbb{E}(S_k) + \sum_s \mathbb{P}(S_k = s)\frac{(R-s)^2}{R^2 - k} \\ &\geq \mathbb{E}(S_k) + \sum_s \mathbb{P}(S_k = s)\left(1 - \frac{s}{R}\right)^2 \\ &= \mathbb{E}(S_k) + \mathbb{E}\left(\left(1 - \frac{S_k}{R}\right)^2\right) \\ &\geq \mathbb{E}(S_k) + \left(\mathbb{E}\left(1 - \frac{S_k}{R}\right)\right)^2, \end{aligned}$$

and thus

$$\frac{\mathbb{E}(S_{k+1})}{R} \geq \frac{\mathbb{E}(S_k)}{R} + \frac{1}{R}\left(1 - \frac{\mathbb{E}(S_k)}{R}\right)^2. \tag{2}$$

We now use induction to complete the proof. Evidently $S_0 = y_0 = 0$. Assume that $\mathbb{E}(S_k)/R \geq y_k$. Define $f(x) = x + \frac{1}{R}(1-x)^2$, so that $f(\mathbb{E}(S_k)/R)$ is the right hand side of (2). Provided $R \geq 2$, f is an increasing function, since $f'(x) = 1 - \frac{2}{R}(1-x) > 0$. It then follows from (2) and the induction hypothesis that

$$\frac{\mathbb{E}(S_{k+1})}{R} \geq f(\mathbb{E}(S_k)/R) \geq f(y_k) = y_{k+1} \qquad \square$$

Lemma 6. $\lim_{R\to\infty} y_{\lfloor(1-\varepsilon)R\rfloor} = \frac{1-\varepsilon}{2-\varepsilon}$.

Proof. The sequence y_k is the result of applying Euler's method to approximate the solution to the differential equation $y'(x) = (1-y)^2$, with initial condition $y(0) = 0$, using step size $1/R$. This has a unique solution on the interval $(0,1)$, given by $y(x) = \frac{x}{x+1}$.

To prove this Lemma, we cite Theorems 1.1 and 1.2 of [10], which prove that the error terms for Euler's method converge uniformly to zero. The only difficulty is that this proof assumes that the DE is of the form $y' = F(x,y)$, with $\frac{\partial F}{\partial y}$ being bounded for all $y \in \mathbb{R}$. In our case, $\frac{\partial}{\partial y}(1-y)^2$ is not bounded. However, a careful examination of the proof shows that, if $y_k, y(x) \in [a,b]$ for all k and $x \in [0,1]$, it is only required that $|\frac{\partial F}{\partial y}| < M$ for $y \in [a,b]$. Clearly $y(x) = \frac{x}{x+1} \in [0,1]$ for $x \in [0,1]$, and it can be shown by induction that $y_k \in [0,1]$ for $0 \leq k \leq \lfloor R(1-\varepsilon)\rfloor$. Thus, since $\frac{\partial}{\partial y}(1-y)^2$ is bounded on $[0,1]$, the proof still applies.

In this case, the k^{th} error term is $|y_k - y(k/R)|$, so that

$$\lim_{R\to\infty} y_{\lfloor(1-\varepsilon)R\rfloor} - y\left(\frac{\lfloor(1-\varepsilon)R\rfloor}{R}\right) = 0.$$

Since $y(1-\varepsilon-\frac{1}{R}) \leq y\left(\frac{\lfloor(1-\varepsilon)R\rfloor}{R}\right) \leq y(1-\varepsilon)$, this proves that $\lim_{R\to\infty} y_{\lfloor(1-\varepsilon)R\rfloor} = y(1-\varepsilon) = \frac{1-\varepsilon}{2-\varepsilon}$. □

Finally, combining Lemmas 4, 5, and 6, we get

$$\liminf_{n\to\infty} \frac{E(L_{n,m})}{R} \geq \liminf_{R\to\infty} \frac{E(S_{(1-\varepsilon)R})}{R} \geq \lim_{R\to\infty} y_{\lfloor(1-\varepsilon)R\rfloor} = \frac{1-\varepsilon}{2-\varepsilon}.$$

Since this holds for all $\varepsilon > 0$, this implies $\liminf_{n\to\infty} \frac{E(L_{n,m})}{R} \geq \frac{1}{2}$. Since $\lim_{n\to\infty} n^{\frac{m}{2m-1}}/R = 1$ (recall that $R = \lfloor n^{\frac{1}{2m-1}} \rfloor^m$), we finally have that $\liminf_{n\to\infty} E(L_{n,m})/n^{\frac{m}{2m-1}} \geq \frac{1}{2}$. □

This lower bound on $\liminf_{n\to\infty} E(L_{n,m})/n^{\frac{m}{2m-1}}$ can actually be improved by adjusting the preceding proof slightly. At the beginning of the proof, we divided I^2 into a $r^m \times r^m$ grid of smaller squares; if we instead use a $c_m r^m \times c_m r^m$ grid, for some constant c_m, then the same proof yields the lower bound

$$\liminf_{n\to\infty} \frac{E(L_{n,m})}{n^{m/2m-1}} \geq \frac{c_m}{1+(c_m)^{2m-1}}.$$

This is maximized when $c_m = \left(\frac{1}{2m-2}\right)^{\frac{1}{2m-1}}$, in which case we obtain that

$$\liminf_{n\to\infty} \frac{E(L_{n,m})}{n^{m/2m-1}} \geq \frac{2m-2}{2m-1}\left(\frac{1}{2m-2}\right)^{\frac{1}{2m-1}}.$$

This is approximately 0.529 when $m = 2$, a slight improvement over $\frac{1}{2}$, and converges to 1 as $m \to \infty$. We have also been able to obtain improvements on the lower bounds using (i) Poisson approximation [5]; (ii) coding the problem using large alphabet results [12]; (iii) exploiting the possibility of multiple matchings within cells; and (iv) exploiting the theory of perfect matchings in random bipartite graphs [9].

4. Open problems

We have shown that $EL_{n,m}$ grows as $n^{m/(2m-1)}$, so the next natural question is whether or not

$$\lim_{n\to\infty} \mathbb{E}(L_{n,m})/n^{\frac{m}{2m-1}}$$

exists. We conjecture it does, and further conjecture (inspired by work in [12]) that

$$\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}(L_{n,m})/n^{\frac{m}{2m-1}} = 2.$$

Talagrand's in equality can be used as in [3] to show that $L_{n,m}$ is concentrated in an interval of length $O(n^{m/4m-2}) = O(\mathbb{E}(L_{n,m})^{1/2})$. It would be interesting to study whether or not this is the true order of the concentraion interval. Ultimately, one would want to determine, à la Baik, Deift and Johansson [4], the limiting distribution of $L_{n,m}$ after an appropriate normalization.

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References

[1] M. Albert (2007). On the length of the longest subsequence avoiding an arbitrary pattern in a random permutation. *Rand. Structures Alg.* **31**, 227–238. MR2343720

[2] D. Aldous and P. Diaconis (1999). Longest increasing subsequences: from patience sorting to the Baik-Deift-Johansson theorem. *Bull. Amer. Math. Soc.* **36**, 413–432. MR1694204

[3] N. Alon and J. Spencer (2000). *The Probabilistic Method, 2nd Edition.* Wiley, New York. MR1885388

[4] J. Baik, P. Deift, and K. Johansson (1999). On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.* **12**, 1119–1178. MR1682248

[5] A. Barbour, L. Holst, and S. Janson (1992). *Poisson Approximation.* Oxford University Press. MR1163825

[6] M. Bouvel and D. Rossin (2006). The longest common pattern problem for two permutations. *Pure Mathematics and Application* **17**, 55–69. MR2380348

[7] M. Bouvel, D. Rossin, and S. Vialette (2007). Longest common separable pattern among permutations, in *Combinatorial Pattern Matching, Lecture Notes in Computer Science* **4580**, 316–327.

[8] A. Frieze (1991). On the length of the longest monotone subsequence in a random permutation. *Ann. Appl. Probab.* **1**, 301–305. MR1102322

[9] A. Frieze (2005). Perfect matchings in random bipartite graphs with minimal degree at least 2. *Rand. Structures Alg.* **26**, 319–358. MR2127373

[10] D. Greenspan (2006). *Numerical Solutions of Ordinary Differential Equations*. Wiley, Weinheim.

[11] J.-H. Kim (1996). On increasing subsequences of random permutations. *J. Combinatorial Theory, Series A* **76**, 148–155. MR1405997

[12] M. Kiwi, M. Loebl, and J. Matoušek (2004). Expected length of the longest common subsequence for large alphabets. *Lecture Notes in Computer Science* **2976**, 302–311. MR2095205

[13] B. F. Logan and L. A. Shepp (1977). A variational problem for random Young tableaux. *Advances in Mathematics* **26**, 206–222. MR1417317

[14] G. Lueker (2009). Improved bounds on the average length of longest common subsequences. *J. Assoc. Computing Machinery* **56**, Article 17, 38 pages. MR2536132

[15] M. Molloy and B. Reed (2002). *Graph Colouring and the Probabilistic Method*. Springer Verlag, Berlin. MR1869439

[16] R. Stanley (2008). Longest alternating subsequences of permutations. *Michigan Math. J.* **57**, 675–687. MR2492475

[17] J. M. Steele (1987). *Probability Theory and Combinatorial Optimization*. SIAM, Philadelphia. MR1422018

[18] A. M. Vershik and S. V. Kerov (1977). Asymptotics of Plancherel measure on the symmetric group and limiting form of Young tables. *Soviet Math. Dokl.* **233**, 527–531. Translation of *Dokl. Acad. Nauk. SSSR* **233**, 1024–1027, 1977. MR0480398

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