Journal of Combinatorics
Volume 7, Number 2–3, 509–530, 2016

Distance matching in punctured planar triangulations

R. E. L. Aldred and Michael D. Plummer

Distance matching extension with prescribed and proscribed edges in planar triangulations has been previously studied. In the present work, matching extension behavior is investigated when the graph families are slightly more general than triangulations. More particularly, we replace the triangulation hypothesis with the weaker hypotheses that (a) the graph is locally connected and (b) the graph has at most two non-triangular faces. We investigate which distance matching properties enjoyed by triangulations are retained and which are lost.

1. Introduction

A graph G with at least $2m + 2n + 2$ vertices which contains a perfect matching is said to satisfy property $E(m, n)$ (or simply "G is $E(m, n)$") if, given any two matchings M and N with $|M| = m$ and $|N| = n$ such that $M \cap N = \emptyset$, there is a perfect matching F in G such that $M \subseteq F$ and $N \cap F = \emptyset$. This property is a generalization of the widely studied concept of matching extension in that a graph is m-extendable if and only if it is $E(m, 0)$.

Matching extension has its roots in the problem of counting perfect matchings, a problem of some interest in chemistry. (Cf. [6].) For surveys of the topic of matching extension, see [9, 10, 11]. See also the book [13].

The $E(m, n)$ property was first introduced in [12]. In the same paper, certain implications and non-implications were shown to exist among the $E(m, n)$ properties for different values of m and n. These will be of interest to us in the present paper. A portion of the implication lattice for $E(m, n)$ is shown in the following figure.

Let us summarize what is known heretofore about $E(m, n)$ for planar triangulations. In general, no planar graph, triangulation or not, is $E(3, 0)$ ([7]), or even $E(2, 1)$ ([1]). If a planar even triangulation is only 3-connected, it may not even contain a perfect matching, so we proceed immediately to the case when the graph is (at least) 4-connected.

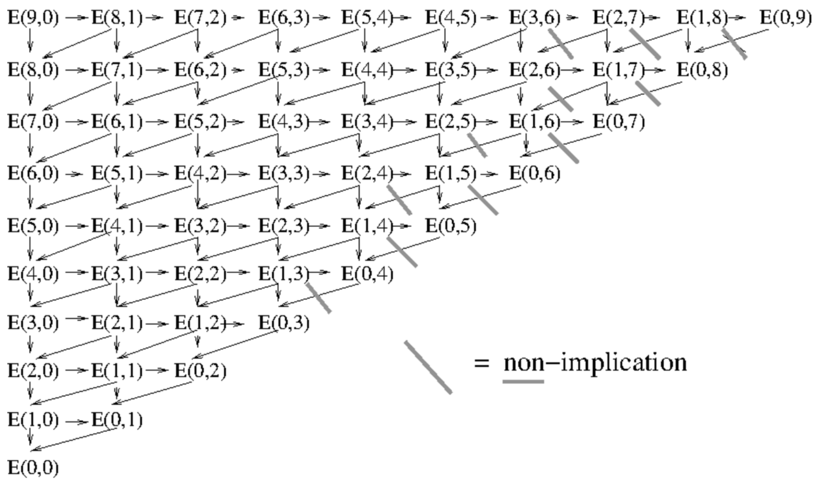

Figure 1: The lattice on implications for $E(m,n)$.

If G is a 4-connected planar even triangulation, it is $E(1,1)$ ([1]), but not necessarily $E[1,2]$ ([3]). It is also $E(0,3)$ ([1]), and hence by [12], also $E(0,2)$ and $E(0,1)$. But it is not necessarily $E(0,4)$ ([3]).

If G is a 5-connected planar even triangulation, it is $E(2,0)$. (In fact $E(2,0)$ holds for all 5-connected even planar graphs, and not just triangulations. (See [5, 8].)) Graphs which are 5-connected planar even triangulations, also have the property $E(1,3)$, but not necessarily $E(1,4)$ ([2]). Finally, by [2], they also satisfy properties $E(0,n)$ for $0 \leq n \leq 7$.

In [2] the authors first showed that the distance between edges could affect whether or not they could be extended to a perfect matching. In particular, although three independent edges in a planar 5-connected even triangulation do not necessarily extend to a perfect matching, if they lie at mutual distance at least 2, then they do in fact so extend.

We define the property $E_d(m,n)$ as follows. Let d be a positive integer and m and n, non-negative integers. A graph G is said to have the property $E_d(m,n)$ (or simply "G is $E_d(m,n)$") if given any two disjoint matchings M with $|M| = m$ and N with $|N| = n$ in G, where the distance between every two edges in M and every two edges in N is at least d, there is a perfect matching F in G such that $M \subseteq F$ and $N \cap F = \emptyset$.

Work in the present paper is motivated by the question: What is the effect of relaxing the condition that the planar graphs are triangulations?

In particular, we study relaxations in two directions by demanding that (a) the planar graph be only locally connected and (b) the graph have only a small number of non-triangular faces.

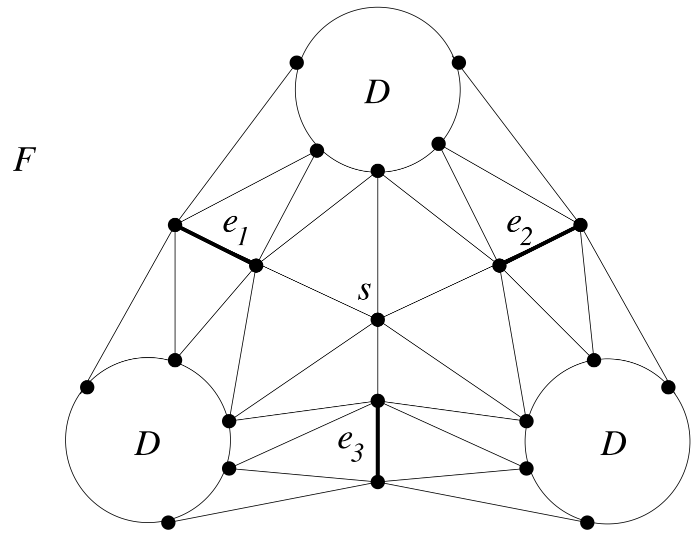

Figure 2: A graph which is not $E_2(3,0)$.

For general graph theoretic terminology, the reader is referred to [4]. In addition, however, we shall need the following. Suppose a graph G contains two disjoint matchings E and F, such that $G' = G - V(E) - F$ does not contain a perfect matching. Then by Tutte's classical result, G' must contain a set of vertices S (usually called a *Tutte set* or *barrier*) such that the number of odd components of $G' - S$ exceeds $|S|$ in number. We shall denote the number of odd components of $G' - S$ by $o(G' - S)$. We shall make use of the idea of the *bipartite distillation* G^* obtained from G via G' based upon E, S and F which we define as follows. (1) Contract each odd component of $G' - S$ to a separate singleton and delete any multiple edges and loops thus formed, (2) delete all even components of $G' - S$, and (3) delete all edges in $E \cup G[S] \cup F$. Then let G^* be the bipartite graph thus obtained having as $S \cup V(E)$ as the vertices of one partite set and the contracted components of $G' - S$ as the vertices in the other partite set. Clearly, G^* will be planar if the original graph G is planar.

2. Main results

We begin our current investigations by treating $E_2(m, n)$.

As noted above, all 5-connected planar even triangulations are $E_2(3, 0)$. (Cf. [2].) In fact these graphs are also $E_2(2, 1)$. (Cf. [3].) However, the graph in Figure 2 is 5-connected, locally connected, planar, even and has exactly one non-triangular face, but is *not* $E_2(3,0)$. (The infinite face F is the only non-triangular face.)

Here D is any suitable graph. For example, D could be any D_{2d+2}, $d \geq 2$, where D_{2d+2} is the graph shown in Figure 3.

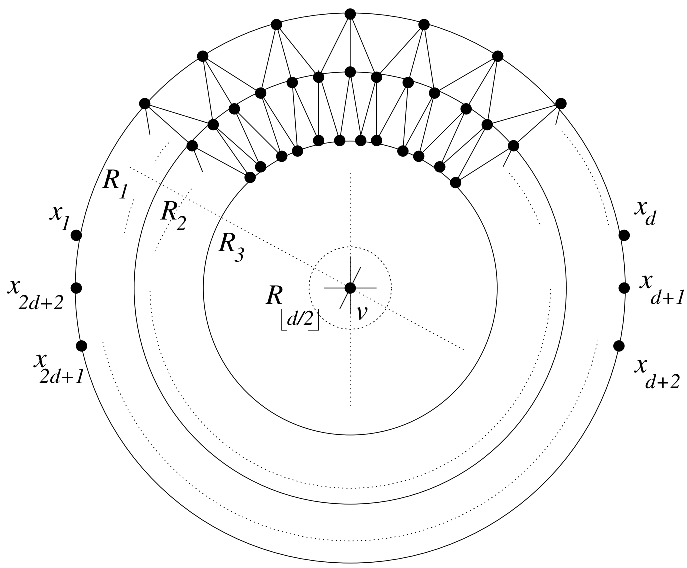

Figure 3: The graph D_{2d+2}.

Note that D_{2d+2} has $(2d+2)(2\lfloor d/2 \rfloor - 1) + 1$ vertices, an odd number.

The graph in Figure 4 is also 5-connected, locally connected, planar, even and has exactly two non-triangular faces F_1 and F_2, but is *not* $E_2(2,1)$. In fact, it is not $E_d(2,1)$, for any $d \geq 2$.

On the other hand, we have the next result.

Theorem 2.1: Let G be a 5-connected locally connected even planar graph. Then G satisfies (a) $E_2(1,3)$ and (b) $E_2(0,n)$, for $0 \leq n \leq 6$.

Proof: In part (a), by way of contradiction, let us suppose that G contains an edge $e = uv$ and an induced matching F consisting of three edges $\{f_1, f_2, f_3\}$, where each f_i is disjoint from e, such that there is no perfect matching in G containing e, but none of the three f_is. Then $G' = G - V(e) - F$ contains a barrier S such that $o(G - S) \geq |S| + 2$.

Forming the bipartite distillation G^* of G via G', e, F and S, we have

$$|E(G^*)| \leq 2(2|S| + 4) - 4 = 4|S| + 4. \quad (1)$$

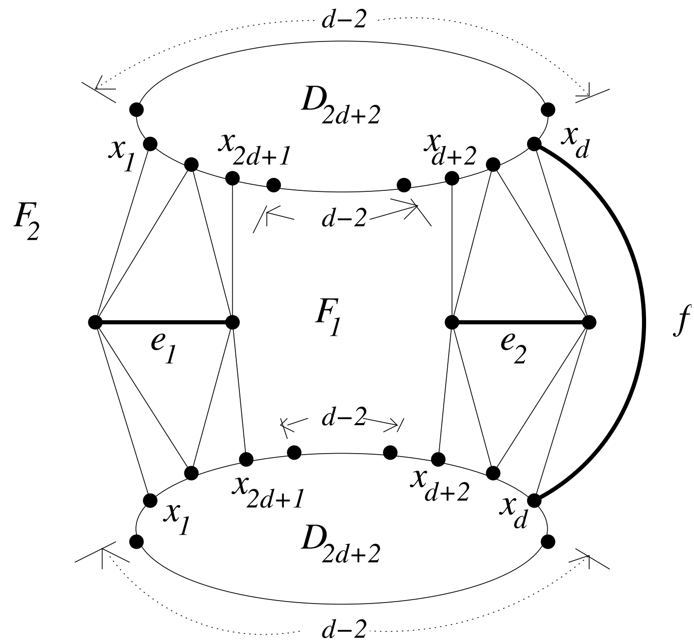

Figure 4: A graph which is not $E_d(2,1)$, $d \geq 2$.

By the 5-connectivity of G, we also have

$$|E(G^*)| \geq 5(|S|+2) - 6 = 5|S| + 4. \tag{2}$$

Combining (1) and (2), we have $|S| = 0$.

By Corollary 3.2 of [1] we know that G is $E(1,2)$, so $o(G'-S) = |S|+2 = 2$ and each edge f_i in F has one endvertex in one odd component of $G'-S$ and the other endvertex in the other odd component of $G'-S$. Since G is locally connected, for each edge $f_i = x_i y_i \in F$, either $x_i y_i u x_i$ or $x_i y_i v x_i$ is a triangular face of G. But $uv = e \in E(G)$ and so G is non-planar, a contradiction.

Now let us consider part (b). Let G be as in the statement of the theorem and let n be the smallest integer such that we can find a set $F = \{f_1, f_2, \ldots, f_n\} \subseteq E(G)$ such that the distance between any f_i and f_j, $i \neq j$, is at least two, but such that every perfect matching in G contains an edge in F.

Thus $G' = G - F$ has no perfect matching and by Tutte's theorem, there is a set $S \subseteq V(G) = V(G')$ such that $o(G'-S) \geq |S|+2$. By the minimality of n, we see that $o(G'-S) = |S|+2$ and that each edge $f_i \in F$

has its endvertices in two distinct odd components of $G' - S$. Moreover, if we choose S to be a smallest possible barrier set for G', each vertex $s \in S$ must have neighbors in at least three distinct odd components of $G' - S$. Forming the bipartite distillation G^* of G based on F and S, we use the 5-connectivity of G to determine that $|E(G^*)| \geq 5(|S| + 2) - 2n$.

Since G^* is planar and bipartite on $2|S| + 2$ vertices, we have $|E(G^*)| \leq 2(2|S| + 2) - 4 = 4|S|$. Consequently, $0 \leq |S| \leq 2n - 10$ and it follows that $n \geq 5$.

If $n = 5$, we have $|S| = 0$ and $G' - S$ has precisely two odd components, call them C_1 and C_2. Since each $f_i \in F$ has one endvertex in C_1 and one in C_2, by local connectivity, f_i must form one edge of a triangular face. But then by the distance criterion, the third vertex of this triangular face must lie in S, a contradiction.

Suppose then that $n = 6$. Then $0 \leq |S| \leq 2$. Note that by arguing as in the preceding paragraph, $S \neq \emptyset$. If $|S| = 1$, then $o(G' - S) = 3$ and the single vertex in S has neighbors in all three odd components of $G' - S$. We denote these three odd components by C_1, C_2 and C_3. Moreover, since G is 5-connected, each C_i must have at least, and hence exactly, four f_is incident with it. We thus may assume the configuration and labelling shown in Figure 5. But then neither endvertex of edge f_1 has a connected neighborhood and hence G is not locally connected, a contradiction.

Thus we may assume that $|S| = 2$ and $G' - S$ has precisely four odd components C_1, C_2, C_3 and C_4. Moreover, since G is 5-connected, each C_i has at least three f_is incident with it. Hence each has *exactly* three such f_is. Note now that no two different C_is can be joined by more than one f_i, or else we contradict the 5-connectivity of G, a contradiction. So each C_i is joined to each of the other three C_js by a single f_i and the four C_is and the six f_js together form a K_4 configuration. But now by planarity it is impossible for each of the two vertices in S to be adjacent to each of the four C_is. Hence G is not 5-connected, a contradiction. □

Remark: Note that a 5-connected, locally connected, planar even graph is $E_2(1, 2)$ because it is $E(1, 2)$ (cf. Corollary 3.2, [1]) and $E(1, 2)$ implies $E_2(1, 2)$. It therefore follows that it is also $E_2(1, 1)$ since $E(1, 2)$ implies $E(1, 1)$ in general (cf. Theorem 2.4 of [12]) and $E(1, 1)$ implies $E_2(1, 1)$.

We note that Theorem 2.1 is best possible in several respects. First, we cannot drop local connectivity as evidenced by the graphs in Figures 6 and 5.

The graph in Figure 6 is 5-connected, planar and even, but not $E_3(1, 3)$, and in fact is not $E_2(1, 3)$. Here F_1 and F_2 are the only non-triangular faces. Note, however, it is *not* locally connected. See either endvertex of edge f_3.

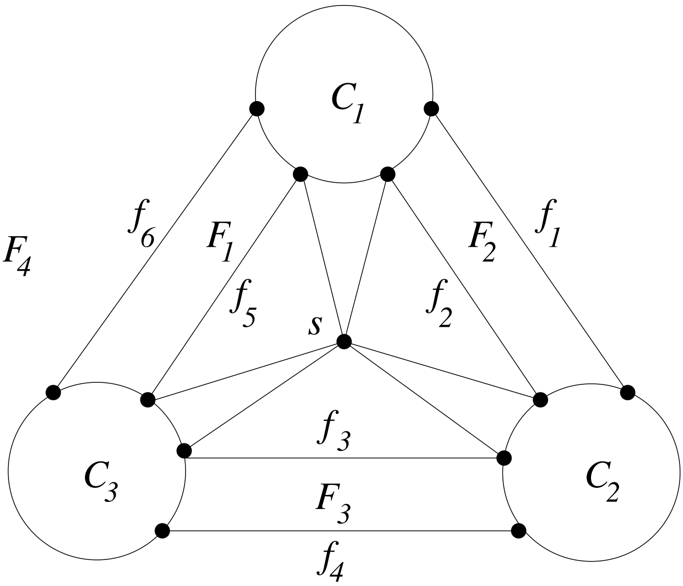

Figure 5.

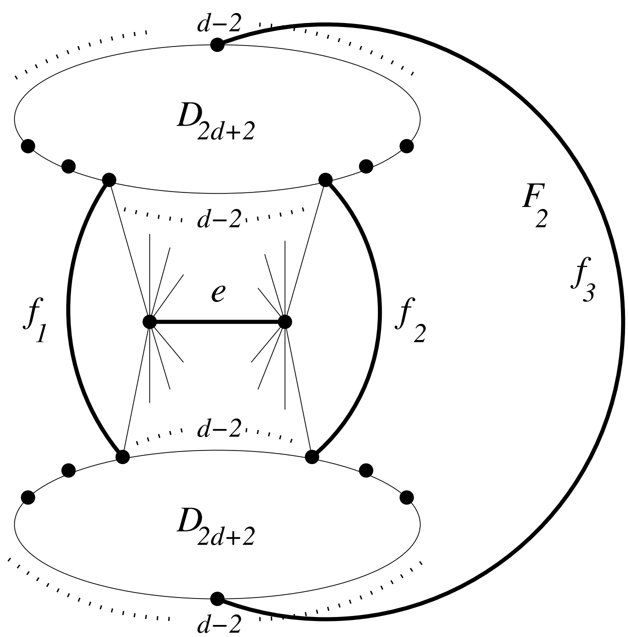

Figure 6: A graph which is not $E_3(1, 3)$.

Similarly, the graph in Figure 2.4 is 5-connected, planar and even, but not $E_2(0,6)$. However, as noted in the preceding proof, it fails to be locally connected. Moreover, we cannot hope to strengthen the conclusions of the theorem, for the graph in Figure 7 below, where D is an appropriate member of the family D_{2d+2}, is 5-connected, planar and even, but not $E_2(1,4)$. (In fact, this counterexample graph is a triangulation.)

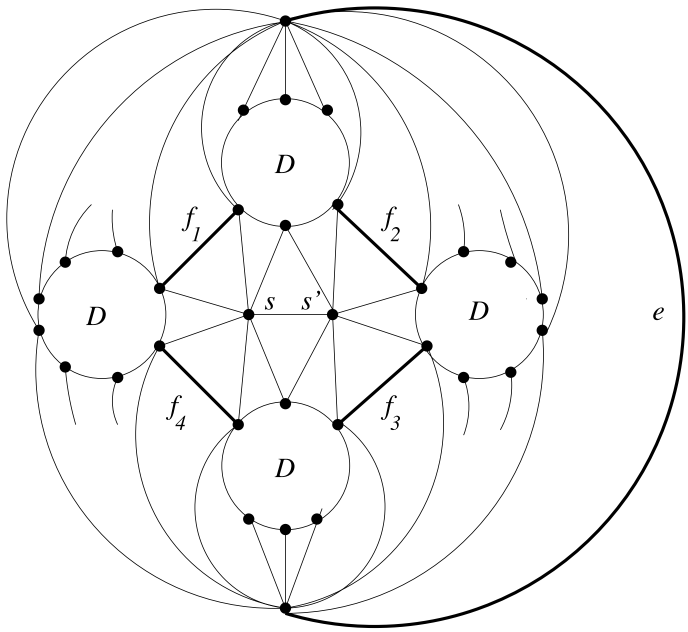

Figure 7: A graph which is not $E_2(1,4)$.

The graph in Figure 8 is 5-connected, locally connected, planar and even, but not $E_2(0,7)$. (Here D can be suitably chosen from the family D_{2d+2}.) However, it has 3 non-triangular faces. It has occurred to the authors that perhaps if one further limited the number of non-triangular faces, a stronger conclusion might be obtained. For example, if G is 5-connected, locally connected, planar and even, but has at most *two* non-triangular faces, perhaps G is $E_2(0,7)$.

The graph in Figure 9, where D can be suitably chosen from the family D_{2d+2}, is 5-connected, locally connected, planar, even and has exactly two non-triangular faces, but is *not* $E_2(0,8)$.

The examples above suggest that restricting the number of non-triangular faces in a 5-connected, planar, even, locally connected graph may

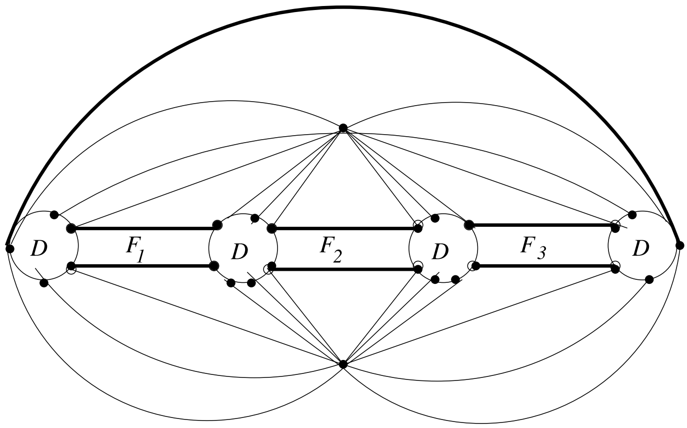

Figure 8: A graph which is not $E_2(0, 7)$.

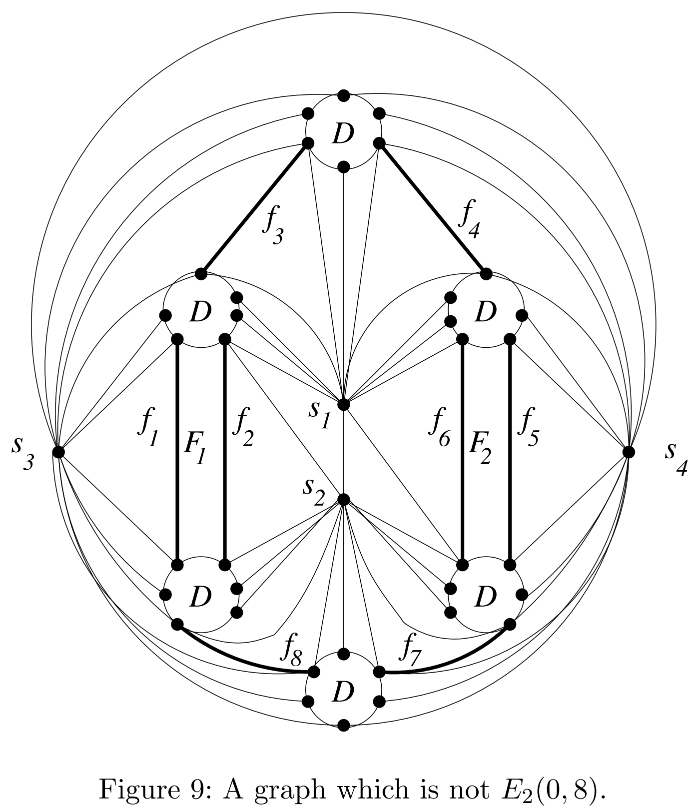

Figure 9: A graph which is not $E_2(0, 8)$.

well strengthen the distance extendability properties. In the following theorem we confirm just that.

Theorem 2.2: Let G be a 5-connected, locally connected, even, planar graph in which at most two faces are not triangular. Then G is $E_3(3,0)$.

Proof: Suppose to the contrary that G is such a graph and suppose further that $E = \{e_i = u_iv_i\}_{i=1}^3$ is a set of three edges which are pairwise distance at least three apart, but which do not lie in a perfect matching in G. Thus $G' = G - V(E)$ has no perfect matching and hence by Tutte's theorem there must exist a set $S \subseteq V(G')$ such that $o(G' - S) \geq |S| + 2$.

Since every 5-connected, even, planar graph is 2-extendable, as we observed in the introduction, it follows that $o(G' - S) = |S| + 2$. Moreover, in the graph G, each edge $e_i = u_iv_i \in E$ has neighbors in at least two odd components of $G' - S$. In addition, if we choose S to be as small a Tutte barrier set as possible, each vertex $s \in S$ has neighbors in at least three odd components of $G' - S$. Let the odd components of $G' - S$ be denoted by $C_1, \ldots, C_{|S|+2}$. Since G is 5-connected, each C_i has at least five neighbors in $V(E) \cup S$. Thus if we let G^* denote the bipartite distillation of G based upon E and S, there are at least $5(|S| + 2) = 5|S| + 10$ edges in G^*. On the other hand, G^* is planar, bipartite and has $2|S| + 8$ vertices, so by Euler's formula, G^* can have at most $2(2|S|+8)-4 = 4|S|+12$ edges. Consequently, $0 \leq |S| \leq 2$.

Suppose first that $|S| = 0$. Then we have precisely two odd components C_1 and C_2 in $G' - S$. Since G is 5-connected, C_1 and C_2 each have neighbors on all three of the edges e_1, e_2 and e_3. Clearly, since by our distance 3 hypothesis there can be no edges between endvertices of distinct edges in E, this results in at least three non-triangular faces in G, a contradiction. (See by way of example regions R_1, R_2 and R_3 in Figure 10.)

So we may assume that $1 \leq |S| \leq 2$.

Let s be a vertex in S. As observed earlier, s has neighbors in at least three odd components of $G' - S$. Since G is locally connected, $G[N_G(s)]$ is connected. Hence, scanning the neighbors of s clockwise about s, we must, without loss of generality, encounter neighbors of s in $(S - s) \cup V(E)$ between the last neighbor of s in $V(C_1)$ and the first neighbor of s in $V(C_2)$, and another between the last neighbor of s in $V(C_2)$ and the first in $V(C_3)$.

By our distance hypothesis, one of these neighbors of s is in $S - \{s\}$, for if both are ends of different e_is, then these two e_is are at distance no more than 2 apart, a contradiction, whereas if both are ends of the same e_i, then su_iv_is is a separating 3-cycle, contradicting the fact that G is 5-connected. Thus $|S| = 2$ and there are precisely four odd components of $G' - S$.

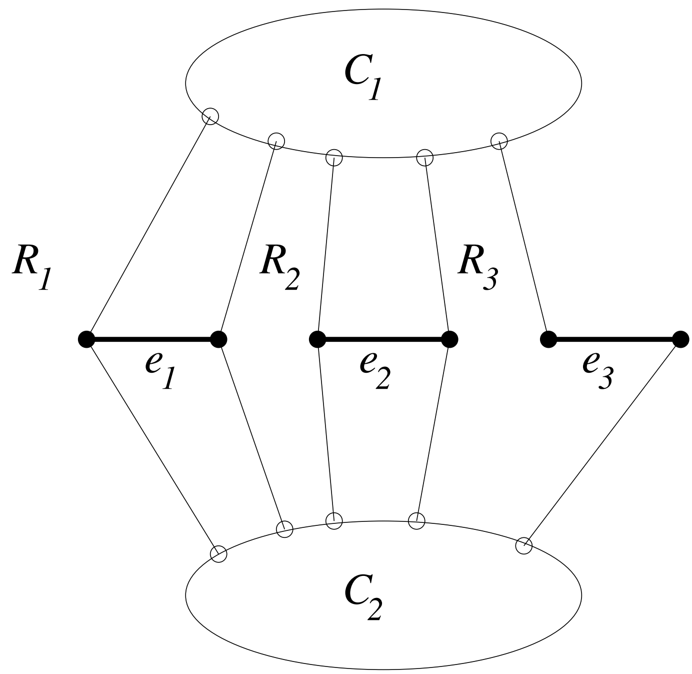

Figure 10.

So without loss of generality, let us assume that as we proceed clockwise about s from C_1 toward C_2 we encounter a second member of S which we denote by s' and then proceding further clockwise from C_2 toward C_3 we encounter a vertex u_1 where $e_1 = u_1v_1$ belongs to the matching E. (See Figure 11.)

Now consider the neighborhood of s'. Again as observed earlier, s' has neighbors in at least three odd components. We already have neighbors of s' in C_1 and C_2. If s' also has a neighbor in C_3, then we have a closed curve Γ running from C_3 through s, s' and back to C_3. Component C_1 lies on one side of Γ, while C_2 and e_1 lie on the other side. By 5-connectivity, both e_2 and e_3 lie on the same side of Γ as does C_1 and hence $\{s, s', u_1, v_1\}$ is a cutset of size 4 in G, a contradiction.

So s' has neighbors in C_1, C_2 and in the fourth odd component C_4. (See Figure 12.)

As we argued for vertex s, vertex s' must have a neighbor in $V(E)$ either between its last neighbor in C_1 and the first in C_4 or between its last neighbor in C_4 and its first in C_2, where again we are scanning the neighbors of s' in a clockwise manner. By the 5-connectivity hypothesis, this neighbor

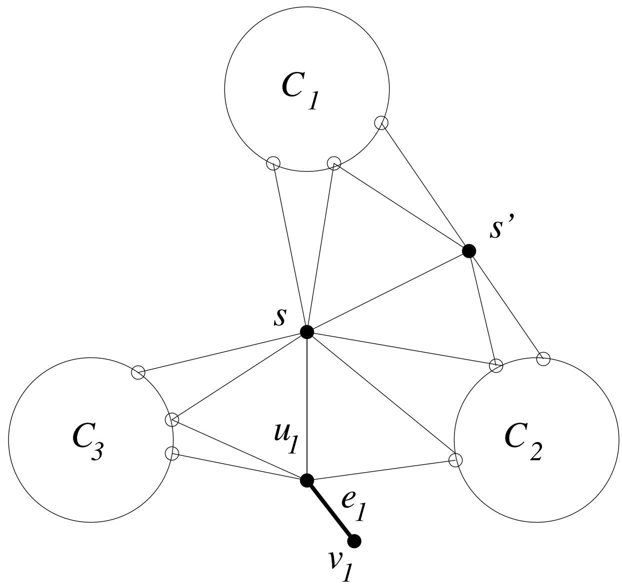

Figure 11.

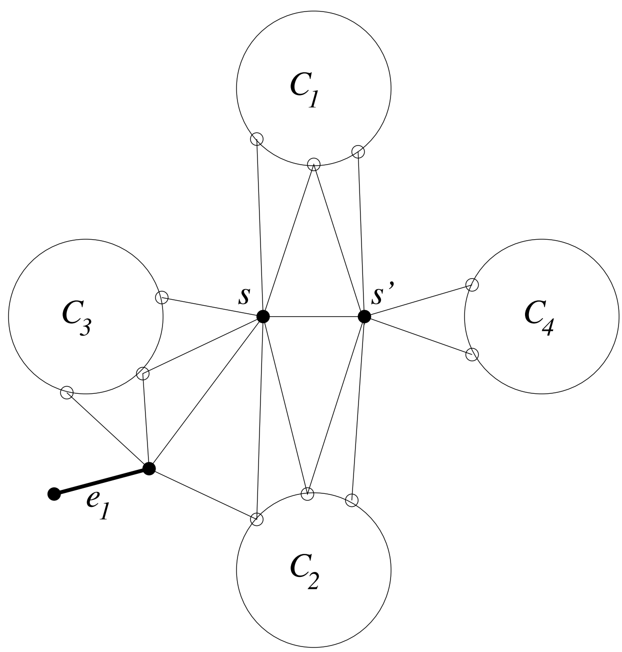

Figure 12.

cannot be an endvertex of e_1, so, without loss of generality, we may suppose that it is an endvertex of e_2.

We thus have two possibilities for the location of edge e_2. These are shown in Figure 13 below.

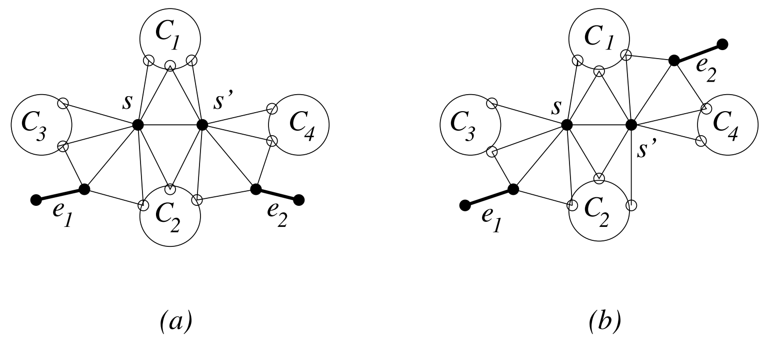

Figure 13.

In Case (a), by 5-connectivity component C_1 must have a neighbor on e_1 or e_2, say e_1 without loss of generality. Thus we have a closed curve Γ through C_1, s, u_1 and possibly v_1, separating C_3 from C_2, C_4, s' and e_2. By 5-connectivity, Γ includes v_1 and e_3 lies on the same side of Γ in the plane as C_3. In particular, neither C_2 nor C_4 has neighbors on e_3. Consequently we have that C_1 cannot have a neighbor on e_2, for otherwise the closed curve Γ' through C_1, s', u_2 (and possibly v_2) would separate C_4 from C_2, C_3, e_1, e_3 and s; that is, $\{s', u_2, v_2\}$ would be a 3-cut, a contradiction. So $\{s, s', u_1, v_1\}$ is a 4-cut separating C_3 from C_4 which is again a contradiction and so Case (a) cannot occur.

In Case (b) we consider the neighbors of C_1 in $S \cup V(E)$. Since there are at least five such neighbors, C_1 has either a neighbor on e_1 or e_3. Suppose C_1 has a neighbor of e_1. As in Case (a) we conclude that e_3 is inside a closed curve through C_1, s, u_1 and v_1 on the same side of the curve as C_3. Consequently C_4 has no neighbors on e_3 and hence has neighbors s', u_2, v_2, u_1 and v_1. This leaves only s, s' and u_1 as neighbors of C_2, contradicting 5-connectivity. So C_1 has no neighbor on e_1 and by symmetry C_2 has no neighbor on e_2. Thus both C_1 and C_2 have neighbors on e_3. This yields a closed curve through C_1, C_2, s and at least one vertex on e_3 separating $C_3 \cup e_1$ from $C_4 \cup e_2$. As a result, both C_3 and C_4 each have both ends of e_3 as neighbors and we must have the configuration in Figure 14. But then $\{s, s', u_3, v_3\}$ is a 4-cut which is a contradiction. This completes the proof. □

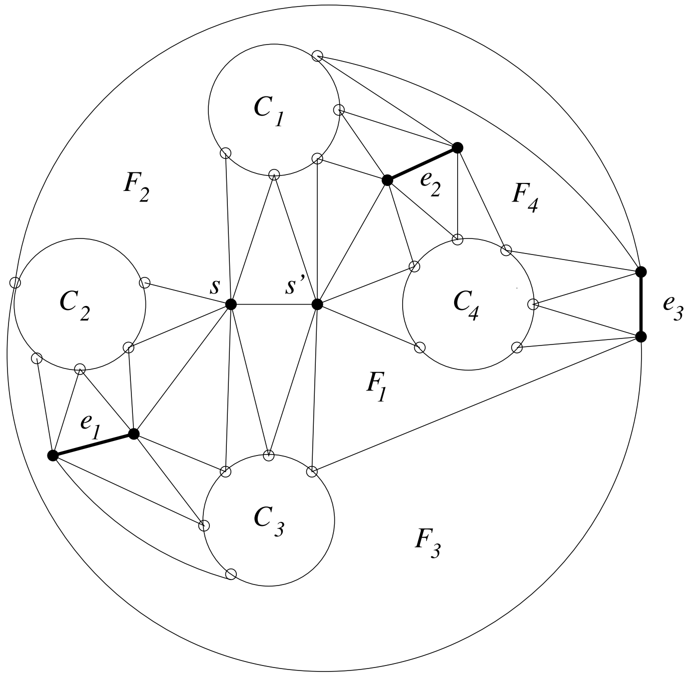

Figure 14.

The graph in Figure 15, with $D = D_6$ say, is 5-connected, locally connected, planar and even, but not $E_3(8,0)$. It has exactly one non-triangular face, namely the exterior face.

The graph in Figure 16 is 5-connected, locally connected, planar and even, but not $E_3(6,0)$. It has exactly two non-triangular faces.

Remark: We do not know if the conclusion of Theorem 2.2 can be strengthened to $E_3(4,0)$ and/or $E_3(5,0)$.

As remarked earlier, the graph shown in Figure 3 is 5-connected, locally connected, planar, even and has only two non-triangular faces. Hence it is $E_3(3,0)$ by Theorem 2.2, but is not $E_3(2,1)$. Thus $E_3(3,0) \not\longrightarrow E_3(2,1)$. On the other hand, for distance 1 this implication *does* hold via the lattice of implications.

Theorem 2.3: Let G be a 5-connected, locally connected, planar, even graph. Then

(i) if $|V(G)| \geq 10$, G is $E_3(1,n)$, for $n \leq 3$ and (ii) if $|V(G)| \geq 2n+4$, G is $E_3(0,n)$, for $0 \leq n \leq 9$.

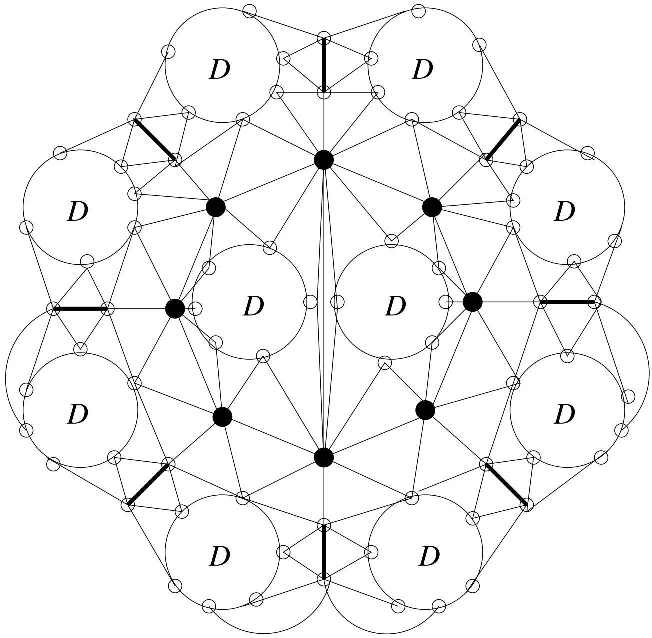

Figure 15: A graph which is not $E_3(8,0)$.

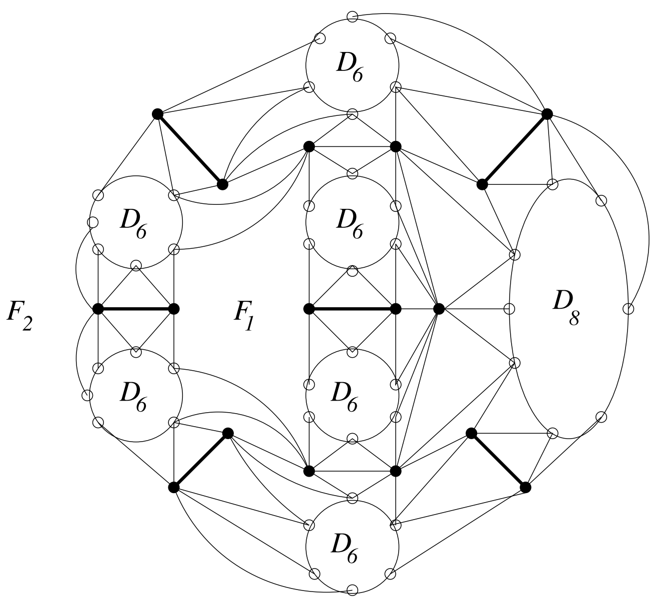

Figure 16: A graph which is not $E_3(6,0)$.

Proof: (i) follows immediately from Theorem 2.1.

(ii) Now let n be the smallest non-negative integer such that G is not $E_3(0,n)$ and suppose $F = \{f_1, \ldots, f_n\}$ is a matching with edges pairwise at distance at least 3 from each other and suppose that every perfect matching in G contains at least one f_i. Then $G' = G - F$ contains no perfect matching and thus there is a barrier $S \subseteq V(G')$. By the minimality of n, $o(G' - S) = |S| + 2$ and each edge $f_i \in F$ has its endvertices in two distinct odd components of $G' - S$.

By the same counting procedure as used previously, we have

$$5|S| + 10 - 2n \leq 2(2|S| + 2) - 4$$

and hence $|S| \leq 2n - 10$.

Now local connectivity of G requires that each of the edges f_i in F must lie in the boundary of a triangular face. The boundary of such a triangular face must also include a vertex from S and no vertex from S can lie in the boundaries of faces with f_i and f_j, for $i \neq j$, by the distance hypothesis for edges in F. Thus $|S| \geq n$.

Hence $n \geq 10$ and the result follows. □

The conclusion that G is $E_3(1,3)$ in part (i) of the preceding theorem is best possible in that Figure 17 exhibits a 5-connected, locally connected, planar, even graph which is *not* $E_3(1,4)$. Note, however, that it has precisely four non-triangular faces.

In Figure 18 a graph is exhibited which is 5-connected, locally connected, planar and even, but which is not $E_3(0,10)$. It possesses ten non-triangular faces. Here we use$D = D_6$, for example.

We know that a 5-connected, locally connected, planar and even graph is necessarily $E_4(2,0)$ by virtue of being $E(2,0)$, but Figure 19 shows a graph of this type which is *not* $E_4(3,0)$.

On the other hand, the graph in Figure 4 shows that these graphs are *not* necessarily $E_4(2,1)$ either.

In the positive direction, however, we have the next theorem for the distance 4 case.

Theorem 2.4: Let G be a 5-connected, locally connected, planar, even graph. Then

(i) if $|V(G)| \geq 2n + 4$, G is $E_4(1,n)$, for $0 \leq n \leq 4$; while (ii) if $|V(G)| \geq 2n + 2$, then G is $E_4(0,n)$, for $0 \leq n \leq 10$.

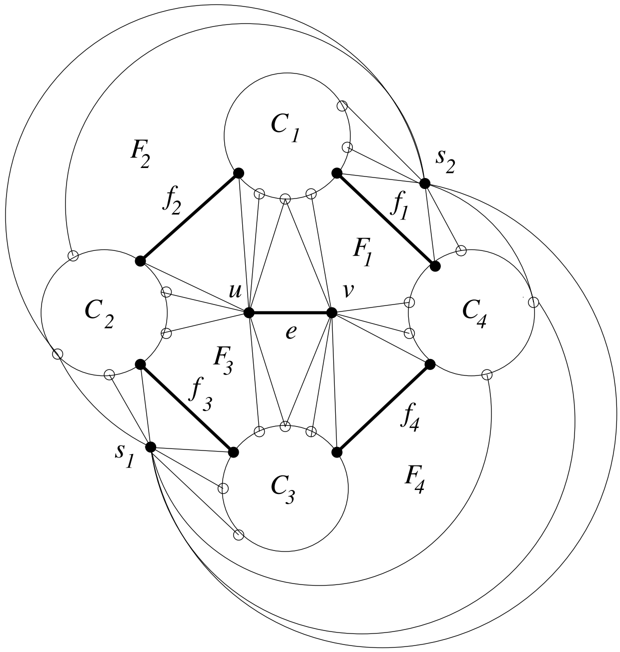

Figure 17: A graph which is not $E_3(1, 4)$.

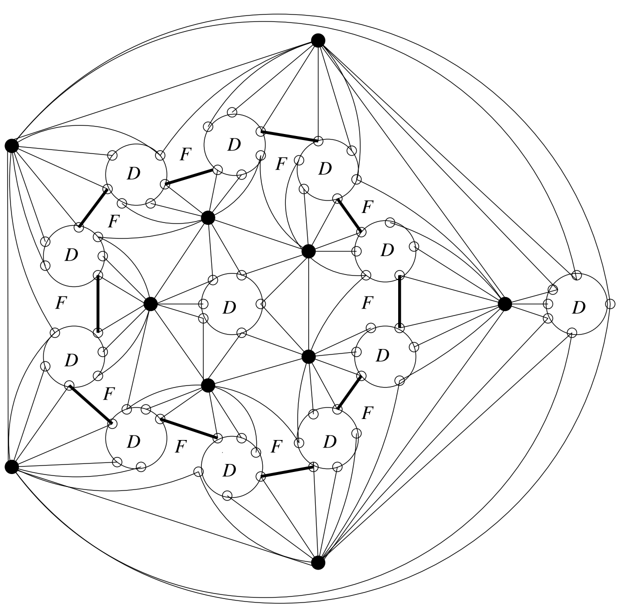

Figure 18: A graph which is not $E_3(0, 10)$.

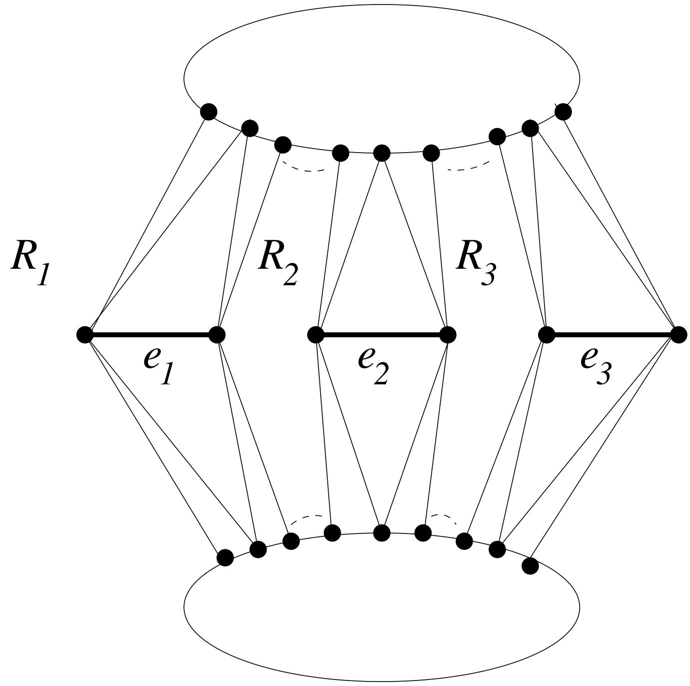

Figure 19: A graph which is not $E_4(3,0)$.

Proof: (i) Suppose G is as in the statement of the theorem, $e = uv \in E(G)$, $F = \{f_1, \ldots, f_n\} \subseteq E(G)$ such that $e \notin F$, the f_is are at mutual distance at least 4 and there is no perfect matching in G containing e, but no f_i. Moreover, suppose n is as small as possible.

Then $G' = G - V(e) - F$ has no perfect matching, so G' has a barrier $S \subseteq V(G')$. By the minimality of n, $G' - S$ has precisely $|S| + 2$ odd components. Form G^*, the bipartite distillation of G via G' based on e, F and S. Then by planarity, G^* has at most $2(2|S| + 4) - 4 = 4|S| + 4$ edges. By the 5-connectivity of G, G^* has at least $5(|S| + 2) - 2n = 5|S| + 10 - 2n$ edges. So

$$0 \leq |S| \leq 2n - 6. \qquad (1)$$

Since G is locally connected and by the minimality of n, each $f_i \in F$ has endvertices lying in two different odd components of $G' - S$, so there is a vertex $w \in K = V(e) \cup S$ such that w is a common neighbor of both endvertices of f_i.

Note that if $w \in S$, w has no neighbors on any $f_j \in F$, $j \neq i$, by our distance 4 hypothesis. Similarly, if $w \in \{u, v\}$, then w has no neighbors on

$f_j \in F, j \neq i$. Moreover, if vertex u has neighbors on f_i, vertex v cannot have neighbors on f_j, j/nei, by the distance 4 hypothesis. Thus

$$|S| + 2 = |K| \geq n + 1. \tag{2}$$

Combining (1) and (2), we have that $2n - 4 \geq n + 1$ and thus $n \geq 5$, a contradiction.

(ii) Now suppose G is as in the statement of the theorem and that $F = \{f_1, \ldots, f_n\}$ is a set of n edges at mutual distance 4 such that every perfect matching in G contains an edge of F. Moreover, assume that such an n is as small as possible. Then the standard counting yields $5|S| + 10 - 2n \leq 4|S|$, and so $|S| \leq 2n - 10$.

By the minimality of n, each $f_i \in F$ has endvertices lying in two different odd components of $G - F - S$.

Since G is locally connected, each edge $f_i \in F$ forms part of at least one triangular face. So there must be an $s_i \in S$ lying on that triangular face. By our distance requirement, no $s \in S$ has neighbors on two different edges in F. Thus $|S| \geq n$.

Now if S is chosen as small as possible, each $s \in S$ must have neighbors in at least three different odd components of $G - F - S$. Consider scanning the neighbors of such an s clockwise. Between the last neighbor in odd component C_1 and the first neighbor in the next odd component C_2, we must encounter either an edge in F or a neighbor of s in S. Similarly, without loss of generality, between the last neighbor in C_2 and the first in the next odd component C_3 we must encounter either an edge of F or a neighbor of $s \in S$. But the distance hypothesis says that it cannot be that both gaps in our scan can be filled by edges in F. Thus s has a neighbor $s' \in S$. Note by the distance 4 hypothesis, s' has no neighbor on an edge in F, if s does have such a neighbor. Consequently, $|S| \geq n + 1$. So we have $n + 1 \leq 2n - 10$ and hence $n \geq 11$, a contradiction. □

Neither $E_4(1, 4)$ nor $E_4(0, 10)$ is known to be a best possible conclusion in the above theorem.

Let us now turn to the distance 5 case. Of course 5-connected, locally connected, planar, even graphs are $E_5(2, 0)$, since they are $E(2, 0)$. On the other hand, these graphs are *not* necessarily $E_5(3, 0)$ by the graph shown in Figure 19. The graph shown in Figure 4 suffices to show that we do *not* necessarily have property $E_5(2, 1)$ for these graphs either. However, we do have the following result which is interesting in that there is no limit on the parameter n in the conclusions.

Theorem 2.5: Let n be a non-negative integer and let G be a 5-connected, locally connected, planar, even graph. Then

(i) if $|V(G)| \geq 2n+4$, then G is $E_5(1,n)$, while (ii) if $|V(G)| \geq 2n+2$, G is $E_5(0,n)$.

Proof: We give the proof of (i). The proof of (ii) is quite similar, but somewhat less complicated.

Suppose to the contrary that $G, e = uv$ and $F = \{f_1, \ldots, f_n\}$ constitute a counterexample, where n is as small as possible. then $G' = G - V(e) - F$ has a barrier S which we may choose to be as small as possible.

Now the minimality of n ensures that $o(G'-S) = |S|+2$. The bipartite distillation of G, based on e, F and S has $2|S|+4$ vertices and hence by planarity at most $4|S|+4$ edges. Since G is 5-connected, $|E(G^*)| \geq 5(|S|+2) - 2n = 5|S| + 10 - 2n$, and so $|S| \leq 2n - 6$.

Since G is locally connected, and by the minimality of n, each f_i in F has endvertices in two different odd components of $G' - S$, we must have both endvertices of f_i adjacent to a vertex in $K = S \cup V(e)$. The distance hypothesis guarantees that for $s_i \in S$ with s_i in a triangle including edge f_i, vertex s_i has no neighbors in $V(f_j)$, for $j \neq i$. Similarly, for u and f_i forming a triangle, neither u nor v has neighbors in f_j, for $j \neq i$.

Moreover, if either u or v lies in a triangular face including edge f_i for some i, then no $s_j \in S$ lying in a triangular face including f_j, for any $j \neq i$, can be adjacent to u or v. If, on the other hand, for each $i = 1, \ldots, n$, there is a vertex $s_i \in S$ such that s_i and f_i lie in the same triangular face boundary, then $s_i s_j \notin E(G)$, for $i \neq j$.

By the minimality of S, each $s_i \in S$ has neighbors in at least three odd components of $G' - S$. So suppose s_i forms a triangle with f_i and f_i has endvertices in odd components C_1 and C_2 of $G' - S$. Then s_i also has neighbors in some C_3, say, where $C_3 \neq C_1, C_2$. Considering the neighbors of s_i lying in odd components when scanning clockwise about s_i, to be successively in C_1, C_2 and C_3, and then in C_1 again, local connectivity requires, without loss of generality, that there is a neighbor of $s_i \in K$ between the last neighbor of s_i in C_2 and its first neighbor in C_3. By the distance 5 hypothesis, this vertex in K forms no triangle with any $f_j, j \neq i$, nor can this vertex in K be u (respectively v) if v (respectively u) forms a triangle with f_j, for $j \neq i$.

Thus if neither u nor v forms a triangle with any edge in F, we have at least two vertices of K at distance at most 2 from f_i for each $f_i \in F$. This yields $|K| \geq 2n$. But as we have seen earlier, $|K| = |S| + 2 \leq 2n - 4$, and we have a contradiction.

So we may assume that, without loss of generality, u and f_n form a triangle. But then each $f_i, 1 \leq i \leq n-1$, has two vertices in S at distance at most 2 from it giving $|S| \geq 2n-2$ and again $|K| \geq 2n$, a contradiction. □

References

[1] R. E. L. Aldred and M. D. Plummer, On restricted matching extension in planar graphs, *Discrete Math.* **231** (2001) 73–79. MR1821947

[2] R. E. L. Aldred and M. D. Plummer, Edge proximity and matching extension in planar triangulations, *Australas. J. Combin.* **29** (2004) 215–224. MR2037348

[3] R. E. L. Aldred and M. D. Plummer, Distance-restricted matching extension in planar triangulations, *Discrete Math.* **310** (2010) 2618–2636. MR2672209

[4] J. Bondy and U. S. R. Murty, *Graph Theory with Applications*, American Elsevier, New York, 1976. MR0411988

[5] D.-J. Lou, 2-extendability of planar graphs, *Acta Sci. Natur. Univ. Sunyatseni* **29** (1990) 124–126 (in Chinese). MR1089453

[6] L. Lovász and M. D. Plummer, *Matching Theory*, North-Holland Math. Studies **121** Ann. Discrete Math. **29**, North-Holland Publ. Co., Amsterdam, 1986. (Reprinted: AMS Chelsea Publishing, Providence, RI, 2009.) MR2536865

[7] M. D. Plummer, A theorem on matchings in the plane, *Discrete Math.* **41** (1989) 347–354. MR0976014

[8] M. D. Plummer, Extending matchings in planar graphs IV, *Discrete Math.* **109** (1992) 207–219. MR1192384

[9] M. D. Plummer, Extending matchings in graphs: a survey, *Discrete Math.* **127** (1994) 277–292. MR1273609

[10] M. D. Plummer, Extending matchings in graphs: an update, *Congr. Numer.* **116** (1996) 3–32. MR1411246

[11] M. D. Plummer, Recent progress in matching extension, *Building bridges* Bolyai Soc. Math. Stud. **19**, Springer, Berlin, 2008, 427–454. MR2484649

[12] M. Porteous and R. E. L. Aldred, Matching extensions with prescribed and forbidden edges, *Australasian J. Combin.* **13** (1996) 163–174. MR1383415

[13] Q. R. Yu and G. Liu, *Graph Factors and Matching Extensions*, Higher Education Press, Beijing, China, 2009. MR2569624

R. E. L. Aldred
Department of Mathematics and Statistics
University of Otago
P.O. Box 56
Dunedin 9054
New Zealand
E-mail address: raldred@maths.otago.ac.nz

Michael D. Plummer
Department of Mathematics
Vanderbiilt University
Nashville, TN 37240
USA
E-mail address: michael.d.plummer@vanderbilt.edu

Received 28 November 2014