Journal of Combinatorics
Volume 7, Number 2–3, 495–508, 2016

Cyclable sets of vertices in 3-connected graphs

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Given a graph G, a set $S \subseteq V(G)$ is called cyclable (in G) if G has a cycle containing every vertex of S; G is hamiltonian if $V(G)$ is cyclable in G. Beginning with a result of Dirac in 1952, many results on sufficient conditions that relate degree sum and neighborhood conditions for hamiltonicity and cyclability, have been obtained. We give a new sufficient condition on degree sums and neighborhoods of any four independent vertices in a graph. We also study the extremal cases of this condition.

1. Introduction and notation

All the graphs considered in this paper are undirected and simple. We use the notation and terminology in [3]. In addition, for a graph $G = (V(G), E(G))$ and a subgraph H of G, the neighborhood in H of a vertex $u \in V(G)$ is $\{v \in V(H) : uv \in E(G)\}$ and is denoted by $N_H(u)$ and the degree of u in H is $d_H(u) := |N_H(u)|$. In the case $H = G$, we use $N(u), d(u)$ instead of $N_G(u), d_G(u)$, respectively.

If $C = c_1c_2 \cdots\cdots c_pc_1$ is a cycle, we let $C[c_i, c_j]$ ($\overline{C}[c_i, c_j]$, resp.) be the sub-path $c_ic_{i+1} \cdots c_j$ ($c_ic_{i-1} \cdots c_j$, resp.), where the indices are taken modulo p. We will consider $C[c_i, c_j]$ both as paths and as vertex sets. Define $C(c_i, c_j] = C[c_{i+1}, c_j]$, $C[c_i, c_j) = C[c_i, c_{j-1}]$ and $C(c_i, c_j) = C[c_{i+1}, c_{j-1}]$. For any i, we put $c_i^+ = c_{i+1}$, $c_i^- = c_{i-1}$, $c_i^{+2} = c_{i+2}$ and $c_i^{-2} = c_{i-2}$. For $V(A) \subseteq V(C)$, we set $V(A)^+ = \{v^+ | v \in V(A)\}, V(A)^- = \{v^- | v \in V(A)\}, V(A)^{+2} = (V(A)^+)^+$ and $V(A)^{-2} = (V(A)^-)^-$. We will use similar definitions for a path.

For any subset S of $V(G)$ and any integer $k \geq 1$, denote by

$\sigma_k(S) = min\{\sum_{i=1}^k d(v_i) : \{v_1, v_2, \cdots, v_k\}$ is an independent set in $S\}$,
$\overline{\sigma}_k(S) = min\{\sum_{i=1}^k d(v_i) - |\bigcap_{i=1}^k N(v_i)| : \{v_1, v_2, \cdots, v_k\}$ is an independent set in $S\}$ and
$\sigma_k^*(S) = min\{\sum_{i=1}^k d(v_i) + |\bigcup_{i=1}^k N(v_i)| - |\bigcap_{i=1}^k N(v_i)| : \{v_1, v_2, \cdots, v_k\}$ is an independent set in $S\}$.

So $\sigma_1(S) = \delta(S)$ which we often use for the minimum degree (in G) of the vertices of S. If $S = V(G)$, we denote $\sigma_k = \sigma_k(G)$, $\overline{\sigma}_k = \overline{\sigma}_k(G)$ and

$\sigma_k^* = \sigma_k^*(G)$ respectively. A vertex v is called an *S-vertex* if $v \in S$. By $G[S]$ we denote the subgraph of G induced by S. Let $\alpha(S)$ be the number of vertices of a maximum independent set of $G[S]$, obviously $\alpha(S) \geq k$.

A subset S of $V(G)$ is called *cyclable* in G if all the vertices of S belong to a common cycle in G. A cycle C of G is called *S-maximum* if $|V(C) \cap S|$ is maximum. Obviously, a $V(G)$-maximum cycle is a longest cycle in G. A graph G is *hamiltonian* if $V(G)$ is cyclable in G, *i.e.*, there is a cycle that contains all vertices of G. For example, the complete bipartite graph $K_{s,t}$ with $s < t$ is not hamiltonian and the subset of vertices in the part of s vertices is cyclable and the subset of vertices in the part of t vertices is not cyclable. A cycle C is called *S-weak-dominating* if every component in $G - V(C)$ contains at most one S-vertex. A cycle C is called *S-dominating* if every component in $G - V(C)$ that has S-vertex is of cardinality one. It is clear that when $S = V(G)$, an S-weak-dominating cycle is a dominating cycle of G, (where a cycle C is *dominating* if no component in $G - V(C)$ has more than one vertex).

The first important result in extremal hamiltonian graph theory is due to Dirac.

Theorem 1 (Dirac 1952 [5])**.** If G is a graph of order $n \geq 3$ such that $\sigma_1 \geq \frac{n}{2}$, then G is hamiltonian. The bound is sharp.

Dirac's theorem concerns a condition of $\sigma_1(G)$, *i.e.*, a degree condition on every vertex. It is natural to generalize them into degree conditions on more independent vertices for hamiltonicity. We summarize some of them in the following theorem.

Theorem 2.

–(1) (Ore, 1960 [14]) Let G be a graph of order $n \geq 3$ such that $\sigma_2 \geq n$. Then G is hamiltonian. The bound is sharp.

–(2) (Schmeichel and Hayes, 1985 [15]) Let G be a 2-connected graph of order n such that $\sigma_2 \geq n - 1$. Then G is hamiltonian unless G is the class of graphs that can be obtained from $K_{(n-1)/2,(n+1)/2}$ by adding some edges in the $\frac{n-1}{2}$-part.

–(3) (Bondy, 1980 [2]) Let G be a k-connected graph of order $n \geq 3$. If $\sigma_{k+1}(G) > \frac{1}{2}(k+1)(n-1)$, then G is hamiltonian. The bound is sharp.

–(4) (Harkat, Li and Tian, 2000 [10]) Let G be a 3-connected graph of order n. If $\sigma_4 \geq n + 2\alpha - 2$, then G is hamiltonian. The bound is sharp.

By an observation, the degree sums or neighborhood unions count roughly $\frac{1}{2}n$ for each vertex. For more results on sufficient conditions with degree sums or neighborhood unions, please see [6],[7],[9] and [10].

Cyclability is a natural generalization of hamiltonicity since clearly, if $S = V(G)$, "S is cyclable" is equivalent to "G is hamiltonian". We summarize the results related to the above ones in the following theorem.

Theorem 3.

–(1) (Bollobás and Brightwell, 1993 [1] and independently Shi,1992 [16]) Let G be a 2-connected graph of order n and let $S \subseteq V(G)$. If $\sigma_1(S) \geq \frac{1}{2}n$, then S is cyclable. (in fact, this is a special case of a more general result obtained in [1]).

–(2) (Ota, 1995 [13]) Let G be a 2-connected graph of order n and let $S \subseteq V(G)$. If $\sigma_2(S) \geq n$, then S is cyclable.

–(3) (Favaron, Flandrin, Li, Liu, Tian and Wu, 1996 [8]) Let G be a 2-connected graph of order n and let $S \subseteq V(G)$. If $\overline{\sigma}_3(S) \geq n$, then S is cyclable in G.

–(4) (Harkat, Tian and Li, 2000 [10]) Let G be a 3-connected graph of order n. If $S \subseteq V(G)$ such that $\sigma_4(S) \geq n + 2\alpha(S) - 2$, then S is cyclable in G.

–(5) (Li, 2000 [11]) Let G be a 3-connected graph of order n. If $S \subseteq V(G)$ such that $\overline{\sigma}_4(S) \geq n + 3$, then G has an S-weak-dominating S-maximum cycle.

The main result of this paper is about sufficient conditions on four independent vertices. We first define

– $\mathcal{A}_S$ is the class of graphs that can be obtained from $K_{(n-1)/2,(n+1)/2}$ in which the $\frac{n+1}{2}$-part entirely and exactly belongs to S, and then by adding some edges (possibly no) to the $\frac{n-1}{2}$-part.

– $\mathcal{B}_S$ is the class of graphs in each G of which there are three vertices v_1, v_2, v_3 such that $G - \{v_1, v_2, v_3\}$ is an union of four disjoint subgraphs H_i $(1 \leq i \leq 4)$ such that for $1 \leq i \leq 4$, there exists at least one S-vertex in H_i, and for every S-vertex u in H_i, $N(u) = (H_i - \{u\}) \cup \{v_1, v_2, v_3\}$.

Theorem 4 (the main theorem)**.** Let G be a 3-connected graph of order n. If $S \subseteq V(G)$, then

– if $\sigma_4^*(S) \geq 2n - 1$, S is cyclable in G and

– if $\sigma_4^*(S) = 2n - 2$, S is cyclable in G or $G \in \mathcal{A}_S \cup \mathcal{B}_S$ or G is the Petersen graph with $S = V(G)$.

2. Proof of the main theorem

We shall use the following lemma.

Lemma 1 ([11])**.** Let $G = (V, E)$ be a graph, $P = v_1v_2...v_p$ a path in G and u_1, u_2, u_3 three vertices in $V(G) - V(P)$ such that for every $i \in \{2, 3\}$, $N_P(u_i) \cap N_P(u_i)^+ = \emptyset$ and for $1 \le i < j \le 3, N_P(u_i) \cap N_P(u_j)^+ = \emptyset$. Then

$$\sum_{i=1}^{3} d_P(u_i) \le \begin{cases} p + \lambda(P) + 1, & \text{if } v_p \notin \bigcap_{i=1}^{3} N_P(u_i) \\ p + \lambda(P) + 2, & \text{if } v_p \in \bigcap_{i=1}^{3} N_P(u_i), \end{cases}$$

where $\lambda(P) = |\Lambda_{u_1,u_2,u_3}(P)|$ with $\Lambda_{u_1,u_2,u_3}(P) = (\bigcap_{i=1}^{3} N_P(u_i))^+ \bigcap (N_P(u_1) \cup N_P(u_2))^-$.

Corollary A. Under the conditions of Lemma 1, we have

$$\sum_{i=1}^{3} d_P(u_i) \le \begin{cases} |P| + |V(P) - \{v_p\} - \bigcup_{i=1}^{3} N_P(u_i)| + 1, & \text{if } v_p \notin \bigcap_{i=1}^{3} N_P(u_i) \\ |P| + |V(P) - \{v_p\} - \bigcup_{i=1}^{3} N_P(u_i)| + 2, & \text{if } v_p \in \bigcap_{i=1}^{3} N_P(u_i). \end{cases}$$

Proof of Corollary A. Since for every $i \in \{2, 3\}$, $N_P(u_i) \cap N_P(u_i)^+ = \emptyset$ and for $1 \le i < j \le 3, N_P(u_i) \cap N_P(u_j)^+ = \emptyset$, the corollary follows directly from the fact that $(\bigcap_{i=1}^{3} N_P(u_i))^+ \bigcap (N_P(u_1) \cup N_P(u_2))^- \subseteq V(P) - \{v_p\} - \bigcup_{i=1}^{3} N_P(u_i)$. □

In the following two corollaries, we discuss the extremal cases.

Corollary B. Under the conditions of Lemma 1 and if
(1)

$$\sum_{i=1}^{3} d_P(u_i) = \begin{cases} p + |V(P) - \{v_p\} - \bigcup_{i=1}^{3} N_P(u_i)| + 1, & \text{if } v_p \notin \bigcap_{i=1}^{3} N_P(u_i) \\ p + |V(P) - \{v_p\} - \bigcup_{i=1}^{3} N_P(u_i)| + 2, & \text{if } v_p \in \bigcap_{i=1}^{3} N_P(u_i). \end{cases}$$

then there are $1 = i_1 \le j_1 < i_2 \le j_2 < \cdots < i_q \le j_q = p$ with $i_{f+1} \ge j_f + 2$ for all f such that $v_{j_f}^+ \notin \cup_{i=1}^{3} N_P(u_i)$ for $1 \le f \le q-1$ and for any subpath $P_f = P[v_{i_f}, v_{j_f}]$, when $1 \le f \le q-1$, we have $P_f \subset N_P(u_1)$, $N_P(u_2) \cap P_f = N_P(u_3) \cap P_f = \{v_{j_f}\}$ and for P_q one of the following six cases occurs:

–(a) $P_q = N_P(u_1)$ and $N_P(u_2) = N_P(u_3) = \{v_p\}$;
–(b) $P_q = N_P(u_1)$, $N_P(u_2) = \{v_p\}$ and $N_P(u_3) = \emptyset$;
–(c) $P_q = N_P(u_1)$, $N_P(u_3) = \{v_p\}$ and $N_P(u_2) = \emptyset$;
–(d) $P_q - \{v_p\} = N_P(u_1)$ and $N_P(u_2) = N_P(u_3) = \{v_p\}$;
–(e) $P_q - \{v_p\} = N_P(u_1)$ and $N_P(u_2) = N_P(u_3) = \{v_{p-1}\}$; and
–(f) $P_q - \{v_p\} = N_P(u_1)$, $N_P(u_2) = \{v_{p-1}\}$ and $N_P(u_3) = \{v_p\}$.

Proof of Corollary B. Suppose that (1) holds. It follows that $(\bigcap_{i=1}^{3} N_P(u_i))^+ \bigcap (N_P(u_1) \cup N_P(u_2))^- = V(P) - \{v_p\} - \bigcup_{i=1}^{3} N_P(u_i)$.

We use induction on p. When $p = 1$ and $p = 2$ the corollary can be directly verified. Assume that it is also true for any path of fewer than p vertices.

Suppose that $u_3v_d \in E(G)$ for some $1 \le d \le p-2$ and d is minimum. Then by the condition of Lemma 1, $v_{d+1} \notin \bigcup_{j=1}^{3} N_P(u_j)$ and hence $v_{d+1} \in (\bigcap_{i=1}^{3} N_P(u_i))^+ \bigcap (N_P(u_1) \cup N_P(u_2))^-$.

Since $v_d \in \bigcap_{i=1}^{3} N_P(u_i)$, from the condition that for every $i \in \{2,3\}$, $N_P(u_i) \cap N_P(u_i)^+ = \emptyset$, it follows that either $v_{d-1} \notin \bigcup_{i=1}^{3} N_P(u_i)$ or $v_{d-1} \in N_P(u_1) - N_P(u_2) \cup N_P(u_3)$. In the case that $v_{d-1} \in N_P(u_1) - N_P(u_2) \cup N_P(u_3)$, we deduce that $v_{d-2} \in N_P(u_2) \cup N_P(u_3)$. If $v_{d-2} \notin N_P(u_1)$ then $v_{d-3} \notin N_P(u_2) \cup N_P(u_3)$. We continue and let $f = min\{t : v_t, v_{t+1}, \cdots, v_d \in N_P(u_1)$ and either $t = 1$ or $v_{t-1} \notin \bigcup_{i=1}^{3} N_P(u_i)\}$.

Put $P_1 = v_1v_2 \cdots v_d$ and $P_2 = v_{d+2}v_{d+3} \cdots v_p$. Since $\sum_{i=1}^{3} d_{P_1}(u_i) = |P_1| + 2$, it gives

$$\sum_{i=1}^{3} d_{P_2}(u_i) = \sum_{i=1}^{3} d_P(u_i) - \sum_{i=1}^{3} d_{P_1}(u_i)$$
$$= \begin{cases} p + |V(P) - \{v_p\} - \bigcup_{i=1}^{3} N_P(u_i)| + 1 - d - 2, & \text{if } v_p \notin \bigcap_{i=1}^{3} N_P(u_i) \\ p + |V(P) - \{v_p\} - \bigcup_{i=1}^{3} N_P(u_i)| + 2 - d - 2, & \text{if } v_p \in \bigcap_{i=1}^{3} N_P(u_i). \end{cases}$$
$$= \begin{cases} |P_2| + |V(P_2) - \{v_p\} - \bigcup_{i=1}^{3} N_{P_2}(u_i)| + 1, & \text{if } v_p \notin \bigcap_{i=1}^{3} N_P(u_i) \\ |P_2| + |V(P_2) - \{v_p\} - \bigcup_{i=1}^{3} N_{P_2}(u_i)| + 2, & \text{if } v_p \in \bigcap_{i=1}^{3} N_P(u_i). \end{cases}$$

Then by using the induction hypothesis, we obtain the conclusion. Hence we assume that $N_P(u_3) \cap \{v_1, v_2, \cdots, v_{p-2}\} = \emptyset$.

If $u_3v_{p-1} \in E(G)$, $v_p \notin N_P(u_1) \cup N_P(u_2) \cup N_P(u_3)$. We use Lemma 1 on the subpath $P' = P[v_1, v_{p-1}]$ and the induction hypothesis. By the equality condition in the corollary, we deduce that the only case (e) follows. So we assume $N_P(u_3) \subseteq \{v_p\}$.

It gives that $\lambda(P) = 0$ and so $V(P) \subseteq N_P(u_1) \cup N_P(u_2) \cup N_P(u_3)$. If $u_2v_d \in E(G)$ with $1 \le d \le p-2$ it implies that $v_{d+1} \notin N_P(u_1) \cup N_P(u_2) \cup N_P(u_3)$, a contradiction. We obtain that $P[v_1, v_{p-2}] \subseteq N_P(u_1)$. Moreover, we can easily see that $|N_P(u_2) \cap \{v_{p-1}, v_p\}| + |N_P(u_3) \cap \{v_{p-1}, v_p\}| \le 2$. We deduce that $N_P(u_1) \cap \{v_{p-1}, v_p\} \ne \emptyset$. When $N_P(u_1) \cap \{v_{p-1}, v_p\} = \{v_{p-1}\}$ we have the cases (d), (e) and (f). When $v_p \in N_P(u_1)$, $v_{p-1} \notin N_P(u_2) \cup N_P(u_3)$ and thus $v_{p-1} \in N_P(u_1)$. We finally obtain cases (a), (b) and (c). □

The following corollary follows directly from Corollary B.

Corollary C. Under the conditions of Lemma 1 and (1), if $N_P(u_1) \cap N_P(u_1)^+ = \emptyset$, then

- $\{v_1, v_3, \cdots, v_{p-4}, v_{p-2}\} \subseteq N_P(u_1) \cap N_P(u_2) \cap N_P(u_3)$ and v_p is adjacent to at least two of the u_1, u_2, u_3's, when p is odd;

– $N_P(u_1) = \{v_1, v_3, \cdots, v_{p-5}, v_{p-3}, v_{p-1}\}$ and either $N_P(u_2) = N_P(u_3) = \{v_1, v_3, , \cdots, v_{p-5}, v_{p-3}, v_p\}$ or $N_P(u_2) = N_P(u_3) = \{v_1, v_3, \cdots, v_{p-5}, v_{p-3}, v_{p-1}\}$ or $N_P(u_2) = \{v_1, v_3, \cdots, v_{p-5}, v_{p-3}, v_{p-1}\}$ and $N_P(u_3) = \{v_1, v_3, \cdots, v_{p-5}, v_{p-3}, v_p\}$, when p is even.

Proof of Theorem 4. To prove Theorem 4 by contradiction, suppose that a graph $G = (V, E)$ and a subset S of V that verify the condition of Theorem 4 are given. Assume that S is not cyclable.

The inserting method was be introduced for use on longest cycle problems in claw-free graphs by Zhang [17] and then has been widely developed by many colleagues. It was generalized into the inserting methods on an S-maximal cycle by in [8] and [10]. We follow [10] and [11] to have some basic definitions and facts on the structures. The definitions and the lemmas 2, 3, 4 with their proofs can be found in [10] and [11].

Suppose that $C = c_1c_2c_3 \cdots c_pc_1$ is a cycle with an implicit orientation according to the increasing subscripts in G such that

(I) C is an S-maximum cycle of G (*i.e.*, C contains maximum number of S-vertices).

Since S is not cyclable, there exists a component H of $G - V(C)$ with $V(H) \cap S \neq \emptyset$. Pick a vertex $x_0 \in V(H) \cap S$. Suppose that there are t paths $P'_1[x_0, v'_1], P'_2[x_0, v'_2]$,..., and $P'_t[x_0, v'_t]$ from x_0 to C having only x_0 in common pairwisely. We have $t \geq 3$ since S is 3-connected. Let $V(P'_i) \cap V(C) = \{v'_i\}$ for each i, and $v'_1, v'_2, ..., v'_t$ occur in this order along the orientation of C. For each $i \in \{1, 2, ..., t\}$, let u_i be the last S-vertex of $P'_i[x_0, v'_i)$ (the vertex u_i may be x_0), and let v_i be the last vertex of $C[v'_i, v'_{i+1})$ adjacent to $P'_i(u_i, v'_i)$ if $P'_i(u_i, v'_i) \neq \emptyset$; otherwise let $v_i = v'_i$, where the indices are taken modulo t. Denote $P_i[x_0, v_i] = P'_i[x_0, u'_i]v_i$ where $u'_i \in N(v_i) \cap P'_i(u_i, v'_i)$, if $P'_i(u_i, v'_i) \neq \emptyset$; otherwise let $P_i[x_0, v_i] = P'_i[x_0, v'_i]$. We assume that

(II) Subject to (I), the path system is chosen in such a way that t is as large as possible.

Since C is an S-maximum cycle of G, we have the followings.

Lemma 2 ([10])**.** Suppose that $\{u, v\} \subset V(C)$. If there is a path $Q[u, v]$ such that $Q(u, v) \cap V(C) = \emptyset$ and $Q(u, v) \cap S \neq \emptyset$, then $C(u, v) \cap S \neq \emptyset$. In particular, $C(v_i, v'_{i+1}) \cap S \neq \emptyset$.

Now we give some definitions.

A segment $C[u, v]$ is a set of consecutive vertices between u and v on C. Two segments of C are intersecting if their intersection contains at least two vertices. A segment is called a non-S-segment if it contains no S-vertices. If x_1, x_2, y_1, y_2 are vertices of C such that $y_2 \in C(y_1, x_1)$, $x_2 \in C(x_1, y_1)$ and $C(y_1, y_2)$ is a non-S-segment, the two edges x_1y_1 and x_2y_2 are called crossing diagonals at x_1 and x_2. An S-vertex u of a segment $C_i = C(v_i, v_{i+1})$ defined above is said to be insertible if there is a non-S-segment $C(x, y) \subseteq C(v_{i+1}, v_i)$ such that ux and uy belong to $E(G)$. In this case, the segment $C[x, y]$ is called an inserting segment for u.

Lemma 3 ([10])**.** Let $i \in \{1, 2, \cdots, t\}$ and $u \in C(v_i, v_{i+1}]$. If there exists a path $Q[u, v_i]$ such that $Q(u, v_i) \cap V(C) = \emptyset$ and $Q(u, v_i) \cap S \neq \emptyset$, then $C(v_i, u)$ contains at least one non-insertible S- vertex. In particular, $C(v_i, v'_{i+1})$ contains at least one non-insertible S-vertex.

For each $i \in \{1, 2, \cdots, t\}$, let x_i (y_i, resp.) be the first (last, resp.) non-insertible S-vertex of $C(v_i, v_{i+1})$. Obviously, $x_i \in C(v_i, v'_{i+1})$ and hence $N(x_i) \cap P_i(u_i, v_i) = \emptyset$.

Remark 1 ([10])**.** For each $w_i \in C(v_i, x_i], 1 \leq i \leq t$, w_i has no neighbor in $\cup_{j=1}^{t} P_i[x_0, v_j)$. In particular, $x_0x_i \notin E(G)$.

Remark 2 ([10])**.** For each $w_i \in C(v_i, x_i], 1 \leq i \leq t$, G contains no path $P[x_0, w_i]$ of length at most 2 such that $P[x_0, w_i) \cap V(C) = \emptyset$, *i.e.*, $N(w_i) \cap N(x_0) \cap H = \emptyset$ and $w_ix_0 \notin E(G)$.

Remark 3 ([10])**.** $C(v_j, y) \cap S \neq \emptyset$ for any $v \in C(v_i, x_i)$ and $y \in N(v) \cap C(v_j, v_{j+1}), i \neq j$.

Lemma 4 ([10])**.** Let $1 \leq i \neq j \leq t$, then for each $w_i \in C(v_i, x_i]$ and each $w_j \in C(v_j, x_j]$

(1) G contains no path $P[w_i, w_j]$ of length at most 2 that is internally disjoint from C. In particular, $w_iw_j \notin E(G)$;

(2) There are no crossing diagonals at w_i and w_j.

Remark 4. Let x'_i, $1 \leq i \leq t$, be any vertex in $C[x_i, v_{i+1}^{-2}]$ such that there is a hamiltonian path $J_i[x'_i, v_{i+1}^{-}]$ of $G[C[x_i, v_{i+1}^{-}]]$ connecting x'_i and v_{i+1}^{-}. Then by using J_i instead of $C[x_i, v_{i+1}^{-}]$ and by the same proof, we can show that x'_i has the same properties as x_i in the above Lemma 4 and Remarks 1, 2, 3.

Remark 5. $X := \{x_0, x'_1, \cdots, x'_t\}$ is an independent set of G and any pair of them do not have a common neighbor in $G - V(C)$.

Now since $t \geq 3$, by Remark 4 and by putting $x_{i_0} = x_0$, we consider an independent set $\{x_{i_0}, x_{i_1}, x_{i_2}, x_{i_3}\}$. Directly, we have

$$(2) \qquad \sum_{h=0}^{3} |N_{G-V(C)}(x_{i_h})| \leq |G - V(C) - \{x_{i_0}\}| = n - |V(C)| - 1.$$

Take any segment $C(v_f, v_{f+1}]$, $1 \leq f \leq t$ (with $v_{t+1} = v_1$) and without loss of generality we assume $H_f = C(v_f^+, v_{f+1}] := c_d c_{d+1} \cdots c_{d+s} \subseteq C(v_3, v_1]$. Then $x_{i_1}, x_{i_2}, x_{i_3}$ are not in H_f. Since they are non-insertible, for every $h \in \{2, 3\}$, $N_{H_f}(x_{i_h}) \cap N_{H_f}(x_{i_h})^+ = \emptyset$. From Lemma 4 (2), we have that for $1 \leq h < k \leq 3, N_{H_f}(x_{i_h}) \cap N_{H_f}(x_{i_k})^+ = \emptyset$. It follows from Corollary A that

(3)

$$\sum_{h=1}^{3} d_{H_f}(x_{i_h})$$
$$\leq \begin{cases} |H_f| + |V(H_f) - \{v_{f+1}\} - \bigcup\limits_{h=1}^{3} N_{H_f}(x_{i_h})| + 1, \text{ if } c_{d+s} \notin \bigcap_{h=1}^{3} N_{H_f}(x_{i_h}) \\ |H_f| + |V(H_f) - \{v_{f+1}\} - \bigcup\limits_{h=1}^{3} N_{H_f}(x_{i_h})| + 2, \text{ if } c_{d+s} \in \bigcap_{h=1}^{3} N_{H_f}(x_{i_h}). \end{cases}$$

Note that by Lemma 4(1), $v_f^+ \notin \bigcup_{h=0}^{3} N(x_{i_h})$. It is easy to see that

$$(4) \qquad |\{v_i : 1 \leq i \leq t, v_i \in \bigcap_{h=1}^{3} N(x_{i_h})\}| + |N(x_{i_0}) \cap \{v_1, v_2, ..., v_t\}|$$
$$\leq |\bigcap_{h=0}^{3} N(x_{i_h})| + t.$$

By using (2), (3) and (4) and by the fact that every vertex not in C is adjacent to at most one of the x_{i_h}'s, we finally get

$$\sum_{h=0}^{3} d(x_{i_h}) \leq n - |V(C)| - 1$$
$$+ \sum_{f=1}^{t} (|H_f| + |V(H_f) - \{v_{f+1}\} - \bigcup_{h=1}^{3} N_{H_f}(x_{i_h})| + 1)$$

$$
\begin{aligned}
&+|\{v_i : 1 \le i \le t, v_i \in \bigcap_{h=1}^{3} N(x_{i_h})\}| \\
&+|N(x_{i_0}) \cap \{v_1, v_2, ..., v_t\}| \\
\le\ & n - |V(C)| - 1 + 2\sum_{f=1}^{t} |H_f| - \sum_{f=1}^{t} |\bigcup_{h=1}^{3} N_{H_f}(x_{i_h})| \\
&+|\bigcap_{h=0}^{3} N(x_{i_h})| + t \\
\le\ & n - |V(C)| - 1 + 2(|V(C) - \{v_1^+, v_2^+, \cdots, v_t^+\}|) \\
&-|\bigcup_{h=1}^{3} N_C(x_{i_h})| + |\bigcap_{h=0}^{3} N(x_{i_h})| + t \\
\le\ & n - 1 + |V(C)| - t - |\bigcup_{h=1}^{3} N_C(x_{i_h})| + |\bigcap_{h=0}^{3} N(x_{i_h})| \\
\le\ & n - 1 + n - 1 - (n - 1 - |V(C)| + t + |\bigcup_{h=1}^{3} N_C(x_{i_h})|) \\
&+|\bigcap_{h=0}^{3} N(x_{i_h})| \\
\le\ & 2n - 2 - |\bigcup_{h=0}^{3} N_C(x_{i_h})| + |\bigcap_{h=0}^{3} N(x_{i_h})|.
\end{aligned}
$$

It follows that the equalities above hold and hence the equality (3) holds for every H_f and $\forall i_1, i_2, i_3$. Moreover we deduce

Remark 6. If $t \ge 4$ or $t = 3$ and either $x_i^+ x_j \in E(G)$ for $i \ne j$ or $\bigcup_{i=1}^{3}(\Lambda_{x_1,x_2,x_3}(C(x_i, v_{i+1}])) \ne \emptyset$, then $G - C = \{x_0\}$, $x_i = v_i^+$ and $y_i = v_{i+1}^-$.

We give a short proof of Remark 6. When $t \ge 4$, pick any $C(x_i, v_{i+1}]$ and we have $i_1, i_2, i_3 \ne i$. Then by the equalities above, one of the i_1, i_2, i_3, say i_1 is adjacent to x_i^+, then we obtain a cycle obtained from $P_i[x_0, v_i)\overline{C}[v_i, x_{i_1})x_{i_1}x_i^+C(x_i^+, v_{i_1})\overline{P}_i[v_{i_1}, x_0]$ by inserting the vertices in $C(v_i, x_i^-] \cup C(v_{i_1}, x_{i_1}^-]$, which contains $V(C) \cup \{x_0\} - \{x_i\}$. By the choice of C and $G - C = \cup_{j=0}^{3} N_{G-C}(x_{i_j})$, we deduce $C(v_i, x_i^-] = \emptyset$ and $N(x_i) \cap (G - C) = \emptyset$. When $t = 3$ and $\bigcup_{i=1}^{3}(\Lambda_{x_1,x_2,x_3}(C(x_i, v_{i+1}])) \ne \emptyset$, we have a similar proof from a cycle containing $V(C) \cup \{x_0\} - \{v\}$ for a vertex $v \in \bigcup_{i=1}^{3}(\Lambda_{x_1,x_2,x_3}(C(x_i, v_{i+1}])) \ne \emptyset$.

We will use the following claim:

Claim.

–(a) For any $1 \leq i < j \leq t$, $|\{x'_i v_j, y_{j-1}x_j\} \cap E(G)| \leq 1$ since otherwise the cycle

$$P_j[x_0, v_j)v_j x'_i Q(x'_i, y_i)C[y_i, y_{j-1})y_{j-1}x_j C(x_j, v_i)\overline{P}_i[v_i, x_0]$$

(where $Q(x'_i, y_i)$ is a hamiltonian path of $C[x_i, y_i]$ between x'_i and y_i) contains more S vertices than C, a contradiction.

–(b) For any $1 \leq i \leq t$, there don't have two consecutive vertices w', $w"$ $\notin C[v_i, v_{i+1}]$ such that $w'x'_i, w"y_i \in E(G)$ since otherwise the cycle

$$P_{i+1}[x_0, v_{i+1})C[v_{i+1}, w')w'x'_i Q(x'_i, y_i)y_i w" C[w", v_i)\overline{P}_i[v_i, x_0]$$

contains more S vertices than C, a contradiction.

We also have

Remark 7 (Claims 2 and 3 in [10])**.** $X \bigcup (\bigcup_{j=1}^{t} \Lambda_{x_{i_1}, x_{i_2}, x_{i_3}}(C(x_j, v_{j+1}])$ is independent.

We first discuss the case that $t \geq 4$.

Pick up any segment, say $H_1 = C[x_1^+, x_2^-] = r_1 r_2 \cdots r_p$. By Corollary C and since x_2,x_{t-1} and x_t are not insertible, then either $|H_1|$ is odd and $\{r_1, r_3, \cdots, r_{p-4}, r_{p-2}\} \subseteq N_H(x_t) \cap N_H(x_{t-1}) \cap N_H(x_2)$ and v_p is adjacent to at least two of the x_t, x_{t-1}, x_2's, or p is even, $N_H(x_t) = \{r_{p-1}, r_{p-3}, r_{p-5}, \cdots, r_3, r_1\}$ and either $N_H(x_{t-1}) = N_H(x_2) = \{r_p, r_{p-3}, r_{p-5}, \cdots, r_3, r_1\}$ or $N_H(x_{t-1}) = N_H(x_2) = \{r_{p-1}, r_{p-3}, r_{p-5}, \cdots, r_3, r_1\}$ or $N_H(x_{t-1}) = \{r_{p-1}, r_{p-3}, r_{p-5}, \cdots, r_3, r_1\}$ and $N_H(x_2) = \{r_p, r_{p-3}, r_{p-5}, \cdots, r_3, r_1\}$.

Now we take x_1, x_{t-1}, x_2 and from the positions of the vertices adjacent to x_{t-1} or x_2 we deduce that when $|H_1|$ is odd, $\{r_1, r_3, \cdots, r_{p-4}, r_{p-2}\} \subseteq N_H(x_1)$ and $|H_1|$ is even, $N_H(x_1) = \{r_{p-1}, r_{p-3}, r_{p-5}, \cdots, r_3, r_1\}$.

We will show that if $|H_1|$ is even, then $|H_1| = 2$. Suppose to the contrary that $|H_1| \geq 4$. Then $v_2^- x_1 \in E(G)$ and by considering $(1, t, t-1)$ and $(t, t-1, 2)$ respectively, we obtain either $x_{t-1}v_2 \in E(G)$ and $v_2 x_1^+ \in E(G)$ or $x_{t-1}v_2^-, x_t v_2^- \in E(G)$ and $v_2 x_1^+, v_{t-1}x_1^+, v_t x_1^+ \in E(G)$. When $x_{t-1}v_2 \in E(G)$ and $v_2 x_1^+ \in E(G)$ we get a contradiction from Claim (b). When $x_{t-1}v_2^-, x_t v_2^- \in E(G)$ and $v_2 x_1^+, v_{t-1}x_1^+, v_t x_1^+ \in E(G)$, we consider y_i's instead of x_i's and since H_1 is even and $|H_1| \geq 4$, we deduce that $y_{t-1}x_1 \in E(G)$. We have a contradiction from Claim (a).

Suppose $|H_1| = 2$. By Corollary C and by considering $(t, t-1, 2)$, we have $x_t y_1 \in E(G)$ and either $x_{t-1}v_2 \in E(G)$ or $x_{t-1}v_2^- \in E(G)$. Similarly, we have $y_2 x_1 \in E(G)$ and either $y_{t-1}v_1 \in E(G)$ or $y_{t-1}x_1 \in E(G)$. But if $x_{t-1}v_2^- \in E(G)$, by Claim (a) we have $y_{t-1}x_1 \notin E(G)$ which implies $y_{t-1}v_1 \in E(G)$. So without loss of generality we assume $x_{t-1}v_2 \in E(G)$. By Claim (a), we have $x_2 y_{t-1} \notin E(G)$. By considering x_t, x_1, x_2 and by using Corollary C, we deduce that $v_t x_1 \in E(G)$. Now the cycle

$$P_1[x_0, v_1)\overline{C}[v_1, x_t)x_t y_1 x_1 v_t \overline{C}[v_t, x_{t-1})x_{t-1}v_2 C(v_2, v_{t-1})\overline{P}_{t-1}[v_{t-1}, x_0],$$

gives a contradiction.

We assume now that $|H_i|$ is odd for $\forall 1 \le i \le t$. Let $H_f = C[x_f^+, v_{f+1}] = a_1 a_3 \cdots a_p$, with p odd. Put $O_f = \{a_1, a_2, \cdots, a_{p-4}, a_{p-2}\}$. By Corollary C, for any $i_1, i_2, i_3 \neq f$, $O_f \subseteq N(x_{i_1}) \cap N(x_{i_2}) \cap N(x_{i_3})$. Then by considering x_f, x_{i_1}, x_{i_2}, and by using Corollary C, we deduce $O_f \subseteq N(x_f)$.

Let $O = \cup_{i=1}^t O_i$. We have $O \subseteq N(x_i)$ for $\forall i$ and by the definition $O^+ \subseteq \cup_{i=1}^t \Lambda_{x_1,x_2,x_3}(C(x_i, v_{i+1}])$. By Remark 7, $\{x_0, x_1, \cdots, x_t\} \cup O^+$ is an independent set of $\frac{n+1}{2}$ vertices and we obtain that $G \in \mathcal{A}_S$ and $\{x_0, x_1, \cdots, x_t\} \cup O^+ \subseteq S$.

Suppose now that $t = 3$.

Suppose that $|\bigcup_{i=1}^3 (\Lambda_{x_1,x_2,x_3}(C(x_i, v_{i+1}]))| \ge 2$. Without loss of generality we have w_1 is the first vertex (direction of the cycle C) in $\Lambda_{x_1,x_2,x_3}(C(x_1, v_2])$ and $w_2 \in \bigcup_{i=1}^3 (\Lambda_{x_1,\underline{x_2},x_3}(C(x_i, v_{i+1}]))$. Then we have $C(x_1, w_1^-] \subseteq N(x_1)$. Since there is a path $\overline{C}[w_1, x_1)x_1 w_1^+ C(w_1^+, v_2]$, we can use w_1 instead of x_1 and consider x_0, w_1, x_2, x_3. It can be obtained that either $C(x_1, w_1^{-2}] \subseteq N(w_1)$ or it is empty. Let $P = C[x_1^+, v_2] = z_1 z_2 \cdots z_p$ with $w_1 = z_d$. Suppose now that there is a vertex $z_i \in \Lambda_{x_1,x_2,x_3}(C[x_1, x_2^-])$. A path $\overline{C}(z_d^-, z_{i+1})z_{i+1}z_1 C(z_1, z_i)z_i z_d C(z_d, z_p]$ and the edge $z_d x_2$ contradict Remark 5. So $\Lambda_{x_1,x_2,x_3}(C[x_1, x_2^-]) = \emptyset$ and hence by Remark 6, $d(x_1) = d(w_1) = d(w_2) = d(x_0) = 3$. It follows that

$$\begin{aligned} 2n - 2 &\le d(x_1) + d(w_1) + d(w_2) + d(x_0) \\ &\quad + |N(x_1) \cup N(w_1) \cup N(w_2) \cup N(x_0)| \\ &= 12 + (n-4), \end{aligned}$$

and hence $n \le 10$. But we have 11 vertices $x_0, v_1, v_2, v_3, x_1, x_2, x_3, w_1, w_1^-, w_2$ and w_2^-, a contradiction.

Assume that $|\bigcup_{i=1}^3 (\Lambda_{x_1,x_2,x_3}(C(x_i, v_{i+1}]))| = 1$ and without loss of generality, $\exists w_1 \in \Lambda_{x_1,x_2,x_3}(C(x_1, v_2])$ (so $w_1 = x_1^{+2}$). By the same argument as above, we have $d(x_1) = d(w_1) = d(x_0) = 3$. And without loss of generality, we assume $|C(x_2, v_3]| \le |C(x_3, v_1]|$. Then by Corollary C, $|N(x_2) \cap$

$C(x_3, v_1]| \leq 1$ and $|N(x_2) \cap C[x_1, v_2]| \leq \frac{|C[x_1,v_2]|}{2}$. These gives that $d(x_2) \leq |C[x_2, v_3]| - 1 + \frac{|C[x_1,v_2]|}{2} + 1 \leq \frac{|C[x_2,v_3]|+|C[x_2,v_1]|+|C[x_1,v_2]|}{2} = \frac{n-1}{2}$. We see that $v_2 \in N(x_0) \cap N(w_1) \cap N(x_1) \cap N(x_2)$. It follows that

$$\begin{aligned} 2n - 2 \leq d(x_0) &+ d(w_1) + d(x_1) + d(x_2) \\ &+ |N(x_0) \cup N(w_1) \cup N(x_1) \cup N(x_2)| \\ &- |N(x_0) \cap N(w_1) \cap N(x_1) \cap N(x_2)| \\ = 9 &+ \frac{n-1}{2} + (n-5) - 1 \end{aligned}$$

and $n \leq 9$, which implies $v_2 = x_1^+$, $v_3 = x_2^+$ and then $w_1 v_3 \in E(G)$. So $v_3 \in N(x_0) \cap N(w_1) \cap N(x_1) \cap N(x_2)$. Now

$$\begin{aligned} 2n - 2 \leq d(x_0) &+ d(w_1) + d(x_1) + d(x_2) \\ &+ |N(x_0) \cup N(w_1) \cup N(x_1) \cup N(x_2)| \\ &- |N(x_0) \cap N(w_1) \cap N(x_1) \cap N(x_2)| \\ = 9 &+ \frac{n-1}{2} + (n-5) - 2 \end{aligned}$$

and $n \leq 7$ which is not possible.

So we may suppose that $\bigcup_{i=1}^{3} (\Lambda_{x_1,x_2,x_3}(C(x_i, v_{i+1}])) = \emptyset$. It follows from Corollaries B and C, that $C(x_i, v_i^-] \subset N(x_i)$ for $i = 1, 2, 3$. Similarly we have $C[x_i, y_i^-] \subset N(y_i)$ for $i = 1, 2, 3$.

Suppose that $C[x_1, y_1]$ contains at least three vertices. Since G is 3-connected, it is easy to prove that either there is a vertex $w_1 \in C[x_1^+, y_1^-]$ which is adjacent to a vertex w_2 in $C[x_2, y_3]$ or one of y_1 and x_1, say y_1 is adjacent to a vertex w_2 in $C[x_2, y_3]$ and v_2 is adjacent to one vertex $z \in C[x_1, y_1^-]$. By Remark 4, $w_2 \notin C[x_2, y_2] \cup C[x_3, y_3]$ and hence $w_2 = v_3$. If $y_1 v_3 \in E(G)$, since y_1 is not insertible, we have $y_1 x_3 \notin E(G)$. If $y_1 x_3, v_2 z \in E(G)$, by Claim (a), we have a contradiction from a hamiltonian path $\overline{C}[y_1, z^+) z^+ x_1 C(x_1, z]$ of $C[x_1, y_1]$. So we get $y_1 x_3 \notin E(G)$. Similarly from $C[x_1, z^-) z^- y_1 \overline{C}(y_1, z]$ and by Claim (a), we have $y_2 x_1 \notin E(G)$.

Suppose $G \notin \mathcal{B}_{\mathcal{S}}$. Since $G - \{v_1, v_2, v_3\}$ is not a union of the 4 connected components $G(C[x_1, y_1]), G(C[x_2, y_2]), G(C[x_3, y_3])$ and $G - C$, we deduce $|C[x_2, y_2]| \geq 2$, $|C[x_3, y_3]| \geq 2$ and $x_2 y_3 \in E(G)$. By Claim (a), no vertex in $C[x_3, y_3^-]$ is adjacent to v_2. If v_1 is adjacent to a vertex $z \in C[x_3, y_3^-]$, then a cycle

$$P_2[x_0, v_2)\overline{C}[v_2, v_1) v_1 z \overline{C}(z, x_3) x_3 z^+ C(z^+, y_3) y_3 x_2 C(x_2, v_3) \overline{P}_3[v_3, x_0]$$

gives a contradiction. So we have shown that there is no edge between

$C[x_3, y_3^-]$ and $C[v_1, y_2]$ and therefore $\{v_3, y_3\}$ is a cutset, a contradiction. So we may assume that for any $1 \leq i \leq 3$, $C[x_i, y_i]$ contains at most two vertices.

Suppose that $G \notin \mathcal{B}_{\mathcal{S}}$. It is easy to see, without loss of generality, that $|C[x_1, y_1]| = |C[x_2, y_2]| = 2$ and $x_1y_2 \in E(G)$. By a similar proof as above, we may obtain that x_2 is not adjacent to x_1, y_1, v_1, x_3, v_3. Since $\{v_2, y_2\}$ is not a cutset, it follows that $x_2y_3 \in E(G)$. By the same argument, we get $x_3y_1 \in E(G)$. Now G is the Petersen graph.

This completes the proof of Theorem 4. □

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Received 30 January 2015