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Narrowing down the gap on cycle-star Ramsey numbers*

Yanbo Zhang, Hajo Broersma, and Yaojun Chen

Given two graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer N such that, for any graph G of order N, either G_1 is a subgraph of G, or G_2 is a subgraph of the complement of G. Let C_m denote a cycle of order m, $K_{1,n}$ a star of order $n+1$ and W_n a wheel of order $n+1$. Already back in the 1970s, exact values of the Ramsey numbers $R(C_m, K_{1,n})$ have been determined for all $m \geq 2n$ and for all odd $m \leq 2n-1$, but for even $m < 2n$ not many exact values are known. In this paper, we use a result of Bondy on pancyclicity to fill a considerable part of this gap. We show that $R(C_m, K_{1,n}) = 2n$ for even m with $n < m < 2n$, and that $R(C_m, K_{1,n}) = 2m-1$ for even m with $3n/4+1 \leq m \leq n$. In addition, we determine another six formerly unknown exact values of Ramsey numbers, namely $R(C_6, K_{1,n})$ for $7 \leq n \leq 11$, and $R(C_6, W_9)$.

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1. Introduction

In this note we deal with finite simple graphs only. We refer to the textbook of Bondy and Murty [4] for any undefined terminology and notation. For convenience, we repeat some of the key definitions and notation.

A complete bipartite graph with bipartition classes of cardinalities m and n is denoted by $K_{m,n}$. Let C_m be a cycle of order m, $K_{1,n}$ a star of order $n+1$ and W_n a wheel of order $n+1$. We use $G_1 \cup G_2$ to denote the disjoint union of two vertex-disjoint graphs G_1 and G_2, and we use kK_n to denote the disjoint union of $k \geq 2$ copies of K_n. The minimum degree, the maximum degree, the length of a shortest cycle, the length of a longest cycle, and the number of components of G are denoted by $\delta(G)$, $\Delta(G)$, $g(G)$, $c(G)$, and $\omega(G)$, respectively. Given two graphs G_1 and G_2, the Ramsey number

*We dedicate this paper to our all-time star in cycle theory Adrian Bondy on the occasion of his 70th birthday.

$R(G_1, G_2)$ is the smallest integer N such that, for any graph G of order N, either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ is the complement of G.

The concept of pancyclicity for undirected graphs was introduced and studied by Bondy [2] in the early 1970s. A graph G is pancyclic if it contains cycles of every length between 3 and $|V(G)|$. Bondy established several sufficient conditions for a graph to be pancyclic. A typical degree condition is given below.

Theorem 1 (Bondy [2]). *If G is a graph with $\delta(G) \geq |V(G)|/2$, then G is pancyclic, or $G = K_{r,r}$ with $r = |V(G)|/2$.*

The above result is interesting by itself, but also played a key role in determining exact values of cycle-star Ramsey numbers. We will use it as one of the ingredients to deduce new exact values of cycle-star Ramsey numbers.

From the early 1970s, cycles and stars have been well-studied in graph Ramsey theory. The following well-known theorem on cycle-star Ramsey numbers is due to Lawrence [7] and dates back to 1973; a proof of this result can also be found in [10].

Theorem 2 (Lawrence [7]).

$$R(C_m, K_{1,n}) = \begin{cases} 2n+1 & \text{for odd } m \leq 2n-1, \\ m & \text{for } m \geq 2n. \end{cases}$$

In fact, $K_{n,n}$ and K_{m-1} establish the lower bounds on $R(C_m, K_{1,n})$ for odd $m \leq 2n-1$ and $m \geq 2n$, respectively. For the upper bounds, both cases may be viewed as a direct corollary of Theorem 1.

For even $m < 2n$, not many results on exact values of these Ramsey numbers are known. In fact, all generic results we know of deal with the case that $m = 4$. Parsons [9] established the following theorem.

Theorem 3 (Parsons [9]). *$R(K_{1,n}, C_4) \leq n + \lfloor\sqrt{n-1}\rfloor + 2$ for all $n \geq 2$, and if $n = q^2 + 1$ and $q \geq 1$, then $R(K_{1,n}, C_4) \leq n + \lfloor\sqrt{n-1}\rfloor + 1$. If q is a prime power, then $R(K_{1,q^2}, C_4) = q^2 + q + 1$ and $R(K_{1,q^2+1}, C_4) = q^2 + q + 2$.*

In the majority of the other cases, to the best of our knowledge exact values of $R(C_4, K_{1,n})$ are still unknown.

In this paper, we study the Ramsey numbers $R(C_m, K_{1,n})$ for values of m that are even and not too small relative to n. In particular, we prove the following theorem in Section 3.

Theorem 4.

$$R(C_m, K_{1,n}) = \begin{cases} 2n & \text{for even } m \text{ with } n < m \le 2n, \\ 2m-1 & \text{for even } m \text{ with } 3n/4+1 \le m \le n. \end{cases}$$

Our techniques cannot be used to obtain exact values of $R(C_m, K_{1,n})$ for even m below $3n/4 + 1$, but we can give a lower bound on $R(C_m, K_{1,n})$ for even m in the interval $\lfloor n/2 \rfloor + 2 \le m \le 3(n+1)/4$. This bound is based on the following graphs.

Let $G_1 = G_2 = K_{\lceil n/2 \rceil}$, $G_3 = K_{\lfloor n/2 \rfloor + 1}$, and $v_i \in V(G_i)$ for $i = 1, 2, 3$. Consider the graph G obtained from $G_1 \cup G_2 \cup G_3$ by identifying v_1, v_2, v_3 (merging them into one vertex, while keeping the remaining parts of the graphs G_1, G_2, G_3 mutually disjoint). It is straightforward to check that G is a graph of order $\lceil 3n/2 \rceil - 1$, that G contains no cycle of length $m \ge \lfloor n/2 \rfloor + 2$, and that $\delta(G) = \lceil n/2 \rceil - 1$. This implies that $\Delta(\overline{G}) = n - 1$, hence that $\overline{G}$ contains no $K_{1,n}$. Thus, for $\lfloor n/2 \rfloor + 2 \le m \le 3(n+1)/4$, $R(C_m, K_{1,n}) \ge \lceil 3n/2 \rceil$. In fact, we expect that equality holds in the latter inequality. This motivates the following conjecture.

Conjecture 1. *$R(C_m, K_{1,n}) = \lceil 3n/2 \rceil$ for even m with $\lfloor n/2 \rfloor + 2 \le m \le 3(n+1)/4$.*

Note that, for proving the statement in the above conjecture, by the above examples it suffices to show $R(C_m, K_{1,n}) \le \lceil 3n/2 \rceil$ for these values of m. Since $R(C_4, K_{1,5}) = 8$ by Parsons [9], Conjecture 1 is true for $m = 4$ (and $n = 5$; this is the only value of n for which $m = 4$ lies in the specified interval).

We also confirm that Conjecture 1 holds for $m = 6$. From $\lfloor n/2 \rfloor + 2 \le 6 \le 3(n+1)/4$, it follows that $7 \le n \le 9$. We prove the following results in Sections 4 and 5.

Theorem 5. *$R(C_6, K_{1,n}) = n + 4$ for $n = 7, 8$.*

Since $\lceil 3n/2 \rceil = 11, 12$ for $n = 7, 8$, respectively, the above result shows that Conjecture 1 holds for $m = 6$ and $n = 7, 8$.

Theorem 6. *$R(C_6, K_{1,n}) = n + 5$ for $n = 9, 10, 11$.*

Since $\lceil 3n/2 \rceil = 14$ for $n = 9$, the above result shows that Conjecture 1 holds for $m = 6$ and $n = 9$. From the above observations, we deduce that Conjecture 1 holds for $m = 6$.

We can summarize the known exact values of $R(C_6, K_{1,n})$ as follows. The values for $4 \le n \le 11$ are obtained from the results in this note.

Table 1: Exact values of $R(C_6, K_{1,n})$ for $1 \leq n \leq 11$

n	1	2	3	4	5	6	7	8	9	10	11
$R(C_6, K_{1,n})$	6	6	6	8	10	11	11	12	14	15	16

Luo et al. [8] reported that they calculated 11 exact values of the Ramsey numbers $R(C_m, W_n)$ by using an efficient algorithm which they called the one-vertex extension method. The 11 values include $R(C_6, W_n)$ for $n = 6, 7, 8$. Together with the fact that $R(C_6, C_n) = \max\{n + 2, 11\}$ for odd n [11], we can show that $R(C_6, W_n) = 16$ for $n = 3, 5, 7, 9$ as an immediate corollary of Theorem 6. In particular, this implies that $R(C_6, W_9) = 16$, and establishes a new exact value of $R(C_m, W_n)$ to the best of our knowledge.

Corollary 1. $R(C_6, W_n) = 16$ *for* $n = 3, 5, 7, 9$.

Proof. Since $3K_5$ contains no C_6 and its complement contains no W_n for odd n, we have $R(C_6, W_n) \geq 16$ for $n = 3, 5, 7, 9$. Let G be a graph of order 16 and suppose that G contains no C_6. We will show that $\overline{G}$ contains W_n. By Theorem 6, $\overline{G}$ contains $K_{1,11}$. In particular, this implies there exists a vertex v such that $d(v) \geq 11$ in $\overline{G}$. Since $G[N_{\overline{G}}(v)]$ contains no C_6, and $R(C_6, C_n) = \max\{n+2, 11\}$ for odd n, it follows that $\overline{G}[N_{\overline{G}}(v)]$ contains C_n for $n = 3, 5, 7, 9$. These cycles together with v form W_ns in $\overline{G}$ for $n = 3, 5, 7, 9$. This completes the proof. □

Recall that to prove Conjecture 1 it suffices to show $R(C_m, K_{1,n}) \leq \lceil 3n/2 \rceil$ for the values of m stated in Conjecture 1. Notice that to prove $R(C_m, K_{1,n}) \leq \lceil 3n/2 \rceil$ one actually has to show that any graph G with $|V(G)| = \lceil 3n/2 \rceil$ and $\delta(G) \geq \lceil n/2 \rceil$ contains cycles of every even length between $\lfloor n/2 \rfloor + 2$ and $3(n + 1)/4$. Allen [1] proved the following extension of Theorem 1: there exists a positive integer n_0 such that any graph G with $|V(G)| = n \geq n_0$ and $\delta(G) \geq n/3$ contains cycles of every even length between 4 and $\lceil n/2 \rceil$. It is not difficult to check that this implies that there exists a positive integer m_0, such that Conjecture 1 holds for all $m \geq m_0$.

2. Preliminaries

Apart from Theorem 1, we need the following auxiliary results for our proofs of Theorem 4, Theorem 5 and Theorem 6.

Lemma 1 (Brandt et al. [5]). *Let G be a nonbipartite graph with $\delta(G) \geq (|V(G)| + 2)/3$. Then G contains cycles of every length between $g(G)$ and $c(G)$, and $g(G) = 3$ or 4.*

Corollary 2. *Let G be a graph with $\delta(G) \geq (|V(G)|+2)/3$. Then G contains cycles of every even length between 4 and $c(G)$.*

Proof. If G is a nonbipartite graph, the result follows directly from Lemma 1. If G is a bipartite graph, let X and Y denote the partition classes of G, and consider a vertex $x \in X$. Since the conditions imply that $\delta(G) \geq 2$, x has at least two neighbors x_1 and x_2, and obviously $x_1, x_2 \in Y$. Now construct a new graph G' from G by adding the edge x_1x_2. Then, clearly G' is not bipartite. Since $\delta(G') \geq (|V(G)|+2)/3$ and $|V(G')| = |V(G)|$, by Lemma 1, G' contains cycles of every length between 4 and $c(G')$. Since $c(G') \geq c(G)$, it remains to prove that every even cycle in G' is also a cycle in G. If not, then G' has an even cycle C containing x_1x_2 as an edge, say, $C = x_1x_2x_3 \ldots x_{2k}x_1$. But then $x_2 \in Y$, $x_3 \in X, \ldots, x_{2k-1} \in X, x_{2k} \in Y$. Since $x_1 \in Y$, this implies that $G'[Y]$ contains $x_{2k}x_1$ as an edge, contradicting the fact that $G'[Y]$ contains exactly one edge x_1x_2. We conclude that every even cycle in G' is also a cycle in G. Therefore, G contains cycles of every even length between 4 and $c(G)$. □

Lemma 2 ([6])**.** *For a 2-connected graph G, $c(G) \geq \min\{2\delta(G), |V(G)|\}$.*

The closure of a graph G is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least $|V(G)|$ until no such pair remains. This closure operation was introduced by Bondy and Chvátal [3]. They showed that the closure is unique and that it preserves the existence of Hamilton cycles: a graph is hamiltonian if and only if its closure is hamiltonian. One of the consequences of this nice result is expressed in the following lemma.

Lemma 3 ([3])**.** *Let G be a simple graph on at least three vertices whose closure is complete. Then G is hamiltonian.*

3. Proof of Theorem 4

We prove the two statements of Theorem 4 separately.

For the purpose of proving the first statement of the theorem, let F be the graph obtained from $2K_n$ by identifying precisely one vertex of each K_n. It is easy to check that $|V(F)| = 2n-1$, $\delta(F) = n-1$, F contains no C_m for $m > n$, and $\overline{F}$ contains no $K_{1,n}$. Thus, $R(C_m, K_{1,n}) \geq 2n$ for $n < m \leq 2n$.

It is sufficient for proving the first statement to prove that $R(C_m, K_{1,n}) \leq 2n$ for even m with $n < m \leq 2n$. Let G be a graph of order $2n$. If $\overline{G}$ contains no $K_{1,n}$, then $\Delta(\overline{G}) \leq n-1$, implying that $\delta(G) \geq n$. By Theorem 1, G

contains C_m for even m and $n < m \leq 2n$. This completes the proof of the first statement of Theorem 4.

We continue with the proof of the second statement. First observe that for even m and $3n/4 + 1 \leq m \leq n$, the inequalities imply that $n \geq 4$, and hence that $m \geq 4$. It is obvious that for these values of m and n, $2K_{m-1}$ contains no C_m and its complement contains no $K_{1,n}$. Thus, $R(C_m, K_{1,n}) \geq 2m - 1$.

To prove that $R(C_m, K_{1,n}) \leq 2m - 1$, let G be a graph of order $2m - 1$. Suppose to the contrary that neither G contains a C_m nor $\overline{G}$ contains a $K_{1,n}$. Then $\Delta(\overline{G}) \leq n - 1$, hence $\delta(G) \geq 2m - n - 1$. Using $m \geq 3n/4 + 1$, we get that $\delta(G) \geq (|V(G)| + 2)/3$. By Corollary 2, G contains cycles of every even length between 4 and $c(G)$. It remains to prove that $c(G) \geq m$. We complete the proof by proving three claims.

Claim 1. *Suppose G_1 is a graph obtained from G by deleting at most two vertices. Then $\omega(G_1) \leq 2$.*

Proof. If $\omega(G_1) \geq 3$, let G_2 be the smallest component of G_1. Then $\delta(G_1) \leq \delta(G_2) \leq |V(G_2)| - 1 \leq |V(G_1)|/3 - 1$. Thus, $\Delta(\overline{G_1}) \geq 2|V(G_1)|/3 \geq 2(2m - 3)/3 \geq n - 2/3$, that is, $\Delta(\overline{G_1}) \geq n$. Since $\overline{G_1}$ is a subgraph of $\overline{G}$, then $\Delta(\overline{G}) \geq n$, which contradicts the fact that $\overline{G}$ contains no $K_{1,n}$. This proves our claim that $\omega(G_1) \leq 2$. □

Claim 2. *Suppose H is a graph obtained from G by deleting at most one vertex. If $\omega(H) = 2$, then each component of H is a 2-connected (sub)graph.*

Proof. Let H_1, H_2 be the two components of H. Then $\delta(H_i) \geq \delta(H) \geq \delta(G) - 1 \geq 2m - n - 2$ for $i = 1, 2$. Since $m \geq 3n/4 + 1$ and $n \geq 4$, this implies that $|V(H_i)| \geq \delta(H_i) + 1 \geq 3$ for $i = 1, 2$. If H_1 is not 2-connected, then there exists a vertex u such that $H_1 - u$ is disconnected. Hence, $H - u$ is a graph obtained from G by deleting at most two vertices, and $\omega(H - u) \geq 3$, contradicting Claim 1. We conclude that H_1 is 2-connected. For the same reason, H_2 is 2-connected, proving our claim. □

Claim 3. *$c(G) \geq m$.*

Proof. Recall that G is a graph of order $2m - 1$ and that $\delta(G) \geq (|V(G)| + 2)/3$. If G is 2-connected, by Lemma 2, $c(G) \geq \min\{2\delta(G), |V(G)|\} \geq m$.

Next assume that G is not 2-connected. Then there exists a vertex $v \in V(G)$ such that $G - v$ is disconnected. By Claim 1, $\omega(G - v) = 2$. Let H_1, H_2 be the two components of $G - v$. Then $\delta(H_i) \geq \delta(G) - 1 \geq 2m - n - 2$. By Claim 2, H_i is 2-connected for $i = 1, 2$. Assuming that $|V(H_1)| \geq |V(H_2)|$, we get that $|V(H_1)| \geq m - 1$. If $|V(H_1)| \geq m$, then, since $3n/4 + 1 \leq m \leq n$, using Lemma 2, we obtain that $c(G) \geq c(H_1) \geq \min\{2\delta(H_1), |V(H_1)|\} \geq \min\{2(2m-n-2), m\} \geq m$.

Finally, assume that $|V(H_1)| = m-1$. Then $|V(H_2)| = m-1$. Since $d_G(v) \geq \delta(G) \geq 2m-n-1 \geq 3$, then either $d_{H_1}(v) \geq 2$ or $d_{H_2}(v) \geq 2$. Without loss of generality, assume that $d_{H_1}(v) \geq 2$. Let $H_3 = G[V(H_1) \cup \{v\}]$. Since $\delta(H_1) \geq 2m-n-2 > |V(H_1)|/2$, H_1 is Hamilton-connected, implying H_3 contains a C_m.

This completes the proof of Claim 3 and of Theorem 4. □

4. Proof of Theorem 5

By the construction preceding the statement of Conjecture 1, we know that $R(C_6, K_{1,n}) \geq n+4$ for $n = 7, 8$. Let G be a graph of order $n+4$. Suppose to the contrary that G contains no C_6 and $\overline{G}$ contains no $K_{1,n}$. Then $\delta(G) \geq 4$. By Theorem 3, G contains C_4. Let C be a longest cycle with $|V(C)| \leq 6$ in G. Then $4 \leq |V(C)| \leq 5$. We distinguish two cases and reach contradictions in all subcases.

Case 1. $|V(C)| = 4$.

First suppose that G contains $K_{2,3}$, and assume that both v_1 and v_2 are adjacent to each vertex of v_3, v_4, v_5. We see that $v_3v_4 \notin E(G)$; otherwise $v_1v_3v_4v_2v_5v_1$ is a C_5, a contradiction. For the same reason, $v_4v_5, v_3v_5 \notin E(G)$. Since $\delta(G) \geq 4$, for $i = 3, 4, 5$, each v_i has another neighbor in $V(G) \setminus \{v_1, v_2, v_3, v_4, v_5\}$, denoted by u_i. Since G contains no C_6, u_3, u_4, u_5 are three distinct vertices. Let $V_1 = \{v_i, u_j \mid 1 \leq i \leq 5, 3 \leq j \leq 5\}$. Then for $i = 3, 4, 5$, u_i is nonadjacent to $V_1 \setminus \{v_i\}$; otherwise G contains C_5 or C_6, a contradiction. Since $\delta(G) \geq 4$, each of u_3, u_4, u_5 has at least three neighbors in $V(G) \setminus V_1$. Since $11 \leq |V(G)| \leq 12$, we have $3 \leq |V(G) \setminus V_1| \leq 4$, and it follows that u_3 and u_4 have a common neighbor in $V(G) \setminus V_1$, say w. Now $wu_3v_3v_1v_4u_4w$ is a C_6, a contradiction. Thus,

$$G \text{ contains no } K_{2,3}. \tag{1}$$

Let $C = v_1v_2v_3v_4v_1$. Since $\delta(G) \geq 4$, each v_i has another neighbor in $V(G) \setminus \{v_1, v_2, v_3, v_4\}$, denoted by u_i, where $1 \leq i \leq 4$. Observe that u_1, u_2, u_3, u_4 are four distinct vertices; otherwise G contains C_5 or $K_{2,3}$. Let $V_1 = \{v_i, u_i \mid 1 \leq i \leq 4\}$. Then for $1 \leq i \leq 4$, u_i is nonadjacent to $V_1 \setminus \{v_i\}$; otherwise G contains C_5 or C_6 or $K_{2,3}$, a contradiction. Since $\delta(G) \geq 4$, each u_i has at least three neighbors in $V(G) \setminus V_1$. Since $11 \leq |V(G)| \leq 12$, we have $3 \leq |V(G) \setminus V_1| \leq 4$, and it follows that u_1 and u_2 have a common neighbor in $V(G) \setminus V_1$, say w. Now $wu_1v_1v_2u_2w$ is a C_5, a contradiction.

Case 2. $|V(C)| = 5$.

First suppose that G contains a subgraph H isomorphic to K_5. Let $G' = G - V(H)$. Since $11 \leq |V(G)| \leq 12$, we have $6 \leq |V(G')| \leq 7$. For any $v \in V(G')$, v has at most one neighbor in H; otherwise G contains C_6. Since $\delta(G) \geq 4$, we have $\delta(G') \geq 3$. Assume that G' is not 2-connected. Then there exists a vertex u in G' such that $G' - u$ is disconnected, $5 \leq |V(G' - u)| \leq 6$ and $\delta(G' - u) \geq 2$. Thus, the subgraph $G' - u$ is a disjoint union of two triangles, denoted by $v_1v_2v_3v_1$ and $v_4v_5v_6v_4$. Since $\delta(G) \geq 4$, both v_1 and v_6 are adjacent to a vertex of H. By distinguishing and analyzing the cases that v_1, v_6 are adjacent to the same vertex of H or not, we can always find a C_6, a contradiction. We conclude that G' is 2-connected. By Lemma 2, $c(G') \geq 2\delta(G') \geq 6$. By Corollary 2, G' contains a C_6, a contradiction. Therefore,

(2) $$G \text{ contains no } K_5.$$

Next suppose that G contains a $C_5 + e$, and let $v_1v_2v_3v_4v_5v_1$ be a C_5 in G with at least one chord. Since G contains $C_5 + e$ but no K_5, we can always find some i such that $v_iv_{i+2} \in E(G)$ and $v_{i+1}v_{i+3} \notin E(G)$, where the indices are taken modulo 5. Without loss of generality, we may assume that $v_1v_3 \in E(G)$ and $v_2v_4 \notin E(G)$. Since $\delta(G) \geq 4$, both v_2 and v_4 have another neighbor in $V(G) \setminus \{v_1, v_2, v_3, v_4, v_5\}$, denoted by u_2 and u_4, respectively. Observe that u_2, u_4 are distinct; otherwise $u_2v_2v_3v_1v_5v_4u_2$ is a C_6. For the same reason, u_2 is nonadjacent to $\{v_1, v_3, v_4, v_5, u_4\}$ and u_4 is nonadjacent to $\{v_2, v_3, v_5, u_2\}$. Let $V_1 = \{v_1, v_2, \ldots, v_5, u_2, u_4\}$. Since $\delta(G) \geq 4$, u_2 has at least three neighbors in $V(G) \setminus V_1$, three of which are denoted by $\{w_1, w_2, w_3\}$; u_4 has at least two neighbors in $V(G) \setminus V_1$, two of which are denoted by $\{w_4, w_5\}$. We deduce that w_1, w_2, w_3, w_4, w_5 are pairwise distinct; otherwise, for some $w \in N(u_2) \cap N(u_4)$, $wu_2v_2v_3v_4u_4w$ is a C_6. Since $|V(G)| \leq 12$, then $V(G) = \{v_i, w_i, u_j \mid 1 \leq i \leq 5, j = 2, 4\}$. Since $\delta(G) \geq 4$ and G contains no C_6, u_4 has to be adjacent to v_1. Moreover, w_4 is nonadjacent to $\{w_1, w_2, w_3\}$; otherwise, say that $w_1w_4 \in E(G)$, then $w_1u_2v_2v_1u_4w_4w_1$ is a C_6. For the same reason, w_4 is nonadjacent to $\{v_1, v_2, v_3, v_5, u_2\}$. Thus, $d(w_4) \leq 3$, a contradiction. Hence,

(3) $$G \text{ contains no } C_5 + e.$$

Now let $C = v_1v_2v_3v_4v_5v_1$. Since G contains no $C_5 + e$ and $\delta(G) \geq 4$, each v_i has at least two neighbors in $V(G) \setminus V(C)$. Because $|V(G) \setminus V(C)| \leq 7$, there exist v_i, v_j such that v_i and v_j have at least one common neighbor in $V(G) \setminus V(C)$. Without loss of generality, assume that u_1 is adjacent to

v_1 and v_3, where $u_1 \in V(G) \setminus V(C)$. We distinguish the following two subcases.

Subcase 2.1. $N(v_2) \cap (N(v_4) \cup N(v_5)) \nsubseteq V(C)$.

There exists $u_2 \in V(G) \setminus V(C)$ such that $u_2 \in N(v_2)$ and $u_2 \in N(v_4) \cup N(v_5)$. By symmetry, we may assume that u_2 is adjacent to v_2 and v_4. It is obvious that u_1 and u_2 are distinct and $u_1u_2 \notin E(G)$; otherwise G contains C_6. Since $\delta(G) \geq 4$, u_1 has at least two neighbors in $V(G) \setminus V(C) \setminus \{u_1, u_2\}$, two of which are denoted by w_1, w_2; u_2 has at least two neighbors in $V(G) \setminus V(C) \setminus \{u_1, u_2\}$, two of which are denoted by w_3, w_4; v_5 has at least two neighbors in $V(G) \setminus V(C) \setminus \{u_1, u_2\}$, two of which are denoted by w_5, w_6. We prove that $w_1, w_2, \ldots, w_6$ are six distinct vertices. If u_1 and v_5 have a common neighbor in $V(G) \setminus V(C) \setminus \{u_1, u_2\}$, say w, then $wu_1v_3v_2v_1v_5w$ is a C_6, a contradiction. By symmetry, u_2 and v_5 have no common neighbor in $V(G) \setminus V(C) \setminus \{u_1, u_2\}$. If u_1 and u_2 have a common neighbor in $V(G) \setminus V(C) \setminus \{u_1, u_2\}$, say w', then $w'u_1v_1v_5v_4u_2w'$ is a C_6, a contradiction. Thus, $V(G)$ contains $\{v_i, u_j, w_k \mid 1 \leq i \leq 5, 1 \leq j \leq 2, 1 \leq k \leq 6\}$ and hence $|V(G)| \geq 13$, which contradicts $|V(G)| \leq 12$.

Subcase 2.2. $N(v_2) \cap (N(v_4) \cup N(v_5)) \subseteq V(C)$.

Since G contains no $C_5{+}e$ and $\delta(G) \geq 4$, each of v_2, v_4, v_5 has at least two neighbors in $V(G) \setminus V(C) \setminus \{u_1\}$. Let $w_1, w_2 \in N(v_2) \cap (V(G) \setminus V(C) \setminus \{u_1\})$, $w_3, w_4 \in N(v_4) \cap (V(G) \setminus V(C) \setminus \{u_1\})$ and $w_5, w_6 \in N(v_5) \cap (V(G) \setminus V(C) \setminus \{u_1\})$. Since $N(v_2) \cap (N(v_4) \cup N(v_5)) \subseteq V(C)$, then $w_1, w_2, \ldots, w_6$ are six distinct vertices. Because $|V(G)| \leq 12$, $V(G) = \{v_i, w_j, u_1 \mid 1 \leq i \leq 5, 1 \leq j \leq 6\}$. We see that w_1 is nonadjacent to v_4 or v_5; otherwise $N(v_2) \cap (N(v_4) \cup N(v_5)) \nsubseteq V(C)$, which is Subcase 2.1. And w_1 is nonadjacent to w_3; otherwise $w_1w_3v_4v_5v_1v_2w_1$ is a C_6. For the same reason, w_1 is nonadjacent to w_4, w_5, w_6. Hence, $d(w_1) \leq 3$, which contradicts $\delta(G) \geq 4$.

Therefore, we conclude that $R(C_6, K_{1,n}) = n + 4$ for $n = 7, 8$. □

5. Proof of Theorem 6

The proof runs along the same lines as the proof of Theorem 5, with small differences in the details.

For $i = 1, 2, 3$, let G_i be formed from three disjoint K_5s by identifying exactly one vertex of i of the copies. So, $G_1 = 3K_5$, G_2 is the disjoint union of K_5 and the join of K_1 and $2K_4$, and G_3 is the join of K_1 with $3K_4$. Then $|V(G_i)| = 16 - i$, G_i contains no C_6 and $\delta(G_i) \geq 4$. Thus, $\overline{G_i}$ contains no $K_{1,12-i}$ for $i = 1, 2, 3$. It follows that $R(C_6, K_{1,n}) \geq n + 5$ for $n = 9, 10, 11$.

Now let G be a graph of order $n+5$. Suppose to the contrary that G contains no C_6 and $\overline{G}$ contains no $K_{1,n}$. Then $\delta(G) \geq 5$. By Theorem 3, G contains C_4. Let C be a longest cycle with $|V(C)| \leq 6$ in G. Then $4 \leq |V(C)| \leq 5$. We distinguish two cases and complete the proof of Theorem 6 by reaching contradictions in all subcases.

Case 1. $|V(C)| = 4$.

First suppose that G contains $K_{2,3}$, and assume that both v_1 and v_2 are adjacent to each vertex of v_3, v_4, v_5. Observe that $v_3v_4 \notin E(G)$; otherwise $v_1v_3v_4v_2v_5v_1$ is a C_5, a contradiction. For the same reason, $v_4v_5, v_3v_5 \notin E(G)$. Since $\delta(G) \geq 5$, for $i = 3, 4, 5$, each v_i has another neighbor in $V(G) \setminus \{v_1, v_2, v_3, v_4, v_5\}$, denoted by u_i. Since G contains no C_6, u_3, u_4, u_5 are three distinct vertices. Let $V_1 = \{v_i, u_j \mid 1 \leq i \leq 5, 3 \leq j \leq 5\}$. Then, for $i = 3, 4, 5$, u_i is nonadjacent to $V_1 \setminus \{v_i\}$; otherwise G contains C_5 or C_6, a contradiction. Since $\delta(G) \geq 5$, each of u_3, u_4, u_5 has at least four neighbors in $V(G) \setminus V_1$. Since $14 \leq |V(G)| \leq 16$, we have $6 \leq |V(G) - V_1| \leq 8$, and it follows that at least two vertices of u_3, u_4, u_5 have a common neighbor in $V(G) \setminus V_1$, say both u_3 and u_4 are adjacent to $w \in V(G) \setminus V_1$. Then, $wu_3v_3v_1v_4u_4w$ is a C_6, a contradiction. Thus,

$$\text{(4)} \qquad G \text{ contains no } K_{2,3}.$$

Let $C = v_1v_2v_3v_4v_1$. Since $\delta(G) \geq 5$, each v_i has another neighbor in $V(G) \setminus \{v_1, v_2, v_3, v_4\}$, denoted by u_i, where $1 \leq i \leq 4$. Observe that u_1, u_2, u_3, u_4 are four distinct vertices; otherwise G contains C_5 or $K_{2,3}$. Let $V_1 = \{v_i, u_i \mid 1 \leq i \leq 4\}$. Then, for $1 \leq i \leq 4$, u_i is nonadjacent to $V_1 \setminus \{v_i\}$; otherwise G contains C_5 or C_6 or $K_{2,3}$, a contradiction. Since $\delta(G) \geq 5$, each u_i has at least four neighbors in $V(G) \setminus V_1$. Since $14 \leq |V(G)| \leq 16$, we have $6 \leq |V(G) - V_1| \leq 8$, and it follows that at least two vertices of u_1, u_2, u_3 have a common neighbor in $V(G) \setminus V_1$. If both u_1 and u_2 are adjacent to $w \in V(G) \setminus V_1$, then $wu_1v_1v_2u_2w$ is a C_5, a contradiction. By symmetry, u_2 and u_3 have no common neighbor in $V(G) \setminus V_1$. If both u_1 and u_3 are adjacent to $w \in V(G) \setminus V_1$, then $wu_1v_1v_2v_3u_3w$ is a C_6, also a contradiction.

Case 2. $|V(C)| = 5$.

First suppose G contains K_5, and let $V(K_5) = \{v_1, v_2, v_3, v_4, v_5\}$. Since $\delta(G) \geq 5$, each v_i has a neighbor not in $V(K_5)$, denoted by u_i. It is easy to check that u_1, u_2, u_3, u_4, u_5 are pairwise distinct. Let $V_1 = \{v_i, u_i \mid 1 \leq i \leq 5\}$. Then u_i has at most one neighbor in V_1. So, u_i has at least four neighbors in $V(G) \setminus V_1$, denoted by w_{ij}, where $1 \leq i \leq 5$, $1 \leq j \leq 4$. Any two vertices of $\{u_i \mid 1 \leq i \leq 5\}$ have no common neighbor in $V(G) \setminus V_1$; otherwise G contains C_6. We conclude that $|V(G)| \geq 30$, contradicting that $|V(G)| \leq 16$. Therefore,

$$(5) \qquad G \text{ contains no } K_5.$$

Now suppose G contains $C_5 + e$, and let $v_1v_2v_3v_4v_5v_1$ be a C_5 with at least one chord. Since G contains $C_5 + e$ but no K_5, we can always find some i such that $v_iv_{i+2} \in E(G)$ and $v_{i+1}v_{i+3} \notin E(G)$, where the indices are taken modulo 5. Without loss of generality, we may assume that $v_1v_3 \in E(G)$ and $v_2v_4 \notin E(G)$. Each of v_2, v_4, v_5 has another neighbor in $V(G) \setminus \{v_1, v_2, v_3, v_4, v_5\}$, denoted by u_2, u_4, u_5, respectively. Observe that u_2, u_4 are distinct; otherwise $u_2v_2v_3v_1v_5v_4u_2$ is a C_6. For the same reason, u_2, u_4, u_5 are pairwise distinct. We assert that either $v_1u_4 \notin E(G)$ or $v_3u_5 \notin E(G)$; otherwise $v_1u_4v_4v_5u_5v_3v_1$ is a C_6. We only deal with the case that $v_1u_4 \notin E(G)$; the other case is similar. Let $V_1 = \{v_1, v_2, \ldots, v_5, u_2, u_4, u_5\}$. It is easy to check that both u_2 and u_4 have at most one neighbor in V_1. Since $\delta(G) \geq 5$, both u_2 and u_4 have at least four neighbors in $V(G) \setminus V_1$. Let $\{w_1, w_2, w_3, w_4\}$ be a subset of $N(u_2) \cap (V(G) \setminus V_1)$ and $\{w_5, w_6, w_7, w_8\}$ be a subset of $N(u_4) \cap (V(G) \setminus V_1)$. We deduce that $w_1, w_2, \ldots, w_8$ are pairwise distinct; otherwise for some $w \in N(u_2) \cap N(u_4)$, $wu_2v_2v_3v_4u_4w$ is a C_6. Since $|V(G)| \leq 16$, $V(G) = \{v_i, u_j, w_k \mid 1 \leq i \leq 5, 1 \leq k \leq 8, j = 2, 4, 5\}$. Since $\delta(G) \geq 5$, G contains no C_6 and $v_2v_4 \notin E(G)$, it follows that v_2 has to be adjacent to at least one vertex in $\{w_1, w_2, w_3, w_4\}$. Since $\delta(G) \geq 5$, G contains no C_6 and $v_1u_4 \notin E(G)$, it follows that v_1 has to be adjacent to at least one vertex in $\{w_1, w_2, w_3, w_4\}$. If v_1 and v_2 have a common neighbor in $\{w_1, w_2, w_3, w_4\}$, say w_1, then $w_1v_2v_3v_4v_5v_1w_1$ is a C_6. If v_1 and v_2 have no common neighbor in $\{w_1, w_2, w_3, w_4\}$, say $v_1w_1, v_2w_2 \in E(G)$, then $v_1w_1u_2w_2v_2v_3v_1$ is a C_6. Hence,

$$(6) \qquad G \text{ contains no } C_5 + e.$$

Now let $C = v_1v_2v_3v_4v_5v_1$. Since G contains no $C_5 + e$ and $\delta(G) \geq 5$, each v_i has at least three neighbors in $V(G) \setminus V(C)$. Because $|V(G) \setminus V(C)| \leq 11$, there exist v_i, v_j such that v_i and v_j have at least one common neighbor in $V(G) \setminus V(C)$. Without loss of generality, assume that u_1 is adjacent to v_1 and v_3, where $u_1 \in V(G) \setminus V(C)$. We distinguish two subcases and reach contradictions.

Subcase 2.1. *$N(v_2) \cap (N(v_4) \cup N(v_5)) \nsubseteq V(C)$.*

There exists $u_2 \in V(G) \setminus V(C)$ such that $u_2 \in N(v_2)$ and $u_2 \in N(v_4) \cup N(v_5)$. By symmetry, we may assume that u_2 is adjacent to v_2 and v_4. It is obvious that u_1 and u_2 are distinct and $u_1u_2 \notin E(G)$; otherwise G contains C_6. Since G contains no $C_5 + e$ and $\delta(G) \geq 5$, each of u_1, u_2, v_5 has at least three neighbors in $V(G) \setminus V(C) \setminus \{u_1, u_2\}$, v_1 has at least two neighbors in $V(G) \setminus V(C) \setminus \{u_1, u_2\}$. It is not difficult to check that any two vertices of $\{u_1, u_2, v_1, v_5\}$ have no common neighbor in $V(G) \setminus V_1$; otherwise G contains C_6. We conclude that $|V(G)| \geq 18$, contradicting that $|V(G)| \leq 16$.

Subcase 2.2. $N(v_2) \cap (N(v_4) \cup N(v_5)) \subseteq V(C)$.

We claim that either v_1, v_4 have no common neighbor in $V(G) \setminus V(C) \setminus \{u_1\}$, or v_3, v_5 have no common neighbor in $V(G) \setminus V(C) \setminus \{u_1\}$. If not, let $wv_1, wv_4, w'v_3w'v_5 \in E(G)$, where $w, w' \in V(G) \setminus V(C) \setminus \{u_1\}$. Obviously, w, w' are distinct. Then $wv_1v_5w'v_3v_4w$ is a C_6, a contradiction. Without loss of generality, we assume that v_1, v_4 have no common neighbor in $V(G) \setminus V(C) \setminus \{u_1\}$. Since G contains no $C_5 + e$ and $\delta(G) \geq 5$, each of v_2, v_4, v_5 has at least three neighbors in $V(G) \setminus V(C) \setminus \{u_1\}$, and v_1 has at least two neighbors in $V(G) \setminus V(C) \setminus \{u_1\}$. It is not difficult to check that any two vertices of $\{v_1, v_2, v_4, v_5\}$ have no common neighbor in $V(G) \setminus V(C) \setminus \{u_1\}$. We conclude that $|V(G)| \geq 17$, contradicting that $|V(G)| \leq 16$.

Therefore, we conclude that $R(C_6, K_{1,n}) = n + 5$ for $n = 9, 10, 11$. $\square$

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Yanbo Zhang
Faculty of EEMCS
University of Twente
P.O. Box 217
7500 AE Enschede
The Netherlands

Department of Mathematics
Nanjing University
Nanjing 210093
P.R. China
E-mail address: ybzhang@163.com

Hajo Broersma
Faculty of EEMCS
University of Twente
P.O. Box 217
7500 AE Enschede
The Netherlands
E-mail address: h.j.broersma@utwente.nl

Yaojun Chen
Department of Mathematics
Nanjing University
Nanjing 210093
P.R. China
E-mail address: yaojunc@nju.edu.cn

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