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Decomposable and indecomposable critical hypergraphs

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A hypergraph is k-critical if it has chromatic number k, but each of its proper subhypergraphs has a coloring with $k-1$ colors. In 1963 T. Gallai [10] proved that every k-critical graph of order at most $2k-2$ is decomposable, that is, its complement is disconnected. We shall prove a counterpart of this result for critical hypergraphs. Based on this result we shall determine the minimum number of edges of a k-critical hypergraph of order n, provided that $k \leq n \leq 2k-1$.

1. Introduction

The coloring theory for graphs and hypergraphs plays a central role in discrete mathematics. The coloring problem is to determine the chromatic number $\chi(H)$ of a given hypergraph H, that is, the minimum number of colors needed to color the vertices of H such that each vertex receive a color and no edge has the same color on all its vertices. Since graphs are just 2-uniform hypergraphs, this hypergraph coloring concept, introduced by Erdős and Hajnal [8] in the 1960s, generalizes the usual graph coloring concept. The study of hypergraph coloring problems leads very natural to the concept of critical hypergraphs. A hypergraph is k-critical if it has chromatic number k, but each of its proper subhypergraphs has chromatic number at most $k-1$. Critical graphs were introduced and investigated first by Dirac in his Ph.D. thesis and the resulting papers [4] and [5]. Critical hypergraphs were introduced by Lovász [17].

In 1963 Gallai [9, 10] published two fundamental papers about the structure of critical graphs. In the first paper he proved an extension of Brooks theorem and established a lower bound for the number of edges in a k-critical graph of order n. Several results of the first paper [9] were later extended to critical hypergraphs, see e.g. [13], [16], [26] and [27]. In the second paper Gallai proved that if a k-critical graph has order at most $2k-2$, then its complement is disconnected. This result is used by Gallai to determine the minimum number of edges in a k-critical graph of order n and to give a complete description of the extremal graphs, provided that $k \leq n \leq 2k-1$. In this paper we shall prove the following two results for critical hypergraphs.

Main Theorem 1. *Every k-critical hypergraph whose order is at most $2k-2$ is obtained from the disjoint union of two subhypergraphs by adding all graph edges between these two subhypergraphs.*

Main Theorem 2. *Let n and k be integers with $n = k+p$ and $2 \leq p \leq k-1$. If $\text{ext}(k,n)$ is the minimum number of edges in a k-critical hypergraph of order n, then $\text{ext}(k,n) = \binom{n}{2} - (p^2+1)$.*

Gallai's proofs of Main Theorem 1 for graphs is not directly applicable to hypergraphs. Proof ideas from an alternative proof by Stehlik [24] may however be extended to the hypergraph case. Our proof of Main Theorem 2 resembles Gallai's original proof for graphs, but we are not able to give a complete characterization of the extremal hypergraphs in case $n = 2k-1$.

The rest of the paper is organized as follows. The second section gives a brief introduction to hypergraphs. The following three sections give a brief introduction to critical graphs and hypergraphs collecting the concepts and results to be used later in this paper. In the third section, we discuss some well known basic properties of critical hypergraphs. In the fourth section we introduce two fundamental construction for critical hypergraphs, the Hajós sum and the Dirac sum. In the fifth section we give some background information to Gallai's decomposition theorem. The proof of our first main result is given in the sixth section. This result is used in the seventh section to describe the structure of critical hypergraphs whose order is near to its chromatic number. The proof of the second main result is given in the last section.

2. Preliminaries

For terminology and notation for hypergraphs we took inspiration from the book of Claude Berge [1].

A *hypergraph* $H = (V, E)$ is a pair of two finite sets, V and E, satisfying $E \subseteq 2^V$ and $|e| \geq 2$ for all $e \in E$. The set $V = V(H)$ is the *vertex set* of H and its elements are the *vertices* of H. The set $E = E(H)$ is the *edge set* of H and its elements are the *edges* of H. A hypergraph H is *empty* if $V(H) = E(H) = \varnothing$; in this case we write $H = \varnothing$. A hypergraph H is *simple* if no edge of H is contained in another edge of H. It is notable that multiple edges are not allowed and that the union of all edges may be a proper subset of the vertex set.

Let $H = (V, E)$ be a hypergraph. The number of vertices of H is its *order*, written $|H|$. An edge e with $|e| \geq 3$ is called a *hyperedge*, and an edge e with $|e| = 2$ is called an *ordinary edge*. As usual, for an ordinary edge $e = \{u, v\}$ we also write $e = uv$ and $e = vu$. If $E \subseteq \binom{V}{p}$, then H is said to be *p-uniform*. So a *graph* is a 2-uniform hypergraph, that is, a hypergraph in which each edge is ordinary. A vertex v is *incident* with an edge e if $v \in e$. For a vertex v of H, let $E_H(v) = \{e \in E(H) \mid v \in e\}$. The *degree* of v in H is $d_H(v) = |E_H(v)|$. The hypergraph H is said to be *regular* and *r-regular* if each vertex of H has degree r. As usual, $\delta(H) = \min_{v \in V(H)} d_H(v)$ is the *minimum degree* of H and $\Delta(H) = \max_{v \in V(H)} d_H(v)$ is the *maximum degree* of H. If $H = \varnothing$, then we define $\delta(H) = \Delta(H) = 0$.

A hypergraph H' is a *subhypergraph* of H, written $H' \subseteq H$, if $V(H') \subseteq V(H)$ and $E(H') \subseteq E(H)$. If $H' \subseteq H$ and $H' \neq H$, then H' is said to be a *proper subhypergraph* of H. For a vertex set $X \subseteq V(H)$, the subhypergraph of H *induced* by X is $H[X] = (X, E(H) \cap 2^X)$. A subhypergraph $H' \subseteq H$ is called an *induced subhypergraph* if $H' = H[V(H')]$. Furthermore, we define $H - X = H[V(H) \setminus X]$. If $X = \{v\}$ is a singleton, then we also write $H - v$ instead of $H - X$. For a set $F \subseteq 2^{V(H)}$, we define $H - F = (V(H), E(H) \setminus F)$ and $H + F = (V(H), E(H) \cup F)$. When $F = \{e\}$ is a singleton, we denote $H - F$ by $H \setminus e$ and $H + F$ by $H + e$. A vertex set $X \subseteq V(H)$ is called an *independent set* of H if the hypergraph $H[X]$ has no edge; and it is called a *clique* of H if the hypergraph $H[X]$ contains all ordinary edges of $\binom{X}{2}$. We call H a *complete p-uniform hypergraph*, where $p \geq 2$ is an integer, if $E(H) = \binom{V(H)}{p}$. If H is a complete p-uniform hypergraph of order n, we write $H = K_n^p$. Note that the hypergraph K_n^n with $n \geq 2$ has exactly one edge. We write $H = C_n$ for an ordinary cycle as a 2-uniform hypergraph of order n. A cycle is called *odd* or *even* depending on whether its order is odd or even.

A nonempty hypergraph H is called *connected* if for every vertex set X with $\varnothing \neq X \subset V(H)$ at least one edge of H contains a vertex of X as well as a vertex of $V(G) \setminus X$. Equivalently, H is connected if and only if there is a path in H between any two of its vertices. A *path of length p* in H is a sequence $(v_1, e_1, v_2, e_2, \ldots, v_p, e_p, v_{p+1})$ of distinct vertices $v_1, v_2, \ldots, v_{p+1}$ of H and distinct edges $e_1, e_2, \ldots, e_p$ of H such that $\{v_i, v_{i+1}\} \subseteq e_i$ for $i = 1, 2, \ldots, p$. A (connected) *component* of a nonempty hypergraph H is a maximal connected subhypergraph.

Let H be a hypergraph and let Γ be a set. A *coloring* of H with *color set* Γ is a mapping $\varphi : V(H) \to \Gamma$ that assigns to each vertex $v \in V(H)$ a *color* $\varphi(v) \in \Gamma$ such that $|\varphi(e)| \geq 2$ for every edge $e \in E(H)$. If $|\Gamma| = k$ with $k \in \mathbb{N}_0$, then we also say that φ is a *k-coloring* of H. We say that H is

k-colorable if H has a k-coloring. Obviously, H is k-colorable if and only if H has a coloring with color set $\Gamma = [1, k]$. Here $[1, k]$ is the set $\{1, 2, \ldots, k\}$. The least integer $k \in \mathbb{N}_0$ for which H has a k-coloring is called the *chromatic number* of H, denoted by $\chi(H)$. A hypergraph with chromatic number k is also said to be *k-chromatic*. So $\chi(H) \leq k$ if and only if H is k-colorable, and $\chi(H) = k$ if and only if H is k-colorable, but not $(k-1)$-colorable. Obviously, any coloring of H induces a coloring with the same color set of each of its subhypergraphs. Consequently, the chromatic number is a monotone hypergraph parameter, that is, $H' \subseteq H$ implies $\chi(H') \leq \chi(H)$ Furthermore, the components of a hypergraph can be colored independently, so if $H \neq \varnothing$, then

$$\chi(H) = \max\{\chi(H') \mid H' \text{ is a component of } H\}. \tag{1}$$

Obviously, a mapping $\varphi : V(H) \to \Gamma$ is a coloring of H if and only if for every color $c \in \Gamma$ the preimage $\varphi^{-1}(c) = \{v \in V(H) \mid \varphi(v) = c\}$ is an independent set of H (possibly empty). These preimages of a coloring are also referred to as *color classes*. So there is a one-to-one correspondence between colorings of H with a set of k colors and sequences $(I_1, I_2, \ldots, I_k)$ of disjoint independent sets of H whose union is $V(H)$. The maximum cardinality of an independent set of H is the *independence number* of H, denoted by $\alpha(H)$. Thus any color class has at most $\alpha(H)$ vertices, which implies that any coloring of H with a set of k colors satisfies $|H| \leq k\alpha(H)$, and so $|H| \leq \chi(H)\alpha(H)$. Evidently, $\chi(H) \leq |H|$ and

$$\chi(H) = |H| \text{ if and only if } V(H) \text{ is a clique of } H. \tag{2}$$

The largest cardinality of a clique of H is the *clique number* of H, denoted by $\omega(H)$. So $\omega(H) = \max\{n \mid K_n^2 \subseteq H\}$ and we obtain $\omega(H) \leq \chi(H)$. Evidently, $\chi(K_n^2) = n$ for all $n \in \mathbb{N}_0$ and for the cycle C_n we have $\chi(C_n) = 2$ if n is even and $\chi(C_n) = 3$ if n is odd. Furthermore, it is easy to show that $\chi(K_n^p) = \lceil n/(p-1) \rceil$, where $p \geq 2$. In particular, we have $\chi(K_n^n) = 2$ for all $n \geq 2$.

As discussed above, the chromatic number $\chi(H)$ is the least integer k for which $V(H)$ can be partitioned into k independent sets. So $\chi(H) \leq 1$ if and only if H is edgeless, and $\chi(H) \leq 2$ if and only if H is bipartite. To decide whether a graph is bipartite, we can use König's theorem. So a graph is bipartite if and only if it contains no odd cycle as a subgraph. However, a good characterization for the class of bipartite hypergraphs is unknown. To recognize whether a hypergraph is bipartite is a NP-complete decision problem as first noted by Lovász [19]. Consequently, the determination of the chromatic number of hypergraphs is a NP-hard optimization problem.

3. Critical hypergraphs

In studying the hypergraph coloring problem, critical hypergraphs play an important role. A hypergraph H is *critical* or *k-critical* if $\chi(H') < \chi(H) = k$ for every proper subhypergraph H' of H. So k-critical hypergraphs are minimal k-chromatic hypergraphs with respect to the hypergraph relation $\subseteq$ (to be a subhypergraph).

To see why critical hypergraphs form a useful concept, let us consider a *hypergraph property* $\mathcal{H}$, that is, a class of hypergraphs closed under taking isomorphic hypergraphs. Suppose that $\mathcal{H}$ is *monotone* in the sense that $H' \subseteq H \in \mathcal{H}$ implies $H' \in \mathcal{H}$. Furthermore, consider a *hypergraph parameter* ρ defined for $\mathcal{H}$, that is, a mapping that assigns to each hypergraph of $\mathcal{H}$ a real number such that $\rho(H') = \rho(H)$ whenever H' and H are isomorphic hypergraphs of $\mathcal{H}$. If we want to bound the chromatic number for the hypergraphs of $\mathcal{H}$ from above by the parameter ρ, then we can apply the critical hypergraph method, provided that ρ is *monotone*, that is, $H' \subseteq H \in \mathcal{H}$ implies $\rho(H') \leq \rho(H)$. The proof of the following proposition is easy and is left to the reader.

Proposition 3.1. *Let $\mathcal{H}$ be a monotone hypergraph property and let ρ be a monotone hypergraph parameter defined for $\mathcal{H}$. Then the following statements hold:*

(a) *For every hypergraph $H \in \mathcal{H}$ there exists a critical hypergraph $H' \in \mathcal{H}$ such that $H' \subseteq H$ and $\chi(H') = \chi(H)$.*
(b) *If $\chi(H') \leq \rho(H')$ for every critical hypergraph $H' \in \mathcal{H}$, then $\chi(H) \leq \rho(H)$ for every hypergraph $H \in \mathcal{H}$.*

The chromatic number is a monotone hypergraph parameter and it is easy to check that if we delete a vertex or an edge from a hypergraph, then the chromatic number decreases by at most one. So if H is a hypergraph and $t \in V(H) \cup E(H)$, then

$$\chi(H) - 1 \leq \chi(H - \{t\}) \leq \chi(H). \tag{3}$$

As a consequence we obtain the following well known result saying that the class of $(k-1)$-colorable hypergraphs can be characterized in terms of forbidden k-critical subhypergraphs.

Proposition 3.2. *Let H be a hypergraph and let $k \in \mathbb{N}$. Then $\chi(H) \leq k-1$ if and only if there is no k-critical hypergraph H' with $H' \subseteq H$.*

Proof. If H contains a k-critical hypergraph H' as a subgraph, then $\chi(H) \geq \chi(H') = k$. Conversely, if $\chi(H) \geq k$, then it follows from (3) that there is a subhypergraph G of H with $\chi(G) = k$. By Lemma 3.1(a), G and hence H contains a k-critical subhypergraph H'. □

The following proposition is a list of some basic properties of critical hypergraphs (see also [1]).

Proposition 3.3. *Let H be a k-critical hypergraph, where $k \geq 2$. Then the following statements hold:*

(a) *$|H| \geq k$ and equality holds if and only if $H = K_k^2$.*
(b) *Every vertex v of H is contained in $k-1$ edges having pairwise only the vertex v in common.*
(c) *H is connected and $\delta(H) \geq k-1$.*
(d) *H is a simple hypergraph.*

Proof. Statement (a) follows from (2) and the fact that K_k^2 is k-critical. Statement (b) is a consequence of the fact that if we delete a vertex v, then the resulting hypergraph $H - v$ has a coloring with a set of $k-1$ colors and for every color c there exists an edge $e_c \in E_H(v)$ such that all vertices of e_c except v have color c. Statement (c) is a consequence of (1) and (b). Statement (d) follows from the fact that if we delete an edge e of H, then there is a coloring of $H \setminus e$ with a set of $k-1$ colors and all vertices of e have the same color. So no other edge can be contained in e. □

Let H be a k-critical hypergraph. By Proposition 3.3(c), $\delta(H) \geq k-1$, which leads to a natural way of classifying the vertices of H into two classes. The vertices of H having degree $k-1$ are called *low vertices* of H, and the remaining vertices are called *high vertices* of H. So any high vertex of H has degree at least k in H. Furthermore, the subhypergraph of H induced by its low vertices is called the *low vertex subhypergraph* of H. For critical graphs, this classification is due to Gallai [9]. The following result due to Kostochka, Stiebitz and Wirth [16] generalizes Gallai's theorem about the structure of the low vertex subgraph of critical graphs. A hypergraph B is called a *hyper-brick* if $B = C_{2p+1}$ for $p \geq 1$, or $B = K_n^n$ for $n \geq 3$, or $B = K_n^2$ for $n \geq 1$. Note that any hyper-brick B is a critical hypergraph consisting only of low vertices and $\chi(B) = \Delta(B) + 1$.

Theorem 3.4. *If H is a critical hypergraph, then any block of its low vertex subhypergraph is a hyper-brick.*

Since any critical hypergraph H satisfies $\chi(H) \leq \delta(H)+1 \leq \Delta(H)+1$ (by Proposition 3.3(c)), it follows from Proposition 3.1(b) that $\chi(H) \leq \Delta(H)+1$ for every hypergraph H. As proved by Rhys [22] the hyper-bricks are the only connected hypergraphs for which equality holds. This generalization of Brooks' theorem is an immediate consequence of Theorem 3.4

Corollary 3.5. *If H is a connected hypergraph and H is not a hyper-brick, then $\chi(H) \leq \Delta(H)$.*

Proof. Suppose $\chi(H) \geq k$, where $k = \Delta(H)+1$. By Proposition 3.2, H contains a k-critical subhypergraph H'. Then $\Delta(H') \leq \Delta(H) = k-1$ and, by Proposition 3.3, it follows that H' is $(k-1)$-regular and connected. By Theorem 3.4 any block of H' is a hyper-brick, from which we conclude that H' is a hyper-brick, since every hyper-brick is regular. Since H is connected and has maximum degree $k-1$, we obtain that $H = H'$, and so H is a hyper-brick, a contradiction. □

If we want to check whether a given hypergraph is critical, it suffices to investigate all edge deleted subhypergraphs. This follows from the following well known result.

Proposition 3.6. *Let H be a connected hypergraph and let $k \geq 2$ be an integer. Then H is k-critical if and only if $\chi(H \setminus e) < k \leq \chi(H)$ for every edge $e \in E(H)$.*

For integers $k, n \in \mathbb{N}_0$, let $\mathrm{Cri}(k)$ denote the class of k-critical hypergraphs and let

$$\mathrm{Cri}(k,n) = \{H \in \mathrm{Cri}(k) \mid |H| = n\}.$$

Since a hypergraph H satisfies $\chi(H) = 0$ if and only if $H = \varnothing$, and $\chi(H) \leq 1$ if and only if $E(H) = \varnothing$, it follows from Proposition 3.2 that

$$\mathrm{Cri}(0) = \{\varnothing\}, \mathrm{Cri}(1) = \{K_1^2\} \text{ and } \mathrm{Cri}(2) = \{K_n^n \mid n \geq 2\}.$$

While König's characterization of bipartite graphs implies that the odd cycles are the only 3-critical graphs, investigations of the class of 3-critical hypergraphs have received significant attention in the literature. For any fixed $k \geq 3$, a good characterization of the class $\mathrm{Cri}(k)$ is unlikely.

A hypergraph is *vertex-critical* or *k-vertex-critical* if $\chi(H') < \chi(H) = k$ for every proper induced subhypergraph H' of H. While k-critical hypergraphs are k-chromatic hypergraphs that are minimal with respect to the relation 'to be a subhypergraph', k-vertex-critical hypergraphs are k-chromatic hypergraphs that are minimal with respect to the relation to be an induced subhypergraph. Clearly, every critical hypergraph is vertex-critical, but not conversely. Examples of vertex-critical graphs that are not critical were given by Dirac. For $k \leq 3$, however, a graph is k-critical if and only if it is k-vertex-critical. It follows from (3) that a hypergraph H is k-vertex-critical if and only if $\chi(H - v) < k \leq \chi(H)$ for every vertex $v \in V(H)$. Results about critical hypergraphs can be often transformed into results about the larger class of vertex-critical hypergraphs.

Proposition 3.7. *Let H be a k-vertex-critical hypergraph. Then H contains a k-critical subhypergraph and any such subhypergraph has the same vertex set as H.*

Proof. That H contains a k-critical subhypergraph follows from Proposition 3.1(a). Now let H' be such a subhypergraph. If a vertex v of H does not belong to H', then $H' \subseteq H - v$ and, since H is k-vertex-critical, we obtain that $\chi(H') \leq \chi(H - v) < k$, which is impossible. □

4. Constructions for critical hypergraphs

In this section we focus on the problem of how to decompose a critical hypergraph into smaller critical hypergraphs. Clearly, to assemble and disassemble critical hypergraphs are two sides of the same coin.

There are two well known constructions for critical graphs that can be easily extended to hypergraphs, known as *the Hajós sum* and *the Dirac sum*. The first construction was invented by Hajós [11], and the second construction was invented by Dirac (see Gallai [9, (2.1)]). A third construction for critical hypergraphs is the *enlarging operation*. Theorem 4.1 was proved in [26]. The proofs of Theorems 4.1, 4.2 and 4.3 are all straightforward.

Theorem 4.1. *Let H_1 and H_2 be two vertex disjoint hypergraphs. For $i = 1, 2$, let v_i be a vertex of H_i, and let $e_i \in E_{H_i}(v_i)$ be an edge. Let H be the hypergraph obtained from $H_1 \backslash e_1$ and $H_2 \backslash e_2$ by identifying the vertices v_1 and v_2 to a new vertex $v_1 * v_2$ and by adding the new edge $e^* = (e_1 \cup e_2) \setminus \{v_1, v_2\}$. We call H a Hajós sum of H_1 and H_2 and write $H = H_1 \wedge H_2$. If H_1 and H_2 are k-critical, where $k \geq 3$, then H is k-critical.*

Theorem 4.2. *Let H_1 and H_2 be two vertex disjoint hypergraphs, and let H be the hypergraph obtained from the union $H_1 \cup H_2$ by adding all ordinary edges between H_1 and H_2, that is, $V(H) = V(H_1) \cup V(H_2)$ and $E(H) = E(H_1) \cup E(H_2) \cup \{uv \mid u \in V(H_1), v \in V(H_2)\}$. We call H the Dirac sum or the join of H_1 and H_2 and write $H = H_1 + H_2$. Then $\chi(H) = \chi(H_1) + \chi(H_2)$, and H is critical if and only if H_1 and H_2 are critical.*

Theorem 4.3. *Let H be a k-critical hypergraph and let $e \in E(H)$ be an edge of H. Let $e' \subseteq V(H)$ be a set such that $e \subseteq e'$ and e' is monochromatic with respect to any coloring of $H \setminus e$ with a set of $k-1$ colors. Then the hypergraph $H' = (H \setminus e) + e'$ is k-critical. We then say that H' is obtained from H by enlarging the edge e to e'.*

The advantage of the Hajós sum is that it not only preserves the chromatic number, but also criticality. If $k \geq 3$, then

$$H_1 \in \mathrm{Cri}(k, n_1), H_2 \in \mathrm{Cri}(k, n_2) \Rightarrow H_1 \wedge H_2 \in \mathrm{Cri}(k, n_1 + n_2 - 1).$$

Since $K_k^2 \in \mathrm{Cri}(k, k)$, we have

$$H \in \mathrm{Cri}(k, n) \Rightarrow H \wedge K_k^2 \in \mathrm{Cri}(k, n + k - 1).$$

For the Dirac sum we have

$$H_1 \in \mathrm{Cri}(k_1, n_1), H_2 \in \mathrm{Cri}(k_2, n_2) \Rightarrow H_1 + H_2 \in \mathrm{Cri}(k_1 + k_2, n_1 + n_2),$$

which gives, in particular, that

$$H \in \mathrm{Cri}(k, n) \Rightarrow K_p^2 + H \in \mathrm{Cri}(k+p, n+p) \text{ and } K_p^p + H \in \mathrm{Cri}(k+2, n+p).$$

The enlarging operation, increasing edges of the hypergraph, preserve the chromatic number as well as the order. Using the Hajós sum and the Dirac sum, we conclude from Proposition 3.3 that if $k \geq 3$, then

$$\mathrm{Cri}(k, k) = \{K_k^2\} \text{ and } \mathrm{Cri}(k, n) \neq \varnothing \text{ if and only if } n \geq k.$$

5. Indecomposable hypergraphs

Following Gallai, a hypergraph is called *decomposable* if it is the Dirac sum of two nonempty vertex disjoint subhypergraphs; otherwise the hypergraph is called *indecomposable*. By Theorem 4.2 it follows that a decomposable critical hypergraph is the Dirac sum of its indecomposable critical subhypergraphs. So the indecomposable critical hypergraphs are building elements of critical hypergraphs. In 1963 Gallai [10] proved the following beautiful result about indecomposable critical graphs.

Theorem 5.1. (Gallai) *If G is an indecomposable k-vertex-critical graph, then $|G| \geq 2k - 1$.*

Let G be a graph. To decide whether G is decomposable we can use its complement. The complement of G, denoted by $\overline{G}$, is the graph with

$$V(\overline{G}) = V(G) \text{ and } E(\overline{G}) = \binom{V(G)}{2} \setminus E(G).$$

Then G is decomposable if and only if its complement $\overline{G}$ is disconnected. Furthermore, for a vertex set $X \subseteq V(G)$, we obtain that X is an independent set (respectively, a clique) of G if and only if X is a clique (respectively, an independent set) of $\overline{G}$. Obviously, $\overline{\overline{G}} = G$.

Let H be a hypergraph. A *k-cover* of H is a mapping $\varphi : V(H) \to [1, k]$ such that $\varphi^{-1}(c)$ is a clique of H for every $c \in [1, k]$. The *cover number* of H, denoted by $\overline{\chi}(H)$, is the least integer k for which H admits a k-covering. A hypergraph H is *k-cover-critical* if $\overline{\chi}(H - v) < k \leq \overline{\chi}(H)$ for every vertex $v \in V(H)$.

Let G be a graph. Then a mapping $\varphi : V(G) \to [1, k]$ is a k-cover of G if and only if φ is a k-coloring of $\overline{G}$. Consequently, $\overline{\chi}(G) = \chi(\overline{G})$, and G is k-cover-critical if and only if $\overline{G}$ is k-vertex-critical. So Theorem 5.1 is equivalent to the following result.

Theorem 5.2. (Gallai) *If G is a connected k-cover-critical graph, then $|G| \geq 2k - 1$.*

In the graph theory literature, there are three different proofs of Gallai's result. Gallai's original proof is an application of matching theory to cover-critical graphs, so he first proved Theorem 5.2 and obtained Theorem 5.1 as a corollary. A second proof of Gallai's result was given by Molloy [20]. Molloy also applies matching theory to the complement of vertex-critical graphs; he uses Berge's version of Tutte's perfect matching theorem to the complement of a vertex-critical graph. A third proof was given by Stehlík [24]. He also prefers to work with cover-critical graphs, but his proof is self-contained and uses no matching theory. We shall adopt Stehlík's proof to establish a counterpart of Gallai's result for hypergraphs.

The concept of a complement does not apply to hypergraphs. However, we can define the relative complement of a hypergraph. So let H be a hypergraph. The *relative complement* of H, denoted by $R(H)$, is the graph with

$$V(R(H)) = V(H) \text{ and } E(R(H)) = \binom{V(H)}{2} \setminus E(H).$$

So if H is a graph, then $R(H) = \overline{H}$. Furthermore, $R(R(R(H))) = R(H)$. If H is a simple hypergraph, then H is decomposable if and only if $R(H)$ is disconnected. We shall prove the following result.

Theorem 5.3. *Let H be a k-critical hypergraph whose relative complement $R(H)$ is connected, then $|H| \geq 2k-1$. Equivalently, if H is a k-critical hypergraph with $|H| \leq 2k-2$, then H is decomposable.*

6. Proof of Theorem 5.3

Before proving the theorem, we need to introduce some definitions. In this section, we shall identify a coloring of a hypergraph with the set of its nonempty color classes. In particular, we suppress unused colors and do not distinguish between equivalent colorings.

Let H be a hypergraph and let $\varphi : V(H) \to \Gamma$ be a coloring of H with color set Γ. We shall use the following notation: $\Gamma_\varphi = \mathrm{im}(\varphi)$ is the *set of used colors*, $c(\varphi) = |\Gamma_\varphi|$ is the *number of used colors* and $\mathcal{X}_\varphi = \{\varphi^{-1}(c) \mid c \in \Gamma_\varphi\}$ is the set of nonempty color classes. Clearly, $c(\varphi) \geq \chi(H)$ and φ is said to be an *optimal coloring* of H if $c(\varphi) = \chi(H)$. Note that $\mathcal{X}_\varphi$ is a partition of the vertex set $V(H)$ into independent sets of H. Furthermore, we denote by $F(\varphi)$ the hypergraph whose vertex set is $V(F(\varphi)) = V(H)$ and whose edge set is

$$E(F(\varphi)) = \{U \in \mathcal{X}_\varphi \mid |U| \geq 2\}.$$

The set of isolated vertices of $F(\varphi)$ is denoted by $I(\varphi)$. Note that a vertex v of H belongs to $I(\varphi)$ if and only if $\{v\}$ is a color class with respect to φ. The hypergraph $F(\varphi)$ has maximum degree $\Delta(F(\varphi)) \leq 1$ and

$$c(\varphi) = |I(\varphi)| + |E(F(\varphi))|.$$

Let $X \subseteq V(H)$. Then we put $\overline{X} = V(H) \setminus X$. We denote by $\varphi|_X$ the coloring φ' of $H[X]$ with $\mathcal{X}_{\varphi'} = \{U \cap X \mid U \in \mathcal{X}_\varphi \text{ and } U \cap X \neq \varnothing\}$. The set X is said to be *φ-closed* if X is the union of a set of color classes with respect to φ, that is, each color class $U \in \mathcal{X}_\varphi$ satisfies $U \subseteq X$ or $U \subseteq \overline{X}$. From the definition it follows, that X is φ-closed if and only if $\overline{X}$ is φ-closed. Furthermore, X is φ-closed if and only if $\varphi' = \varphi|_X$ satisfies $\mathcal{X}_{\varphi'} \subseteq \mathcal{X}_\varphi$.

Proposition 6.1. *Let H be hypergraph, let φ be an optimal coloring of H, and let $X \subseteq V(H)$ be φ-closed. Then $\overline{X}$ is φ-closed and, for $Y \in \{X, \overline{X}\}$, the restriction $\varphi|_Y$ is an optimal coloring of $H[Y]$ satisfying $\varphi = \varphi|_X \cup \varphi|_{\overline{X}}$ and $c(\varphi) = c(\varphi|_X) + c(\varphi|_{\overline{X}})$. Furthermore, $I(\varphi)$ is a clique of H.*

Proof. Since X is φ-closed, $\overline{X}$ is φ-closed, too, and $\mathcal{X}_\varphi$ is the disjoint union of the sets $\mathcal{X}_{\varphi|_X}$ and $\mathcal{X}_{\varphi|_{\overline{X}}}$. This implies that $\varphi = \varphi|_X \cup \varphi|_{\overline{X}}$ and $c(\varphi) = c(\varphi|_X) + c(\varphi|_{\overline{X}})$. Hence, if $Y \in \{X, \overline{X}\}$, then the restriction $\varphi|_Y$ is an optimal coloring of $H[Y]$. For otherwise, there would be a coloring φ_1 of $H[Y]$ with $c(\varphi_1) < c(\varphi|_Y)$. But then $\varphi_2 = \varphi_1 \cup \varphi|_{\overline{Y}}$ would be a coloring of H satisfying

$$c(\varphi_2) = c(\varphi_1) + c(\varphi|_{\overline{Y}}) < c(\varphi|_Y) + c(\varphi|_{\overline{Y}}) = c(\varphi),$$

which is impossible. If $I(\varphi)$ is not a clique of H, then $I(\varphi)$ contains two vertices u and v such that $uv \notin E(H)$ and so $X = \{u, v\}$ is an independent set of H. But then there is a coloring φ' of $H[X]$ with $c(\varphi') = 1$ implying that $\varphi|_X$ is no optimal coloring of $H[X]$. Since X is φ-closed, this is a contradiction. □

Proposition 6.2. *For a vertex v of a hypergraph H there exists an optimal coloring φ of H with $v \in I(\varphi)$ if and only if $\chi(H - v) < \chi(H)$.*

Proof. If φ is an optimal coloring of H with $v \in I(\varphi)$, then the set $\{v\}$ is φ-closed and it follows from Proposition 6.1 that $\varphi' = \varphi|_{V(H)\setminus\{v\}}$ is an optimal coloring of $H - v$ with $c(\varphi') = c(\varphi) - 1$ implying that $\chi(H - v) = \chi(H) - 1$. If conversely $\chi(H - v) < \chi(H)$, then (3) implies that $\chi(H - v) = \chi(H) - 1$. Thus, there is an optimal coloring φ' of $H - v$ with $c(\varphi') = \chi(H) - 1$. This coloring can be extended to a coloring φ of H by assigning to v an additional color, so that $c(\varphi) = \chi(H)$. Then φ is an optimal coloring of H satisfying $v \in I(\varphi)$. □

For two colorings φ_1 and φ_2 of H, let $F(\varphi_1, \varphi_2)$ denote the hypergraph with $V(F(\varphi_1, \varphi_2)) = V(H)$ and $E(F(\varphi_1, \varphi_2)) = E(F(\varphi_1)) \cup E(F(\varphi_2))$. Obviously, this hypergraph has maximum degree at most 2.

Proposition 6.3. *Let φ_1 and φ_2 be two distinct optimal colorings of a hypergraph H, and let $F = F(\varphi_1, \varphi_2)$. Then the following statements hold:*

(a) *If X is the vertex set of a component of the hypergraph F, then X is both φ_1-closed and φ_2-closed, and $\varphi_3 = \varphi_1|_X \cup \varphi_2|_{\overline{X}}$ is an optimal coloring of H with $I(\varphi_3) = (I(\varphi_1) \cap X) \cup (I(\varphi_2) \cap \overline{X})$.*
(b) *If v_1v_2 is an edge of $R(H)$ satisfying $v_1 \in I(\varphi_1)$ and $v_2 \in I(\varphi_2)$, then v_1 and v_2 belong to the same component of the hypergraph F.*

Proof. First we prove (a). Let U be any nonempty color class of φ_1 or φ_2. Then either $|U| = 1$ or U is an edge of F and, since X is the vertex set of a component of F, it follows that either $U \subseteq X$ or $U \subseteq \overline{X}$. So X is both φ_1-closed and φ_2-closed. From Proposition 6.1 it then follows that also $\overline{X}$ is both φ_1-closed and φ_2-closed. Furthermore, it follows that $\varphi_1' = \varphi_1|_X$ is an optimal coloring of $H[X]$ and $\varphi_2' = \varphi_2|_{\overline{X}}$ is an optimal coloring of $H[\overline{X}]$, where $\chi(H) = \chi(H[X]) + \chi(H[\overline{X}])$. From this we conclude that $\varphi_3 = \varphi_1' \cup \varphi_2'$ is an optimal coloring of G. It is easy to check that $I(\varphi_3) = I(\varphi_1') \cup I(\varphi_2') = (I(\varphi_1) \cap X) \cup (I(\varphi_2) \cap \overline{X})$. This proves (a).

For the proof of (b), let X be the vertex set of the component of F containing v_1. Then it follows from (a) that $\varphi_3 = \varphi_1|_X \cup \varphi_2|_{\overline{X}}$ is an optimal coloring of H. If $v_2 \notin X$, then we conclude from (a) and the assumption of (b) that $\{v_1, v_2\} \subseteq I(\varphi_3)$. But then Proposition 6.1 implies that v_1v_2 is an edge of H and not of $R(H)$ as assumed. This contradiction proves (b). □

If a hypergraph H has an optimal coloring φ such that $|I(\varphi)| = 1$, then $|H| \geq 2c(\varphi) - 1$. So Theorem 5.3 is an immediate consequence of the following result.

Theorem 6.4. *Let H be a critical hypergraph whose relative complement $R(H)$ is connected. Then for every vertex v of H there is an optimal coloring φ of H such that $I(\varphi) = \{v\}$.*

Proof. Let v be an arbitrary vertex of H. Since H is critical, $\chi(H - v) < \chi(H)$ and Proposition 6.2 implies that there exists an optimal coloring φ of H with $v \in I(\varphi)$. An optimal coloring φ of H is called a *v-extreme coloring* of H if $v \in I(\varphi)$ and $|I(\varphi)|$ is minimum subject to this condition. To complete the proof we must show that a v-extreme coloring φ of H satisfies $|I(\varphi)| = 1$. The proof is arranged in a series of three claims.

Claim 6.4.1. *Let v be a vertex of H, let φ_1 be a v-extreme coloring of H and let φ_2 be any coloring of H. Then any component of the hypergraph $F = F(\varphi_1, \varphi_2)$ contains at most one vertex of $I(\varphi_1)$.*

Proof. Suppose, to the contrary, that some component F' of F contains at least two vertices of $I(\varphi_1)$. Then let

$$P = (v_0, e_0, v_1, e_1, \ldots, v_{p-1}, e_{p-1}, v_p)$$

be a shortest path in the subhypergraph F' such that v_0 and v_p are two distinct vertices of $I(\varphi_1)$. Clearly, such a path P exists and we may choose P so that v_p is distinct from v. By the minimality of P, the only vertices of P belonging to $I(\varphi_1)$ are v_0 and v_p. Clearly, the vertices v_i are distinct and the edges e_i are distinct, where $\{v_i, v_{i+1}\} \subseteq e_i$. Also, by the minimality of P, the edges of P alternately lie in $F(\varphi_1)$ and $F(\varphi_2)$. Since v_0 and v_p belong to $I(\varphi_1)$, the edges e_0 and e_{p-1} both belong to $F(\varphi_2)$, and so the length p of P is odd, say $p = 2q + 1$ with $q \geq 1$.

Let $X = \{v_0, v_1, \ldots, v_p\}$ be the vertex set of P. Then $\varphi_1|_X$ is a coloring of $H[X]$ with $c(\varphi_1|_X) = q + 2$ and $\varphi_2|_X$ is a coloring of $H[X]$ with $c(\varphi_2|_X) = q + 1$. So $\varphi_1|_X$ is no optimal coloring of $H[X]$. Since φ_1 is an optimal coloring of H, it then follows from Proposition 6.1 that X is not φ_1-closed. Consequently, P contains a hyperedge of $F(\varphi_1)$. Then there is a largest integer j such that v_j belongs to a hyperedge of $F(\varphi_1)$, where $0 < j < p$. Let

$$P' = (v_j, e_j, v_{j+1}, \ldots, v_{p-1}, e_{p-1}, v_p)$$

be the subpath of P between v_j and v_p, and let $Y = \{v_j, v_{j+1}, \ldots, v_p\}$ be the vertex set of P'. By definition of v_j, the edge e_j belongs to $F(\varphi_2)$, so the length $p - j$ of P' is odd. For the restricted colorings we obtain that

$$\text{(i) } c(\varphi_1|_Y) + c(\varphi_1|_{\overline{Y}}) = c(\varphi_1) + 1 \text{ and (ii) } I(\varphi_1|_{\overline{Y}}) = I(\varphi_1) - \{v_p\}.$$

Equation (i) follows from the fact that $U = e_{j-1}$ is a color class of φ_1 whose size is at least 3 and this is the only color class of φ_1 which is divided into two color classes, namely $U \cap Y = \{v_j\}$ and $U \cap \overline{Y} = U \setminus \{v_j\}$. Equation (ii) follows from the fact that v_p is the only isolated vertex of $F(\varphi_1)$ belonging to Y. Furthermore, we conclude that $c(\varphi_1|_Y) = c(\varphi_2|_Y) + 1$ and $I(\varphi_2|_Y) = \varnothing$. Then $\varphi_3 = \varphi_2|_Y \cup \varphi_1|_{\overline{Y}}$ is a coloring of H satisfying

$$c(\varphi_3) = c(\varphi_2|_Y) + c(\varphi_1|_{\overline{Y}}) = c(\varphi_1|_Y) + c(\varphi_1|_{\overline{Y}}) - 1 = c(\varphi_1)$$

and

$$I(\varphi_3) = I(\varphi_2|_Y) \cup I(\varphi_1|_{\overline{Y}}) = I(\varphi_1) \setminus \{v_p\}.$$

This implies that φ_3 is an optimal coloring of H such that v belongs to $I(\varphi_3)$ and $|I(\varphi_3)| < |I(\varphi_1)|$. However, this contradicts the assumption that φ_1 is a v-extreme coloring of H. △

Claim 6.4.2. *Let v_1v_2 be an edge of $R(H)$ and let φ_1 be a v_1-extreme coloring of H. Then there exists a v_2-extreme coloring φ_2 of H such that $I(\varphi_1) \setminus \{v_1\} = I(\varphi_2) \setminus \{v_2\}$.*

Proof. There is a v_2-extreme coloring φ_3 of H. By Proposition 6.3(b), the vertices v_1 and v_2 belong to the same component of the hypergraph $F(\varphi_1, \varphi_3)$. Let X be the vertex set of this component, and let

$$\varphi_2 = \varphi_3|_X \cup \varphi_1|_{\overline{X}}.$$

By Proposition 6.3(a), φ_2 is an optimal coloring of H. By Claim 6.4.1, we conclude that $I(\varphi_1) \cap X = \{v_1\}$ and $I(\varphi_3) \cap X = \{v_2\}$, which gives $I(\varphi_1) \cap \overline{X} = I(\varphi_1) \setminus \{v_1\}$ and $I(\varphi_3) \cap \overline{X} = I(\varphi_3) \setminus \{v_2\}$. By Proposition 6.3(a), we conclude that $v_2 \in I(\varphi_2)$ and $I(\varphi_1) \setminus \{v_1\} = I(\varphi_2) \setminus \{v_2\}$. So φ_2 is an optimal coloring of H such that $v_2 \in I(\varphi_2)$ and $|I(\varphi_2)| = |I(\varphi_1)|$. To show that φ_2 is a v_2-extreme coloring of H, it suffices to show that $|I(\varphi_2)| \leq |I(\varphi_3)|$. By symmetry, the coloring $\varphi_4 = \varphi_1|_X \cup \varphi_3|_{\overline{X}}$ is an optimal coloring of H such that $v_1 \in I(\varphi_4)$ and $|I(\varphi_4)| = |I(\varphi_3)|$. Since φ_1 is a v_1-extreme coloring of H, we then conclude that $|I(\varphi_3)| = |I(\varphi_4)| \geq |I(\varphi_1)| = |I(\varphi_2)|$ as required. This completes the proof. △

Claim 6.4.3. *Let v be a vertex of H, and let φ be a v-extreme coloring of H. Then for every vertex v' of H there is a v'-extreme coloring φ' such that $I(\varphi) \setminus \{v\} = I(\varphi') \setminus \{v'\}$.*

Proof. The statement is evident if $v = v'$, otherwise, since $R(H)$ is a connected graph, there is a path $P = (v_0, v_1, \ldots, v_p)$ in $R(H)$ with $p \geq 1$, $v_0 = v$ and $v_p = v'$. Then $v_i v_{i+1}$ is an ordinary edge of $R(H)$ for $i = 0, 1, \ldots, p-1$. Starting with $\varphi_0 = \varphi$, it follows from Claim 6.4.2 that, for $i \in [1, p]$, there exists a v_i-extreme coloring φ_i of H such that $I(\varphi_{i-1}) \setminus \{v_{i-1}\} = I(\varphi_i) \setminus \{v_i\}$. Consequently, we obtain that $I(\varphi) \setminus \{v\} = I(\varphi_0) \setminus \{v_0\} = I(\varphi_p) \setminus \{v_p\}$, which proves the claim. △

To conclude the proof of Theorem 6.4, let v be an arbitrary vertex of H. Suppose, to the contrary, that there exists a v-extreme coloring φ of H such that $I(\varphi) \neq \{v\}$. Then there exists a vertex $v' \in I(\varphi) \setminus \{v\}$ and, by Claim 6.4.3, there is a v'-extreme coloring φ' such that $I(\varphi) \setminus \{v\} = I(\varphi') \setminus \{v'\}$, which is a contradiction since $v' \in I(\varphi) \setminus \{v\}$. Thus the proof of the theorem is complete. □

7. Hypergraphs whose order is near to χ

Let H be a simple hypergraph. A vertex v of H is called *universal* if $vw \in E(H)$ for all vertices $w \in V(H) \setminus \{v\}$; an edge e of H is called *universal* if $vw \in E(H)$ whenever $v \in e$ and $w \in V(H) \setminus e$. Let X be the set of universal vertices of H with $|X| = p$, let $e_1, e_2, \ldots, e_q$ be the universal hyperedges of H, where $|e_i| = n_i \geq 3$, and let $Y = V(H) \setminus (X \cup e_1 \cup e_2 \cup \cdots \cup e_q)$. Then $H[X]$ is a complete graph, $H[e_i] = K_{n_i}^{n_i}$ and $n_i \geq 3$ for $i = 1, 2, \ldots, q$, and

$$H = K_p + K_{n_1}^{n_1} + \cdots + K_{n_t}^{n_t} + H[Y],$$

where the remaining hypergraph $H[Y]$ has neither universal vertices nor universal edges. We shall apply Theorem 5.3 to deduce the following result.

Theorem 7.1. *Let H be a k-critical hypergraph, let p be the number of universal vertices of H, and let q be the number of universal hyperedges of H. Then the following statements hold:*

(a) *$0 \le p \le k$ and there exists a $(k-p)$-critical hypergraph H' such that $H = K_p + H'$, H' has no universal vertices, and $|H'| \ge \frac{3}{2}(k-p)$. Furthermore, $p \ge 3k - 2|H|$ and equality holds if and only if H' is the Dirac sum of $\frac{1}{2}(k-p)$ disjoint K_3^3's.*

(b) *$0 \le p + 2q \le k$ and there exists a $2q$-critical hypergraph H_1 and a $(k-p-2q)$-critical hypergraph H_2 such that $H = K_p + H_1 + H_2$, H_1 is the Dirac sum of q 2-critical hypergraphs each of order at least 3, H_2 has no universal vertices and no universal edges, and $|H_2| \ge \frac{5}{3}(k-p-2q)$. Furthermore, $2p + q \ge 5k - 3|H|$ and equality holds if and only if H_1 is the Dirac sum of q disjoint K_3^3's and $\frac{1}{3}(k-p-2q)$ disjoint hypergraphs belonging to* Cri$(3,5)$.

Proof. In what follows, let H be an arbitrary k-critical hypergraph. Then H is a simple hypergraph and

$$H = H_1 + H_2 + \cdots + H_t,$$

where $R(H_1), R(H_2), \dots, R(H_t)$ are the components of $R(H)$. For $i \in [1,t]$, let $k_i = \chi(H_i)$ and $n_i = |H_i|$. By Theorem 4.2, we obtain that

(i) $k = k_1 + k_2 + \cdots + k_t$ and H_i is a k_i-critical hypergraph for $i \in [1,t]$.

Since $R(H_i)$ is connected, Theorem 5.3 implies that

(ii) $|H_i| \ge 2k_i - 1$ for $i \in [1,t]$.

Since Cri$(1) = \{K_1\}$ and Cri$(2) = \{K_n^n \mid n \ge 2\}$, we then conclude that either $k_i = 1$ and $H_i = K_1$, or $k_i = 2$ and $H_i = K_{n_i}^{n_i}$ with $n_i \ge 3$, or $k_i \ge 3$ and $|H_i| \ge 5$. For a subset X of $[1,t]$, let

$$H_X = \sum_{i \in X} H_i \text{ and } k_X = \sum_{i \in X} k_i,$$

where $H_\varnothing = \varnothing$ and $k_\varnothing = 0$. By Theorem 4.2, H_X is a k_X-critical hypergraph. Let $P = \{i \in [1,t] \mid k_i = 1\}$, $Q = \{i \in [1,t] \mid k_i = 2\}$, $R = [1,t] \setminus (P \cup Q)$, $p = |P|$, $q = |Q|$, and $r = |R|$. Then P, Q and R are pairwise disjoint sets whose union is $[1,t]$. Thus we obtain that

(iii) $H = H_P + H_Q + H_R$, where $H_P = K_p$ and $H_Q = \sum_{i \in Q} K_{n_i}^{n_i}$.

Note that p is the number of universal vertices of H and q is the number of universal hyperedges of H. First we want to establish a lower bound for p. So let $\overline{P} = [1,t] \setminus P$. Then $\overline{P} = R \cup Q$ and $H = H_P + H_{\overline{P}}$. For $i \in \overline{P}$, we have that $k_i \geq 3$ and so, by (ii), $|H_i| \geq 2k_i - 1 \geq \frac{3}{2}k_i$, where equality holds if and only if $H_i = K_3^3$. For the order of H, it follows from (i) that

$$|H| = p + \sum_{i \in \overline{P}} |H_i| \geq p + \tfrac{3}{2} \sum_{i \in \overline{S}} k_i = p + \tfrac{3}{2}(k - p),$$

which is equivalent to $p \geq 3k - 2|H|$. Clearly, $p = 3k - 2|H|$ if and only if $H_{\overline{P}}$ is the Dirac sum of $\frac{1}{2}(k-p)$ disjoint K_3^3's. This proves (a).

For $i \in R$ we have that $k_i \geq 3$ and so, by (ii), $|H_i| \geq 2k_i - 1 \geq \frac{5}{3}k_i$, where equality holds if and only if $H_i \in \mathrm{Cri}(3,5)$. For the order of H we then obtain that

$$|H| = p + \sum_{i \in Q} |H_i| + \sum_{i \in R} |H_i| \geq p + 3q + \tfrac{5}{3} \sum_{i \in R} k_i = p + 3q + \tfrac{5}{3} k_R.$$

Since $k = k_P + k_Q + k_R = p + 2q + k_R$ (by (i)), it follows that $|H| \geq p + 3q + \frac{5}{3}(k - p - 2q)$, which is equivalent to $2p + q \geq 5k - 3|H|$. Clearly, $2p + q = 5k - 3|H|$ if and only if $H_i = K_3^3$ for all $i \in Q$ and $H_i \in \mathrm{Cri}(3,5)$ for all $i \in R$. Thus, (b) is proved. □

For a hypergraph K and a class of hypergraphs $\mathcal{H}$, define $K + \mathcal{H} = \{K + H \mid H \in \mathcal{H}\}$ if $\mathcal{H} \neq \varnothing$, and $K + \mathcal{H} = \varnothing$ otherwise. If $\mathcal{H}$ is a hypergraph property, then we do not distinguish between isomorphic hypergraphs, so we are only interested in the number of isomorphism types of $\mathcal{H}$, that is, the number of equivalence classes of $\mathcal{H}$ with respect to the isomorphism relation for hypergraphs.

The number of isomorphism types of the class $\mathrm{Cri}(k,n)$ is finite, where $\mathrm{Cri}(k,n) = \varnothing$ if $k > n$ and $\mathrm{Cri}(k,k) = \{K_k^2\}$. Furthermore, $\mathrm{Cri}(1,n) = \varnothing$ if $n > 1$ and $\mathrm{Cri}(2,n) = \{K_n^n\}$ if $n \geq 2$. From Theorem 7.1(a) we conclude that $\mathrm{Cri}(k,k+1) = K_1^2 + \mathrm{Cri}(k-1,k)$ if $k \geq 3$, which implies by induction on k that

$$\mathrm{Cri}(k,k+1) = \{K_{k-2}^2 + K_3^3\} \tag{4}$$

if $k \geq 2$. For the class $\mathrm{Cri}(4,6)$ we conclude from Theorem 7.1(b) that

$$\mathrm{Cri}(4,6) = (K_1^2 + \mathrm{Cri}(3,5)) \cup (K_3^3 + \mathrm{Cri}(2,3)).$$

By Theorem 7.1(a), it follows that $\mathrm{Cri}(k,k+2) = K_1^2 + \mathrm{Cri}(k-1,k+1)$ if $k \geq 5$, which implies by induction on k that

$$\mathrm{Cri}(k,k+2) = \{K_{k-4}^2 + K_3^3 + K_3^3\} \cup (K_{k-3}^2 + \mathrm{Cri}(3,5))$$

if $k \geq 4$. If $n = k+3$, then we conclude from Theorem 7.1(b) that

$$\begin{aligned}\mathrm{Cri}(5,8) &= (K_1^2 + \mathrm{Cri}(4,7)) \cup \bigcup_{q=3}^{5} (K_q^q + \mathrm{Cri}(3, 8-q)) \\ &= (K_1^2 + \mathrm{Cri}(4,7)) \cup (K_3^3 + \mathrm{Cri}(3,5)) \cup \mathrm{Cri}',\end{aligned}$$

where $\mathrm{Cri}' = \{K_1^2 + K_3^3 + K_4^4, K_3^2 + K_5^5\}$, and from Theorem 7.1(a) we conclude that

$$\mathrm{Cri}(6,9) = (K_1^2 + \mathrm{Cri}(5,8)) \cup \{K_3^3 + K_3^3 + K_3^3\}.$$

If $k \geq 7$, then Theorem 7.1(a) implies that

$$\mathrm{Cri}(k, k+3) = K_1^2 + \mathrm{Cri}(k-1, k+2).$$

By induction on k, we then conclude that if $k \geq 6$, then

$$\mathrm{Cri}(k, k+3) = (K_{k-4}^2 + \mathrm{Cri}(4,7)) \cup (K_{k-5}^2 + K_3^3 + \mathrm{Cri}(3,5)) \cup \mathrm{Cri}'',$$

where

$$\mathrm{Cri}'' = K_{k-6}^2 + \{K_2^2 + K_3^3 + K_4^4, K_4^2 + K_5^5, K_3^3 + K_3^3 + K_3^3\}.$$

The first interesting classes of critical hypergraphs that are indecomposable are the classes $\mathrm{Cri}(3,5)$, $\mathrm{Cri}(3,6)$ and $\mathrm{Cri}(4,7)$. Based on a computer search, the second authors established 9 isomorphism types for the class $\mathrm{Cri}(3,5)$ and 64 isomorphism types for the class $\mathrm{Cri}(3,6)$. That $\mathrm{Cri}(3,5)$ has indeed exactly 9 isomorphism types can be proved by a simple case analysis. Let $H_1, H_2, \ldots, H_8$ be the graphs shown in Figure 1, and let $H_9 = K_5^3$. The hypergraph H_3 is obtained from $H_1 = C_5$ by the enlarging operation. The hypergraphs H_4 and H_5 are obtained from H_3 by the enlarging operation, and H_6 is obtained from H_4 respectively H_5 by the enlarging operation.

Proposition 7.2. *The isomorphism types of the class* $\mathrm{Cri}(3,5)$ *are the hypergraphs* $H_1, H_2, \ldots, H_9$.

Sketch of Proof. Using Proposition 3.6 it is straightforward to show that $H_i \in \mathrm{Cri}(3,5)$ for $i = 1, 2, \ldots, 9$. Now let $H \in \mathrm{Cri}(3,5)$. Our aim is to show that H is isomorphic to H_i for some i. Since both H and H_i belong to $\mathrm{Cri}(3,5)$, it suffices to show that $H \subseteq H_i$ or $H_i \subseteq H$.

To this end, let m be the number of ordinary edges of H, and let G be the subhypergraph of H whose vertex set is $V(H)$ and whose edge set

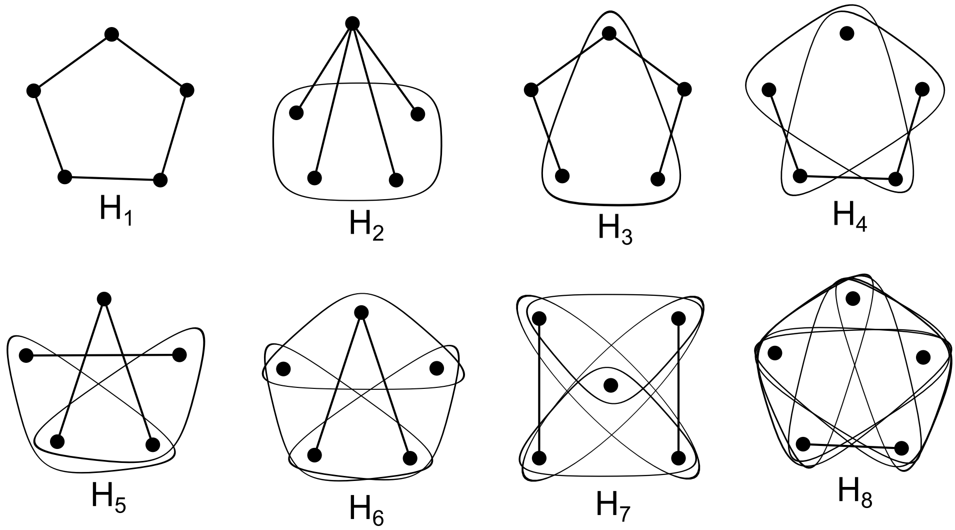

Figure 1: Hypergraphs of Cri$(3,5)$.

consists of the ordinary edges of H. So G is a graph. First assume that G contains a cycle C. If $|C|$ is odd, then C is a 3-critical subhypergraph of H and so $H = C$, implying that $H = H_1$. If $|C|$ is even, then let uv be an edge of C. Then there is a 2-coloring φ of $H \setminus uv$. Then φ is also a 2-coloring of $C \setminus uv$, and so $\varphi(u) \neq \varphi(v)$. But then φ is a 2-coloring of H, which is impossible. Now assume that G contains no cycle. Then G is a forest and so $m \leq |G| - 1 = 4$.

By Proposition 3.3(b)(d), H is a simple hypergraph with $\delta(H) \geq 2$ and every vertex v of H is contained in two edges having only vertex v in common. So if e is a hyperedge of H, then $3 \leq |e| \leq 4$. If $|e| = 4$ for an hyperedge e of H, then for every vertex $v \in e$ the ordinary edge vw, where w is the remaining vertex of $V(H) \setminus e$, must belong to H, and so $H_2 = K_1 + K_4^4 \subseteq H$, and we are done. So it remains to consider the case where $|e| = 3$ for every hyperedge e of H. To complete the proof, we distinguish three cases.

Case 1: $m = 4$. Then G is a tree of order 5 and either $K_{1,3} \subseteq G$ or $P_5 \subseteq G$. Clearly, H contains a hyperedge e with $|e| = 3$ and e is an independent set of G. So if $K_{1,3} \subseteq G$, we conclude that $H' = K_1^2 + K_3^3 \subseteq H$. Since H' is a 3-critical hypergraph, $H' = H$ and so $|H| = 4$, which is impossible. If $P_5 \subseteq G$, we conclude that $H_3 \subseteq H$ and we are done.

Case 2: $m = 3$. Then the three edges of G form a path, a path plus an independent edge, or a star. If G consists of the edges v_1v_2, v_2v_3 and v_3v_4, then let v_5 the remaining vertex. Then v_5 must be in two hyperedges having only this vertex in common, which leads to $H_4 \subseteq H$. If G consists of the edges v_1v_2, v_2v_3 and v_4v_5, then it follows from Proposition 3.3(b) that the two hyperedges $\{v_4, v_1, v_3\}$ and $\{v_5, v_1, v_3\}$ belong to H, and so $H_5 \subseteq H$. It remains to consider the case when the three edges of G form a star, say v_1v_2, v_1v_3 and v_1v_4. Let v_5 the remaining vertex of H. If $\{v_2, v_3, v_4\}$ is a hyperedge of H, then $H' = K_1^2 + K_3^3$ is a 3-critical hypergraph contained in H, and so $H = H'$ and $|H| = |H'| = 4$, a contradiction. Otherwise, the two sets $\{v_1, v_5\}$ and $\{v_2, v_3, v_4\}$ are independent sets of H, and so $\chi(H) = 2$, a contradiction.

Case 3: $m \leq 2$. Since H is simple, it is easy to check that $H_6 \subseteq H$ (if $m = 2$ and the two edges of G form a path), $H \subseteq H_7$ (if $m = 2$ and the two edges of G are independent), $H_8 \subseteq H$ (if $m = 1$), and $H \subseteq K_5^3 = H_9$ (if $m = 0$). □

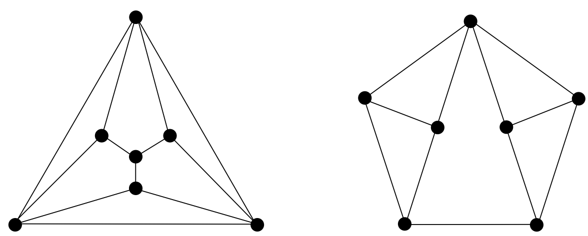

Figure 2: The only two 4-critical graphs on 7 vertices.

Let $\mathrm{Cri}^*(k, n)$ denote the subclass of $\mathrm{Cri}(k, n)$ containing all hypergraphs having no universal vertices and no universal edges. By Proposition 7.2, $\mathrm{Cri}^*(3, 5)$ has 8 isomorphism types. Furthermore, we obtain that

$$\mathrm{Cri}(3, 6) = \mathrm{Cri}^*(3, 6) \cup \{K_1^2 + K_5^5\}$$

and

$$\mathrm{Cri}(4, 7) = \mathrm{Cri}^*(4, 7) \cup (K_1^2 + \mathrm{Cri}^*(3, 6)) \cup \{K_2^2 + K_5^5, K_3^3 + K_4^4\}.$$

Lists of small critical graphs were determined by Toft [27], by Jensen and Royle [12], and by Royle (see his web page on small graphs). In particular, there are exactly two (non-isomorphic) 4-critical graphs G_1 and G_2 on 7 vertices, neither of which contains a universal vertex (see Figure 2). The graph $G_2 = K_4^2 \wedge K_4^2$ is a Hajós sum of two K_4^2's, so in this sense it is also decomposable.

8. Critical hypergraphs with few edges

One interesting feature of critical hypergraphs is the fact that their edge number increases with their order. So, it is an interesting task to investigate the extremal function $\mathrm{ext}(\cdot,\cdot)$ defined by

$$\mathrm{ext}(k,n) = \min\{|E(H)| \mid H \in \mathrm{Cri}(k,n)\}$$

and the corresponding class of extremal hypergraphs defined by

$$\mathrm{Ext}(k,n) = \{H \in \mathrm{Cri}(k,n) \mid |E(H)| = \mathrm{ext}(k,n)\},$$

where k and n are positive integers. For an integer $r \geq 2$, let $\mathrm{Cri}_r(k,n)$ be the subclass consisting of all r uniform hypergraphs of $\mathrm{Cri}(k,n)$. Furthermore, let $\mathrm{ext}_r(k,n) = \min\{|E(H)| \mid H \in \mathrm{Cri}_r(k,n)\}$ and let $\mathrm{Ext}_r(k,n)$ be the corresponding set of extremal r-uniform hypergraphs. Note that the function $\mathrm{ext}(k,n)$ is well defined for $n \geq k \geq 3$, while for the function $\mathrm{ext}_r(k,n)$ the situation is more complicated. For instance, there is no k-critical graph of order $k+1$. Provided that $\mathrm{Cri}_r(k,n) \neq \varnothing$, we obtain that $\mathrm{ext}(k,n) \leq \mathrm{ext}_r(k,n)$. Since k-critical hypergraphs have minimum degree at least $k-1$ (Proposition 3.3), $\mathrm{ext}_r(k,n) \geq \frac{1}{r}(k-1)n$. Over the years many improvements of this trivial lower bound have been presented in the graph theory literature, and we refer the reader to the survey by Kostochka [13]. That it is worthwhile to study the function $\mathrm{ext}_2(k,n)$ was first emphasized by Dirac [6] and subsequently by Gallai [9], [10] and by Ore [21]. Very recently Kostochka and Yancey [14] succeeded in determining the best linear approximation to the function $\mathrm{ext}_2(k,n)$.

Theorem 8.1. *If $n \geq k \geq 4$ and $n \neq k+1$, then*

$$\mathrm{ext}_2(k,n) \geq \frac{(k+1)(k-2)n - k(k-3)}{2(k-1)},$$

where equality holds if $n \equiv 1 \pmod{k-1}$.

The proof of this pioneering result, which is long and sophisticated, is based on the potential method. To see that equality can hold, let $g(k,n)$ denote the Kostochka–Yancey bound, that is,

$$g(k,n) = \tfrac{1}{2}(k - \frac{2}{k-1})n + c_k \text{ with } c_k = -\frac{k(k-3)}{2(k-1)}.$$

Then it is easy to check that the function g satisfies the following recursion

$$g(k,k) = \binom{k}{2} \text{ and } g(k, n+k-1) = g(k,n) + \binom{k}{2} - 1. \tag{5}$$

This recursion can be used to show that if $n \geq k$ and $n \equiv 1 \,(\mathrm{mod}\ k-1)$, then there is a k-critical graph of order n and with $g(k,n)$ edges. To this end, let G_1 be a k-critical graph and, for $\ell \geq 1$, let $G_{\ell+1} = G_\ell \wedge K_k$ be a Hajós sum of G_ℓ and K_k. By Theorem 4.1, G_ℓ is a k-critical graph for every $\ell \geq 1$. Let $n_\ell = |V(G_\ell)|$ and $m_\ell = |E(G_\ell)|$. For $\ell \geq 1$, we then obtain that

$$n_{\ell+1} = n_\ell + k - 1 \text{ and } m_{\ell+1} = m_\ell + \binom{k}{2} - 1.$$

So, if we take $G_1 = K_k$, we get $n_1 = k$ and $m_1 = \binom{k}{2} = g(k, n_1)$. Using the recursion (5), it follows by induction on ℓ that $m_\ell = g(k, n_\ell)$ and so $G_\ell \in \mathrm{Cri}_2(k, n_\ell)$ for all $\ell \geq 1$. Clearly, $\mathrm{Cri}_2(k,k) = \{K_k\}$. If we start the sequence G_ℓ with a graph $G_1^{(p)}$ having $k+p$ vertices and $m(k,k+p)$ edges, where $0 \leq p \leq k-2$, then we obtain an upper bound $h(k,n)$ of $\mathrm{ext}_2(k,n)$, where

$$h(k,n) = \tfrac{1}{2}(k - \frac{2}{k-1})n + c_{k,p}$$

with

$$c_{k,p} = m(k, k+p) - \tfrac{1}{2}(k - \frac{2}{k-1})(k+p).$$

It is notable that the additive term $c_{k,p}$ of the function h depends both on k and on the residue class $p+1$ of n (mod $(k-1)$). So, we get an upper bound for $\mathrm{ext}_2(k,n)$, that is, $\mathrm{ext}_2(k,n) \leq h(k,n)$. The best upper bound that we can achieve in this way is when we choose $G_1^{(p)}$ from the class $\mathrm{Ext}_2(k, k+p)$ so that $m(k,k+p) = \mathrm{ext}_2(k,k+p)$. Let us denote the resulting function in this case by $\mathrm{ore}(k,n)$, since this function was implicitly introduced by Ore. Then $\mathrm{ore}(k,n) = g(k,n)$, provided that $n \equiv 1 \,(\mathrm{mod}\ k-1)$. There is a small gap, since there is no k-critical graph of order $k+1$. In this case we can start the recursion with a graph of $\mathrm{Ext}_2(k, 2k)$. As proved by Kostochka and Stiebitz [15] $\mathrm{ext}_2(k,2k) = k^2 - 3$, which leads to

$$m(k,k+1) = \mathrm{ext}_2(k,2k) - \binom{k}{2} + 1 = \binom{k+1}{2} - 2 = \mathrm{ext}(k,k+1),$$

since $\mathrm{Cri}(k,k+1) = \mathrm{Ext}(k,k+1) = \{K_{k-2}^2 + K_3^3\}$. This implies that $\mathrm{ore}(k,n)$ is also an upper bound for $\mathrm{ext}(k,n)$. Gallai [10] used Theorem 5.1 in order to establish the exact values for the function $\mathrm{ext}_2(k,n)$ including a description of the extremal classes $\mathrm{Ext}_2(k,n)$, provided that $k+2 \le n \le 2k-1$.

For an integer $k \ge 3$, let $\mathcal{DG}(k)$ denote the family of all graphs G whose vertex set consists of three nonempty pairwise disjoint sets X, Y_1 and Y_2 with

$$|Y_1| + |Y_2| = |X| + 1 = k - 1$$

and two additional vertices v_1 and v_2 such that X and $Y_1 \cup Y_2$ are cliques in G not joined by any edge, and $N_G(v_i) = X \cup Y_i$ for $i = 1, 2$. This class of k-critical graphs with order $2k-1$ was introduced and investigated by Dirac [7] and by Gallai [9]. Note that all graphs in this class are indecomposable.

Theorem 8.2. (Gallai) *Let $n = k + p$ be an integer, where $k, p \in \mathbb{N}$ and $2 \le p \le k-1$. Then*

$$\mathrm{ext}_2(k,n) = \binom{n}{2} - (p^2+1) = \frac{1}{2}((k-1)n + p(k-p) - 2)$$

and $\mathrm{Ext}_2(k,n) = K_{k-p-1} + \mathcal{DG}(p+1)$.

It seems much more difficult to establish a lower bound for the function $\mathrm{ext}(k,n)$. In 1970 Lovász [18] proved that $\mathrm{ext}(3,n) = n$ for all $n \ge 3$. Other proofs were given by Woodall [28] in 1972, by Seymour [23] in 1974 and by Burstein [3] in 1976. Seymour's proof is particularly simple, based on elementary linear algebra. As pointed out in [25] the Kostochka–Yancey bound also holds for hypergraphs. This can be easily proved by induction on the number of hyperedges using the recursion (5) as well as a construction of Toft [26].

Theorem 8.3. *If $n \ge k \ge 4$, then* $\mathrm{ext}(k,n) \ge \frac{(k+1)(k-2)n-k(k-3)}{2(k-1)}$*, where equality holds if* $n \equiv 1 \pmod{k-1}$.

As an immediate consequence of Theorem 8.3 and Seymour's bound we obtain the following lower bound. It is notable that we cannot prove this trivial lower bound without using the Kostochka–Yancey bound and its extension to hypergraphs.

Corollary 8.4. *If $n \ge k \ge 3$, then* $\mathrm{ext}(k,n) \ge \frac{1}{2}(k-1)n$.

Based on Theorem 5.3 and Corollary 8.4 we shall prove a counterpart of Theorem 8.2.

Theorem 8.5. *Let $n = k+p$ be an integer, where $k, p \in \mathbb{N}$ and $1 \le p \le k-1$. Then*

$$\mathrm{ext}(k,n) = \binom{n}{2} - (p^2+1) = \frac{1}{2}((k-1)n + p(k-p) - 2)$$

and $\mathrm{Ext}(k,k+p) = K_{k-p-1} + \mathrm{Ext}(p+1, 2p+1)$.

Proof. Let H be a k-critical hypergraph of order n, where $n = k+p$ and $1 \leq p \leq k-1$. The number of edges of a hypergraph H' is denoted by $m(H')$. Furthermore, let

$$e(k,p) = \frac{1}{2}((k-1)(k+p) + p(k-p) - 2).$$

Our aim is to show that $m(H) \geq e(k,p)$ and that equality holds if and only if $H \in K_{k-p-1} + \mathrm{Ext}(p+1, 2p+1)$. The proof is by induction on k. The statement is evident if $k = 2$, since then $p = 1$, $H = K_3^3$ and $m(H) = e(2,3) = 1$. So assume that $k \geq 3$. If $p = 1$, then $H = K_{k-2}^2 + K_3^3$ (by (4)) and $2m(H) = 2e(k,1) = k^2 - 4 + k$, so we are done. If $p = k-1$, then $n = 2k-1$ and, by using Theorems 8.3 if $k \geq 4$ and Corollary 8.4 if $k = 3$, we obtain that $m(H) \geq \mathrm{ext}(k,n) \geq k^2 - k - 1 = e(k, k-1)$, which yields the desired result. So it remains to consider the case when $2 \leq p \leq k-2$. By Theorem 5.3 it follows that H is decomposable.

Case 1: *H has a universal vertex v.* Then $H' = H - v$ is a $(k-1)$-critical hypergraph of order $n' = k + p - 1$ with $m(H) = m(H') + n'$. From the induction hypothesis it follows that

$$m(H) = m(H') + k + p - 1 \geq e(k-1, p) + (k+p-1) = e(k,p).$$

Furthermore, $m(H) = e(k,p)$ is equivalent to $m(H') = e(k-1,p)$, which is equivalent to $H' \in K_{k-p-2} + \mathrm{Ext}(p+1, 2p+1)$ and hence to $H \in K_{k-p-1} + \mathrm{Ext}(p+1, 2p+1)$. So we are done.

Case 2: *H has a universal hyperedge e, but no universal vertex.* Then $H = K_q^q + H'$, where $q \geq 3$ and H' is a $(k-2)$-critical hypergraph of order $n' = k+p-q$. Since $k \geq 4$ and H' is a $(k-2)$-critical hypergraph of order n' with no universal vertex, $n' \geq k$ and so $q \leq p$. Then $n' = k+p-q \leq 2k-5$ and the induction hypothesis implies that $m(H') \geq e(k-2, p+2-q)$. Since $m(H) = m(H') + qn' + 1$ and $3 \leq q \leq p$, this leads to

$$\begin{aligned} 2m(H) &\geq 2e(k-2, p+2-q) + 2q(k+p-q) + 2 \\ &= 2e(k,p) + (q-2)(4p - 3q + 3) > 2e(k,p). \end{aligned}$$

So we are done.

Case 3: *H has no universal hyperedge and no universal vertex.* Then H is a simple hypergraph and

$$H = H_1 + H_2 + \cdots + H_t,$$

where $R(H_1), R(H_2), \ldots, R(H_t)$ are the components of $R(H)$. It follows from Theorem 5.3 that $t \geq 2$. For $i \in [1, t]$, let $k_i = \chi(H_i)$, $n_i = |H_i|$, $m_i = |E(H_i)|$ and $\overline{m}_i = \binom{n_i}{2} - m_i$. By Theorem 4.2, we obtain that

$$k = k_1 + k_2 + \cdots + k_t \text{ and } H_i \text{ is a } k_i\text{-critical hypergraph for } i \in [1, t].$$

By the assumption of the case, $k_i \geq 3$ for $i \in [1, t]$. Theorem 5.3 implies that $n_i \geq 2k_i - 1 \geq 5$ for $i \in [1, t]$. For a subset X of $[1, t]$, let

$$H_X = \sum_{i \in X} H_i, k_X = \sum_{i \in X} k_i, n_X = \sum_{i \in X} n_i \text{ and } \overline{m}_X = \sum_{i \in X} \overline{m}_i,$$

where the sum over the empty set is zero. By Theorem 4.2, H_X is a k_X-critical hypergraph whose order is n_X.

Our aim is to show that $m(H) \geq e(k, p) + 1$. To this end, we split the set $[1, t]$ into two subsets, namely $A = \{i \in [1, t] \mid n_i = 2k_i - 1\}$ and $B = \{i \in [1, t] \mid n_i \geq 2k_i\}$. Let $a = |A|$ and $b = |B|$. Since $A \cap B = \varnothing$ and $A \cup B = [1, t]$, we obtain that

$$H = H_A + H_B \text{ and } a + b = t \geq 1. \tag{6}$$

Using Theorem 5.3, we conclude from (6) that

$$k = k_A + k_B \text{ and } n = n_A + n_B. \tag{7}$$

The definition of A implies that

$$n_i = 2k_i - 1, k_i \geq 3 \text{ and } n_i \geq 5 \text{ whenever } i \in A, \tag{8}$$

from which we conclude that

$$k_A = \sum_{i \in A} k_i \geq 3a \text{ and } n_A = \sum_{i \in A} n_i = 2k_A - a. \tag{9}$$

Since H_i is a k_i-critical hypergraph with $k_i \geq 3$, we deduce from Corollary 8.4 that $2m_i \geq (k_i - 1)n_i$. Since $n_i = 2k_i - 1$ for $i \in A$ (by (8)), this yields

$$2\overline{m}_i = 2\binom{n_i}{2} - 2m_i \leq n_i(n_i - k_i) = \binom{n_i}{2} \text{ whenever } i \in A. \tag{10}$$

Using (8) and (9), we obtain that

$$2\overline{m}_A = \sum_{i \in A} 2\overline{m}_i \leq \sum_{i \in A} \binom{n_i}{2} \leq (a-1)\binom{5}{2} + \binom{n_A - 5(a-1)}{2},$$

which is equivalent to

$$2\overline{m}_A \leq 2(k_A - 3a)^2 + 9(k_A - 3a) + 10a. \tag{11}$$

Note that this inequality also holds if $a = 0$. If $b \geq 1$, then H_B is a k_B-critical hypergraph of order n_B, where

$$k_B \geq 3 \tag{12}$$

and

$$n_B \geq 2k_B. \tag{13}$$

By Corollary 8.4, we obtain that $2m(H_B) \geq (k_B - 1)n_B$. Let

$$\overline{m}(H_B) = \binom{n_B}{2} - m(H_B) \text{ and } \overline{m} = \overline{m}_A + \overline{m}(H_B).$$

Then

$$2\overline{m}(H_B) \leq n_B(n_B - k_B) = (n_B - k_B)^2 + k_B(n_B - k_B). \tag{14}$$

Note that (13) and (14) also holds if $b = 0$. Using (11) and (14), we obtain that

$$2\overline{m} \leq 2(k_A - 3a)^2 + 9(k_A - 3a) + 10a + (n_B - k_B)^2 + k_B(n_B - k_B). \tag{15}$$

Since $n \leq 2k - 2$, we deduce that $(a, b) \neq (1, 0)$. Next we show that

$$\overline{m} \leq p^2. \tag{16}$$

By (6) and the definition of $\overline{m}$, this is equivalent to $m(H) \geq \binom{n}{2} - p^2$ and hence to $m(H) \geq e(k, p) + 1$. Using (7) and (9), we obtain that

$$p = n - k = n_A - k_A + n_B - k_B = k_A - a + n_B - k_B,$$

which yields

$$\begin{aligned} 2p^2 &= 2((k_A-3a)+2a)^2+2(n_B-k_B)^2+4(k_A-a)(n_B-k_B) \\ &= 2(k_A-3a)^2+8a(k_A-3a)+8a^2+2(n_B-k_B)^2+ \\ &\quad 4(k_A-a)(n_B-k_B). \end{aligned}$$

Together with (15), this leads to

$$\begin{aligned} 2(p^2-\overline{m}) &\geq (8a-9)(k_A-3a)+a(8a-10)+ \\ &\quad (n_B-k_B)((n_B-2k_B)+4(k_A-a)). \end{aligned} \tag{17}$$

If $a \geq 2$, then (16) follows from (17), (9) and (13). If $a=1$ and $b \geq 1$, then $n_B - k_B \geq k_B \geq 3$ (by (12) and (13)). From (17), (9) and (13) we then conclude that

$$2(p^2-\overline{m}) \geq 11(k_B-1) > 0.$$

If $a=0$ and $b \geq 1$, then (16) follows from (17) and (13). Since $(a,b) \neq (1,0)$, this shows that (16) holds and hence $m(H) \geq e(k,p)+1$. This completes the proof of the theorem. □

Gallai [10] and Dirac [7] proved that $\mathrm{Ext}_2(k,2k-1) = \mathcal{DG}(k)$ if $k \geq 3$. By Theorem 8.2 and Theorem 8.5, it follows that $\mathrm{ext}(k,n) = \mathrm{ext}_2(k,n)$ for $k+2 \leq n \leq 2k-1$ and hence $\mathcal{DG}(k) \subseteq \mathrm{Ext}(k,2k-1)$ if $k \geq 3$. However, a complete description of the class $\mathrm{Ext}(k,2k-1)$ is unknown. Clearly, $\mathrm{Ext}(2,3) = \{K_3^3\}$ and $\mathrm{Ext}(3,5) = \{H_1,H_2,H_3,H_4,H_5,H_6\}$. It is also unknown whether every hypergraph of $\mathrm{Ext}(k,2k-1)$ belongs to $\mathcal{DG}(k)$ or is obtained from such a graph by the enlarging operation. Ore [21] conjectured that $\mathrm{ext}_2(k,n) = \mathrm{ore}(k,n)$ for all possible values (k,n), and our results support the conjecture that $\mathrm{ext}(k,n) = \mathrm{ext}_2(k,n)$.

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