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Inflations of anti-cycles and Hadwiger's Conjecture

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Dedicated to Adrian Bondy on his 70th birthday

Let G be any graph. An *inflation* of a graph G is obtained from G first by replacing each of its vertices with either the empty set or a complete graph, and then whenever two vertices x and y are adjacent in G we join each vertex of the replacement clique for x to each vertex of the replacement clique for y by an edge.

Let G be any inflation of the complement of an odd cycle. It is shown that (1) $\chi(G) = \max\{\omega(G), \lceil |V(G)|/2 \rceil\}$ and (2) $G \succeq K_{\chi(G)}$ by a short direct proof; that is to say, Hadwiger's Conjecture holds for G. We present a comprehensive survey of the area surrounding these results.

Keywords and phrases: Hadwiger's Conjecture, inflation, coloring, chromatic number, perfect graph, anti-cycle, quasi-line graph, vertex-critical, edge-critical.

1. Introduction

Hadwiger's Conjecture [9] from 1943 suggests a far reaching generalization of the Four Color Theorem, namely that any k-chromatic graph has the complete graph K_k as a minor.

Wolfgang Mader and Paul Seymour were the first persons to stress that an important special case of Hadwiger's Conjecture is obtained by adding the assumption that the independence number of the graph is no greater than 2. At first glance, this class of graphs may seem too restricted to be of interest until one realizes that the complements of such graphs are the triangle-free graphs, a graph class which has received considerable attention over the years.

By a result of Kim [11] it is known that there is a constant c such that there exist graphs G on n vertices with $\alpha(G) \leq 2$, and having clique number $\omega(G) \leq c\sqrt{n \log n}$. But since $\chi(G) \geq n/2$ for these graphs, Hadwiger's Conjecture would imply that we can contract G to $K_{\lceil n/2 \rceil}$, a clique much larger in size than a largest clique in G.

The special case of Hadwiger's Conjecture in which $\alpha(G) \leq 2$ seems to have been first considered in print by Duchet and Meyniel [6].

Chudnovsky and Seymour [5] proved that Hadwiger's Conjecture is true when $\alpha(G) \leq 2$ and G contains a clique on at least $|V(G)|/4$ vertices if $|V(G)|$ is even, and at least $(|(V(G)| + 3)/4$ vertices if $|V(G)|$ is odd.

In the present paper, we shall be concerned with *inflations* of a graph G. Such a graph is obtained from G by replacing each of its vertices with either the empty set or a complete graph, and then whenever two vertices x and y are adjacent in G we join each vertex of the replacement clique for x to each vertex of the replacement clique for y by an edge. The replacement cliques will be called "blobs". If $\alpha = 2$ in the original graph, then any inflation of the graph also satisfies $\alpha = 2$.

The k-chromatic graphs which have K_k as a subgraph and which, for all $t < k$, have K_t-subgraphs in all their induced t-chromatic subgraphs are called *perfect*. Lovász [15] proved that inflations of perfect graphs are perfect. It is not known if a similar inflation result holds for graphs satisfying Hadwiger's Conjecture. However, if G is a path, an even cycle or the complement of a path or even cycle, then G is perfect. Hence any inflation H of such a graph is also perfect and satisfies $\chi(H) = \omega(H)$ and Hadwiger's Conjecture is thus trivially satisfied.

In [18] it was proved that Hadwiger's Conjecture holds for all inflations of graphs G having independence number $\alpha(G) \leq 2$ and $|V(G)| \leq 11$. In [17] it was proved that Hadwiger's Conjecture holds for all inflations of the Petersen graph, and in [2] more generally that it holds for all inflations of 3-chromatic graphs.

A graph G is a *quasi-line graph* if the neighborhood of every vertex can be partitioned into two cliques. (Note that there may be edges joining the two cliques.) These graphs are characterized in [4] and constitute a superclass of the class of line graphs. Chudnovsky and Ovetsky Fradkin [3] proved that Hadwiger's Conjecture holds for all quasi-line graphs thus generalizing an earlier result of Reed and Seymour [19] for line graphs of multi-graphs. The class of quasi-line graphs is closed under making inflations, and hence Hadwiger's Conjecture holds for any inflation of a quasi-line graph. Since every cycle and complement of a cycle (and inflation of such a graph) is a quasi-line graph, it follows that any inflation of a cycle or a complement of a cycle also satisfies Hadwiger's Conjecture.

If G is any inflation of an odd cycle C_n, then $\chi(G) = \max\{\omega(G), \lceil(2|V(G)|/(n-1)\rceil\}$. That $\chi(G)$ is at least this maximum follows from the fact that $\alpha(G) \leq (n-1)/2$. The equality was first obtained by T. Gallai (oral communication, 1969). This result appeared in print as Exercise $5i$ (page 35) in [7], as Theorem 14.1.4 in [16] and it also follows from Theorem 1 in [14]. This theorem, due to T. Gallai, characterizes when an inflation

G of an odd cycle C_n is an edge-critical graph and in fact the characterization for vertex-criticality is the same. In the case when all blobs are non-empty, if we write $n = 2k + 1$, this characterization amounts to saying that any two neighboring blobs together have size at most $k - 1$ and $|V(G)| = (k-1)((n-1)/2) + 1$.

In this paper we consider inflations of the other type of minimal non-perfect graphs, namely the complements $\overline{C_{2p+1}}$ of the odd cycles (also known as *anti-cycles*). We obtain a formula for their chromatic number and give a direct proof that they satisfy Hadwiger's Conjecture. Moreover, we characterize when such a graph is vertex-critical.

We shall adopt the following terminology and notation. A *clique* in a graph G is a maximal set of pairwise adjacent vertices in G. The size of a largest clique in G is denoted by $\omega(G)$. The chromatic number of G is denoted by $\chi(G)$. We write $G \succeq H$ if G contracts to H, that is if H can be obtained from G by deleting edges and vertices and contracting edges. A graph G is *vertex-(chromatic)-critical* (or simply *vertex-critical*) if $\chi(G - x) < \chi(G)$ for every vertex $x \in V(G)$. A graph G is *perfect* if $\chi(H) = \omega(H)$, for every induced subgraph H of G.

2. Main results

In this section we begin by deriving a formula for the chromatic number of an inflation of the complement of an odd cycle of length at least 5. Such a graph G clearly has $\alpha(G) \leq 2$.

Let G be any graph with $\alpha(G) \leq 2$ and $\chi(G) = k$. Then for such a graph, we note two obvious facts: (i) $k \geq \omega(G)$ and (ii) $k \geq \lceil |V(G)|/2 \rceil$.

Theorem 2.1. *Suppose $p \geq 2$ and suppose G is an inflation of $\overline{C_{2p+1}}$. Then $\chi(G) = \max\{\omega(G), \lceil |V(G)|/2 \rceil\}$.*

Proof: Let H be k-vertex-critical subgraph of G, where $k = \chi(G)$. Then H is also an inflation of $\overline{C_{2p+1}}$.

Case 1: Suppose one of the blobs in H is empty. Then H is an inflation of the complement of a path. But a path, being bipartite, is perfect. Hence by [15] the complement of this path is also perfect. But also by [15] the inflation of a perfect graph is itself perfect. That is, H is perfect. Hence $\chi(H) = \omega(H)$ and hence $k = \omega(H) \leq \omega(G)$ and hence by (i), $k = \omega(G)$ and, since $p \geq 2$, by (ii) we also have $k = \max\{\omega(G), \lceil |V(G)|/2 \rceil\}$ and the result is true in Case 1.

Case 2: Suppose, then, that no blob in H is empty.

Let x be any vertex of H. Then $H-x$ has at most $2\chi(H-x) = 2(k-1) = 2k-2$ vertices, since at most two vertices can be colored the same color.

Case 2.1: Suppose first that $|V(H)| = 2k-1$. Then $k = \lceil |V(H)|/2 \rceil \leq \lceil |V(G)|/2 \rceil$. But we know that $k \geq \omega(G)$ and by (ii), $k = \lceil |V(G)|/2 \rceil$ and it follows that $\chi(G) = \max\{\omega(G), \lceil |V(G)|/2 \rceil\}$ as desired.

Case 2.2: So suppose $|V(H)| \leq 2k-2$. Then $\overline{H}$ is disconnected by Gallai [8] (Satz E_2.1). and hence H consists of the join of two non-empty critical subgraphs H_1 and H_2. Let the blobs of H be designated $B_1, B_2, \ldots, B_{2p+1}$ in clockwise cyclic order.

Clearly each blob B_i lies either entirely within H_1 or H_2. Suppose, without loss of generality, that $B_1 \subseteq H_1$. Then it is easy to see that $B_2 \subseteq H_1$ and, continuing, all the B_i must lie in H_1; i.e., $H_2 = \emptyset$, a contradiction. This completes the proof of Theorem 2.1. □

Remark 2.2. *The above result clearly fails to hold when $p = 1$, for just consider the trivial inflation of $\overline{K_3}$ in which each blob has size 1.*

Remark 2.3. *The fractional chromatic number $\chi^*(G)$ of a graph G is defined by $\chi^*(G) = \min_q(\chi(G[K_q])/q$, where $G[K_q]$ represents the inflation of G obtained by replacing each vertex by a copy of K_q. A perfect graph G has $\chi^*(G) = \chi(G)$; for example, $\chi^*(C_{2p}) = 2$ and $\chi^*(\overline{C_{2p}}) = p$. For $2p+1 \geq 5$, Gallai's result mentioned earlier implies that $\chi^*(C_{2p+1}) = 2 + 1/p$ and 2.1 implies that $\chi^*(\overline{C_{2p+1}}) = p + 1/2$.*

Although the next theorem follows from the characterization of quasi-line graphs found in [3] and also from the result in [5] mentioned above, we present a direct proof, as it is brief.

Theorem 2.4. *If $p \geq 1$ and G is an inflation of $\overline{C_{2p+1}}$, then $G \succeq K_{\chi(G)}$.*

Proof: If $p = 1$, the result is clearly true, so henceforth we suppose that $p \geq 2$.

The proof now proceeds by induction on $|V(G)|$. Suppose $|V(G)| \leq 2p$. Then there must be an empty blob and, as argued in the proof of Theorem 2.1, G must be perfect. But then $\chi(G) = \omega(G)$ and hence G contains $K_{\omega(G)}$, so $G \succeq K_{\omega(G)} = K_{\chi(G)}$.

Now suppose that $|V(G)| \geq 2p+1$ and that the theorem is true for all graphs with fewer than $|V(G)|$ vertices. Let K_ω be a maximum clique in G. Relabelling the blobs if necessary, without loss of generality we may assume that $V(B_1) \subseteq V(K_\omega)$. Hence $V(B_2) \cap V(K_\omega) = \emptyset$. Choose $x \in V(B_2)$. Let B_{p+2} and B_{p+3} be the "opposite" pair of blobs from B_2. At least one of these two blobs is not in K_ω. Choose y in such a blob. Then $K_\omega \subseteq G - x - y$.

If any of $B_1, B_2, \ldots, B_{2p+1}$ is empty, then arguing as before, G is perfect and therefore $\chi(G) = \omega(G)$. But then G contracts to $K_{\chi(G)}$ as desired. So suppose that all $2p+1$ blobs are non-empty.

Then edge $xy \in E(G)$. Moreover, the graph $G-x-y$ is also an inflation of $\overline{C_{2p+1}}$ and $\omega(G-x-y) = \omega(G)$.

Let us first suppose that $p \geq 3$. Applying the induction hypothesis to $G-x-y$, we have that $G-x-y \succeq K_{\chi(G-x-y)}$. But then $G \succeq K_{\chi(G-x-y)+1}$ where the extra vertex results from the contraction of the edge xy. (Note that here we use the assumption that $p \geq 3$.)

It remains to show that $\chi(G-x-y)+1 \geq \chi(G)$. By 2.1, this inequality may be rewritten as

$$\max\{\omega(G-x-y), \lceil |V(G)|/2 \rceil - 1\} + 1 \geq \max\{\omega(G), \lceil |V(G)|/2 \rceil\},$$

or, since $\omega(G-x-y) = \omega(G)$,

$$\max\{\omega(G), \lceil |V(G)|/2 \rceil - 1\} + 1 \geq \max\{\omega(G), \lceil |V(G)|/2 \rceil\},$$

that is,

$$\max\{\omega(G)+1, \lceil |V(G)|/2 \rceil\} \geq \max\{\omega(G), \lceil |V(G)|/2 \rceil\}.$$

But this is obvious.

Finally, suppose that $p = 2$. That is to say, G is an inflation of $\overline{C_5}$. But $\overline{C_5} \cong C_5$. So it remains to show that any inflation G of C_5 contracts to $K_{\chi(G)}$. However, in Pedersen's paper [17] there is a complete proof that (a) if G is any inflation of the 5-cycle, then $\chi(G) = \max\{\omega(G), \lceil |V(G)|/2 \rceil\}$ and (b) $G \succeq K_{\chi(G)}$; that is, Hadwiger's Conjecture holds for G. (Lemma 3 and Corollary 5, respectively.)

(Alternatively, the result in this $p = 2$ case follows from much stronger results in [18].) □

Remark 2.5. *The Hadwiger number of an inflation of $\overline{C_{2p+1}}$, that is, the largest h for which the graph has K_h as a minor, is not always equal to the chromatic number of the graph. The graph shown in Figure 1 is 5-chromatic, but if the edges ek, bh, bd and dh are contracted, a K_6-minor is obtained.*

One can ask also when the inflation of the complement of an odd cycle is k-vertex-critical. The next result provides an answer to this question.

Theorem 2.6. *Suppose $p \geq 2$ and let G be an inflation of $\overline{C_{2p+1}}$. Then G is k-vertex-critical and different from K_k if and only if*

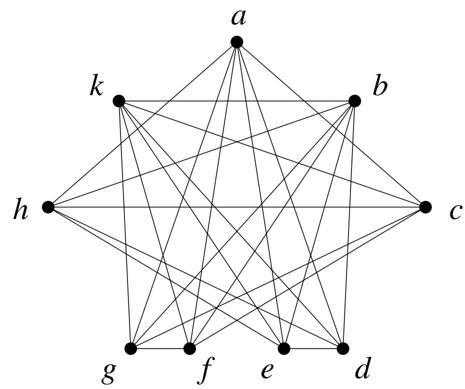

Figure 1: An inflation of $\overline{C_7}$.

(a) $k > \omega(G)$ *and*

(b) $|V(G)| = 2k - 1$.

Proof: Suppose that G is k-vertex-critical and $G \neq K_k$.

Part (a) is clear, so let us consider part (b).

From Theorem 2.1 and part (a), we have that $k = \max\{\omega(G), \lceil |V(G)|/2 \rceil\} = \lceil |V(G)|/2 \rceil$, and hence $|V(G)| = 2k - 1$ or $|V(G)| = 2k$. Now choose any $x \in V(G)$. Since $\chi(G - x) = k - 1$ and $\alpha(G - x) \leq 2$, we have $|V(G - x)| \leq 2k - 2$. Hence $|V(G)| \leq 2k - 1$. This proves part (b).

Let us now consider the converse. $G \neq K_k$ by part (b). Since $k > \omega(G)$, we have by Theorem 2.1 that $k = \chi(G) = \max\{\omega(G), \lceil |V(G)|/2 \rceil\} = \lceil |V(G)|/2 \rceil$. Now choose $x \in V(G)$. Then, again using Theorem 2.1 and the fact that $G - x$ is also the inflation of an anti-cycle, $\chi(G - x) = \max\{\omega(G - x), \lceil |V(G - x)|/2 \rceil\} \leq \max\{\omega(G), k - 1\} < k$. That is, G is k-vertex-critical. □

Remark 2.7. *One might further ask if the vertex-critical inflations of* $\overline{C_{2p+1}}$ *are also edge-critical. This is true for inflations of* C_{2p+1}*. However, this is not true for inflations of* $\overline{C_{2p+1}}$ *and the graph in Figure 1 is a counterexample. This graph is 5-vertex-critical, but not 5-edge-critical. However, if the four edges* af, ag, bh *and* ck *are deleted, a 5-edge-critical graph is obtained. In fact, the graph so obtained is a 9-vertex inflation of* $\overline{C_5}$ *in which four of the five vertices of* $\overline{C_5}$ *are replaced by complete 2-graphs.*

Remark 2.8. *The dodecahedron is a 3-chromatic graph and hence by the theorem of Casselgren and Pedersen [2], Hadwiger's Conjecture holds for all inflations of the dodecahedron. Is the chromatic number of an inflation of* *the dodecahedron always equal to the chromatic number of one of the inflated 5-cycles? Equivalently, if an inflation of the dodecahedron is k-chromatic, is there a k-critical subgraph equal to one of the inflated 5-cycles? This question seems to be unsolved.*

More generally, one could try to characterize those k-chromatic inflations of 3-chromatic graphs which have a k-critical inflated odd cycle as a subgraph. There are 3-chromatic graphs with k-chromatic inflations not having this property. An example may be found in [2].

Appendix

One referee suggested a different approach for proving Theorems 2.1 and 2.6 based on matching theory. This alternative proof is based upon the following lemma.

Lemma: Let G be an inflation $\overline{C_{2p+1}}$, for some $p \geq 2$. if $\omega \leq |V(G)|/2$, then $\overline{G}$ contains either a hamiltonian cycle or a 2-factor with only even cycles.

The proof, based on Hall's Theorem, is elementary, but not trivial. Our Theorems 2.1 and 2.6 can then be derived as corollaries of this lemma.

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