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Group-colouring, group-connectivity, claw-decompositions, and orientations in 5-edge-connected planar graphs

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In honour of Adrian Bondy's 70th birthday

Let G be a graph, let Γ be an Abelian group with identity 0_Γ, and, for each vertex v of G, let $p(v)$ be a prescription such that $\sum_{v\in V(G)} p(v) = 0_\Gamma$. A (Γ, p)-*flow* consists of an orientation D of G and, for each edge e of G, a label $f(e)$ in $\Gamma \setminus \{0_\Gamma\}$ such that, for each vertex v of G,

$$\sum_{e \text{ points in to } v} f(e) - \sum_{e \text{ points out from} v} f(e) = p(v).$$

If such an orientation D and labelling f exists for all such p, then G is Γ-*connected*.

Our main result is that if G is a 5-edge-connected planar graph and $|\Gamma| \geq 3$, then G is Γ-connected. This is equivalent to a dual colourability statement proved by Lai and Li (2007): planar graphs with girth at least 5 are "Γ-colourable". Our proof is considerably shorter than theirs. Moreover, the Γ-colourability result of Lai and Li is already a consequence of Thomassen's (2003) 3-list-colour proof for planar graphs of girth at least 5.

Our theorem (as well as the girth 5 colourability result) easily implies that every 5-edge-connected planar graph for which $|E(G)|$ is a multiple of 3 has a claw decomposition, resolving a question of Barát and Thomassen. It also easily implies the dual of Grötzsch's Theorem, that every planar graph without 1- or 3-cut has a 3-flow; this is equivalent to Grötzsch's Theorem.

1. Introduction

Barát and Thomassen [1] considered whether there is a particular edge-connectivity k_c so that every k_c-edge-connected graph G with $|E(G)|$ a mul-

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tiple of 3 has a claw-decomposition (for a simple graph, this means its edge set partitions into sets of size 3, each inducing a $K_{1,3}$). Thomassen [13] recently showed edge-connectivity 8 suffices. (This has recently been reduced to 6 by Lovász et al. [9].) Barát and Thomassen further thought that restricting attention to planar graphs would be more tractable. For example, Dehn [4] proved that every planar triangulation (minus a 3-cycle) has a claw-decomposition.

Jaeger et al. [5] introduced the notions of group colouring and group connectivity to generalize colourings and flows in graphs. Let Γ be an Abelian group with identity 0_Γ. Assign to each edge of a graph G an orientation and some element $w(e) \in \Gamma$. A (Γ, w)-*colouring* is a function $c : V(G) \to \Gamma$ such that, for each oriented edge uv of G, $c(u) \neq c(v) + w(uv)$. The graph G is Γ-*colourable* if, for every w, G has a (Γ, w)-colouring.

Likewise, if, for each vertex v of G, we assign an element $p(v) \in \Gamma$ such that $\sum_{v \in V(G)} p(v) = 0_\Gamma$, then a (Γ, p)-*flow* consists of an orientation D of G and a function $f : E(G) \to \Gamma \setminus \{0_\Gamma\}$ such that, for every vertex v of G,

$$\sum_{e \in \delta^+(v)} f(e) - \sum_{e \in \delta^-(v)} f(e) = p(v)\,.$$

Here $\delta^+(v)$ and $\delta^-(v)$ are the sets of edges oriented to point in to v and out from v, respectively. The graph G is Γ-*connected* if, for every p, there is a (Γ, p)-flow.

A consequence of a principal result in [5] is that a 2-connected planar graph G is Γ-colourable if and only if its dual is Γ-connected. This generalizes the usual duality between colourings of a planar graph and flows in its dual.

Thomassen [12] proved that planar graphs with girth 5 are 3-list colourable. As we discuss in the next section, his proof adapts almost verbatim, with only small changes in how the lists are manipulated, to prove that girth 5 planar graphs are Γ-colourable, as long as Γ is an Abelian group with at least three elements. Lai and Li [6] also prove this result, with a minor strengthening.

Combined with the duality result of [5], the colourability result implies that every 5-edge-connected planar graph G is Γ-connected for every Abelian group Γ with at least three elements. In particular, a 5-edge-connected planar graph G has a claw-decomposition if $|E(G)|$ a multiple of 3. Lai [7] presented a 4-edge-connected example that shows that 5-edge-connected cannot be reduced to 4-edge-connected.

The main purpose of this work is to give a substantially simpler proof of the dual form of the group colourability result. That is, we give a direct proof of the fact that every 5-edge-connected planar graph is Γ-connected, as long as Γ has at least three elements. This implies the claw-decomposition result mentioned in the preceding paragraph and it implies Tutte's 3-flow conjecture for planar graphs. This latter is the dual of Grötzsch's Theorem; a proof of the flow version can be found in Steinberg and Younger [10].

Our proof is found in Section 3. Some reflections on connections between list-colourability and group-colourabilty, and their dual statements, are in the next section. Our final Section 4 relates our theorem to nowhere-zero 3-flows and claw-decompositions in planar graphs.

2. List-colouring, group-colouring and group-connection

In this section, we provide some remarks and questions relating list-colouring and group-colouring and comments on group-connectivity. We begin by recalling the following two theorems due to Thomassen.

Theorem 2.1 ([11]). *Every simple planar graph is 5-list-colourable.*

Theorem 2.2 ([12]). *Every planar graph of girth at least* 5 *is 3-list-colourable.*

Lai and Li [6] adapted the proof of Theorem 2.2 to show that if Γ is an Abelian group with at least 3 elements, then G is Γ-colourable. Chuang, Lai, Omidi, Wang, and Zakeri [3] did the same type of adaption to Theorem 2.1. We point out below that virtually no adaption is required; the proofs in [11] and [12] apply directly.

The original version of this paper only proved that 5-edge-connected planar graphs are $\mathbb{Z}_3$-connected. At that time, we were unaware of [6, 3]. After learning of [6, 3], we recognized a few points that seem to be quite general.

The arguments in [11, 12] have the character of describing an ordering to colour the vertices such that, when a particular vertex is to be coloured, a colour available for that vertex is not used on any of its coloured neighbours. The mechanism is to use the sizes of the lists to guarantee such an available colour.

Theorem 2.3. *If Γ is a group with at least five elements, then every simple planar graph is Γ-colourable. Dually, every 3-edge-connected planar graph is Γ-connected.*

Theorem 2.4. *If Γ is a group with at least three elements, then every planar graph with girth at least 5 is Γ-colourable. Dually, every 5-edge-connected planar graph is Γ-connected.*

In fact, Theorems 2.1 and 2.2 generalize even further. The lists of size 5 or 3 can be chosen from an Abelian group of size at least 5 or 3 and there is still a group-colouring. It is not necessary to allow all group elements at each vertex, as long as there are enough of them.

We illustrate the essential idea using the last paragraph (page 181 in [11]) of the proof of Theorem 2.3. The vertex v_p in the boundary is deleted. Two colours x, y in $L(v_p) \setminus \{1\}$ are chosen (1 being the colour of v_1). In the (Γ, w)-colouring context, we have a label $w(v_1v_p)$ on the oriented edge v_1v_p and a colour $c(v_1)$ for v_1. We choose x, y as two colours in $L(v_p)$ such that $x \neq c(v_1) + w(v_1v_p)$ and $y \neq c(v_1) + w(v_1v_p)$. This guarantees that we can, at least with respect to v_1, freely colour v_p with either x or y.

The other choices to be made in this case are to remove two elements of each $L(u_i)$ depending on x, y. We may assume v_pu_i has this orientation, and we delete $x + w(v_pu_i)$ and $y + w(v_pu_i)$ from $L(u_i)$. We are then assured that the inductive colourings of the u_i will permit the use of either x or y at v_p.

For Theorem 2.4, the case of a separating pentagon in the proof from [12] also requires a modification of the edge-weights.

With respect to our proof in the next section, a similar phenomenon occurs. We proved our result originally only thinking about $\mathbb{Z}_3$-connection. The significant point in the proof is that, anytime you have two or more unoriented and unlabelled edges at a vertex u, there is an orientation and non-zero labelling for these edges so as to realize $p(u)$.

This fact is true as long as the group Γ has at least two non-zero elements. Therefore, it was a very simple matter to edit our argument to the form presented here.

A very natural question is motivated by the well-known fact that the number of nowhere-zero Γ-flows on G depends only on $|\Gamma|$. It was already raised by Jaeger et al. [5], even for the smallest case $\Gamma = \mathbb{Z}_4$ and $\Omega = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Question 2.5 ([5]). Suppose G is a graph and Γ and Ω are finite Abelian groups such that $|\Gamma| = |\Omega|$. Is it true that if G is Γ-connected, then G is Ω-connected?

It is our belief that many theorems about k-list colourings will generalize to Γ-colourability for the reasons mentioned above.

We can easily show that the following variation of Brooks' Theorem holds.

Theorem 2.6. *Let G be a connected, simple graph with maximum degree Δ and let Γ be any group with at least Δ elements. Then either: G is Γ-colourable; or $|\Gamma| = 2$ and G is a cycle (possibly even); or G is $K_{|\Gamma|+1}$.*

Proof. The proof of Brooks' Theorem in [2] applies in much the same way. One first uses induction to show that we may assume G is $|\Gamma|$-regular, so $|\Gamma| = \Delta$. The cases $\Delta \le 2$ are trivial, so we assume $\Delta \ge 3$.

If G is 2-connected, then we use the extension of the Lovász proof [8], described in detail in [2]: either G is complete or there exist vertices x, y, z such that xz, yz are edges of G, xy is not an edge, and $G - \{x, y\}$ is connected. The Γ-colouring is assured in this case by first colouring x and y so as to both forbid the same colour at z. Then colour the vertices of $G - \{x, y\}$ starting with those furthest from z in $G - \{x, y\}$ and finishing with z. For a vertex of $G - \{x, y, z\}$, when it is coloured it has an uncoloured neighbour, so fewer than Δ colours are forbidden. When z is coloured, the choices of the colours at x and y imply that fewer than Δ colours are forbidden. Thus, in all cases, some colour is always available to colour each vertex of $G - \{x, y\}$, yielding the desired Γ-colouring of G.

In the case G has a cut-vertex, let H and K be subgraphs of G, each with at least one edge, such that $G = H \cup K$, and $H \cap K$ consists of a single vertex v. Since G is Δ-regular and connected, neither H nor K is Δ-regular and, therefore, each has an appropriate Γ-colouring, say c_H and c_K.

If $c_H(v) \ne c_K(v)$, we adjust the colouring of c_K by replacing each $c_K(w)$ with $c_K(w) + (c_H(v) - c_K(v))$. This is still a Γ-colouring of K and now $c_H(v) = c_K(v)$, so c_H and c_K combine to give the required Γ-colouring of G. □

We remark that graphs with multiple edges (but no loops) are interesting in the context of group-colouring. The proof above shows that, if G is a connected graph such that, for some weights $w(e)$, G has no (Γ, w)-colouring, where $|\Gamma| = \Delta$, the simplification of G is either a cycle or complete, and G is Δ-regular. Is there a non-trivial characterization of the triples (G, Γ, w) (where $|\Gamma| = \Delta$) such that G has no (Γ, w)-colouring?

3. Group-connection of 5-edge-connected planar graphs

In general, our graphs are loopless, but may have multiple edges. A cut $\delta(A)$ in a graph G is *non-peripheral* if both $|A| \ge 2$ and $|V(G) \setminus A| \ge 2$. For an orientation D of G and subset A of $V(G)$, we use $\delta^+(A)$ for $\{e \in E(G) \mid e \text{ points in to } A \text{ in } D\}$ and $\delta^-(A)$ for $\{e \in E(G) \mid e \text{ points out from } A \text{ in } D\}$. Clearly, $\delta(A) = \delta^+(A) \cup \delta^-(A)$.

For an Abelian group Γ, we set $\Gamma^* = \Gamma \setminus \{0_\Gamma\}$. For an orientation D of G and labelling $f : E(G) \to \Gamma^*$, for each $v \in V(G)$, we set

$$F(v) = \sum_{e \in \delta^+(v)} f(e) - \sum_{e \in \delta^-(v)} f(e)\,.$$

Theorem 3.1. *Let Γ be any Abelian group with at least three elements and let G be a 3-edge-connected graph embedded in the plane with at most two specified vertices d and t such that:*

1. *if d exists, then it has degree 3, 4, or 5, has its incident edges oriented and labelled with elements in Γ^*, and is in the boundary of the unbounded face;*
2. *if t exists, then it has degree 3 and is in the boundary of the unbounded face;*
3. *there are at most two 3-cuts, which can only be $\delta(\{d\})$ and $\delta(\{t\})$;*
4. *if d has degree 5, then t does not exist; and*
5. *every vertex not in the boundary of the unbounded face has five edge-disjoint paths to the boundary of the unbounded face.*

If every vertex v has a prescribed flow $p(v) \in \Gamma$ such that $\sum_v p(v) = 0_\Gamma$, and $F(d) = p(d)$, then the given orientation and flow extends to a (Γ, p)-flow.

In the proof, we refer to the subgraph of G consisting of those vertices and edges incident with the unbounded face as the *boundary*. A vertex or edge not in the boundary is *interior*.

Proof. The proof is by induction on the number of edges.

We start with an elementary observation that is used throughout the remainder of this work. Suppose x is a vertex such that all edges incident with x except e and f are oriented and labelled with elements of Γ^*. For any orientations of e and f, and any choice of element of Γ^* on e, there is an element of Γ that may be assigned to f to realize $p(x)$. Since there are at least two choices from Γ^* to assign to e, at least one of them will also give a non-zero assignment to f.

We start with some **prior reductions**.

(PR1) *A 2-cycle consisting of unoriented edges.*
Contract the entire set of multiple edges, any two of which make a 2-cycle. The result has a flow f by induction. Complete the flow on G by orienting and placing flows on the contracted edges to realize the prescription at one of their incident vertices. The other incident vertex will automatically realize its prescription, since there cannot be only one vertex not realizing its prescription.
Henceforth, we assume G has no 2-cycle of unoriented edges.

(PR2) *A cut-vertex.*
If v is a cut-vertex, then the edge-connectivity implies v has degree at least 6 and so is neither t nor d. Let H and K be

non-trivial subgraphs, each with at least two vertices, so that $G = H \cup K$ and $H \cap K$ is just v. If $\deg_H(v) \geq 4$, then the induction applies directly to H. If $\deg_H(v) = 3$, then H is just v and one of d and t. (In fact, H is just v and d by (PR1)). The prescription for v in both H and K is determined by the remaining vertices in the appropriate subgraph. In all cases, both H and K have (Γ, p)-flows, yielding the desired flow for G. Henceforth, we assume G is 2-connected.

(PR3) *A non-peripheral 4-cut* $\delta(A)$.
Choose the labelling of A so that d, if it exists, is in A; contract $G - A$ to a vertex to get G_1 and use induction to obtain a (Γ, p)-flow on G_1. Now contract $G[A]$ in G to become the vertex v in the graph G_2. The edges incident with v inherit their orientations and flow values from G_1. Use induction to obtain a (Γ, p)-flow on G_2. The flows combine to yield the desired flow on G.

(PR4) *A non-peripheral 5-cut that does not separate d from t.*
This is essentially the same as for a non-peripheral 4-cut.

(PR5) *d has degree 5.*
Let e be one of the boundary edges incident with d, with v its other end.
Let $G' = G - e$, let α in Γ^* be the label of e, and set the prescription p' on G' to be p except that the tail x of e satisfies $p'(x) = p(x) + \alpha$ and the head y of e satisfies $p'(y) = p(y) - \alpha$. Note that v is the only possible degree 3 vertex in G'; all other vertices have degree at least 4 and every interior vertex of G' has (in G and therefore in G') five edge-disjoint paths to the boundary of G'.
Since every non-peripheral cut in G has size at least 4, clearly every non-peripheral cut in G' has size at least 3, so G' is 3-edge-connected. Moreover, if $|\delta_{G'}(A)| = 3$, then $e \in \delta_G(A)$, so that G has a non-peripheral 4-cut, handled by (PR3).
Henceforth we assume that, if d exists, then $\deg(d) \leq 4$.

(PR6) *A non-peripheral 5-cut $\delta(A)$ such that $\delta(A) \cap \delta(\{t\})$ has precisely one edge, which is in the boundary.*
Suppose $\delta(A)$ is such a 5-cut in G such that $d \in A$. By (PR4), we know $t \notin A$. Suppose some edge incident with t is in $\delta(A)$.
First contract $V \setminus A$ to orient $G/(V \setminus A)$ by induction. Then contract A to a, with the edges incident with a inheriting their orientations from $G/(V \setminus A)$. In G/A, a has degree 5 and t has

degree 3. Orient the edges incident with t (two are not incident with a) and assign elements from Γ^* to them to achieve $p(t)$, delete t, and adjust the labels on the neighbours appropriately. The vertex a now has degree 4 in $(G/A) - t$ and at most one vertex in $(G/A) - t$ has degree 3. Since any non-peripheral cut of size s in $(G/A) - t$ yields a non-peripheral cut in $G - t$ of size s, there is a non-peripheral cut in G of size at most $s + 1$. Since $s + 1 \geq 5$, $s \geq 4$, so the induction applies to $(G/A) - t$.

(PR7) *An undirected chord of the cycle bounding the unbounded face and incident with a vertex of degree 3 or 4.*
Let uv be a chord with u having degree 3 or 4. Let H and K be the two subgraphs of G so that $G = H \cup K$ and $H \cap K$ is just uv, u, and v; the labelling is chosen so that d, if it exists, is in H. By (PR1), K has some vertex other than u and v and, therefore, at least two such vertices. Thus, (PR3) shows $|\delta(V(H))| \geq 5$.
Contract uv in H. The prescription at the vertex of contraction is determined so as to make the sum of prescriptions in H/uv add up to 0_Γ. The induction yields an orientation of H/uv. To apply the induction to K, we first orient and put non-zero flows on the (2 or 3) edges of K incident with u to combine with the orientations and flows on the edges of H incident with u to realize $p(u)$, then add one new edge directed from u to v, with flow $\alpha \in \Gamma^*$, and apply the induction to $K + uv$ (with $p'(u) = p(u) + \alpha$ and $p'(v)$ selected to make the sum of prescriptions in $K + uv$ equal to 0_Γ). The combination of (Γ, p)-flows yields the desired flow on G.

We now consider all the various possibilities for the existence of d and t.

1. *d does not exist.*
   If some boundary vertex has degree at most 4, then orient and label its edges to achieve its prescription. Otherwise, orient a boundary edge e with label $\alpha \in \Gamma^*$, add α to the prescription of its tail, subtract α from the prescription of its head, and delete e.
   In the case e is deleted, no non-peripheral 3-cut is introduced, as such a 3-cut $\delta_{G-e}(A)$ implies $\delta_G(A)$ is a non-peripheral 4-cut in G, allowing the reduction (PR3). Thus Conditions 1–4 are evidently satisfied in $G - e$. Any vertex v not in the boundary of the infinite face of $G - e$ is not in the boundary of the infinite face of G and, therefore, there are five edge-disjoint paths from v to the boundary of the infinite face of G. These paths are also five edge-disjoint paths from v to the boundary

of the infinite face of $G - e$, yielding 5. There is a (Γ, p)-flow on $G - e$ and, therefore, including the orientation and labelling of e, on G. Henceforth, we assume d exists. In our future reductions, the verifications of Conditions 1–5 all follow the same routine. Only when non-peripheral 3-cuts may be introduced is there something significant to check; we consider this possibility when relevant below.

2. *d and t are adjacent.*
   Orient and label the two edges at t other than dt to realize $p(t)$ and contract dt to become d (with a new prescription) in the new graph. The induction completes the orientation of G.
3. *d has degree 4 and t does not exist.*
   Delete one of the boundary edges e incident with d, and adjust the prescriptions on the ends of e.
4. *d has degree 3 and t does not exist.*
   Delete d and appropriately change the prescriptions at its neighbours (for example, if e is incident with d and points in to u with label α, then $p(u)$ changes to $p(u) - \alpha$). The only possible degree 3 vertices in $G - d$ are the neighbours of d incident with the unbounded face. If both have degree 3, then appropriately orient the edges incident with one of the two. This oriented vertex becomes d while the other becomes t. In this case, $G - d$ has no non-peripheral 3-cut, since d may be reintroduced so as to increase the size of the cut to at most 4 in G, yielding reduction (PR3).

   Henceforth, we may assume d and t both exist. By hypothesis, d has degree either 3 or 4. The boundary neighbours of t are u and v, while its third neighbour is w.
5. *At least one of u or v has degree 5 or more.*
   By (PR7), no edge incident with t is a chord of the cycle bounding the unbounded face. Orient and label the three edges incident with t to achieve $p(t)$, delete t, and appropriately modify the prescriptions on the neighbours of t. The resulting graph $G - t$ has no non-peripheral 3-cut for the same reason as in 4.
   Henceforth, we assume both u and v have degree 4 in G.
   A principle method we adopt is to *lift* two edges incident with u; there is an analogous lift at v. This means deleting the two edges ux and uy and adding a new edge xy (even if one already exists), which we call e_u. In our context, the two edges at u will always be the boundary edge ux that is not ut, and its interior neighbour uy. If du is an edge, then we orient and label e_u to be consistent at d with du.

A (Γ, p)-flow of the resulting graph will yield a (Γ, p)-flow of G in which the direction and label of e_u are naturally transferred to give directions and labels to ux and uy. These will not affect the net flow $F(u)$.

6. *u is not adjacent to w.*
   As described just above, lift the two edges at u, orient and label the remaining two edges incident with u to realize its prescription, and orient and label the other two edges at t to realize $p(t)$. Delete the four oriented and labelled edges and adjust accordingly the prescriptions on the remaining neighbours of u and t. Let G' be the resulting graph. The only 3-vertex in G' other than d is v.
   There are five edges joining $G[\{u,t\}]$ to $G - \{u,t\}$. If $\delta_{G'}(A)$ is a non-peripheral cut in G', then either $\delta_G(A)$ or $\delta_G(A \cup \{u,t\})$ has size at most two more than $\delta_{G'}(A)$. Since both these cuts in G have size at least 5, $|\delta_{G'}(A)| \geq 3$.
   Suppose that $\delta_{G'}(A)$ is a non-peripheral 3-cut in G' such that $d \in A$. If $|\delta_G(A \cup \{u,t\})| = 5$, then $\delta_G(A \cup \{u,t\})$ is a 5-cut that does not separate d and t, allowing reduction (PR4). Thus, we may assume $|\delta_G(A \cup \{u,t\})| \geq 6$. Therefore, at least three of the five edges incident with u and t, other than ut, have an end in $V(G) \setminus (A \cup \{u,t\})$.
   It follows that at most two of the five edges incident with u and t, other than ut, have an end in A. Thus, $|\delta_G(A)| \leq 5$. Reduction (PR3) allows us to assume $|\delta_G(A)| \geq 5$, in which case $|\delta_G(A)| = 5$.
   If t is incident with an edge of $\delta_G(A)$, then we have reduction (PR6). Therefore, both edges incident with t but not with u have their ends in $V(G) \setminus (A \cup \{u,t\})$. In this case, $|\delta(A \cup \{u\})| = 5$ and t is incident with an edge of $\delta(A \cup \{u\})$, again allowing reduction (PR6).

   This leaves one final case.
7. *Both u and v are adjacent to w.*
   Let e_1 and e_2 be the two edges incident with v but not incident with either t or w and, for $i = 1, 2$, let v_i be the other end of e_i. Orient and label e_1 and e_2 to realize $p(v)$. Delete e_1 and e_2 and adjust the prescriptions at v_1 and v_2 accordingly. The new prescription at v is 0_Γ. Choose the labelling of e_1 and e_2 so that e_1 and v_1 are in the boundary. Let G_v be obtained from $G - \{e_1, e_2\}$ by contracting the subgraph induced by $\{t, u, v, w\}$ to a single vertex x. Note that $\deg_{G_v}(x) = (\deg_G(u) - 2) + (\deg_G(w) - 3) \geq 4$. Thus, v_1 is the only possible degree 3 vertex in G_v other than d.

The prescription at x is the one required to make the sum of all prescriptions in G_v be 0_Γ.
If $\delta_{G_v}(A)$ is any non-peripheral cut in G_v, labelled so that $x \notin A$, then $\delta_G(A)| \subseteq \delta_{G_v}(A) \cup \{e_1, e_2\}$. By reduction (PR3), we may assume $|\delta_G(A)| \geq 5$, so $|\delta_{G_v}(A)| \geq 3$.
Suppose $\delta_{G_v}(A)$ is a non-peripheral 3-cut in G_v, labelled so that $x \notin A$. Then $|\delta_G(A)| = 5$ and $e_1, e_2 \in \delta_G(A)$.
Evidently, $\deg(v) = 4$ and $vt, vw \notin \delta_G(A)$, so $|\delta_G(A \cup \{v\})| = 5$. Since $u, t, w \notin A$, $\delta_G(A \cup \{v\})$ is non-peripheral. But $tv \in \delta_G(A \cup \{v\})$, yielding reduction (PR6). Thus, we may assume G_v has no non-peripheral 3-cuts. In this case, the induction shows G_v has a (Γ, p)-flow. The flows on the edges in G_v are transferred to G. The orientations and labels on e_1 and e_2 are known. Orient and label the edges uw and ut to satisfy $p(u)$ and then orient and label tw and tv to satisfy $p(t)$. Finally, orient and label vw so that tv and vw make a directed path of length 2 with the same labels. Observe that w is the only vertex for which it is possible that $F(w)$ is not equal to $p(w)$. However, there cannot be only one mismatch, so we have a (Γ, p)-flow on G. □

4. 3-flows and claw-decompositions in planar graphs

In this section, we give the simple applications of Theorem 3.1 to 3-flows and claw-decompositions. The first is the dual of Grötzsch's Theorem.

Theorem 4.1. *Every planar graph without 1- or 3-cut has a balanced mod-3 orientation, or, equivalently, a nowhere-zero 3-flow.*

Proof. It is easy and standard to use the induction to eliminate any 0- or 2-cut. Now, in addition to no 1- or 3-cut, we add the condition that there is an oriented vertex d of degree 4 or 5 whose incident edges are already oriented so that the in- and out-degrees at d are the same modulo 3. The induction now easily applies if there is a non-peripheral 4- or 5-cut.

Next we lift a degree 4 vertex, if there is one, into two edges and apply induction. In the remaining case, Theorem 3.1 applies with all prescriptions $0_{\mathbb{Z}_3}$; the resulting orientation is a nowhere-zero $\mathbb{Z}_3$-flow. In a standard way (see, for example, [14]), this implies there is a nowhere-zero 3-flow. □

In the context of a graph G with multiple edges but not loops, a *claw* is three edges incident with a common vertex and a *claw-decomposition* of G is a partition of $E(G)$ into claws.

Theorem 4.2. *Every 5-edge-connected planar graph G for which $|E(G)|$ is a multiple of 3 has a claw-decomposition.*

Proof. Apply Theorem 3.1 with $\Gamma = \mathbb{Z}_3$ and, for each vertex v of G, set $p(v)$ to be, modulo 3, the degree $d(v)$. Let f be the resulting nowhere-zero $\mathbb{Z}_3$-flow. For each edge e of G with $f(e) = 2$, reverse the orientation and replace the label 2 with 1. The result is a new $\mathbb{Z}_3$-flow f' such that, for every edge e, $f'(e) = 1$.

In f', the number $in(v)$ of edges pointing in to v in f' less the number $out(v)$ of edges pointing out is congruent to $d(v) \pmod 3$. We also trivially have $in(v) + out(v) = d(v)$. From these two congruences, we conclude that $in(v) \equiv d(v) \pmod 3$ and $out(v) \equiv 0 \pmod 3$. Thus, the out-going arcs can be divided into claws. □

In particular, if a planar triangulation is 5-edge-connected, then it has a claw decomposition. More generally, we have the following.

Corollary 4.3. *If G is a simple triangulation of the plane other than K_4, then G has a claw decomposition.*

Proof. We may assume the triangulation is not 5-edge-connected. If it is not 4-edge-connected, then it has a vertex v of degree 3. If it is 4-edge-connected, not 5-edge-connected, and not K_4, then it has a vertex v of degree 4. In both cases we delete v. In order to apply the induction when $\deg(v) = 4$, we may add one of the two edges in the face of length 4 to get a smaller simple triangulation. In either case, if the smaller triangulation is not K_4, then we apply the induction.

In both cases, if the smaller triangulation is K_4, then the original triangulation is $K_5 - e$, which has a claw decomposition, with each claw having a degree 4 vertex as its centre. □

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