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The adjoint representation of a classical Lie algebra and the support of Kostant's weight multiplicity formula

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Even though weight multiplicity formulas, such as Kostant's formula, exist their computational use is extremely cumbersome. In fact, even in cases when the multiplicity is well understood, the number of terms considered in Kostant's formula is factorial in the rank of the Lie algebra and the value of the partition function is unknown. In this paper, we address the difficult question: *What are the contributing terms to the multiplicity of the zero-weight in the adjoint representation of a finite-dimensional classical Lie algebra?* We describe and enumerate the cardinalities of these sets (through linear homogeneous recurrence relations with constant coefficients) for the classical Lie algebras $\mathfrak{so}_{2r+1}(\mathbb{C})$, $\mathfrak{sp}_{2r}(\mathbb{C})$, and $\mathfrak{so}_{2r}(\mathbb{C})$. The $\mathfrak{sl}_{r+1}(\mathbb{C})$ case was computed by the first author in [7]. In addition, we compute the cardinality of the set of contributing terms for non-zero weight spaces in the adjoint representation. In the $\mathfrak{so}_{2r+1}(\mathbb{C})$ case, the cardinality of one such non-zero weight is enumerated by the Fibonacci numbers.

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1. Introduction

The following problem arises in representation theory of complex semisimple Lie algebras: For a dominant weight λ, what is the multiplicity of the weight μ in the irreducible representation with highest weight λ, which we denote by $L(\lambda)$? This problem dates back to Hermann Weyl, [16], and it continues to attract the attention of present day mathematicians.

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The first approaches addressing this question stemmed from formulas such as the Weyl character formula. However in 1948 Kostant developed his well-known formula for computing the multiplicity of a weight in an irreducible highest weight representation, [11]. This formula consists of an alternating sum over the Weyl group and involves the computation of a partition function. Despite the availability of such a formula, using it for computational purposes can be quite daunting.

In terms of computational complexity, we note that in [13], Narayanan proved that the problem of computing Kostka numbers and Littlewood-Richardson coefficients is $\#P$-complete. This implies that "... unless $P = NP$, which is widely disbelieved, there do not exist efficient algorithms that compute these numbers." Since the Kostka number $K_{\lambda,\mu}$ also can be interpreted as the multiplicity of the weight μ in the representation of $\mathfrak{sl}_{r+1}(\mathbb{C})$ with highest weight λ, it is clear that computing weight multiplicities, in much generality, is a computationally complex problem. However, there are cases when computing weight multiplicities can be done in polynomial time. Take for example computing the set of all non-zero Kostka numbers for a particular μ, [1].

This shows us that the existence of formulas, such as that of Kostant, provide a means to compute weight multiplicities, yet the computation itself is difficult and time-consuming. The complications that arise when using Kostant's formula are due to the fact the number of terms appearing in the alternating sum is factorial in the rank of the Lie algebra and the value of the partition function involved is very often unknown. In this paper, we focus on addressing the issue of the support of the partition function and show that even in cases when the multiplicity is well understood, the support is not.

The depth of this approach is appreciated through the following question:

What is the multiplicity of the zero-weight in the adjoint representation of a finite-dimensional classical Lie algebra?

Lie theory provides the answer almost instantly: the multiplicity of the zero-weight is the rank of the Lie algebra. In this paper, we address the difficult question:

What elements of the Weyl group contribute (non-trivially) to the multiplicity of the zero-weight in the adjoint representation of a finite-dimensional classical Lie algebra?

Clearly the benefit of such an approach greatly reduces the computational expense. Moreover, this method of computing weight multiplicities is of interest to both combinatorialists and geometers. In terms of combinatorics, the methods used to determine the cardinality of the support (as we will show) lead to new integer sequences and recurrence relations. For geometers, the act of finding the support of Kostant's weight multiplicity is related to finding the set of elements in the Weyl group (which are reflections about a hyperplane perpendicular to a simple root), which "move" a weight to points on a lattice lying within a specified polytope.

In this paper we take a combinatorial approach and we let $\mathcal{A}_r(\tilde{\alpha}, 0)$ denote the set of elements of the Weyl group which contribute non-trivially to the multiplicity of the zero-weight in the adjoint representation of a classical Lie algebra of rank r. We show that the cardinality of the contributing sets satisfy linear homogeneous recurrence relations with constant coefficients. To provide our main results we define the following recurrence relations:

- For $r \geq 2$, let $a_r = a_{r-1} + a_{r-2}$, with $a_0 = 0$ and $a_1 = 1$
- For $r \geq 4$, let $b_r = b_{r-1} + b_{r-2} + 3b_{r-3} + b_{r-4}$, with $b_0 = b_1 = 0$, $b_2 = 2$ and $b_3 = 3$
- For $r \geq 4$, let $c_r = c_{r-1} + c_{r-2} + 3c_{r-3} + c_{r-4}$, with $c_0 = 0$, $c_1 = c_2 = 1$ and $c_3 = 2$
- For $r \geq 8$, let $d_r = d_{r-1} + d_{r-2} + 3d_{r-3} + d_{r-4}$, with $d_4 = 4$, $d_5 = 7$, $d_6 = 14$ and $d_7 = 34$.

Theorem 1.1 (Cardinality of the supporting set)**.** *Let $\mathfrak{g}$ be a classical Lie algebra of rank r and let $\tilde{\alpha}$ denote its highest root. Then*

- *If $\mathfrak{g} = \mathfrak{sl}_{r+1}(\mathbb{C})$ and $r \geq 2$, then $|\mathcal{A}_(\tilde{\alpha}, 0)| = a_r$.*
- *If $\mathfrak{g} = \mathfrak{so}_{2r+1}(\mathbb{C})$ and $r \geq 4$, then $|\mathcal{A}_(\tilde{\alpha}, 0)| = b_r + b_{r-1} + b_{r-2}$.*
- *If $\mathfrak{g} = \mathfrak{sp}_{2r}(\mathbb{C})$ and $r \geq 4$, then $|\mathcal{A}_(\tilde{\alpha}, 0)| = c_r + c_{r-1}$.*
- *If $\mathfrak{g} = \mathfrak{so}_{2r}(\mathbb{C})$ and $r \geq 8$, then $|\mathcal{A}_(\tilde{\alpha}, 0)| = 2d_r + d_{r-1}$.*

To illustrate the computational benefit of knowing the cardinality of $\mathcal{A}_r(\tilde{\alpha}, 0)$ we provide the first few terms in these sequences:[1]

[1]The sequence of integers for type A consists of the Fibonacci numbers. The sequences of integers for types B, C, and D were added by the authors to The On-Line Encyclopedia of Integer Sequences (OEIS) as A232163, A232165, and A234598, respectively.

Type A_r ($r \geq 2$): 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, ...
Type B_r ($r \geq 2$): 2, 5, 10, 22, 49, 106, 231, 506, 1104, 2409, 5262, 11489, 25082, 54766, ...
Type C_r ($r \geq 2$): 2, 3, 8, 18, 37, 82, 181, 392, 856, 1873, 4086, 8919, 19480, 42530, 92853, ...
Type D_r ($r \geq 4$): 9, 18, 35, 82, 180, 385, 846, 1853, 4034, 8810, 19249, 42014, 91727, 200298, ...

Observe that in the type A case the number of elements of the Weyl group is given by $r!$, in Lie type B and C the order of W is $2^r(r!)$, and in type D the order is $2^{r-1}(r!)$. It is clear that the cardinality of $\mathcal{A}(\tilde{\alpha}, 0)$ is extremely small when compared to the order of the Weyl group and thus greatly reduces the computations. As an example consider the Lie algebra C_8, whose Weyl group order is 110100480, yet only 181 of these elements actually contribute to the multiplicity formula!

In the same spirit, we address the multiplicity of other non-zero weights in the adjoint representation. However, it is fundamental in Lie theory that the non-zero weight spaces of the adjoint representation of $\mathfrak{g}$ are the roots and have multiplicity 1. In Section 7, we compute the cardinality of the set of contributing terms for specific non-zero weight spaces in the adjoint representation. In the type B case, we show that the number of contributing terms to one such non-zero weight is enumerated by the Fibonacci numbers.

Our approach in this paper proves that while the number of terms appearing in Kostant's formula grows factorially with the rank of the Lie algebra, in these cases, the number of terms that contribute non-trivially to the multiplicity formula only grow exponentially. Moreover, the results in this paper are inspired by the following theorem of Kostant, [11]: If $\mathfrak{g}$ is a simple Lie algebra of rank r with highest root $\tilde{\alpha}$, then $m_q(\tilde{\alpha}, 0) = \sum_{i=1}^{r} q^{e_i}$, where m_q denotes the q-analog of Kostant's weight multiplicity formula (as defined in [12]) and $e_1, e_2, \ldots, e_r$ are the exponents of the Lie algebra.

Knowing which elements of the Weyl group contribute non-trivially to the multiplicity of the zero-weight in the adjoint representation will provide a stepping stone for further work on providing a purely combinatorial proof of the result of Kostant's regarding the exponents of the classical Lie algebras of type B, C, and[2] D. This will be the focus of future work.

Sections 2 and 3 provide the necessary background and some general results which facilitate the proof of Theorem 1.1. In Sections 4-6, we describe the elements of $\mathcal{A}_r(\tilde{\alpha}, 0)$ for the classical Lie algebras $\mathfrak{so}_{2r+1}(\mathbb{C})$, $\mathfrak{sp}_{2r}(\mathbb{C})$,

[2]The type A case was fully considered in [7].

and $\mathfrak{so}_{2r}(\mathbb{C})$, which we refer to as the Lie algebras of type B, C, and D, respectively. The Lie algebra $\mathfrak{sl}_{r+1}(\mathbb{C})$ (type A) was considered by the first author in [7].

2. Background

Throughout this article we let G be a simple linear algebraic group over $\mathbb{C}$, T a maximal algebraic torus in G of dimension r, and B, $T \subseteq B \subseteq G$, a choice of Borel subgroup. Then let $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{b}$ denote the Lie algebras of G, T, and B respectively. We let Φ be the set of roots corresponding to $(\mathfrak{g}, \mathfrak{h})$, and let $\Phi^+ \subseteq \Phi$ be the set of positive roots with respect to $\mathfrak{b}$. Let $\Delta \subseteq \Phi^+$ be the set of simple roots. Denote the set of integral and dominant integral weights by $P(\mathfrak{g})$ and $P_+(\mathfrak{g})$, respectively. Let $W = Norm_G(T)/T$ denote the Weyl group corresponding to G and T, and for any $w \in W$, we let $\ell(w)$ denote the length of w.

A finite-dimensional complex irreducible representation of $\mathfrak{g}$ is equivalent to a highest weight representation with dominant integral highest weight λ, which we denote by $L(\lambda)$. To find the multiplicity of a weight μ in $L(\lambda)$, we use Kostant's weight multiplicity formula, [10]:

$$m(\lambda, \mu) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \wp(\sigma(\lambda + \rho) - (\mu + \rho)), \tag{1}$$

where $\wp$ denotes Kostant's partition function and $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$. Recall that Kostant's partition function is the nonnegative integer valued function, $\wp$, defined on $\mathfrak{h}^*$, by $\wp(\xi)$ = number of ways ξ may be written as a nonnegative integral sum of positive roots, for $\xi \in \mathfrak{h}^*$.

With the aim of describing the contributing terms of (1) we introduce.

Definition 2.1. *For λ, μ dominant integral weights of $\mathfrak{g}$ define the* Weyl alternation set *to be*

$$\mathcal{A}(\lambda, \mu) = \{\sigma \in W : \ \wp(\sigma(\lambda + \rho) - (\mu + \rho)) > 0\}.$$

Therefore, $\sigma \in \mathcal{A}(\lambda, \mu)$ if and only if $\sigma(\lambda+\rho)-(\mu+\rho)$ can be written as a nonnegative integral combination of positive roots. Moreover, in the simple Lie algebra cases, the positive roots are made up of certain nonnegative integral sums of simple roots. Hence we can reduce to $\sigma \in \mathcal{A}(\lambda, \mu)$ if and only if $\sigma(\lambda+\rho)-(\mu+\rho)$ can be written as a nonnegative integral combination of simple roots. We recall that by definition $\wp(0) = 1$, and hence if $\sigma \in W$ and $\sigma(\lambda + \rho) = \mu + \rho$, then $\sigma \in \mathcal{A}(\tilde{\alpha}, 0)$.

Our goal is to describe the elements of the Weyl group which contribute to the multiplicity of the zero-weight in the adjoint representation of the classical Lie algebras. Namely, we compute the Weyl alternation sets, $\mathcal{A}(\tilde{\alpha}, 0)$, where $\tilde{\alpha}$ denotes the highest root of a simple Lie algebra $\mathfrak{g}$. To achieve this goal, Subsection 2.1 gives some necessary combinatorial background used in the proofs of our main results.

2.1. Weyl groups acting on root spaces

It is widely established throughout the literature that the finite-dimensional Lie algebras are classified by the Dynkin diagrams in Figure 1, [8]. The action of a simple reflection $s_i \in W$ on the set of simple roots Δ is also characterized by these Dynkin diagrams. We will now recall a short synopsis of this classification for ease of reference and refer the reader to [2], [3], and [9] for more on the combinatorics of Weyl (Coxeter) groups and their actions on roots.

By the definition of a simple reflection, for any two simple roots α_i and α_j we have $s_i(\alpha_i) = -\alpha_i$, $s_j(\alpha_j) = -\alpha_j$, $s_i(\alpha_j) = \alpha_j + c_{ij}\alpha_i$ and $s_j(\alpha_i) = \alpha_i + c_{ji}\alpha_j$. The integers c_{ij} and c_{ji} are in the set $\{0, 1, 2, 3\}$, and their particular values are determined by how many edges (and their direction) connect the nodes α_i and α_j in the Dynkin diagrams as summarized in Figure 2, [3]. It is then a simple exercise to show that s_i permutes the remaining positive roots [2, Lemma 4.4.3].

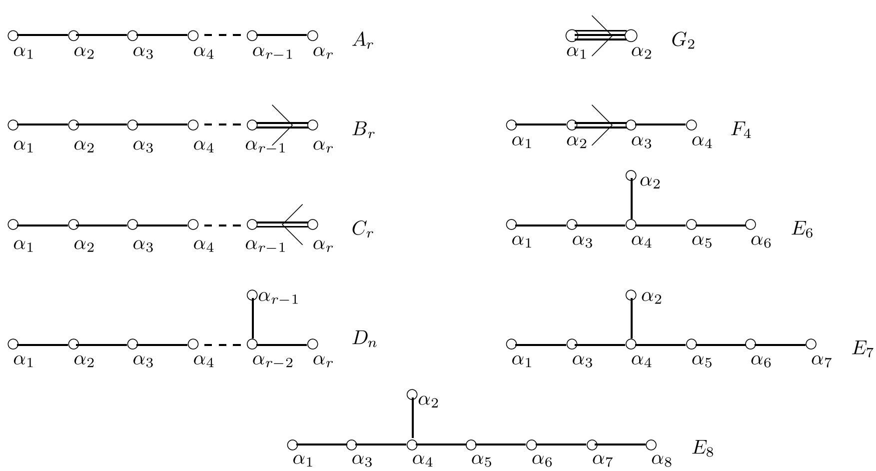

Figure 1: Dynkin diagrams.

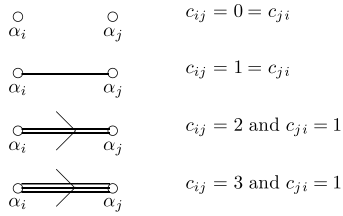

Figure 2: Three cases for the values of c_{ij}.

3. General results for classical Lie algebras

We begin with some general results regarding the classical Lie algebras. First we give some preliminary information for each of the Lie algebras we consider. We follow the notation used by Goodman and Wallach [6].

Type A_r ($\mathfrak{sl}_r(\mathbb{C})$): Let $r \geq 1$ and let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq r$. Then $\Delta = \{\alpha_i \mid 1 \leq i \leq r\}$, is a set of simple roots. The associated set of positive roots is $\Phi^+ = \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq r\}$, where the highest root is $\tilde{\alpha} = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ and $\rho = \frac{1}{2}\sum_{i=1}^{r} i(r-i+1)\alpha_i$. For $1 \leq i \leq r-1$ we have $s_i(\alpha_i) = -\alpha_i$, $s_i(\alpha_{i-1}) = \alpha_{i-1} + \alpha_i$, and $s_i(\alpha_{i+1}) = \alpha_i + \alpha_{i+1}$. For $i = r$, we have that $s_r(\alpha_r) = -\alpha_r$ and $s_r(\alpha_{r-1}) = \alpha_{r-1} + \alpha_r$.

Type B_r ($\mathfrak{so}_{2r+1}(\mathbb{C})$): Let $r \geq 2$ and let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq r-1$ and $\alpha_r = \varepsilon_r$. Then $\Delta = \{\alpha_i \mid 1 \leq i \leq r\}$, is a set of simple roots. The associated set of positive roots is $\Phi^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j : 1 \leq i < j \leq r\} \cup \{\varepsilon_i : 1 \leq i \leq r\}$, where the highest root is $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_r$ and $\rho = \frac{1}{2}\sum_{i=1}^{r} i(2r-i)\alpha_i$. For $1 \leq i \leq r-1$ we have $s_i(\alpha_i) = -\alpha_i$, $s_i(\alpha_{i-1}) = \alpha_{i-1} + \alpha_i$, and $s_i(\alpha_{i+1}) = \alpha_i + \alpha_{i+1}$. For $i = r$ we have that $s_r(\alpha_r) = -\alpha_r$ and $s_r(\alpha_{r-1}) = \alpha_{r-1} + 2\alpha_r$.

Type C_r ($\mathfrak{sp}_{2r}(\mathbb{C})$): Let $r \geq 3$ and let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq r-1$ and $\alpha_r = 2\varepsilon_r$. Then $\Delta = \{\alpha_i \mid 1 \leq i \leq r\}$, is a set of simple roots. The associated set of positive roots is $\Phi^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j : 1 \leq i < j \leq r\} \cup \{2\varepsilon_i : 1 \leq i \leq r\}$, where the highest root is $\tilde{\alpha} = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{r-1} + \alpha_r$ and $\rho = \frac{1}{2}\sum_{i=1}^{r-1} i(2r-i+1)\alpha_i + \frac{r(r+1)}{4}\alpha_r$. For $1 \leq i \leq r$ we have $s_i(\alpha_i) = -\alpha_i$, $s_i(\alpha_{i-1}) = \alpha_{i-1} + \alpha_i$. For $1 \leq i \leq r-2$, $s_i(\alpha_{i+1}) = \alpha_i + \alpha_{i+1}$, while $s_{r-1}(\alpha_r) = 2\alpha_{r-1} + \alpha_r$.

Type D_r ($\mathfrak{so}_{2r}(\mathbb{C})$): Let $r \geq 4$ and let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq r-1$ and $\alpha_r = \varepsilon_{r-1} + \varepsilon_r$. Then $\Delta = \{\alpha_i \mid 1 \leq i \leq r\}$, is a set of simple roots. The associated set of positive roots is $\Phi^+ = \{\varepsilon_i - \varepsilon_j,\ \varepsilon_i + \varepsilon_j \mid 1 \le i < j \le r\}$, where the highest root is $\tilde{\alpha} = \varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$ and $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha = (r-1)\varepsilon_1 + (r-2)\varepsilon_2 + (r-3)\varepsilon_3 + \cdots 2\varepsilon_{r-2} + \varepsilon_{r-1} = \frac{1}{2}\sum_{i=1}^{r-2} 2(ir - \frac{i(i+1)}{2})\alpha_i + \frac{r(r-1)}{4}(\alpha_{r-1} + \alpha_r)$. For $1 \le i \le r$, $s_i(\alpha_i) = -\alpha_i$. If $1 \le i < j \le r-1$ with $|i-j| = 1$ or if $i = r-2$ and $j = r$, then $s_i(\alpha_j) = s_j(\alpha_i) = \alpha_i + \alpha_j$.

Lemma 3.1. *The following simple transpositions do not fix the highest root in each respective Lie type.*

- *In type A_r: $s_1(\tilde{\alpha}) = \tilde{\alpha} - \alpha_1$ and $s_r(\tilde{\alpha}) = \tilde{\alpha} - \alpha_r$,*
- *In type B_r: $s_2(\tilde{\alpha}) = \tilde{\alpha} - \alpha_2$,*
- *In type C_r: $s_1(\tilde{\alpha}) = \tilde{\alpha} - 2\alpha_1$,*
- *In type D_r: $s_2(\tilde{\alpha}) = \tilde{\alpha} - \alpha_2$.*

The rest of simple reflections fix the highest root $s_i(\tilde{\alpha}) = \tilde{\alpha}$.

Notice that Lemma 3.1 follows from how simple transpositions act on simple roots as detailed in Section 2. For any Lie type we can define $\rho = \varpi_1 + \varpi_2 + \cdots + \varpi_r$, where ϖ_i is dual to the coroots. Then by definition if $1 \le i, j \le r$, then $s_i(\varpi_j) = \varpi_j - \delta_{i,j}\alpha_i$, where $\delta_{i,j}$ is the Kronecker delta function whose value is 1 if $i = j$ and 0 otherwise.

Lemma 3.2. *In any Lie type, if s_i is a simple root reflection, then $s_i(\rho) = \rho - \alpha_i$.*

Proof. By definition of ρ and the action of a simple root, we have that $s_i(\rho) = s_i(\varpi_1 + \varpi_2 + \cdots + \varpi_r) = \varpi_1 + \cdots + \varpi_{i-1} + (\varpi_i - \alpha_i) + \varpi_{i+1} + \cdots + \varpi_r = \rho - \alpha_i$. □

If β and γ are elements of $\mathfrak{h}^*$, we say $\beta \le \gamma$ if and only if $\beta = a_1\alpha_1 + \cdots + a_r\alpha_r$ and $\gamma = b_1\alpha_1 + \cdots + b_r\alpha_r$ where $a_i \le b_i$ for all $1 \le i \le r$. The analogous definition holds for a strict inequality. Also recall that for any $\sigma \in W$, $\ell(\sigma)$ denotes the length of σ. Given the above setup we can now prove the following lemma.

Lemma 3.3. *Let $\sigma \in W_r$ for any Lie type. If $\sigma(\rho) = \rho - \sum_{i\in J} c_i\alpha_i$, for some $J \subseteq \{1, 2, 3, \ldots, r\}$, and if there exists $i \in J$ such that $c_i \ge 3$, then $\sigma \notin \mathcal{A}(\tilde{\alpha}, 0)$.*

Proof. It follows from Lemma 3.2 that $\sigma(\rho) = \rho - \sum_{i\in J} c_i\alpha_i$ for some $J \subseteq \{1, 2, \ldots, r\}$ where c_i is a positive integer for every $i \in J$. Now notice that for any $\sigma \in W_r$ we have that $\sigma(\tilde{\alpha}) \le \tilde{\alpha}$. Hence

$$\sigma(\tilde{\alpha} + \rho) - \rho \le \tilde{\alpha} + \sigma(\rho) - \rho = \tilde{\alpha} - \sum_{i\in J} c_i\alpha_i.$$

Since $\tilde{\alpha}$ is the highest root we know that for any $1 \le i \le r$ the coefficient of α_i in $\tilde{\alpha}$ is either 1 or 2. So if there exists an index $i \in J$ for which $c_i \ge 3$, then the coefficient of α_i in the expression $\tilde{\alpha} - \sum_{i \in J} c_i \alpha_i$ would be negative. This would also be the case for the coefficient of α_i in the expression $\sigma(\tilde{\alpha}+\rho)-\rho$, since $\sigma(\tilde{\alpha}+\rho)-\rho \le \tilde{\alpha} - \sum_{i \in J} c_i \alpha_i$. Therefore $\sigma \notin \mathcal{A}(\tilde{\alpha}, 0)$. □

Let $N(\sigma) = \{\alpha \in \Phi^+ : \sigma(\alpha) \in -\Phi^+\}$ denote the set of positive roots that are negated by σ. It is well-known that $|N(\sigma)| = \ell(\sigma)$ for any $\sigma \in W$ [2, Proposition 4.4.4]. Hence the set $N(\sigma) \subset \Phi^+$ is nonempty for any $\sigma \neq e$. We conclude that the weight $\sigma(\rho)$ is always less than or equal to ρ, and hence the weight $\sigma(\rho) - \rho$ can be expressed as a sum of negative roots in the following way:

$$\sigma(\rho) - \rho = \sum_{\alpha \in N(\sigma)} \sigma(\alpha) = -\sum_{\alpha_j \in \Delta} c_j \alpha_j \text{ with } c_j \ge 0 \text{ for all } \alpha_j \in \Delta.$$

The following Proposition proves that if σ contains σ' as a subword then

$$-\left(\sigma(\rho) - \rho\right) > -\left(\sigma'(\rho) - \rho\right).$$

While the result seems intuitive, we could not find a reference in the literature, and even our proof is rather cumbersome. Hence we would welcome an elegant proof.

Proposition 3.4. *Let s_i denote the simple root reflection corresponding to $\alpha_i \in \Delta$. Write the weights $\sigma(\rho) - \rho$, $\sigma s_i(\rho) - \rho$, and $s_i\sigma(\rho) - \rho$ as linear combinations of positive roots as follows:*

- $\sigma(\rho) - \rho = -\sum_{\alpha_j \in \Delta} c_j \alpha_j$ *with* $c_j \ge 0$,
- $\sigma s_i(\rho) - \rho = -\sum_{\alpha_j \in \Delta} d_j \alpha_j$ *with* $d_j \ge 0$, *and*
- $s_i\sigma(\rho) - \rho = -\sum_{\alpha_j \in \Delta} e_j \alpha_j$ *with* $e_j \ge 0$.

Then:

1. *If $\ell(\sigma s_i) > \ell(\sigma)$, then $d_j \ge c_j$ for all j and $d_k > c_k$ for at least one $\alpha_k \in \Delta$.*
2. *If $\ell(s_i\sigma) > \ell(\sigma)$, then $e_j \ge c_j$ for all j with $e_i > c_i$.*

The proof of Proposition 3.4 will use the following lemma. We pull it aside now so that the verification of this fact does not disrupt the proof of Proposition 3.4.

Lemma 3.5. *If $\ell(s_i\sigma) > \ell(\sigma)$ and $\sigma(-\alpha) = \beta$ for some $\alpha \in N(\sigma)$ and $s_i(\beta) < \beta$, then $s_i(\beta) = \sigma(-\gamma)$ for some γ in $N(\sigma)$. In other words, there exists a $\gamma \in N(\sigma)$ such that s_i permutes $\sigma(-\gamma)$ and $\sigma(-\alpha)$.*

Proof. Suppose $\ell(s_i\sigma) > \ell(\sigma)$ and $\sigma(-\alpha) = \beta$ for some $\alpha \in N(\sigma)$. Suppose further that $s_i(\beta) < \beta$ are both positive roots. Then $s_i(\beta) = \beta - c_i\alpha_i$ for some $c_i \geq 0$. Then consider

$$\sigma^{-1}(s_i(\beta)) = \sigma^{-1}(\beta - c_i\alpha_i) = \sigma^{-1}(\beta) - c_i\sigma^{-1}(\alpha_i) = -\alpha - c_i\sigma^{-1}(\alpha_i). \tag{2}$$

Lemma 1.6 in Humphrey's Reflection Groups and Coxeter Groups, states that if $\ell(s_i\sigma) > \ell(\sigma)$ then $\sigma^{-1}(\alpha_i) \in \Phi^+$ [9, Lemma 1.6]. We know that $\sigma^{-1}s_i \in W$ permutes the set of roots, so $\sigma^{-1}s_i(\beta)$ is a root. Equation (2) shows $\sigma^{-1}(s_i\beta) = -\alpha - c_i\sigma^{-1}(\alpha_i)$ is a negative root because α and $\sigma^{-1}(\alpha_i)$ are both positive roots. Hence $\gamma = \alpha + c_i\sigma^{-1}(\alpha_i) \in \Phi^+$. Now we see that

$$s_i(\beta) = \sigma(\sigma^{-1}(s_i(\beta)) = \sigma(\sigma^{-1}(\beta - c_i\alpha_i)) = \sigma\left(-\alpha - c_i\sigma^{-1}(\alpha_i)\right) = \sigma(-\gamma).$$

Since $\sigma(-\gamma) = s_i\beta$ is an element of Φ^+, we see that $\gamma \in N(\sigma)$ as desired. □

We are now ready to prove Proposition 3.4.

Proof. (Proof of Statement 1) Equation (4.25) in Section 4.4 of Bjorner and Brenti states that $\ell(\sigma s_i) > \ell(\sigma) \iff \sigma(\alpha_i) \in \Phi^+$. Hence if $\ell(\sigma s_i) > \ell(\sigma)$, then $\sigma(\alpha_i)$ is a positive root so it can be expressed as $\sigma(\alpha_i) = \sum_{\alpha_j \in \Delta} b_j\alpha_j$ with every $b_j \geq 0$ and at least one $b_k > 0$. Applying Lemma 3.2 we compute that

$$\begin{aligned}\sigma s_i(\rho) - \rho = \sigma(\rho - \alpha_i) - \rho = \sigma(\rho) - \rho - \sigma(\alpha_i) &= -\sum_{\alpha_j \in \Delta} c_j\alpha_j - \sigma(\alpha_i)\\ = -\sum_{\alpha_j \in \Delta}(c_j + b_j)\alpha_j &= -\sum_{\alpha_j \in \Delta} d_j\alpha_j.\end{aligned}$$

The last equality proves that $d_j \geq c_j$ for all j and that there exists at least one k with $d_k > c_k$, which completes the proof of Statement 1.
(Proof of Statement 2) We can write the weight $\sigma(\rho) - \rho$ in the following two ways:

$$\sigma(\rho) - \rho = -\sum_{\alpha_j \in \Delta} c_j\alpha_j = \sum_{\alpha \in N(\sigma)} \sigma(\alpha).$$

Since $s_i(\rho) = \rho - \alpha_i$ we calculate that $s_i(\sigma(\rho) - \rho) = s_i\sigma(\rho) - s_i(\rho) = s_i\sigma(\rho) - \rho + \alpha_i$ or equivalently, that $s_i\sigma(\rho) - \rho = s_i(\sigma(\rho) - \rho) - \alpha_i$. Hence we see that

$$s_i\sigma(\rho) - \rho = s_i(\sigma(\rho) - \rho) - \alpha_i = s_i\left(-\sum_{\alpha_j \in \Delta} c_j\alpha_j\right) - \alpha_i = -\sum_{\alpha_j \in \Delta} e_j\alpha_j.$$

It follows from the way s_i acts on the positive roots that $c_j = e_j$ if $j \neq i$ and $e_i = \sum_{j \in adj(i)} c_j + 1 - c_i$ where $adj(i)$ denotes the set of indices of simple roots adjacent to α_i in the Dynkin diagram. We will now show that if $\ell(s_i\sigma) > \ell(\sigma)$ then $e_i > c_i$. Since $\ell(s_i\sigma) > \ell(\sigma)$ we have $|N(s_i\sigma)| = |N(\sigma)|+1$ [2, Proposition 4.4.4], and in fact $N(s_i\sigma) = N(\sigma) \cup \{\gamma : \sigma(\gamma) = \alpha_i\}$. Hence we get the following expression for $s_i\sigma(\rho) - \rho$:

$$\begin{aligned} s_i(\sigma(\rho)) - \rho &= -\sum_{\alpha_j \in \Delta} e_j\alpha_j = s_i\left(\sum_{\alpha \in N(\sigma)} \sigma(\alpha)\right) - \alpha_i \\ &= s_i\left(-\sum_{\alpha_j \in \Delta} c_j\alpha_j\right) - \alpha_i. \end{aligned} \tag{3}$$

For each $\alpha \in N(\sigma)$ either $s_i\sigma(-\alpha) \geq \sigma(\alpha)$ or $s_i\sigma(-\alpha) < \sigma(\alpha)$. Lemma 3.5 shows that if the root $s_i\sigma(-\alpha) < \sigma(-\alpha)$ then there is another root $\gamma \in N(\sigma)$ such that the following sums are equal

$$s_i(\sigma(-\alpha)) + s_i(\sigma(-\gamma)) = \sigma(-\alpha) + \sigma(-\gamma).$$

In other words, s_i permutes the two positive roots $\sigma(-\alpha)$ and $\sigma(-\gamma)$. This implies that

$$s_i\left(\sum_{\alpha \in N(\sigma)} \sigma(\alpha)\right) \geq \sum_{\alpha \in N(\sigma)} \sigma(\alpha) \text{ and } s_i\left(-\sum_{\alpha_j \in \Delta} c_j\alpha_j\right) \geq -\sum_{\alpha_j \in \Delta} c_j\alpha_j.$$

Therefore Equation (3) can be extended to an inequality as follows

$$\begin{aligned} s_i(\sigma(\rho)) - \rho &= -\sum_{\alpha_j \in \Delta} e_j\alpha_j = s_i\left(\sum_{\alpha \in N(\sigma)} \sigma(\alpha)\right) - \alpha_i \\ &= s_i\left(-\sum_{\alpha_j \in \Delta} c_j\alpha_j\right) - \alpha_i \geq -\sum_{\alpha_j \in \Delta} c_j\alpha_j - \alpha_i. \end{aligned}$$

Notice we have hence shown that

$$-\sum_{\alpha_j \in \Delta} e_j\alpha_j \geq -\sum_{\alpha_j \in \Delta} c_j\alpha_j - \alpha_i.$$

Since $e_j = c_j$ for all $j \neq i$ (as s_i was the only thing acting on the c_j values) the previous line shows that $e_i \geq c_i + 1$ as desired. This completes the proof of Statement 2. $\square$

Proposition 3.4 allows us to rule out a significant number of elements of W from being in the set $\mathcal{A}(\tilde{\alpha}, 0)$. In fact we know that if $\sigma' \in W$ contains σ as a subword then $\sigma' = s_{i_1}s_{i_2}\cdots s_{i_k}\sigma s_{j_1}s_{j_2}\cdots s_{j_l}$. So Proposition 3.4 shows that if $\sigma \notin \mathcal{A}(\tilde{\alpha}, 0)$, then neither is any σ' containing σ in its reduced word decomposition.

We will now compute $\sigma(\rho)$ for all σ consisting of at most four root reflections associated to consecutive simple roots in the Dynkin diagrams. This leads to a classification of the *minimal forbidden subwords*, i.e. the smallest length Weyl group elements which are not in the set $\mathcal{A}(\tilde{\alpha}, 0)$. These minimal forbidden elements are presented in Lemma 3.11.

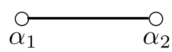

Figure 3: Dynkin diagram of the root system A_2.

Lemma 3.6. *When the Dynkin diagram of α_i and α_{i+1} embeds into that of A_2 (Figure 3) the products of s_i and s_{i+1} have the following effect on ρ*

$$\begin{aligned} s_{i+1}s_i(\rho) &= \rho - \alpha_i - 2\alpha_{i+1}, \\ s_is_{i+1}(\rho) &= \rho - 2\alpha_i - \alpha_{i+1}, \textit{and} \\ s_is_{i+1}s_i(\rho) &= \rho - 2\alpha_i - 2\alpha_{i+1}. \end{aligned}$$

Proof. Observe that

$$\begin{aligned} s_{i+1}s_i(\rho) &= s_{i+1}(\rho - \alpha_i) = \rho - \alpha_{i+1} - (\alpha_i + \alpha_{i+1}) = \rho - \alpha_i - 2\alpha_{i+1}, \\ s_is_{i+1}(\rho) &= s_i(\rho - \alpha_{i+1}) = \rho - \alpha_i - (\alpha_i + \alpha_{i+1}) = \rho - 2\alpha_i - \alpha_{i+1}, \text{ and} \\ s_is_{i+1}s_i(\rho) &= s_i(\rho - \alpha_{i+1} - (\alpha_i + \alpha_{i+1})) = \rho - \alpha_i + \alpha_i - 2(\alpha_i + \alpha_{i+1}) \\ &= \rho - 2\alpha_i - 2\alpha_{i+1}. \end{aligned}$$

$\square$

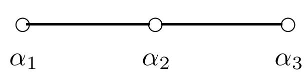

Figure 4: Dynkin diagram of the root system A_3.

Lemma 3.7. *If the Dynkin diagram of α_i, α_{i+1}, and α_{i+2} embeds into the Dynkin diagram of A_3 (Figure 4), then the elements $s_is_{i+1}s_{i+2}$, $s_{i+2}s_{i+1}s_i$, $s_is_{i+2}s_{i+1}$, and $s_{i+1}s_is_{i+2}$ act on ρ as follows:*

$$s_i s_{i+1} s_{i+2}(\rho) = \rho - 3\alpha_i - 2\alpha_{i+1} - \alpha_{i+2},$$
$$s_{i+2} s_{i+1} s_i(\rho) = \rho - \alpha_i - 2\alpha_{i+1} - 3\alpha_{i+2},$$
$$s_i s_{i+2} s_{i+1}(\rho) = \rho - 2\alpha_i - \alpha_{i+1} - 2\alpha_{i+2},$$
$$s_{i+1} s_i s_{i+2}(\rho) = \rho - \alpha_i - 3\alpha_{i+1} - \alpha_{i+2}.$$

Proof. Using the same technique as before notice that

$$\begin{aligned} s_i s_{i+1} s_{i+2}(\rho) &= s_i s_{i+1}(\rho - \alpha_{i+2}) = s_i(\rho - 2\alpha_{i+1} - \alpha_{i+2}) \\ &= \rho - 3\alpha_i - 2\alpha_{i+1} - \alpha_{i+2}, \\ s_{i+2} s_{i+1} s_i(\rho) &= s_{i+2} s_{i+1}(\rho - \alpha_i) = s_{i+2}(\rho - \alpha_i - 2\alpha_{i+1}) \\ &= \rho - \alpha_i - 2\alpha_{i+1} - 3\alpha_{i+2}, \\ s_i s_{i+2} s_{i+1}(\rho) &= s_i s_{i+2}(\rho - \alpha_{i+1}) = s_i(\rho - \alpha_{i+1} - 2\alpha_{i+2}) \\ &= \rho - 2\alpha_i - \alpha_{i+1} - 2\alpha_{i+2}, \\ s_{i+1} s_i s_{i+2}(\rho) &= s_{i+1} s_i(\rho - \alpha_{i+2}) = s_{i+1}(\rho - \alpha_i - \alpha_{i+2}) \\ &= \rho - \alpha_i - 3\alpha_{i+1} - \alpha_{i+2}. \end{aligned}$$ □

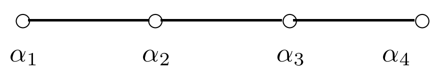

Figure 5: Dynkin diagram of the root system A_4.

Lemma 3.8. *Assume that the Dynkin diagram of α_i, α_{i+1}, α_{i+2}, and α_{i+3} embeds into A_4 (Figure 5). Let i_1, i_2, i_3, and i_4 be distinct elements of the set $\{i, i+1, i+2, i+3, i+4\}$. If $\sigma = s_{i_1} s_{i_2} s_{i_3} s_{i_4}$, then $\sigma \notin \mathcal{A}(\tilde{\alpha}, 0)$.*

Proof. By Lemma 3.7, the only length-three product of s_i, s_{i+1} and s_{i+2} that is in $\mathcal{A}(\tilde{\alpha}, 0)$ is the element $s_i s_{i+2} s_{i+1}$. To obtain a word σ with all four simple transpositions, we can either multiply on the left or right by s_{i+3}. However, if we multiply by s_{i+3} on the left we get $s_{i+3} s_i s_{i+2} s_{i+1} = s_i s_{i+3} s_{i+2} s_{i+1}$. This contains $s_{i+3} s_{i+2} s_{i+1}$ which is not in $\mathcal{A}(\tilde{\alpha}, 0)$ by Lemma 3.7. Thus Proposition 3.4 implies that $s_i s_{i+3} s_{i+2} s_{i+1}$ is also not in $\mathcal{A}(\tilde{\alpha}, 0)$. If we multiply on the right by s_{i+3} we obtain $s_i s_{i+2} s_{i+1} s_{i+3}$. By Lemma 3.7, $s_{i+2} s_{i+1} s_{i+3}$ is not in $\mathcal{A}(\tilde{\alpha}, 0)$ because $s_{i+2} s_{i+1} s_{i+3}(\rho) = \rho - \alpha_{i+1} - 3\alpha_{i+2} - \alpha_{i+3}$. So again by Proposition 3.4, $s_i s_{i+2} s_{i+1} s_{i+3} \notin \mathcal{A}(\tilde{\alpha}, 0)$. An analogous argument shows that $s_{i+1} s_{i+3} s_{i+2}$ cannot be extended to a product containing s_i so that the resulting element will be in $\mathcal{A}(\tilde{\alpha}, 0)$. Thereby completing the proof. □

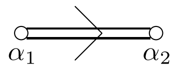

Figure 6: Dynkin diagram of the root system B_2.

Lemma 3.9. *When the Dynkin diagram of α_i and α_{i+1} embeds into the Dynkin diagram of type B_2 (Figure 6) or when $i = r-1$ in type B_r, we have the following*

$$s_i s_{i+1}(\rho) = \rho - 2\alpha_i - \alpha_{i+1} \text{ and } s_{i+1}s_i(\rho) = \rho - \alpha_i - 3\alpha_{i+1}.$$

Proof. Observe that

$$s_i s_{i+1}(\rho) = s_i(\rho - \alpha_{i+1}) = \rho - 2\alpha_i - \alpha_{i+1} \text{ and}$$
$$s_{i+1}s_i(\rho) = s_{i+1}(\rho - \alpha_i) = \rho - \alpha_i - 3\alpha_{i+1}.$$ □

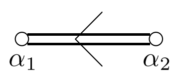

Figure 7: Dynkin diagram of the root system C_2.

Lemma 3.10. *When the Dynkin diagram of α_i and α_{i+1} embeds into the Dynkin diagram of type C_2 (Figure 7) or when $i = r-1$ in type C_r, we have the following*

$$s_i s_{i+1}(\rho) = \rho - 3\alpha_i - \alpha_{i+1} \text{ and } s_{i+1}s_i(\rho) = \rho - \alpha_i - 2\alpha_{i+1}.$$

Proof. Observe that

$$s_i s_{i+1}(\rho) = s_i(\rho - \alpha_{i+1}) = \rho - 3\alpha_i - \alpha_{i+1} \text{ and}$$
$$s_{i+1}s_i(\rho) = s_{i+1}(\rho - \alpha_i) = \rho - \alpha_i - 2\alpha_{i+1}.$$ □

Lemmas 3.7 and 3.8 allow us to identify a set of Weyl group elements which are not in the Weyl alternation set for any classical type. We record this set now for ease of reference in the type specific proofs presented in the following sections.

Lemma 3.11. *Let $\sigma \in W_r$ be a Weyl group element in any classical Lie type. Then*

1. if $\sigma = s_i s_{i+1} s_{i+2}$, $s_{i+2} s_{i+1} s_i$, $s_{i+1} s_i s_{i+2}$ and α_i, α_{i+1}, and α_{i+2} embed into the Dynkin diagram of A_3, or

2. if $\sigma = s_{i_1} s_{i_2} s_{i_3} s_{i_4}$ with i_1, i_2, i_3, i_4 distinct elements of $\{i, i+1, i+2, i+3\}$, where α_i, α_{i+1}, α_{i+2}, and α_{i+3} embed into A_4, or

3. if $\sigma = s_{i+1} s_i$ and α_i and α_{i+1} embed into the Dynkin diagram of B_2, or

4. if $\sigma = s_i s_{i+1}$ and α_i and α_{i+1} embed into the Dynkin diagram of C_2,

then σ is not in the Weyl alternation set $\mathcal{A}(\tilde{\alpha}, 0)$.

Proof. This result follows from applying Lemma 3.3 to Lemmas 3.7, 3.8, 3.9, and 3.10. □

If σ is one of the elements listed in Lemma 3.11, then we say that σ is a *minimal forbidden subword*. Using Proposition 3.4 we have shown that all elements of W containing these listed elements as subwords will not be in the respective Weyl alternation sets $\mathcal{A}(\tilde{\alpha}, 0)$.

4. $\mathcal{A}(\tilde{\alpha}, 0)$ in type B

When we consider the Lie algebra of type B and rank r we denote the Weyl alternation set as:

$$\mathcal{B}_r(\lambda, \mu) := \mathcal{A}(\lambda, \mu) = \{\sigma \in W : \wp(\sigma(\lambda + \rho) - \rho - \mu) > 0\},$$

where W denotes the corresponding Weyl group. Recall that in this case the Weyl group is isomorphic to the group of signed permutations on r letters and has order $2^r \cdot r!$. In order to illustrate the complexity in computing weight multiplicities we present a detailed example.

Example 4.1. *We will use Kostant's weight multiplicity formula to compute the multiplicity of the zero-weight in the adjoint representation of $\mathfrak{so}_7(\mathbb{C})$. In this process we will compute the Weyl alternation set $\mathcal{B}_3(\tilde{\alpha}, 0)$. First note that the Weyl group, W, corresponding to the Lie algebra $\mathfrak{so}_7(\mathbb{C})$ has order $2^3 3! = 48$. This means that Kostant's weight multiplicity formula will be an alternating sum consisting of 48 terms.*

We begin by considering the term corresponding to the identity element of W. First notice that $1(\tilde{\alpha} + \rho) - \rho = \tilde{\alpha}$ and now we must compute the value of Kostant's partition function. To compute the number of ways to express $\tilde{\alpha}$ as a sum of positive roots we use parenthesis to denote which positive roots we are using in this expression. In this way we can see that we may write $\tilde{\alpha}$ in the following eleven ways:

$$\begin{aligned}\tilde{\alpha} &= (\alpha_1) + 2(\alpha_2) + 2(\alpha_3)\\ &= (\alpha_1) + 2(\alpha_2 + \alpha_3)\\ &= (\alpha_1) + (\alpha_2) + (\alpha_2 + 2\alpha_3)\\ &= (\alpha_1) + (\alpha_2) + (\alpha_3) + (\alpha_2 + \alpha_3)\\ &= (\alpha_1 + \alpha_2) + (\alpha_2 + 2\alpha_3)\\ &= (\alpha_1 + \alpha_2) + (\alpha_2 + \alpha_3) + (\alpha_3)\\ &= (\alpha_1 + \alpha_2) + (\alpha_2) + 2(\alpha_3)\\ &= (\alpha_1 + \alpha_2 + \alpha_3) + (\alpha_2 + \alpha_3)\\ &= (\alpha_1 + \alpha_2 + \alpha_3) + (\alpha_2) + (\alpha_3)\\ &= (\alpha_1 + \alpha_2 + 2\alpha_3) + (\alpha_2)\\ &= (\alpha_1 + 2\alpha_2 + 2\alpha_3).\end{aligned}$$

Thus $\wp(1(\tilde{\alpha} + \rho) - \rho) = 11$.

A computation shows that of the 48 elements of the Weyl group only the five elements 1, s_1, s_2, s_3, *and* s_3s_1 *contribute a positive partition function value. Thus* $\mathcal{B}_3(\tilde{\alpha}, 0) = \{1, s_1, s_2, s_3, s_3s_1\}$. *It is worth remarking again that as the rank of the Lie algebra increases the number of terms grows factorially, and thus it is more evident that it is essential to know which elements are contributing non-zero terms to the alternating sum.*

Now we can finally compute the multiplicity of the zero-weight in the adjoint representation by reducing the sum to only the contributing terms. Thus

$$\begin{aligned}m(\tilde{\alpha}, 0) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \wp(\sigma(\tilde{\alpha} + \rho) - \rho) &= \sum_{\sigma \in \mathcal{B}_3(\tilde{\alpha},0)} (-1)^{\ell(\sigma)} \wp(\sigma(\tilde{\alpha} + \rho) - \rho)\\ &= 11 - 4 - 1 - 5 + 2 = 3,\end{aligned}$$

which is the rank of the Lie algebra $\mathfrak{so}_7(\mathbb{C})$, *as we expected.*

We now return to the general case of computing $\mathcal{B}_r(\tilde{\alpha}, 0)$ for all r. First we recall that the Weyl group of type B_r is a poset with order given by inclusion of subwords. To cut down on the number of elements in W_r that we need to consider, we start by describing the set of Weyl group elements which are not in $\mathcal{B}_r(\tilde{\alpha}, 0)$. Proposition 3.4 shows that any Weyl group element that contains one of the elements listed below in its reduced word expression will not be in $\mathcal{B}_r(\tilde{\alpha}, 0)$.

Lemma 4.1. *Let* σ *be a reduced word in* W. *If* σ *contains* s_1s_2, s_2s_1, s_2s_3, s_3s_2, *or* s_rs_{r-1} *in its reduced word decomposition, then* $\sigma \notin \mathcal{B}_r(\tilde{\alpha}, 0)$.

Proof. A simple calculation shows that

$$\begin{aligned}
s_1s_2(\tilde{\alpha}+\rho)-\rho &= \tilde{\alpha}-3\alpha_1-2\alpha_2 = -2\alpha_1+2\alpha_3+\cdots+2\alpha_r,\\
s_2s_1(\tilde{\alpha}+\rho)-\rho &= \tilde{\alpha}-\alpha_1-3\alpha_2 = -\alpha_2+2\alpha_3+\cdots+2\alpha_r,\\
s_2s_3(\tilde{\alpha}+\rho)-\rho &= \tilde{\alpha}-3\alpha_2-\alpha_3 = \alpha_1-\alpha_2+\alpha_3+2\alpha_4+\cdots+2\alpha_r,\\
s_3s_2(\tilde{\alpha}+\rho)-\rho &= \tilde{\alpha}-2\alpha_2-3\alpha_3 = \alpha_1-\alpha_3+2\alpha_4+\cdots+2\alpha_r, \text{ and}\\
s_rs_{r-1}(\tilde{\alpha}+\rho)-\rho &= \tilde{\alpha}-\alpha_{r-1}-3\alpha_r = \alpha_1+2\alpha_2+\cdots+2\alpha_{r-2}+\alpha_{r-1}-\alpha_r.
\end{aligned}$$

Thus Proposition 3.4 implies that no Weyl group element σ containing any of the products of simple reflections s_1s_2, s_2s_1, s_2s_3, or s_rs_{r-1} in its reduced word decomposition is in the Weyl alternation set $\mathcal{B}_r(\tilde{\alpha},0)$. □

We call the subwords described in Lemmas 3.11 and 4.1 the *minimal forbidden subwords* of type B. It is easy to see that the vast majority of elements in W_r contain one of these minimal forbidden subwords. Thus we have greatly reduced the number of elements we must consider. Now that we have described which elements of W_r are not in $\mathcal{B}_r(\tilde{\alpha},0)$, we turn our attention to the elements σ which do not contain a minimal forbidden subword.

The next Proposition and its corollary describe the Weyl group elements in $\mathcal{B}_r(\tilde{\alpha},0)$ as commuting products of short strings of simple root reflections. We shall refer to the elements listed in Proposition 4.2 as the *basic allowable subwords* of type B.

Proposition 4.2. *The following elements of W_r are in $\mathcal{B}_r(\tilde{\alpha},0)$*

(1) $(r\geq 2)$*: 1, i.e. the identity element of* W_r
(2) $(r\geq 3)$*:* s_i *for any* $1\leq i\leq r$
(3) $(r\geq 4)$*:* s_is_{i+1} *for any* $3\leq i\leq r-1$
(4) $(r\geq 5)$*:* $s_{i+1}s_i$ *for any* $3\leq i\leq r-2$
(5) $(r\geq 5)$*:* $s_is_{i+1}s_i$ *for any* $3\leq i\leq r-2$
(6) $(r\geq 6)$*:* $s_is_{i+2}s_{i+1}$ *for any* $3\leq i\leq r-3$*.*

Proof. Recall $\sigma\in\mathcal{B}_r(\tilde{\alpha},0)$ if and only if $\sigma(\tilde{\alpha}+\rho)-\rho$ can be written as a nonnegative integral combination of simple roots. Clearly $1\in\mathcal{B}_r(\tilde{\alpha},0)$ since $1(\tilde{\alpha}+\rho)-\rho=\tilde{\alpha}$ which can be written as a sum of simple roots with nonnegative integer coefficients.

Let $r\geq 3$ and $i\in\{1,3,4,\ldots,r\}$. Then by Lemma 3.1 $s_i(\tilde{\alpha}+\rho)-\rho=\tilde{\alpha}+(\rho-\alpha_i)-\rho=\alpha_1+2\alpha_2+\cdots+2\alpha_{i-1}+\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_r$, and when $i=2$ we have that $s_2(\tilde{\alpha}+\rho)-\rho=(\tilde{\alpha}-\alpha_2)+(\rho-\alpha_2)-\rho=\tilde{\alpha}-2\alpha_2=\alpha_1+2\alpha_3+\cdots+2\alpha_r$. Hence $s_i\in\mathcal{B}_r(\tilde{\alpha},0)$ for all $1\leq i\leq r$.

Let $r \geq 4$ and let $3 \leq i \leq r-1$. Then by Lemmas 3.1 and 3.6 we have that $s_is_{i+1}(\tilde{\alpha}+\rho)-\rho = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{i-1} + \alpha_{i+1} + 2\alpha_{i+2} + \cdots + 2\alpha_r$. Hence $s_is_{i+1} \in \mathcal{B}_r(\tilde{\alpha}, 0)$, for all $3 \leq i \leq r-1$, whenever $r \geq 6$.

Let $r \geq 5$ and let $3 \leq i \leq r-2$. Then by Lemmas 3.1 and 3.6

$$s_{i+1}s_i(\tilde{\alpha}+\rho)-\rho = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{i+1} + \alpha_i + 2\alpha_{i+2} + \cdots + 2\alpha_r, \text{ and}$$
$$s_is_{i+1}s_i(\tilde{\alpha}+\rho)-\rho = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{i-1} + 2\alpha_{i+2} + \cdots + 2\alpha_r.$$

Hence $s_{i+1}s_i$ and $s_is_{i+1}s_i \in \mathcal{B}_r(\tilde{\alpha}, 0)$, for all $3 \leq i \leq r-2$ when $r \geq 5$.

Let $r \geq 6$ and $3 \leq i \leq r-3$. Then by Lemmas 3.1 and 3.7 we have $s_is_{i+2}s_{i+1}(\tilde{\alpha}+\rho)-\rho = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{r-1} + \alpha_{i+1} + 2\alpha_{i+3} + \cdots + 2\alpha_r$. Hence $s_is_{i+2}s_{i+1} \in \mathcal{B}_r(\tilde{\alpha}, 0)$, for all $3 \leq i \leq r-3$, whenever $r \geq 6$. □

Proposition 4.2 gives a list of words that are contained in the alternation set $\mathcal{B}_r(\tilde{\alpha}, 0)$. In fact this list of subwords include the longest (reduced) subwords consisting of consecutively indexed reflections which lie in $\mathcal{B}_r(\tilde{\alpha}, 0)$. To create all other elements of the set $\mathcal{B}_r(\tilde{\alpha}, 0)$ we simply multiply elements from this list that do not share indices of adjacent simple roots in the Dynkin diagram. To make this idea precise we define a product of commuting basic allowable subwords by using the Dynkin diagram.

Definition 4.1. *Let $\tau = s_{i_1}s_{i_2}\cdots s_{i_m}$ and $\pi = s_{j_1}s_{j_2}\cdots s_{j_n}$ be basic allowable subwords and let*

$$sup(\tau) = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_m}\} \text{ and } sup(\pi) = \{\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_n}\}.$$

If $sup(\tau) \cap sup(\pi) = \emptyset$ and for any simple root $\alpha_k \in sup(\tau)$ there does not exists a simple root $\alpha_{k'} \in sup(\pi)$ such that α_k and $\alpha_{k'}$ are adjacent nodes in the Dynkin diagram, then $\tau\pi = \pi\tau$ and we say that τ and π are commuting basic allowable subwords*.*

Proposition 4.2 lists the basic allowable subwords of type B, the following lemma considers what occurs when we multiply two noncommuting basic allowable subwords.

Lemma 4.3. *Let σ and τ be noncommuting basic allowable subwords. Then the product $\sigma\tau$ falls into one of the following three cases:*

1. *$\sigma\tau$ is itself a basic allowable subword with $\ell(\sigma\tau) = \ell(\sigma) + \ell(\tau)$,*
2. *$\sigma\tau$ reduces to a commuting product of basic allowable subwords with $\ell(\sigma\tau) < \ell(\sigma) + \ell(\tau)$,*
3. *$\sigma\tau$ contains a minimal forbidden subword as listed in Lemma 3.11.*

Corollary 4.4. *Let $\sigma \in W_r$ be a reduced word. If σ is a product of commuting basic allowable subwords of type B_r, then $\sigma \in \mathcal{B}_r(\tilde{\alpha}, 0)$.*

Proof. This follows from the fact that all basic allowable subwords are in $\mathcal{B}_r(\tilde{\alpha}, 0)$ and if σ is a product of commuting basic allowable subwords then these subwords act on nonadjacent subsets of the roots in the expression $\tilde{\alpha} + \rho$. Hence $\sigma(\tilde{\alpha} + \rho) - \rho$ will continue to be expressible as a nonnegative integral combination of simple roots, and thus this commuting product of basic allowable subwords will again be in $\mathcal{B}_r(\tilde{\alpha}, 0)$. □

Lemma 4.3, and Corollary 4.4 show that the only elements of W_r that are contained in $\mathcal{C}_r(\tilde{\alpha}, 0)$ are words which reduce to basic allowable subwords, as listed in Proposition 4.2, and products of basic allowable subwords that commute (i.e. their supports are not adjacent in the Dynkin diagram.) With these facts in hand, we are now ready to state a classification of the set $\mathcal{B}_r(\tilde{\alpha}, 0)$ in terms of commuting products of basic allowable subwords.

Theorem 4.5. *Let $\sigma \in W_r$ be a reduced word. Then $\sigma \in \mathcal{B}_r(\tilde{\alpha}, 0)$ if and only if σ is either basic allowable subword or a product of commuting basic allowable subwords of type B.*

Proof. ($\Leftarrow$) From Proposition 4.2 and Corollary 4.4 we see that the basic allowable subwords and all products of commuting basic allowable subwords are in $\mathcal{B}_r(\tilde{\alpha}, 0)$.

($\Rightarrow$) Suppose that $\sigma = b_1 b_2 \cdots b_k$ is a reduced word, and that each b_i is a basic allowable subword of maximal length (i.e. none of the products $b_i b_j$ can be written as a basic allowable subword). Suppose that there are a pair of basic allowable subwords b_i and b_j in the reduced word for σ which are not commuting basic allowable subwords. Lemma 4.3 shows that there are three possible cases: In the first case $b_i b_j$ is a minimal forbidden word, and that means $\sigma \notin \mathcal{B}_r(\tilde{\alpha}, 0)$. In the second case $b_i b_j$ can be combined into one basic allowable subword of the same length. This contradicts our assumption that each of the b_i have maximal length. The third case is that $b_i b_j$ is a product of commuting allowable subwords with $\ell(b_i b_j) < \ell(b_i) + \ell(b_j)$. This contradicts our assumption that $\sigma = b_1 b_2 \cdots b_k$ is a reduced word. Thus σ is not in $\mathcal{B}_r(\tilde{\alpha}, 0)$ if it cannot be written as a product of commuting basic allowable subwords. □

4.1. Cardinality of $\mathcal{B}_r(\tilde{\alpha}, 0)$

To determine the cardinality of $\mathcal{B}_r(\tilde{\alpha}, 0)$, we will begin by considering the elements of $\mathcal{B}_r(\tilde{\alpha}, 0)$ which do not contain a factor of s_r. Let P_r denote the set of all such elements and we define $P_0 = P_1$ to be the empty set.

Proposition 4.6. *The cardinality of the set* P_r *is given by the following recursive formula:*

$$|P_r| = |P_{r-1}| + |P_{r-2}| + 3|P_{r-3}| + |P_{r-4}|,$$

where $|P_0| = |P_1| = 0, |P_2| = 2, |P_3| = 3$.

Proof. We immediately have $|P_0| = |P_1| = 0$, as both P_0 and P_1 are defined to be the empty set. A fast computation of $P_2 = \{1, s_1\}$ and $P_3 = \{1, s_1, s_2\}$ shows us that $|P_2| = 2$ and $|P_3| = 3$.
Now for $r \geq 3$, we note that an allowable subword of $\mathcal{B}_{r-1}(\tilde{\alpha}, 0)$ is also an allowable subword of $\mathcal{B}_r(\tilde{\alpha}, 0)$. Hence $\mathcal{B}_{r-1}(\tilde{\alpha}, 0) \subset \mathcal{B}_r(\tilde{\alpha}, 0)$, and since no element of $\mathcal{B}_{r-1}(\tilde{\alpha}, 0)$ could have a factor of s_r, then $P_{r-1} \subset P_r$ as well. Thus, if $\sigma \in P_{r-1}$, then $\sigma \in P_r$. Proposition 4.2 and the definition of P_r describe the additional elements that appear in P_r as the value of r increases. In particular,

1. For $r \geq 4$, if $\sigma \in P_{r-2}$, then $\sigma s_{r-1} \in P_r$.
2. For $r \geq 5$, if $\sigma \in P_{r-3}$, then P_r will contain $\sigma s_{r-2} s_{r-1}$, $\sigma s_{r-1} s_{r-2}$, and $\sigma s_{r-2} s_{r-1} s_{r-2}$.
3. For $r \geq 6$, if $\sigma \in P_{r-4}$, then $\sigma s_{r-3} s_{r-1} s_{r-2} \in P_r$.

Let $P_j \pi = \{\sigma\pi \mid \sigma \in P_j\}$ for any Weyl group element π and any positive integer j. Then for any $k \geq 5$, P_k is the union of the pairwise disjoint sets

$$\begin{aligned} P_k = P_{k-1} &\cup (P_{k-2} s_{k-1}) \cup (P_{k-3} s_{k-2}) \cup (P_{k-3} s_{k-2} s_{k-1}) \\ &\cup (P_{k-3} s_{k-1} s_{k-2}) \cup (P_{k-4} s_{k-3} s_{k-1} s_{k-2}). \end{aligned}$$

Thus $|P_k| = |P_{k-1}| + |P_{k-2}| + 3|P_{k-3}| + |P_{k-4}|$. □

We now count the elements of $\mathcal{B}_r(\tilde{\alpha}, 0)$ containing a factor of s_r. To do so we note the following.

Lemma 4.7. *Let* $r \geq 3$. *If* $\sigma \in \mathcal{B}_r(\tilde{\alpha}, 0)$ *and* σ *contains a factor of* s_r, *then* $\sigma = \pi s_r$ *for some* $\pi \in P_{r-1}$ *or* $\sigma = \tau s_{r-1} s_r$ *for some* $\tau \in P_{r-2}$.

Proof. By Theorem 4.5 we know that if $\sigma \in \mathcal{B}_r(\tilde{\alpha}, 0)$, then $\sigma = b_1 b_2 \cdots b_m$ for $b_1, b_2, \ldots, b_m$ commuting basic allowable subwords as described by Proposition 4.2. Now if σ contains a factor of s_r, then either σ contains $s_{r-1} s_r$ or it will not contain a factor of s_{r-1} at all. Hence, by definition of P_k, we have that either $\sigma = \tau s_{r-1} s_r$ for some $\tau \in P_{r-2}$ or $\sigma = \pi s_r$ for some $\pi \in P_{r-1}$. □

Corollary 4.8. *For $r \geq 2$, the cardinality of the set $\mathcal{B}_r(\tilde{\alpha}, 0)$ is given by the following recursive formula:*

$$|\mathcal{B}_r(\tilde{\alpha}, 0)| = |P_r| + |P_{r-1}| + |P_{r-2}|,$$

where $|P_0| = |P_1| = 0$ and $|P_2| = 2$.

Proof. Let $r \geq 2$. Then by Lemma 4.7 we know that $\mathcal{B}_r(\tilde{\alpha}, 0)$ is the union of three pairwise disjoint sets. Namely $\mathcal{B}_r(\tilde{\alpha}, 0) = P_r \cup (P_{r-1}s_r) \cup (P_{r-2}s_{r-1}s_r)$. Thus $|\mathcal{B}_r(\tilde{\alpha}, 0)| = |P_r| + |P_{r-1}| + |P_{r-2}|$. □

Beginning with $i = 2$, we give the first few terms of the sequences $|P_i|$ and $|\mathcal{B}_i|$:

$|P_i|$:[3] 2, 3, 5, 14, 30, 62, 139, 305, 660, 1444, 3158, 6887, 15037, 32842, …
$|\mathcal{B}_i|$:[4] 2, 5, 10, 22, 49, 106, 231, 506, 1104, 2409, 5262, 11489, 25082, …

5. $\mathcal{A}(\tilde{\alpha}, 0)$ in type C

When we consider the Lie algebra of type C and rank r we denote the Weyl alternation set as:

$$\mathcal{C}_r(\lambda, \mu) := \mathcal{A}(\lambda, \mu) = \{\sigma \in W : \wp(\sigma(\lambda + \rho) - \rho - \mu) > 0\},$$

where W denotes the corresponding Weyl group. Recall that in this case the Weyl group is isomorphic to the group of signed permutations on r letters and has order $2^r \cdot r!$. Some direct calculations, as those provided in Example 4.1, show that:

$$\mathcal{C}_2(\tilde{\alpha}, 0) = \{1, s_2\}$$
$$\mathcal{C}_3(\tilde{\alpha}, 0) = \{1, s_2, s_3\}$$
$$\mathcal{C}_4(\tilde{\alpha}, 0) = \{1, s_2, s_3, s_4, s_2s_3, s_3s_2, s_2s_3s_2, s_2s_4\}$$
$$\mathcal{C}_5(\tilde{\alpha}, 0) = \left\{\begin{matrix} 1, s_2, s_3, s_4, s_5, s_2s_3, s_2s_3s_5, s_3s_2, s_3s_2s_5, s_2s_5, s_2s_3s_2, \\ s_2s_3s_2s_5, s_2s_4, s_3s_4, s_4s_3, s_3s_4s_3, s_2s_4s_3, s_3s_5 \end{matrix}\right\}$$
$$\mathcal{C}_6(\tilde{\alpha}, 0) = \left\{\begin{matrix} 1, s_2, s_3, s_4, s_5, s_6, s_2s_3, s_2s_4, s_2s_5, s_2s_6, s_3s_2, s_3s_4, s_3s_5, s_3s_6, \\ s_4s_3, s_4s_5, s_4s_6, s_5s_4, s_2s_3s_2, s_2s_3s_5, s_2s_3s_6, s_2s_4s_3, s_2s_4s_5, \\ s_2s_4s_6, s_2s_5s_4, s_3s_2s_5, s_3s_2s_6, s_3s_4s_3, s_3s_4s_6, s_3s_5s_4, s_4s_3s_6, \\ s_4s_5s_4, s_2s_3s_2s_5, s_2s_3s_2s_6, s_2s_4s_3s_6, s_2s_4s_5s_4, s_3s_4s_3s_6 \end{matrix}\right\}$$

[3] The sequence A232162 was added by the authors to The On-Line Encyclopedia of Integer Sequences (OEIS).
[4] The sequence A232163 was added by the authors to The On-Line Encyclopedia of Integer Sequences (OEIS).

As we pointed out previously, it is important to note that the cardinalities of the Weyl alternation sets above are much smaller than the order of the respective Weyl group, see Table 1 on page 96.

Table 1: Cardinality comparison in type C

Rank	Weyl Alternation Set Cardinality	Weyl Group Order
2	2	8
3	3	48
4	8	384
5	18	3840
6	37	46080

We now describe the elements σ of W_r which are not in the Weyl alternation set $\mathcal{C}_r(\tilde{\alpha}, 0)$ by identifying a list of *minimal forbidden subwords* that prohibit σ from being in $\mathcal{C}_r(\tilde{\alpha}, 0)$.

Lemma 5.1. *Let σ be a reduced word in W. If σ contains s_1, $s_{r-1}s_r$, or $s_r s_{r-1}$ in its reduced word decomposition, then $\sigma \notin \mathcal{C}_r(\tilde{\alpha}, 0)$.*

Proof. Recall that by Lemma 3.1, $s_1(\tilde{\alpha}) = \tilde{\alpha} - 2\alpha_1$ and all other simple root reflections fix the highest root, while by Lemma 3.6 we know $s_1(2\rho) = 2\rho - 2\alpha_1$. So s_1 is never in the Weyl alternation set $\mathcal{C}_r(\tilde{\alpha}, 0)$, nor is any word containing s_1. Lemma 3.10 shows that the Weyl group elements $s_{r-1}s_r$ and $s_r s_{r-1}$ are never in $\mathcal{C}_r(\tilde{\alpha}, 0)$ nor is any word containing them. ☐

Recall that Lemma 3.11 also shows that a Weyl group element σ containing a product of simple reflections of the form $s_i s_{i+1} s_{i+2}$, $s_{i+2} s_{i+1} s_i$, or $s_{i+1} s_i s_{i+2}$ where $3 \leq i \leq r-2$ or a product four consecutive simple root reflections $s_i, s_{i+1}, s_{i+2}, s_{i+3}$ is not in the Weyl alternation set $\mathcal{C}_r(\tilde{\alpha}, 0)$. We call the elements listed in Lemmas 3.11 and 5.1 the *minimal forbidden subwords* of type C.

We can now describe the elements of the Weyl alternation set $\mathcal{C}_r(\tilde{\alpha}, 0)$ as products of basic allowable subwords. We note that once again each basic allowable subword listed in the following Proposition is the largest product of consecutive simple reflections that do not result in a minimal forbidden subword.

Proposition 5.2. *The following elements of W_r are in $\mathcal{C}_r(\tilde{\alpha}, 0)$*

(1) $(r \geq 2)$*: 1, i.e. the identity element of* W_r
(2) $(r \geq 2)$*:* s_i *for any* $2 \leq i \leq r$
(3) $(r \geq 4)$*:* $s_i s_{i+1}$ *for any* $2 \leq i \leq r-2$
(4) $(r \geq 4)$*:* $s_{i+1} s_i$ *for any* $2 \leq i \leq r-2$

(5) $(r \geq 4)$*:* $s_i s_{i+1} s_i$ *for any* $2 \leq i \leq r-2$
(6) $(r \geq 5)$*:* $s_i s_{i+2} s_{i+1}$ *for any* $2 \leq i \leq r-3$*.*

We refer to the elements listed in Proposition 5.2 as the *basic allowable subwords* of Type C.

Proof. Recall that in the Type C case the highest root is $\tilde{\alpha} = 2\alpha_1 + \cdots + 2\alpha_{r-1} + \alpha_r$, and that $\sigma \in \mathcal{C}_r(\tilde{\alpha}, 0)$ if and only if $\sigma(\tilde{\alpha}+\rho)-\rho$ can be written as a nonnegative integral combination of simple roots. We will apply Lemma 3.1 and Lemma 3.10 in the statement below. Clearly $1 \in \mathcal{C}_r(\tilde{\alpha}, 0)$ since $1(\tilde{\alpha} + \rho) - \rho = \tilde{\alpha}$ which can be written as a sum of simple roots with nonnegative integer coefficients.

Let $r \geq 2$ and $2 \leq i \leq r$. Then by Lemmas 3.1 and 3.6 $s_i(\tilde{\alpha} + \rho) - \rho = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{i-1} + \alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{r-1} + \alpha_r$. Hence $s_i \in \mathcal{C}_r(\tilde{\alpha}, 0)$ for all $2 \leq i \leq r$, with $r \geq 2$.

Let $r \geq 4$ and let $2 \leq i \leq r-2$. Then by Lemmas 3.1 and 3.6 we have

$$s_i s_{i+1}(\tilde{\alpha} + \rho) - \rho = 2\alpha_1 + \cdots + 2\alpha_{i-1} + \alpha_{i+1} + 2\alpha_{i+2} + \cdots + 2\alpha_{r-1} + \alpha_r,$$
$$s_{i+1} s_i(\tilde{\alpha} + \rho) - \rho = 2\alpha_1 + \cdots + 2\alpha_{i-1} + \alpha_i + 2\alpha_{i+2} + \cdots + 2\alpha_{r-1} + \alpha_r, \text{ and}$$
$$s_i s_{i+1} s_i(\tilde{\alpha} + \rho) - \rho = 2\alpha_1 + \cdots + 2\alpha_{i-1} + 2\alpha_{i+2} + \cdots + 2\alpha_{r-1} + \alpha_r.$$

Hence $s_i s_{i+1}, s_{i+1} s_i, s_i s_{i+1} s_i \in \mathcal{C}_r(\tilde{\alpha}, 0)$, for all $2 \leq i \leq r-2$, with $r \geq 4$.

Let $r \geq 5$ and let $2 \leq i \leq r-3$. Then by Lemmas 3.1 and 3.7

$$s_i s_{i+2} s_{i+1}(\tilde{\alpha} + \rho) - \rho = 2\alpha_1 + \cdots + 2\alpha_{i-1} + \alpha_{i+1} + 2\alpha_{i+3} + \cdots + 2\alpha_{r-1} + \alpha_r.$$

Hence $s_i s_{i+2} s_{i+1} \in \mathcal{C}_r(\tilde{\alpha}, 0)$, for all $2 \leq i \leq r-3$, with $r \geq 5$. ☐

Lemma 5.3. *Let* σ *and* τ *be noncommuting basic allowable subwords. Then the product* $\sigma\tau$ *falls into one of the following three cases:*

1. $\sigma\tau$ *is itself a basic allowable subword with* $\ell(\sigma\tau) = \ell(\sigma) + \ell(\tau)$*,*
2. $\sigma\tau$ *reduces to a commuting product of basic allowable subwords with* $\ell(\sigma\tau) < \ell(\sigma) + \ell(\tau)$*,*
3. $\sigma\tau$ *contains a minimal forbidden subword as listed in Lemma 3.11.*

The proof of Lemma 5.3 is a finite computation that is analogous to the one provided for type B, which is found in Appendix A.

Corollary 5.4. *If* $\sigma \in W$ *can be expressed as a product of commuting basic allowable subwords of Type* C*, then* $\sigma \in \mathcal{C}_r(\tilde{\alpha}, 0)$*.*

Proof. By Proposition 5.2, all basic allowable subwords are in $\mathcal{C}_r(\tilde{\alpha}, 0)$. Moreover, two basic allowable subwords commute if and only if they act on disjoint sets of simple roots. Hence, in a product of commuting basic allowable subwords each subword acts on nonconsecutive indices of the expression $\tilde{\alpha} + \rho$. Hence the expression $\sigma(\tilde{\alpha}+\rho)-\rho$ will continue to be expressible as a nonnegative integral combination of simple roots, and thus a product of commuting basic allowable subwords will again be in $\mathcal{C}_r(\tilde{\alpha}, 0)$. □

Lemma 5.3, and Corollary 5.4 show that the only elements of W_r that are contained in $\mathcal{B}_r(\tilde{\alpha}, 0)$ are words which reduce to basic allowable subwords, as listed in Proposition 5.2, and products of basic allowable subwords that commute (i.e. their supports are not adjacent in the Dynkin diagram.) Thus we give a classification of the set $\mathcal{C}_r(\tilde{\alpha}, 0)$ in terms of commuting products of basic allowable subwords as follows.

Theorem 5.5. *Let $\sigma \in W_r$ be a reduced word. Then $\sigma \in \mathcal{C}_r(\tilde{\alpha}, 0)$ if and only if σ is either basic allowable subword or a product of commuting basic allowable subwords of type C.*

Proof. ($\Leftarrow$) From Proposition 5.2 and Corollary 5.4 we see that the basic allowable subwords and all products of commuting basic allowable subwords are in $\mathcal{C}_r(\tilde{\alpha}, 0)$.

($\Rightarrow$) Suppose that $\sigma = b_1 b_2 \cdots b_k$ is a reduced word, and that each b_i is a basic allowable subword of maximal length (i.e. none of the products $b_i b_j$ can be written as a basic allowable subword). Suppose that there are a pair of basic allowable subwords b_i and b_j in the reduced word for σ which are not commuting basic allowable subwords. Lemma 5.3 shows that there are three possible cases: In the first case $b_i b_j$ is a minimal forbidden subword, and that means $\sigma \notin \mathcal{C}_r(\tilde{\alpha}, 0)$. In the second case $b_i b_j$ can be combined into one basic allowable subword of the same length. This contradicts our assumption that each of the b_i have maximal length. The third case is that $b_i b_j$ is a product of commuting allowable subwords with $\ell(b_i b_j) < \ell(b_i) + \ell(b_j)$. This contradicts our assumption that $\sigma = b_1 b_2 \cdots b_k$ is a reduced word. Thus σ is not in $\mathcal{C}_r(\tilde{\alpha}, 0)$ if it cannot be written as a product of commuting basic allowable subwords. □

5.1. Cardinality of $\mathcal{C}_r(\tilde{\alpha}, 0)$

We will now build $\mathcal{C}_r(\tilde{\alpha}, 0)$ recursively in order to determine the cardinality of this set. For $r \geq 3$, let P_r denote the subset of $\mathcal{C}_r(\tilde{\alpha}, 0)$ of all elements which do not contain a factor of s_r. We define P_0 as the empty set and some simple computations show that $P_1 = P_2 = \{1\}$ and $P_3 = \{1, s_2\}$.

Proposition 5.6. *The cardinality of the set P_r is given by the following recursive formula:*

$$|P_r| = |P_{r-1}| + |P_{r-2}| + 3|P_{r-3}| + |P_{r-4}|,$$

where $|P_0| = 0, |P_1| = |P_2| = 1, |P_3| = 2$.

Proof. We know that P_0 is the empty set, hence $|P_0| = 0$. By definition of P_r and some basic computations we can show that $P_1 = P_2 = \{1\}$ and $P_3 = \{1, s_2\}$, hence $|P_1| = |P_2| = 1$ and $|P_3| = 2$. Now for $r \geq 3$, we note that an allowable subword of $\mathcal{C}_{r-1}(\tilde{\alpha}, 0)$ is also an allowable subword of $\mathcal{C}_r(\tilde{\alpha}, 0)$. Hence $\mathcal{C}_{r-1}(\tilde{\alpha}, 0) \subset \mathcal{C}_r(\tilde{\alpha}, 0)$, and since no element of $\mathcal{C}_{r-1}(\tilde{\alpha}, 0)$ could have a factor of s_r, then $P_{r-1} \subset P_r$ as well. Thus, if $\sigma \in P_{r-1}$, then $\sigma \in P_r$. Theorem 5.2 and the definition of P_r describe the additional elements that appear in P_r as the value of r increases. In particular,

1. For $r \geq 3$, if $\sigma \in P_{r-2}$, then $\sigma s_{r-1} \in P_r$.
2. For $r \geq 4$, if $\sigma \in P_{r-3}$, then P_r will contain $\sigma s_{r-2} s_{r-1}$, $\sigma s_{r-1} s_{r-2}$, and $\sigma s_{r-2} s_{r-1} s_{r-2}$.
3. For $r \geq 5$, if $\sigma \in P_{r-4}$, then $\sigma s_{r-3} s_{r-1} s_{r-2} \in P_r$.

Let $P_j\pi = \{\sigma\pi \mid \sigma \in P_j\}$ for any Weyl group element π and any positive integer j. Then for $k \geq 5$, P_k is the union of the pairwise disjoint sets

$$\begin{aligned} P_k = P_{k-1} &\uplus (P_{k-2} s_{k-1}) \uplus (P_{k-3} s_{k-2} s_{k-1}) \uplus (P_{k-3} s_{k-1} s_{k-2}) \\ &\uplus (P_{k-3} s_{k-2} s_{k-1} s_{k-2}) \uplus (P_{k-4} s_{k-3} s_{k-1} s_{k-2}). \end{aligned}$$

Thus $|P_k| = |P_{k-1}| + |P_{k-2}| + 3|P_{k-3}| + |P_{k-4}|$. □

We now count the elements of $\mathcal{C}_r(\tilde{\alpha}, 0)$ containing a factor of s_r.

Lemma 5.7. *Let $r \geq 2$. If $\sigma \in \mathcal{C}_r(\tilde{\alpha}, 0)$ and σ contains a factor of s_r, then $\sigma = \pi s_r$ for some $\pi \in P_{r-1}$.*

Proof. By Theorem 5.5 we know that if $\sigma \in \mathcal{C}_r(\tilde{\alpha}, 0)$, then $\sigma = b_1 b_2 \cdots b_m$ for $b_1, b_2, \ldots, b_m$ commuting basic allowable subwords as described by Proposition 5.2. Now if σ contains a factor of s_r, then σ will not contain a factor of s_{r-1} at all. Hence, by definition of P_{r-1}, we have that $\sigma = \pi s_r$ for some $\pi \in P_{r-1}$. □

Corollary 5.8. *For $r \geq 2$, the cardinality of the set $\mathcal{C}_r(\tilde{\alpha}, 0)$ is given by the following recursive formula:*

$$|\mathcal{C}_r(\tilde{\alpha}, 0)| = |P_r| + |P_{r-1}|,$$

where $|P_1| = |P_2| = 1$.

Proof. Let $r \geq 2$. Then by Lemma 5.7 we know that $\mathcal{C}_r(\tilde{\alpha}, 0)$ is the union of two pairwise disjoint sets. Namely $\mathcal{C}_r(\tilde{\alpha}, 0) = P_r \cup (P_{r-1}s_r)$. Thus $|\mathcal{C}_r(\tilde{\alpha}, 0)| = |P_r| + |P_{r-1}|$. □

Beginning with $i = 2$, we give the first few terms of the sequences $|P_i|$ and $|\mathcal{C}_i|$:

$|P_i|$:[5] 1, 2, 6, 12, 25, 57, 124, 268, 588, 1285, 2801, 6118, 13362, 29168, ...
$|C_i|$:[6] 2, 3, 8, 18, 37, 82, 181, 392, 856, 1873, 4086, 8919, 19480, 42530, ...

6. $\mathcal{A}(\tilde{\alpha}, 0)$ in type D

When we consider the Lie algebra of type D and rank $r \geq 4$ we denote the Weyl alternation set as follows:

$$\mathcal{D}_r(\lambda, \mu) := \mathcal{A}(\lambda, \mu) = \{\sigma \in W : \wp(\sigma(\lambda + \rho) - \rho - \mu) > 0\}. \tag{4}$$

Direct calculations, as those provided in Example 4.1, show that:

$$\mathcal{D}_4(\tilde{\alpha}, 0) = \{1, s_1, s_2, s_3, s_4, s_1s_3, s_1s_4, s_3s_4, s_1s_3s_4\}$$

$$\mathcal{D}_5(\tilde{\alpha}, 0) = \left\{\begin{matrix} 1, s_1, s_2, s_3, s_4, s_5, s_1s_3, s_1s_4, s_1s_5, s_2s_4, s_2s_5, \\ s_4s_5, s_3s_4, s_3s_5, s_1s_4s_5, s_1s_3s_4, s_1s_3s_5, s_2s_4s_5 \end{matrix}\right\}$$

$$\mathcal{D}_6(\tilde{\alpha}, 0) = \left\{\begin{matrix} 1, s_1, s_2, s_3, s_4, s_5, s_6, s_1s_3, s_1s_4, s_1s_5, s_1s_6, s_2s_4, s_2s_5, \\ s_2s_6, s_3s_4, s_3s_5, s_3s_6, s_4s_3, s_4s_5, s_4s_6, s_5s_6, s_1s_3s_4, \\ s_1s_3s_5, s_1s_3s_6, s_1s_4s_3, s_1s_4s_5, s_1s_4s_6, s_1s_5s_6, s_2s_4s_5, \\ s_2s_4s_6, s_2s_5s_6, s_3s_4s_3, s_3s_5s_6, s_1s_3s_4s_3, s_1s_3s_5s_6 \end{matrix}\right\}$$

[5]The sequence A232164 was added by the authors to The On-Line Encyclopedia of Integer Sequences (OEIS).
[6]The sequence A232165 was added by the authors to The On-Line Encyclopedia of Integer Sequences (OEIS).

$$\mathcal{D}_7(\tilde{\alpha},0)=\left\{\begin{array}{c}1, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_1s_3, s_3s_4, s_1s_4, s_2s_4,\\ s_4s_3, s_4s_5, s_1s_5, s_2s_5, s_3s_5, s_5s_4, s_5s_6, s_5s_7, s_1s_6,\\ s_2s_6, s_3s_6, s_4s_6, s_1s_7, s_2s_7, s_3s_7, s_4s_7, s_7s_6, s_1s_3s_4,\\ s_3s_4s_3, s_1s_4s_3, s_1s_4s_5, s_2s_4s_5, s_4s_5s_4, s_1s_3s_5, s_5s_3s_4,\\ s_1s_5s_4, s_2s_5s_4, s_1s_5s_6, s_2s_5s_6, s_3s_5s_6, s_1s_5s_7, s_2s_5s_7,\\ s_3s_5s_7, s_1s_3s_6, s_3s_4s_6, s_1s_4s_6, s_2s_4s_6, s_4s_3s_6, s_1s_3s_7,\\ s_3s_4s_7, s_1s_4s_7, s_2s_4s_7, s_4s_3s_7, s_1s_7s_6, s_2s_7s_6, s_3s_7s_6,\\ s_4s_7s_6, s_1s_3s_4s_3, s_1s_4s_5s_4, s_2s_4s_5s_4, s_1s_5s_3s_4,\\ s_1s_3s_5s_6, s_1s_3s_5s_7, s_1s_3s_4s_6, s_3s_4s_3s_6, s_1s_4s_3s_6,\\ s_1s_3s_4s_7, s_3s_4s_3s_7, s_1s_4s_3s_7, s_1s_3s_7s_6, s_3s_4s_7s_6,\\ s_1s_4s_7s_6, s_2s_4s_7s_6, s_4s_3s_7s_6, s_1s_3s_4s_3s_6, s_1s_3s_4s_3s_7,\\ s_1s_3s_4s_7s_6, s_3s_4s_3s_7s_6, s_1s_4s_3s_7s_6, s_1s_3s_4s_3s_6s_7\end{array}\right\}$$

We start by identifying a list of minimal forbidden subwords that are not in $\mathcal{D}_r(\tilde{\alpha},0)$.

Lemma 6.1. *Any Weyl group element $\sigma\in W_r$ containing the following subwords is not in the Weyl alternation set $\mathcal{D}_r(\tilde{\alpha},0)$*

$$\begin{aligned}&s_1s_2,\ \ s_2s_1,\ \ s_2s_3,\ \ s_3s_2,\ \ s_{r-1}s_{r-2},\, or\ s_rs_{r-2},\\ &s_is_{i+1}s_{i+2},\ \ s_{i+2}s_{i+1}s_i,\ \ or\ s_{i+1}s_is_{i+2}\ where\ 1\le i\le r-2.\end{aligned}$$

In addition, any σ containing a product of four consecutive simple reflections $s_i,\ s_{i+1},\ s_{i+2},\ s_{i+3}$ in any order, will not be in $\mathcal{D}_r(\tilde{\alpha},0)$.

Proof. We calculate that s_1s_2, s_2s_1, s_2s_3, s_3s_2, $s_{r-1}s_{r-2}$, and s_rs_{r-2} are not in the Weyl alternation set $\mathcal{D}_r(\tilde{\alpha},0)$ because

$$\begin{aligned}s_1s_2(\tilde{\alpha}+\rho)-\rho&=\tilde{\alpha}-3\alpha_1-2\alpha_2\\&=-2\alpha_1+2\alpha_3+\cdots+2\alpha_{r-2}+\alpha_{r-1}+\alpha_r,\\ s_2s_1(\tilde{\alpha}+\rho)-\rho&=\tilde{\alpha}-\alpha_1-3\alpha_2\\&=-\alpha_2+2\alpha_3+\cdots+2\alpha_{r-2}+\alpha_{r-1}+\alpha_r,\\ s_2s_3(\tilde{\alpha}+\rho)-\rho&=\tilde{\alpha}-3\alpha_2-\alpha_3\\&=\alpha_1-2\alpha_2+\alpha_3+2\alpha_4+\cdots+2\alpha_{r-2}+\alpha_{r-1}+\alpha_r,\\ s_3s_2(\tilde{\alpha}+\rho)-\rho&=\tilde{\alpha}-2\alpha_2-3\alpha_3\\&=\alpha_1-\alpha_3+2\alpha_4+\cdots+2\alpha_{r-2}+\alpha_{r-1}+\alpha_r,\\ s_{r-1}s_{r-2}(\tilde{\alpha}+\rho)-\rho&=\tilde{\alpha}-\alpha_{r-2}-2\alpha_{r-1}\\&=\alpha_1+2\alpha_2+\cdots+2\alpha_{r-2}-\alpha_r,\ \text{and}\end{aligned}$$

$$\begin{aligned} s_r s_{r-2}(\tilde{\alpha}+\rho)-\rho &= \tilde{\alpha}-\alpha_{r-2}-2\alpha_r \\ &= \alpha_1+2\alpha_2+\cdots+2\alpha_{r-3}+\alpha_{r-2}+\alpha_{r-1}-\alpha_r. \end{aligned}$$

Thus σ cannot contain any of the above subwords as factors in its reduced word expression.

Now Lemma 3.11 shows that if σ contains any of the subwords $s_i s_{i+1} s_{i+2}$, $s_{i+2} s_{i+1} s_i$, or $s_{i+1} s_i s_{i+2}$ with $1 \le i \le r-2$ or a product of four consecutive simple root reflections, then σ is not in $\mathcal{D}_r(\tilde{\alpha}, 0)$. □

We have identified a large set of elements in W_r which are not in the Weyl alternation set $\mathcal{D}_r(\tilde{\alpha}, 0)$. Now we will show that the remaining elements are in $\mathcal{D}_r(\tilde{\alpha}, 0)$ and describe them as products of basic allowable subwords as follows.

Proposition 6.2. *The following elements of W_r are in $\mathcal{D}_r(\tilde{\alpha}, 0)$*

(1) $(r \ge 2)$: 1, i.e. the identity element of W_r
(2) $(r \ge 3)$: s_i for any $1 \le i \le r$
(3) $(r \ge 4)$: $s_i s_{i+1}$ for any $3 \le i \le r-1$
(4) $(r \ge 6)$: $s_{i+1} s_i$ for any $3 \le i \le r-3$
(5) $(r \ge 6)$: $s_i s_{i+1} s_i$ for any $3 \le i \le r-3$
(6) $(r \ge 7)$: $s_i s_{i+2} s_{i+1}$ for any $3 \le i \le r-4$.

We will refer to the elements listed in Proposition 6.2 as the *basic allowable subwords* of type D.

Proof. Recall that for $1 \le i \le r$, $s_i(\alpha_i) = -\alpha_i$. If $1 \le i < j \le r-1$ with $|i-j| = 1$ or if $i = r-2$ and $j = r$, then $s_i(\alpha_j) = s_j(\alpha_i) = \alpha_i + \alpha_j$. For $i = r-1$ or $i = r$ we have that $s_{r-1}(\alpha_r) = \alpha_r$ and $s_r(\alpha_{r-1}) = \alpha_{r-1}$. The highest root in this case is $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$.

Observe that $\sigma \in \mathcal{D}_r(\tilde{\alpha}, 0)$ if and only if $\sigma(\tilde{\alpha}+\rho)-\rho$ can be written as a nonnegative integral combination of simple roots.

Clearly $1 \in \mathcal{D}_r(\tilde{\alpha}, 0)$ since $1(\tilde{\alpha}+\rho)-\rho = \tilde{\alpha}$ which can be written as a sum of simple roots with nonnegative integer coefficients.

Let $r \ge 3$ and observe that by Lemma 3.1 and Lemma 3.6

$$\begin{aligned} s_1(\tilde{\alpha}+\rho)-\rho &= \tilde{\alpha}+\rho-\alpha_1-\rho \\ &= 2\alpha_2+\cdots+2\alpha_{r-2}+\alpha_{r-1}+\alpha_r, \\ s_2(\tilde{\alpha}+\rho)-\rho &= \tilde{\alpha}-\alpha_2+\rho-\alpha_2-\rho \\ &= \alpha_1+2\alpha_3+\cdots+2\alpha_{r-2}+\alpha_{r-1}+\alpha_r, \\ s_{r-1}(\tilde{\alpha}+\rho)-\rho &= \tilde{\alpha}+\rho-\alpha_{r-1}-\rho \\ &= \alpha_1+2\alpha_+\cdots+2\alpha_{r-2}+\alpha_r, \end{aligned}$$

$$s_r(\tilde{\alpha}+\rho)-\rho = \tilde{\alpha}+\rho-\alpha_r-\rho$$
$$= \alpha_1+2\alpha_2+\cdots+2\alpha_{r-2}+\alpha_{r-1}.$$

Now for $3 \le i \le r$ we have that by Lemma 3.1 and Lemma 3.6

$$s_i(\tilde{\alpha}+\rho)-\rho = \alpha_1+2\alpha_2+\cdots+2\alpha_{i-1}+\alpha_i+2\alpha_{i+1}+\cdots$$
$$+2\alpha_{r-2}+\alpha_{r-1}+\alpha_r.$$

Hence $s_i \in \mathcal{D}_r(\tilde{\alpha},0)$ for all $1 \le i \le r$, with $r \ge 3$.

Now let $r \ge 4$ and $3 \le i \le r-3$. Then by Lemmas 3.1 and 3.6

$$s_is_{i+1}(\tilde{\alpha}+\rho)-2\rho = \alpha_1+2\alpha_2+\cdots+2\alpha_{i-1}+\alpha_{i+1}+2\alpha_{i+2}+\cdots$$
$$+2\alpha_{r-2}+\alpha_{r-1}+\alpha_r.$$

Similarly,

$$s_{r-2}s_{r-1}(\tilde{\alpha}+\rho)-\rho = \tilde{\alpha}+\rho-\alpha_{r-1}-\alpha_r-\rho = \alpha_1+2\alpha_2+\cdots+2\alpha_{r-2}.$$

Hence $s_is_{i+1} \in \mathcal{D}_r(\tilde{\alpha},0)$, for all $3 \le i \le r-3$, with $r \ge 4$.

Now let $r \ge 6$ and $3 \le i \le r-3$. Then by Lemmas 3.1 and 3.6

$$s_{i+1}s_i(\tilde{\alpha}+\rho)-\rho = \alpha_1+2\alpha_2\cdots+2\alpha_{i-1}+\alpha_i+2\alpha_{i+2}+\cdots$$
$$+2\alpha_{r-2}+\alpha_{r-1}+\alpha_r.$$

Hence $s_{i+1}s_i \in \mathcal{D}_r(\tilde{\alpha},0)$, for all $3 \le i \le r-3$, with $r \ge 6$.

Let $r \ge 6$ and let $3 \le i \le r-3$. Then by Lemmas 3.1 and 3.6

$$s_is_{i+1}s_i(\tilde{\alpha}+\rho)-\rho = \alpha_1+2\alpha_2+\cdots+2\alpha_{i-1}+2\alpha_{i+2}+\cdots$$
$$+2\alpha_{r-2}+\alpha_{r-1}+\alpha_r.$$

Hence $s_is_{i+1}s_i \in \mathcal{D}_r(\tilde{\alpha},0)$, for all $3 \le i \le r-3$, with $r \ge 6$.

Let $r \ge 7$ and let $3 \le i \le r-4$. Then by Lemmas 3.1 and 3.7

$$s_is_{i+2}s_{i+1}(\tilde{\alpha}+\rho)-\rho = \alpha_1+2\alpha_2+\cdots+2\alpha_{i-1}+\alpha_{i+1}+2\alpha_{i+3}+\cdots$$
$$+2\alpha_{r-2}+\alpha_{r-1}+\alpha_r.$$

Hence $s_is_{i+2}s_{i+1} \in \mathcal{D}_r(\tilde{\alpha},0)$, for all $3 \le i \le r-4$, with $r \ge 7$. This completes the proof. □

Lemma 6.3. *Let σ and τ be noncommuting basic allowable subwords. Then the product $\sigma\tau$ falls into one of the following three cases:*

1. *$\sigma\tau$ is itself a basic allowable subword with $\ell(\sigma\tau) = \ell(\sigma) + \ell(\tau)$,*
2. *$\sigma\tau$ reduces to a commuting product of basic allowable subwords with $\ell(\sigma\tau) < \ell(\sigma) + \ell(\tau)$,*
3. *$\sigma\tau$ contains a minimal forbidden subword as listed in Lemma 3.11.*

The proof of Lemma 6.3 is analogous to the one provided for type B given in Appendix A.

Corollary 6.4. *If $\sigma \in W$ can be expressed as a product of commuting basic allowable subwords of type D, then $\sigma \in \mathcal{D}_r(\tilde{\alpha}, 0)$.*

Proof. This follows from the fact that all basic allowable subwords are in $\mathcal{D}_r(\tilde{\alpha}, 0)$ by Proposition 6.2, and since we are assuming these basic allowable subwords commute, these subwords act on disjoint sets of simple roots in expression $\tilde{\alpha} + \rho$. Hence the expression $\sigma(\tilde{\alpha} + \rho) - \rho$ will continue to be expressible as a nonnegative integral combination of simple roots, and thus this disjoint product of basic allowable subwords will again be in $\mathcal{D}_r(\tilde{\alpha}, 0)$. □

Lemma 6.3, and Corollary 6.4 show that the only elements of W_r that are contained in the alternation set $\mathcal{D}_r(\tilde{\alpha}, 0)$ are words which reduce to basic allowable subwords (these subwords are listed in Proposition 6.2) and products of basic allowable subwords that commute (i.e. their supports are not adjacent in the Dynkin diagram.) Thus we give a classification of the set $\mathcal{D}_r(\tilde{\alpha}, 0)$ in terms of commuting products of basic allowable subwords as follows.

Theorem 6.5. *Let $\sigma \in W_r$ be a reduced word. Then $\sigma \in \mathcal{D}_r(\tilde{\alpha}, 0)$ if and only if σ is either basic allowable subword or a product of commuting basic allowable subwords of type D.*

Proof. ($\Leftarrow$) From Proposition 6.2 and Corollary 6.4 we see that the basic allowable subwords and all products of commuting basic allowable subwords are in $\mathcal{D}_r(\tilde{\alpha}, 0)$.

($\Rightarrow$) Suppose that $\sigma = b_1 b_2 \cdots b_k$ is a reduced word, and that each b_i is a basic allowable subword of maximal length (i.e. none of the products $b_i b_j$ can be written as a basic allowable subword). Suppose that there are a pair of basic allowable subwords b_i and b_j in the reduced word for σ which are not commuting basic allowable subwords. Lemma 6.3 shows that there are three possible cases: In the first case $b_i b_j$ is a minimal forbidden subword, and that means $\sigma \notin \mathcal{D}_r(\tilde{\alpha}, 0)$. In the second case $b_i b_j$ can be combined into one basic allowable subword of the same length. This contradicts our assumption that each of the b_i have maximal length. The third case is that $b_i b_j$ is a product of commuting allowable subwords with $\ell(b_ib_j) < \ell(b_i) + \ell(b_j)$. This contradicts our assumption that $\sigma = b_1b_2 \cdots b_k$ is a reduced word. Thus σ is not in $\mathcal{D}_r(\tilde{\alpha}, 0)$ if it cannot be written as a product of commuting basic allowable subwords. □

6.1. Cardinality of $\mathcal{D}_r(\tilde{\alpha}, 0)$

To help us recursively count the elements in $\mathcal{D}_r$, we start by defining some special subsets of the support. Letting $\mathcal{D}_r := \mathcal{D}_r(\tilde{\alpha}, 0)$, as denoted in Equation (4), we then let $M_r \subset \mathcal{D}_r$ denote the subset of $\mathcal{D}_r$ consisting of elements that do not contain s_1 in any reduced word decomposition. Let $N_r \subset \mathcal{D}_r$ denote the subset of $\mathcal{D}_r$ consisting of elements that contain s_1. By definition $N_r = \mathcal{D}_r \setminus M_r$, $\mathcal{D}_r = M_r \uplus N_r$ and hence $|\mathcal{D}_r| = |M_r| + |N_r|$. Let $L_r \subset \mathcal{D}_r$ denote the subset of $\mathcal{D}_r$ consisting of elements that do not contain s_1 or s_2. Note that if $\sigma \in N_r$, then there exists $\tau \in L_r$ such that $s_1\tau = \sigma$. Hence $|N_r| = |L_r|$.

With this notation in place, we define a map

$$\phi : \mathcal{D}_{r-1} \to M_r \subset \mathcal{D}_r$$

which sends s_i to s_{i+1} for every simple transposition $s_1, \ldots, s_{r-1}$.

We can now characterize the elements of the set N_r. When $r \geq 8$ the elements of N_r are obtained from the sets L_{r-1},L_{r-2}, L_{r-3}, and L_{r-4} by either multiplying s_1 times a word from $\phi(L_{r-1})$, multiplying s_1s_3 times a word from $\phi^2(L_{r-2})$, multiplying $s_1s_3s_4$, $s_1s_4s_3$, or $s_1s_3s_4s_3$ times a word from $\phi^3(L_{r-3})$, or multiplying $s_1s_3s_5s_4$ times a word from $\phi^4(L_{r-4})$.

Since $|N_r| = |L_r|$ this implies that the cardinality of N_r satisfies the following recursion:

$$|N_r| = |N_{r-1}| + |N_{r-2}| + 3|N_{r-3}| + |N_{r-4}|. \tag{5}$$

Next we characterize the elements of the set M_r. Every element of M_r either contains s_2 or it does not. The ones that contain s_2 are obtained by multiplying s_2 times the elements of $\phi(L_{r-1})$. The elements of M_r by definition do not contain s_1, so if in addition they do not contain s_2 they are, again by definition, all elements of L_r. This implies that $|M_r|$ satisfies the following recursion:

$$|M_r| = |L_r| + |L_{r-1}| = |N_r| + |N_{r-1}| = 2|N_{r-1}| + |N_{r-2}| + 3|N_{r-3}| + |N_{r-4}|. \tag{6}$$

Finally, by the definitions of $\mathcal{D}_r$, M_r, and N_r we see that

$$|\mathcal{D}_r| = |M_r| + |N_r| = 2|N_r| + |N_{r-1}| = 3|N_{r-1}| + 2|N_{r+2}| + 6|N_{r-3}| + 2|N_{r-4}|. \tag{7}$$

We have listed the elements of D_r for $r \leq 7$ in the previous section. From these sets, and the recursions described in Equations (5), (6), (7), we can find the cardinalities of the sets $\mathcal{D}_r$, M_r, N_r, and L_r for $r \geq 4$.[7] The cardinalities of the sets $\mathcal{D}_r$, M_r, N_r for $4 \leq r \leq 16$ are:

$|D_r|$: 9, 18, 35, 82,180, 385, 846, 1853, 4034, 8810, 19249, 42014, 91727, ...
$|M_r|$: 5, 11, 21, 48, 107, 229, 501, 1099, 2394, 5225, 11417, 24923, 54409, ...
$|N_r|$: 4, 7, 14, 34, 73, 156, 345, 754, 1640, 3585, 7832, 17091, 37318, 81490, ...

7. Non-zero weight spaces

It is fundamental in Lie theory that the zero-weight space is a Cartan subalgebra, and that the non-zero weights of the adjoint representation of $\mathfrak{g}$ are the roots and have multiplicity 1. We visit this from our point of view in the case of the Lie algebras of types B, C, and D. First we begin with the following general result.

Theorem 7.1. *Let λ be a dominant integral weight of the simple Lie algebra $\mathfrak{g}$ of rank r. Then $\sigma(\lambda+\rho)-\lambda-\rho$ can be written as a nonnegative integral sum of positive roots if and only if σ is the identity.*

Proof. ($\Rightarrow$) If $\sigma \neq 1$, then there exists nonnegative integers $m_1, \ldots, m_j$ between 1 and r, such that $\sigma(\lambda+\rho) = \lambda+\rho-\sum_{i=1}^{j} m_i\alpha_i$. Then $\sigma(\lambda+\rho)-\lambda-\rho = -\sum_{i=1}^{j} m_i\alpha_i$. Hence $\sigma(\lambda+\rho)-\lambda-\rho$ cannot be written as nonnegative integral sum of positive roots.

($\Leftarrow$) If $\sigma = 1$, then $\sigma(\lambda+\rho)-\lambda-\rho = 0$, which can be written as a nonnegative integral combination of positive roots as desired. □

Recall that the fundamental weights (relative to the choice of simple roots) are the elements $\varpi_1, \ldots, \varpi_r$ of $\mathfrak{h}^*$ which are dual to the coroot basis $\{\check{\alpha}_1, \ldots, \check{\alpha}_r\}$, see [6] for notation. Also recall that in every Lie type the highest root is a dominant weight since it is the highest weight of the adjoint representation. Thus Theorem 7.1 implies the following.

Corollary 7.2. *Let $\tilde{\alpha}$ denote the highest root of the Lie algebra of type A, B, C, or D, respectively. Then, in each respective Lie type, the Weyl*

[7]These sequences of integers, A234576, A234597, A234599, were added by the authors to The On-Line Encyclopedia of Integer Sequences (OEIS).

alternation set associated to the pair of dominant weights $\lambda = \tilde{\alpha}$ *and* $\mu = \tilde{\alpha}$ *is given by* $\mathcal{A}(\tilde{\alpha}, \tilde{\alpha}) = \{1\}$*.*

Recall that given $\mu \in P(\mathfrak{g})$, there exists $w \in W$ and $\xi \in P_+(\mathfrak{g})$ such that $w(\xi) = \mu$ and given that weight multiplicities are invariant under W (Propositions 3.1.20, 3.2.27 in [6]) it suffices to consider $\mu \in P_+(\mathfrak{g})$. Thus Corollary 7.2 implies that for all Lie types, $m(\tilde{\alpha}, \mu) = 1$, whenever $\mu \in \Phi$.

However, it is interesting to consider the remaining cases where there exists a dominant positive root, which is not the highest root. Namely the case $\lambda = \tilde{\alpha}$ and $\mu = \varpi_1$ in type B and the case $\lambda = \tilde{\alpha}$ and $\mu = \varpi_2$ in type C.[8]

Theorem 7.3. *Let* $\sigma \in W$*, then* $\sigma \in \mathcal{B}_r(\tilde{\alpha}, \varpi_1)$ *if and only if* $\sigma = 1$ *or* $\sigma = s_{i_1} s_{i_2} \cdots s_{i_j}$*, where* $i_1, \ldots, i_j$ *are nonconsecutive integers between* 3 *and* r*.*

Proof. Recall $\sigma \in \mathcal{B}_r(\tilde{\alpha}, \varpi_1)$ if and only if $\sigma(\tilde{\alpha} + \rho) - \rho - \varpi_1$ can be written as a nonnegative integral combination of simple roots. Also recall that in the type B case the highest root is $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_r$ and $\varpi_1 = \alpha_1 + \cdots + \alpha_r$.

($\Leftarrow$): Observe that $1(\tilde{\alpha} + \rho) - \rho - \varpi_1 = (\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_r) - (\alpha_1 + \cdots + \alpha_r) = \alpha_2 + \cdots + \alpha_r$, which can be written as a sum of simple roots with nonnegative integer coefficients. Thus, if $\sigma = 1$, then $\sigma \in \mathcal{B}_r(\tilde{\alpha}, \varpi_1)$. Now observe that if $3 \le i \le r$, then by Lemmas 3.1 and 3.6

$$s_i(\tilde{\alpha} + \rho) - \rho - \varpi_1 = \alpha_2 + \cdots + \alpha_{i-1} + \alpha_{i+1} + \cdots + \alpha_r.$$

Hence $s_i \in \mathcal{B}_r(\tilde{\alpha}, \varpi_1)$ for all $3 \le i \le r$. Suppose $\sigma = s_{i_1} s_{i_2} \cdots s_{i_j}$, where $i_1, \ldots, i_j$ are nonconsecutive integers between 3 and r. Then by Lemmas 3.1 and 3.6 we have that

$$s_{i_1} s_{i_2} \cdots s_{i_j}(\tilde{\alpha} + \rho) - \rho - \varpi_1 = (\alpha_2 + \cdots + \alpha_r) - (\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_j}).$$

Thus $\sigma \in \mathcal{B}_r(\tilde{\alpha}, \varpi_1)$ as claimed.

($\Rightarrow$): Suppose that $\sigma \in \mathcal{B}_r(\tilde{\alpha}, \varpi_1)$. If $\sigma = 1$, we are done. So suppose that σ is not the identity element. First notice that $s_1(\tilde{\alpha} + \rho) - \rho - \varpi_1 = -\alpha_1 + \alpha_2 + \cdots + \alpha_r$ and $s_2(\tilde{\alpha} + \rho) - \rho - \varpi_1 = -\alpha_2 + \alpha_3 + \cdots + \alpha_r$. Hence σ cannot contain s_1 and s_2 as a factor. Then by Lemmas 3.1 and 3.6

$$s_i s_{i+1}(\tilde{\alpha} + \rho) - \rho - \varpi_1 = \alpha_2 + \cdots + \alpha_{i-1} - \alpha_i + \alpha_{i+2} + \cdots + \alpha_r \text{ and}$$
$$s_{i+1} s_i(\tilde{\alpha} + \rho) - \rho - \varpi_1 = \alpha_2 + \cdots + \alpha_{i-1} - \alpha_{i+1} + \alpha_{i+2} + \cdots + \alpha_r.$$

[8] It is a simple exercise to show that these are the only other dominant positive roots. In fact, this is exercise 3.2.5 #1(a) in [6].

Therefore σ cannot contain any consecutive factors, as claimed. □

The Fibonacci numbers, denoted by F_n and defined in [14], are given by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, where $F_1 = F_2 = 1$.

Corollary 7.4. *Let $r \geq 2$. Then $|\mathcal{B}_r(\tilde{\alpha}, \varpi_1)| = F_r$.*

The proof of Corollary 7.4 follows from the fact that the r^{th} Fibonacci number, F_r, counts the number of ways to choose nonconsecutive integers from the numbers $3, 4, \ldots, r$.

Now we consider the case $\lambda = \tilde{\alpha}$ and $\mu = \varpi_2$ in the Lie algebra of type C.

Theorem 7.5. *Let $\sigma \in W$. Then $\sigma \in \mathcal{C}(\tilde{\alpha}, \varpi_2)$ if and only if $\sigma = 1$.*

Proof. Recall that the highest root is $\tilde{\alpha} = 2\alpha_1 + \cdots + 2\alpha_{r-1} + \alpha_r$ and $\varpi_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{r-1} + \alpha_r = \tilde{\alpha} - \alpha_1$. (⇒): Let $\sigma \in \mathcal{C}_r(\tilde{\alpha}, \varpi_2)$. If $\sigma = 1$, then we are done. So suppose σ is not the identity. Now observe that by Lemmas 3.1 and 3.6 $s_1(\tilde{\alpha}+\rho)-\rho-\varpi_2 = (\tilde{\alpha}-2\alpha_1)+(\rho-\alpha_1)-\rho-(\tilde{\alpha}-\alpha_1) = -2\alpha_1$ and for any $2 \leq i \leq r$ we have that $s_i(\tilde{\alpha}+\rho)-\rho-\varpi_2 = \tilde{\alpha}+(\rho-\alpha_i)-\rho-(\tilde{\alpha}-\alpha_1) = \alpha_1 - \alpha_i$. So σ cannot contain any factors $s_1, \ldots, s_r$. Thus σ must be the identity.

(⇐): Observe that $1(\tilde{\alpha}+\rho)-\rho-\varpi_2 = \tilde{\alpha}+\rho-\rho-(\tilde{\alpha}-\alpha_1) = \alpha_1$, hence $1 \in \mathcal{C}_r(\tilde{\alpha}, \varpi_2)$. □

Corollary 7.6. *In type C, $m(\tilde{\alpha}, \varpi_2) = 1$.*

This follows directly from Theorem 7.5 which implies that $m(\tilde{\alpha}, \varpi_2) = \wp(1(\tilde{\alpha}+\rho)-\rho-\varpi_2) = \wp(\alpha_1) = 1$. Thus the multiplicity of the weight ϖ_2 in the adjoint representation of the Lie algebra of type C is 1, as expected.

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Appendix A

Lemma 4.3. *Let σ and τ be noncommuting basic allowable subwords. Then the product $\sigma\tau$ falls into one of the following three cases:*

1. $\sigma\tau$ is itself a basic allowable subword with $\ell(\sigma\tau) = \ell(\sigma) + \ell(\tau)$,
2. $\sigma\tau$ reduces to a commuting product of basic allowable subwords with $\ell(\sigma\tau) < \ell(\sigma) + \ell(\tau)$,
3. $\sigma\tau$ contains a minimal forbidden subword as listed in Lemma 3.11.

Proof. We proceed by a case by case analysis, multiplying pairs of basic allowable subwords and classifying which case each product falls into. Let us begin by considering the product of two identical basic allowable subwords:

- $(s_i)(s_i) = 1$ is a basic allowable subword (Proposition 4.2 Part (1)),
- $(s_is_{i+1})(s_is_{i+1}) = (s_{i+1}s_is_{i+1})s_{i+1} = s_{i+1}s_i$ is a basic allowable subword (Proposition 4.2 Part (3)),
- $(s_{i+1}s_i)(s_{i+1}s_i) = (s_is_{i+1}s_i)s_i = s_is_{i+1}$ is a basic allowable subword (Proposition 4.2 Part (2)),
- $(s_is_{i+1}s_i)(s_is_{i+1}s_i) = 1$ is a basic allowable subword (Proposition 4.2 Part (1)),
- $(s_is_{i+2}s_{i+1})(s_is_{i+2}s_{i+1}) = s_i(s_{i+2}s_{i+1}s_i)s_{i+2}s_{i+1}$ contains a minimal forbidden subword (Lemma 3.11 Part 1).

In the case where we multiply two distinct basic allowable subwords we can consider the following cases.
Using (2)(3):

- $(s_{i-1})(s_is_{i+1}) = s_{i-1}s_is_{i+1}$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_i)(s_is_{i+1}) = s_{i+1}$ is a basic allowable subword (Proposition 4.2 Part (2)),
- $(s_{i+1})(s_is_{i+1}) = s_{i+1}s_is_{i+1} = s_1s_{i+1}s_i$ is a basic allowable subword (Proposition 4.2 Part (5)).

Using (3)(2):

- $(s_is_{i+1})(s_{i-1}) = s_is_{i-1}s_{i+1}$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_is_{i+1})(s_i) = s_is_{i+1}s_i$ is a basic allowable subword (Proposition 4.2 Part (5)),
- $(s_is_{i+1})(s_{i+1}) = s_i$ is a basic allowable subword (Proposition 4.2 Part (2)).

Using (2)(4):

- $(s_{i-1})(s_{i+1}s_i) = s_{i-1}s_{i+1}s_i$ is a basic allowable subword (Proposition 4.2 Part (5)),

- $(s_i)(s_{i+1}s_i) = s_is_{i+1}s_i$ is a basic allowable subword (Proposition 4.2 Part (5)),
- $(s_{i+1})(s_{i+1}s_i) = s_i$ is a basic allowable subword (Proposition 4.2 Part (2)).

Using (4)(2):

- $(s_{i+1}s_i)(s_{i-1}) = s_{i+1}s_is_{i-1}$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_{i+1}s_i)(s_i) = s_{i+1}$ is a basic allowable subword (Proposition 4.2 Part (2)),
- $(s_{i+1}s_i)(s_{i+1}) = s_is_{i+1}s_i$ is a basic allowable subword (Proposition 4.2 Part (5)).

Using (2)(5):

- $(s_{i-1})(s_is_{i+1}s_i) = (s_{i-1}s_is_{i+1})s_i$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_i)(s_is_{i+1}s_i) = s_{i+1}s_i$ is a basic allowable subword (Proposition 4.2 Part (4)),
- $(s_{i+1})(s_is_{i+1}s_i) = s_is_{i+1}$ is a basic allowable subword (Proposition 4.2 Part (3)).

Using (5)(2):

- $(s_is_{i+1}s_i)(s_{i-1}) = s_i(s_{i+1}s_is_{i-1})$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_is_{i+1}s_i)(s_i) = s_is_{i+1}$ is a basic allowable subword (Proposition 4.2 Part (3)),
- $(s_is_{i+1}s_i)(s_{i+1}) = s_{i+1}s_i$ is a basic allowable subword (Proposition 4.2 Part (4)).

Using (2)(6):

- $(s_{i-1})(s_is_{i+2}s_{i+1}) = s_{i+2}s_{i-1}s_is_{i+1}$ contains a minimal forbidden subword (Lemma 3.11 Part 2),
- $(s_i)(s_is_{i+2}s_{i+1}) = s_{i+2}s_{i+1}$ is a basic allowable subword (Proposition 4.2 Part (4)),
- $(s_{i+1})(s_is_{i+2}s_{i+1}) = (s_{i+1}s_is_{i+2})s_{i+1}$ contains a minimal forbidden subword (Lemma 3.11 Part 1).

Using (6)(2):

- $(s_is_{i+2}s_{i+1})(s_{i-1}) = s_{i+2}s_is_{i-1}s_{i+1}$ contains a minimal forbidden subword (Lemma 3.11 Part 2),

- $(s_i s_{i+2} s_{i+1})(s_i) = s_i(s_{i+2} s_{i+1} s_i)$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_i s_{i+2} s_{i+1})(s_{i+1}) = s_i s_{i+2}$ a commuting product of allowable basic subwords.

Using (3)(4):

- $(s_{i-2} s_{i-1})(s_{i+1} s_i) = s_{i+1}(s_{i-2} s_{i-1} s_i)$ contains a forbidden subword (Lemma 3.11 Part 1),
- $(s_{i-1} s_i)(s_{i+1} s_i) = s_{i-1}(s_{i+1} s_i s_{i+1}) = s_{i+1}(s_{i-1} s_i s_{i+1})$ contains a forbidden subword (Lemma 3.11 Part 1),
- $(s_i s_{i+1})(s_{i+1} s_i) = 1$ is a basic allowable subword (Proposition 4.2 Part (1)),
- $(s_{i+1} s_{i+2})(s_{i+1} s_i) = s_{i+1}(s_{i+2} s_{i+1} s_i)$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_{i+2} s_{i+3})(s_{i+1} s_i) = (s_{i+2} s_{i+1} s_{i+3}) s_i$ contains a minimal forbidden subword (Lemma 3.11 Part 1).

Using (4)(3):

- $(s_{i+1} s_i)(s_{i-2} s_{i-1}) = s_{i-2} s_{i+1} s_i s_{i-1}$ contains a minimal forbidden subword (Lemma 3.11 Part 2),
- $(s_{i+1} s_i)(s_{i-1} s_i) = (s_{i+1} s_i s_{i-1}) s_i$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_{i+1} s_i)(s_i s_{i+1}) = 1$ is a basic allowable subword (Proposition 4.2 Part (1)),
- $(s_{i+1} s_i)(s_{i+1} s_{i+2}) = s_{i+1}(s_i s_{i+1} s_{i+2})$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_{i+1} s_i)(s_{i+2} s_{i+3}) = (s_{i+1} s_{i+2} s_{i+3} s_i)$ contains a minimal forbidden subword (Lemma 3.11 Part 2).

Using (3)(5):

- $(s_{i-2} s_{i-1})(s_i s_{i+1} s_i) = (s_{i-2} s_{i-1} s_i s_{i+1}) s_i)$ contains a forbidden subword (Lemma 3.11 Part 2),
- $(s_{i-1} s_i)(s_i s_{i+1} s_i) = s_{i-1} s_{i+1} s_i$ is a basic allowable subword (Proposition 4.2 Part (6)),
- $(s_i s_{i+1})(s_i s_{i+1} s_i) = s_{i+1}$ is a basic allowable subword (Proposition 4.2 Part (2)),
- $(s_{i+1} s_{i+2})(s_i s_{i+1} s_i) = (s_{i+1} s_i s_{i+2}) s_{i+1} s_i$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_{i+2} s_{i+3})(s_i s_{i+1} s_i) = (s_i s_{i+2} s_{i+1} s_{i+3}) s_i$ contains a minimal forbidden subword (Lemma 3.11 Part 2).

Using (5)(3):

- $(s_is_{i+1}s_i)(s_{i-2}s_{i-1}) = s_i(s_{i+1}s_is_{i-2}s_{i-1})$ contains a minimal forbidden subword (Lemma 3.11 Part 2),
- $(s_is_{i+1}s_i)(s_{i-1}s_i) = s_i(s_{i+1}s_is_{i-1})s_i$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_is_{i+1}s_i)(s_is_{i+1}) = s_i$ is a basic allowable subword (Proposition 4.2 Part (2)),
- $(s_is_{i+1}s_i)(s_{i+1}s_{i+2}) = s_is_{i+1}(s_is_{i+1}s_{i+2})$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_is_{i+1}s_i)(s_{i+2}s_{i+3}) = s_i(s_{i+1}s_is_{i+2}s_{i+3})$ contains a minimal forbidden subword (Lemma 3.11 Part 2).

Using (3)(6):

- $(s_{i-2}s_{i-1})(s_is_{i+2}s_{i+1}) = s_{i-2}(s_{i-1}s_i)s_{i+2}s_{i+1})$ contains a minimal forbidden subword (Lemma 3.11 Part 2),
- $(s_{i-1}s_i)(s_is_{i+2}s_{i+1}) = (s_{i-1})(s_{i+2}s_{i+1})$ a commuting product of basic allowable subwords,
- $(s_is_{i+1})(s_is_{i+2}s_{i+1}) = s_i(s_{i+1}s_is_{i+2})s_{i+1}$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_{i+1}s_{i+2})(s_is_{i+2}s_{i+1}) = (s_{i+1}s_is_{i+1}) = s_is_{i+1}s_i$ is a basic allowable subword (Proposition 4.2 Part (5)),
- $(s_{i+2}s_{i+3})(s_is_{i+2}s_{i+1}) = s_{i+2}(s_{i+3}s_is_{i+2}s_{i+1})$ contains a minimal forbidden subword (Lemma 3.11 Part 2),
- $(s_{i+3}s_{i+4})(s_is_{i+2}s_{i+1}) = (s_{i+3}s_is_{i+2}s_{i+1})s_{i+4}$ contains a minimal forbidden subword (Lemma 3.11 Part 2).

Using (6)(3):

- $(s_is_{i+2}s_{i+1})(s_{i-2}s_{i-1}) = s_{i+2}(s_is_{i+1}s_{i-2}s_{i-1})$ contains a minimal forbidden subword (Lemma 3.11 Part 2),
- $(s_is_{i+2}s_{i+1})(s_{i-1}s_i) = (s_is_{i+2}s_{i+1}s_{i-1})s_i$ contains a minimal forbidden subword (Lemma 3.11 Part 2),
- $(s_is_{i+2}s_{i+1})(s_is_{i+1}) = s_i(s_{i+2}s_{i+1}s_i)s_{i+1}$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_is_{i+2}s_{i+1})(s_{i+1}s_{i+2}) = s_i$ is a basic allowable subword (Proposition 4.2 Part (2)),
- $(s_is_{i+2}s_{i+1})(s_{i+2}s_{i+3}) = s_{i+2}(s_is_{i+1}s_{i+2}s_{i+3})$ contains a minimal forbidden subword (Lemma 3.11 Part 2),
- $(s_is_{i+2}s_{i+1})(s_{i+3}s_{i+4}) = s_i(s_{i+2}s_{i+1}s_{i+3}s_{i+4})$ contains a minimal forbidden subword (Lemma 3.11 Part 2).

Using (4)(5):

- $(s_{i-1}s_{i-2})(s_is_{i+1}s_i) = (s_{i-1}s_is_{i+1}s_{i-2})s_i$ contains a minimal forbidden subword (Lemma 3.11 Part 2),
- $(s_is_{i-1})(s_is_{i+1}s_i) = s_i(s_{i-1}s_is_{i+1})s_i$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_{i+1}s_i)(s_is_{i+1}s_i) = s_i$ is a basic allowable subword (Proposition 4.2 Part (2)),
- $(s_{i+2}s_{i+1})(s_is_{i+1}s_i) = (s_{i+2}s_{i+1}s_i)s_{i+1}s_i$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_{i+3}s_{i+2})(s_is_{i+1}s_i) = (s_{i+3}s_is_{i+2}s_{i+1})s_i$ contains a minimal forbidden subword (Lemma 3.11 Part 2).

Using (5)(4):

- $(s_is_{i+1}s_i)(s_{i-1}s_{i-2}) = s_i(s_{i+1}s_is_{i-1}s_{i-2})$ contains a minimal forbidden subword (Lemma 3.11 Part 2),
- $(s_is_{i+1}s_i)(s_is_{i-1}) = s_is_{i-1}s_{i+1}$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_is_{i+1}s_i)(s_{i+1}s_i) = s_{i+1}$ is a basic allowable subword (Proposition 4.2 Part (2)),
- $(s_is_{i+1}s_i)(s_{i+2}s_{i+1}) = s_i(s_{i+1}s_is_{i+2})s_{i+1}$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_is_{i+1}s_i)(s_{i+3}s_{i+2}) = s_i(s_{i+3}s_{i+1}s_is_{i+2})$ contains a minimal forbidden subword (Lemma 3.11 Part 2).

Using (4)(6):

- $(s_{i-1}s_{i-2})(s_is_{i+2}s_{i+1}) = (s_{i-1}s_is_{i+2}s_{i+1})s_{i-2}$ contains a minimal forbidden subword (Lemma 3.11 Part 2),
- $(s_is_{i-1})(s_is_{i+2}s_{i+1}) = s_i(s_{i+2}s_{i-1}s_is_{i+1})$ contains a minimal forbidden subword (Lemma 3.11 Part 2),
- $(s_{i+1}s_i)(s_is_{i+2}s_{i+1}) = s_{i+1}s_{i+2}s_{i+1}$ is a basic allowable subword (Proposition 4.2 Part (5)),
- $(s_{i+2}s_{i+1})(s_is_{i+2}s_{i+1}) = (s_{i+2}s_{i+1}s_i)s_{i+2}s_{i+1}$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_{i+3}s_{i+2})(s_is_{i+2}s_{i+1}) = s_{i+3}(s_is_{i+1})$ a commuting product of basic allowable subwords,
- $(s_{i+4}s_{i+3})(s_is_{i+2}s_{i+1}) = s_{i+4}(s_{i+3}s_{i+2}s_is_{i+1})$ contains a minimal forbidden subword (Lemma 3.11 Part 2).

Using (6)(4):

- $(s_is_{i+2}s_{i+1})(s_{i-1}s_{i-2}) = s_{i+2}(s_is_{i-1}s_{i+1}s_{i+2})s_{i+1}$ contains a minimal forbidden subword (Lemma 3.11 Part 2),
- $(s_is_{i+2}s_{i+1})(s_is_{i-1}) = s_i(s_{i+2}s_{i+1}s_is_{i-1})$ contains a minimal forbidden subword (Lemma 3.11 Part 2),
- $(s_is_{i+2}s_{i+1})(s_{i+1}s_i) = s_{i+2}$ is a basic allowable subword (Proposition 4.2 Part (2)),
- $(s_is_{i+2}s_{i+1})(s_{i+2}s_{i+1}) = s_{i+2}(s_is_{i+1}s_{i+2})s_{i+1}$ contains a minimal forbidden subword (Lemma 3.11 Part 1),
- $(s_is_{i+2}s_{i+1})(s_{i+3}s_{i+2}) = (s_is_{i+2}s_{i+1}s_{i+3})s_{i+2}$ contains a minimal forbidden subword (Lemma 3.11 Part 2),
- $(s_is_{i+2}s_{i+1})(s_{i+4}s_{i+3}) = s_i(s_{i+4}s_{i+2}s_{i+1}s_{i+3})$ contains a minimal forbidden subword (Lemma 3.11 Part 2).

Notice that if we use two basic allowable subwords from part (5) and (6), then their product will contain a subword whose indices are four consecutive roots in the Dynkin diagram. Hence they will contain a minimal forbidden subword as listed in Lemma 3.11. ☐

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