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Graph saturation in multipartite graphs

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Let G be a fixed graph and let $\mathcal{F}$ be a family of graphs. A subgraph J of G is *$\mathcal{F}$-saturated* if no member of $\mathcal{F}$ is a subgraph of J, but for any edge e in $E(G)-E(J)$, some element of $\mathcal{F}$ is a subgraph of $J+e$. We let $\text{ex}(\mathcal{F}, G)$ and $\text{sat}(\mathcal{F}, G)$ denote the maximum and minimum size of an $\mathcal{F}$-saturated subgraph of G, respectively. If no element of $\mathcal{F}$ is a subgraph of G, then $\text{sat}(\mathcal{F}, G) = \text{ex}(\mathcal{F}, G) = |E(G)|$.

In this paper, for $k \geq 3$ and $n \geq 100$ we determine $\text{sat}(K_3, K_k^n)$, where K_k^n is the complete balanced k-partite graph with partite sets of size n. We also give several families of constructions of K_t-saturated subgraphs of K_k^n for $t \geq 4$. Our results and constructions provide an informative contrast to recent results on the edge-density version of $\text{ex}(K_t, K_k^n)$ from [A. Bondy, J. Shen, S. Thomassé, and C. Thomassen, Density conditions for triangles in multipartite graphs, *Combinatorica* **26** (2006), 121–131] and [F. Pfender, Complete subgraphs in multipartite graphs, *Combinatorica* **32** (2012), no. 4, 483–495].

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1. Introduction

All graphs in this paper are simple. Let $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v, respectively, and for a set of vertices S, let $N(S) = \bigcup_{x \in S} N(S)$. The set $N[S]$ is defined similarly. Further, $d(v)$ denotes the degree of a vertex v, and $\delta(G)$ denotes the minimum degree of a graph G. Given two sets of vertices X, and Y, we let $E(X, Y)$ denote the set of edges joining X and Y. Central to this paper is K_k^n, the complete balanced

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k-partite graph with partite sets of size n. Throughout, $V_1, V_2, \ldots, V_k$ will be the partite sets of K_k^n such that $V_i = \{v_i^1, v_i^2, \ldots, v_i^n\}$ for each $i \in \{1, \ldots, k\}$. Furthermore, to avoid certain degeneracies, we assume that $k \geq 3$ and that $n \geq 2$.

Given a family of graphs $\mathcal{F}$, a graph G is *$\mathcal{F}$-saturated* if no element of $\mathcal{F}$ is a subgraph of G, but for any edge e in the complement of G, some element of $\mathcal{F}$ is a subgraph of $G + e$. If $\mathcal{F} = \{H\}$, then we say that G is *H-saturated*. The classical extremal function $\text{ex}(H, n)$ is the maximum number of edges in an n-vertex H-saturated graph. Erdős, Hajnal and Moon [8] studied $\text{sat}(H, n)$, the *minimum* number of edges in an n-vertex H-saturated graph, and determined $\text{sat}(K_t, n)$. The value of $\text{sat}(H, n)$ is known precisely for very few choices of H, and the best upper bound on $\text{sat}(H, n)$ for general H appears in [14]. It remains an interesting problem to determine a non-trivial lower bound on $\text{sat}(H, n)$. For a thorough survey of results on the sat function, we refer the reader to [9].

The focus of this paper is the study of $\mathcal{F}$-saturated subgraphs of a general graph. Specifically, let G be a fixed graph and let $\mathcal{F}$ be a family of graphs. A subgraph J of G is *$\mathcal{F}$-saturated* if no member of $\mathcal{F}$ is a subgraph of J, but for any edge e in $E(G) - E(J)$, some element of $\mathcal{F}$ is a subgraph of $J + e$. We let $\text{ex}(\mathcal{F}, G)$ and $\text{sat}(\mathcal{F}, G)$ denote the maximum and minimum size of an $\mathcal{F}$-saturated subgraph of G, respectively. If no element of $\mathcal{F}$ is a subgraph of G, then $\text{sat}(\mathcal{F}, G) = \text{ex}(\mathcal{F}, G) = |E(G)|$. Note as well that $\text{sat}(H, n) = \text{sat}(H, K_n)$ and $\text{ex}(H, n) = \text{ex}(H, K_n)$.

The problem of determining $\text{sat}(\mathcal{F}, G)$ for general G was first proposed in [8], and Erdős notably studied $\text{ex}(K_3, G)$ (amongst other related problems) in [7]. Subsequently Bollobás [2, 3] and Wessel [22, 23] studied a related problem on bipartite graphs. Let $K_{m,n}$ have partite sets V_1 and V_2 with $|V_1| = m$ and $|V_2| = n$. Let $\text{sat}(K_{(a,b)}, K_{(m,n)})$ denote the minimum number of edges in a subgraph J of $K_{m,n}$ such that 1) J does not contain $K_{a,b}$ with the partite set of size a being a subset of V_1 and the partite set of size b being a subset of V_2; and 2) the addition of any edge joining V_1 and V_2 to J completes such a copy of $K_{a,b}$. This parameter is the natural saturation number analogue of the Zarankiewicz number. Bollobás and Wessel independently determined the value of $\text{sat}(K_{(a,b)}, K_{(m,n)})$ for all a, b, m, and n. These results were extended to the setting of k-partite, k-uniform hypergraphs by Alon [1] and were also generalized by Pikhurko in his Ph.D. Thesis [20].

More recently, Moshkovitz and Shapira [17] studied saturation parameters in k-uniform k-partite hypergraphs without the subset requirements of the Bollobás-Wessel problem. For $d = 2$ this is the problem of determining $\text{sat}(K_{a,b}, K_{m,n})$. Moshkovits and Shapira provided constructions showing that $\mathrm{sat}(K_{a,b}, K_{n,n}) \leq (a+b-2)n - \left\lfloor \left(\frac{(a+b-2)}{2}\right)^2 \right\rfloor$ and conjectured that the bound is sharp for n sufficiently large. This construction and the Bollobás-Wessel results demonstrate that for n sufficiently large, $\mathrm{sat}(K_{a,b}, K_{n,n}) < \mathrm{sat}(K_{(a,b)}, K_{(n,n)})$. Gan, Korándi, and Sudakov [11] then showed that $\mathrm{sat}(K_{a,b}, K_{n,n}) \geq (a+b-2)n - (a+b-2)^2$ and proved that the Moshkovitz-Shapira bound is sharp for $K_{2,3}$, the first nontrivial case. Additionally, Dudek and Wojda [6] gave several bounds and exact results for $\mathrm{sat}(P_k, K_{m,n})$. Continuing with multipartite graphs, Sullivan and Wenger [21] determined the saturation number of nearly balanced complete tripartite graphs in sufficiently large tripartite graphs.

There has also been much recent work on saturation in hypercubes. Let Q_n denote the n-dimensional hypercube. Choi and Guan [5] gave an asymptotic upper bound for the special case $\mathrm{sat}(Q_2, Q_n)$ and some exact values for small values of n. Johnson and Pinto [13] improved and generalized this bound, proving that $\mathrm{sat}(Q_m, Q_n) = o(n2^{n-1})$. Finally, Morrison, Noel, and Scott [16] determined the order of growth of $\mathrm{sat}(Q_m, Q_n)$, proving that $\mathrm{sat}(Q_m, Q_n) = \Theta(2^n)$.

In other work, Morrison, Noel, and Scott [15] have also considered saturation numbers in posets, disproving a conjecture of Gerbner et al. [12] regarding the saturation number of chains. Ferrara, Harris, and Jacobson [10] also examined the structure of $\mathcal{F}$-saturated subgraphs of a general graph via a combinatorial game.

In this paper we study $\mathrm{sat}(K_t, K_k^n)$. We determine $\mathrm{sat}(K_3, K_k^n)$ for $k \geq 4$ when n is large enough, and $\mathrm{sat}(K_3, K_3^n)$ for all values of n. For $t \geq 4$, we also provide constructions of K_t-saturated subgraphs of K_k^n with few edges.

The corresponding problem of determining $\mathrm{ex}(K_3, K_k^n)$ has received considerable attention. When determining the maximum size of an H-free subgraph of a complete multipartite graph, frequently one studies the minimum number of edges joining any two partite sets rather than the total number of edges in the subgraph. Consequently, results on the maximum size of H-free subgraphs of multipartite graphs are expressed in terms of edge-densities. In 2006, Bondy, Shen, Thomassé, and Thomassen [4] determined the maximum edge-density of triangle-free subgraphs of complete tripartite graphs. Furthermore, they gave bounds on the edge density that guarantees that a subgraph of an infinite-partite graph with finite parts contains a triangle. Pfender [19] extended these results, determining the maximum density of a K_k-free subgraph of an ℓ-partite graph for large enough ℓ. In contrast to the results on the extremal function in multipartite graphs, our results for K_3-saturated subgraphs of K_k^n cannot be meaningfully expressed in terms

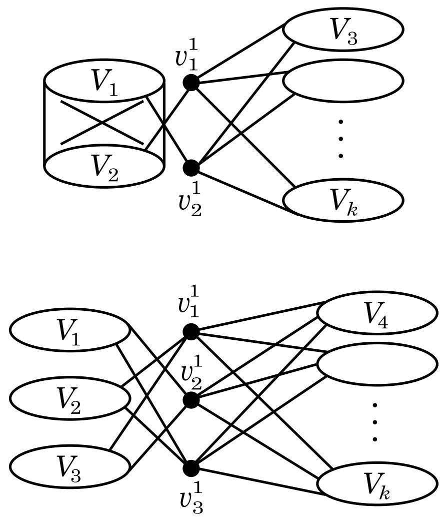

Figure 1: Constructions 1 (above) and 2 (below).

of edge densities, as we demonstrate that the minimum saturated graphs often have edge density tending to zero within certain pairs of partite sets.

2. K_3-saturated subgraphs of K_k^n

In this section we examine K_3-saturated subgraphs of K_k^n. In particular, for n large enough we determine $\mathrm{sat}(K_3, K_k^n)$ for all k, and we determine $\mathrm{sat}(K_3, K_3^n)$ for all values of n. First we provide two constructions for K_3-saturated subgraphs of K_k^n, either of which can be optimal depending on the relative sizes of k and n.

Construction 1. Begin with the complete bipartite graph with partite sets V_1 and V_2 and remove the edge $v_1^1 v_1^2$. Then join each vertex in $V_3 \cup \ldots \cup V_k$ to both v_1^1 and v_2^1 (see Figure 1, left). We call this graph G_1. Thus,

$$E(G_1) = \{v_m^1 v_i^j : 1 \le m \le 2, 3 \le i \le k, 1 \le j \le n\} \cup \{v_1^i v_2^j : i + j \ge 3\}.$$

Construction 2. First join $V_1 \setminus \{v_1^1\}$ to v_2^1 and v_3^1, join $V_2 \setminus \{v_2^1\}$ to v_1^1 and v_3^1, and join $V_3 \setminus \{v_3^1\}$ to v_1^1 and v_2^1. Then join each vertex in $V_4 \cup \ldots \cup V_k$ to v_1^1, v_2^1, and v_3^1 (see Figure 1, right). We call this graph G_2. Thus,

$$E(G_2) = \{v_m^1 v_i^j : 1 \le m \le 3, 4 \le i \le k, 1 \le j \le n\} \\ \cup \left\{v_m^1 v_i^j : (m, i) \in \{(1,2),(1,3),(2,3)\}, j \ge 2\right\}.$$

Lemma 1. *The graphs in Constructions 1 and 2 are K_3-saturated subgraphs of K_k^n, and thus*

$$\mathrm{sat}(K_3, K_k^n) \le \begin{cases} 2kn + n^2 - 4n - 1 & \text{if } k \ge n - 1 + 5/n \\ 3kn - 3n - 6 & \text{if } k < n - 1 + 5/n. \end{cases}$$

Proof. The graphs in Constructions 1 and 2 are clearly K_3-saturated subgraphs of K_k^n, and

$$|E(G_1)| = 2kn + n^2 - 4n - 1$$

and

$$|E(G_2)| = 3kn - 3n - 6.$$

Furthermore, $|E(G_1)| \le |E(G_2)|$ when $k \ge n - 1 + 5/n$. □

We now determine $\mathrm{sat}(K_3, K_k^n)$ when $k \ge 4$ and n is sufficiently large.

Theorem 2. *If $k \ge 4$ and $n \ge 100$, then*

$$\mathrm{sat}(K_3, K_k^n) = \min\{2kn + n^2 - 4k - 1, 3kn - 3n - 6\}.$$

Further, the graphs in Constructions 1 and 2, respectively, are the unique sharpness examples for the appropriate values of n and k.

To prove Theorem 2, we consider two cases, depending on the minimum degree of a K_3-saturated subgraph of K_k^n with the minimum number of edges. Each of the cases is treated in a separate lemma. The first lemma also addresses the case $k = 3$, as it will be useful in addressing the 3-partite case later in the paper.

Lemma 3. *If $k \ge 3$, $n \ge 100$, and G is a K_3-saturated subgraph of K_k^n with minimum degree $\delta(G) \ge 3$, then $|E(G)| \ge 3kn - 3n$.*

Proof. Let G be a K_3-saturated subgraph of K_k^n with minimum degree $\delta(G) \ge 3$, and $|E(G)| < 3kn - 3n$. Before proceeding, we give a brief sketch of the proof, to provide some insight into its various elements. We first show that G cannot have many vertices of low degree that are pairwise at distance at least 3 (Claim 1). Then, we demonstrate the existence of a maximal independent set of vertices, $\tilde{S}$, such that the vertices in $\tilde{S}$ have low degree and have relatively few common neighbors. After obtaining $\tilde{S}$, we show that there is a set Z of at most four vertices such that $N(Z)$ contains most of the vertices in $\tilde{S}$ (Claim 4). We then use the vertices of Z to index a careful counting argument that gives a lower bound on the number of edges induced by the neighborhood of a certain subset of $\tilde{S}$. This lower bound contradicts an upper bound on this number of edges established earlier in the proof (Claim 3).

We are now ready to proceed with the proof. Clearly, G has minimum degree at most five, as otherwise $|E(G)| \geq 3kn$.

Claim 1. G does not contain four independent vertices of degree at most 5 with pairwise disjoint neighborhoods.

Proof of Claim 1. Suppose that u_1, u_2, u_3 and u_4 are independent vertices with pairwise disjoint neighborhoods. Since G is saturated and the addition of the edge u_iu_j cannot create a triangle in G, it must be that $u_1, u_2, u_3, u_4 \in V_i$ for some i. Furthermore every vertex $y \in V(G)\backslash V_i$ has a neighbor in $N[u_j]$ for $1 \leq j \leq 4$. Thus,

$$|E(G)| \geq 4kn - 4n - \tfrac{3}{2}(d(u) + d(v) + d(w) + d(x)) \geq 4kn - 4n - 30,$$

where the last term addresses the double counting of edges between the disjoint neighborhoods of u_1, u_2, u_3, and u_4. For $n \geq 15$, this is a contradiction. □

Throughout the remaining claims, let $\tilde{S}$ be a maximal set of vertices with the following properties:

1. $\tilde{S}$ is an independent set,
2. $\tilde{S}$ contains no vertex of degree 6 or larger, and
3. for every $u \in \tilde{S}$, we have $|N(u) \cap N(\tilde{S} - u)| \leq 5 - d(u)$.

A set $\tilde{S}$ with the above properties can easily be found by a greedy search as follows. First, greedily find a maximal independent set $\tilde{S}$ of vertices of degree 3 without respect for property (3). Then, add vertices of degree 4 from $V \setminus N[\tilde{S}]$ to $\tilde{S}$ with property (3) one-by-one. If the addition of such a vertex u prompts another previously added vertex v of degree $d(v) = 4$ to lose property (3), then v is the only vertex in $\tilde{S}$ with $N(u) \cap N(v) \neq \emptyset$. Delete that vertex v from $\tilde{S}$, with the consequence that then $N(u) \cap N(\tilde{S} - u) = \emptyset$. Note that with each additional vertex, $|\tilde{S}|$ grows by one, or $|\tilde{S}|$ stays the same and the number of vertices u with $N(u) \cap N(\tilde{S} - u) = \emptyset$ increases by one. As there can be at most three such vertices by Claim 1, $|\tilde{S}|$ has to grow by one at least once for every four steps. Thus, this process will terminate, and all vertices u of degree 4 that remain in $V \setminus N[\tilde{S}]$ have at least two neighbors in $N(\tilde{S})$. Now add vertices $u \in V \setminus N[\tilde{S}]$ of degree 5 with $N(u) \cap N(\tilde{S}) = \emptyset$ one-by-one. This does not affect property (3) for any of the previous vertices. Finally, remove all vertices u of degree 3 from $\tilde{S}$ one-by-one for which $N(u) \subseteq N(\tilde{S} - u)$. Note that this does not change $N(\tilde{S})$, as each of these removed vertices has all three of its neighbors in $N(\tilde{S})$. Let $\tilde{X} = N(\tilde{S})$, and $\tilde{L} = V \setminus (\tilde{S} \cup \tilde{X})$.

Claim 2. $|E(\tilde{X})| < 3(|\tilde{X}| - n)$, so that in particular $|\tilde{X}| \geq n + 1$.

Proof of Claim 2. Assume otherwise, and observe that the maximality of $\tilde{S}$ implies that for every $v \in \tilde{L}$, $d_{\tilde{X}}(v) + \frac{1}{2}d_{\tilde{L}}(v) \geq 3$. We therefore have that

$$\begin{aligned}|E(G)| &\geq |E(\tilde{X})| + |E(\tilde{S}, \tilde{X})| + |E(\tilde{L}, \tilde{X})| + |E(\tilde{L})| \\ &\geq 3|\tilde{X}| - 3n + 3|\tilde{S}| + \sum_{v \in \tilde{L}} (d_{\tilde{X}}(v) + \tfrac{1}{2}d_{\tilde{L}}(v)) \\ &\geq 3|\tilde{X}| - 3n + 3|\tilde{S}| + 3|\tilde{L}| \\ &= 3kn - 3n,\end{aligned}$$

a contradiction showing the claim. □

Now, let $S \subseteq \tilde{S}$ be the set of vertices $s \in \tilde{S}$ with $N(s) \cap N(\tilde{S} - s) \neq \emptyset$ and let $X = N(S)$.

Claim 3. $|E(X)| < 3(|X| - n)$.

Proof of Claim 3. Let S' be the set of vertices s satisfying $N(s) \cap N(\tilde{S} - s) = \emptyset$, and let $|S'| = m$. If $S' = \emptyset$, then the claim follows immediately from Claim 2. If $S' \neq \emptyset$, then there is a vertex $s \in S'$ such that $N(s) \cap X = \emptyset$. Thus all vertices in $\tilde{S}$ must be in the same partite set V_i, and all vertices in $N(\tilde{S})$ must be in other partite sets. Furthermore, there is a path of length 2 joining each vertex in X to each vertex in S', so $|E(X, N(S')| \geq m|X|$. If $|E(X)| \geq 3(|X| - n)$, it follows that

$$\begin{aligned}|E(\tilde{X})| &\geq 3(|X| - n) + m|X| \\ &\geq 3(|\tilde{X}| - 5m - n) + m(|\tilde{X}| - 5m) \\ &= 3(|\tilde{X}| - n) + m(|\tilde{X}| - 5m - 15).\end{aligned}$$

By Claim 1, $m \leq 3$, so this is a contradiction. □

Claim 4. There exists a set $Z \subset X$ such that $|Z| \leq 4$ and $S \subseteq N(Z)$.

Proof of Claim 4. Let $Z \subset X$ be minimum with $S \subset N(Z)$ and suppose first that $S \subset V_i$ for some i. By the minimality of Z, for each $x \in Z$ there is some $s \in S$ such that $N_Z(s) = \{z\}$. Hence if $|Z| \geq 5$, then every vertex not in V_i is adjacent to at least 5 vertices in $V_i \cup N(V_i)$, and every vertex in $N(V_i)$ is adjacent to at least one vertex in V_i. By avoiding a double count of edges within $N(V_i)$, we get

$$|E(G)| \geq (1 + \tfrac{4}{2})|N(V_i)| + 5(kn - n - |N(V_i)|) \geq 3kn - 3n,$$

a contradiction. Otherwise suppose that for distinct i and j there are vertices $s_i \in V_i$ and $s_j \in V_j$ in S. By property (3), every vertex in $S \setminus V_i$ is adjacent to one of at most two neighbors of s_i. Similarly, every vertex in $S \setminus V_j$ is adjacent to one of at most two neighbors of s_j. Thus there is a set of at most four vertices in X whose combined neighborhood contains S. □

Let $Z = \{z_1, z_2, \ldots, z_{|Z|}\}$. Let $Y = \{y \in X \setminus Z : |N(y) \cap S| = 1\}$, and $W = X \setminus Y$. For $y \in Y$, let s_y be the unique vertex in S with $y \in N(s_y)$. Let

$$\begin{aligned} &S_i = S \cap N(z_i) \setminus N(\{z_1, \ldots, z_{i-1}\}), \text{ and} \\ &Y_i = Y \cap N(S_i). \end{aligned}$$

For $w \in W \setminus \{z_1, \ldots, z_i\}$, let

$$Y_i(w) = \{y \in Y : \{z_i, w\} \subset N(s_y)\}.$$

Suppose that y and y' are distinct vertices in $Y_i(w)$ and note that since s_y and $s_{y'}$ share two neighbors in X, the conditions imposed on $\tilde{S}$ imply that $d(s_y) = d(s_{y'}) = 3$. Consequently, either $yy' \in E(X)$, or both $s_{y'}, y \in V_\ell$ and $s_y, y' \in V_j$ for some j and ℓ. Otherwise, we would have $N(s_y) \cap N(y') = N(s_{y'}) \cap N(y) = \emptyset$, a contradiction to the assumption that G is K_3-saturated. Note this implies that we can never have both $y, y' \in V_j$ as V_j is an independent set. Therefore each vertex in $Y_i(w)$ is adjacent to all but at most one other vertex in $Y_i(w)$, so we have

$$|E(Y_i(w))| \geq \tfrac{1}{2}|Y_i(w)|(|Y_i(w)| - 2).$$

Partition S_i into sets $S_i^{(1)}, \ldots, S_i^{(d_i)}$ such that s and s' are in the same set if and only if they have a common neighbor in $W \setminus \{z_1, \ldots, z_i\}$. For each $S_i^{(j)}$ pick a vertex $s_i^{(j)} \in S_i^{(j)}$ and a vertex $y_i^{(j)} \in N(s_i^{(j)}) \cap Y$. Finally, assign each pair $(y_i^{(j)}, s_i^{(j)})$ the label (p, q) where $y_i^{(j)} \in V_p$ and $s_i^{(j)} \in V_q$. Given two such pairs $(y_i^{(j)}, s_i^{(j)})$ and $(y_i^{(\ell)}, s_i^{(\ell)})$ with labels (p, q) and (p', q'), respectively, there is an edge joining $y_i^{(j)}$ to $N(s_i^{(\ell)})$ whenever $p \neq q'$. Thus, when we consider $(y_i^{(j)}, s_i^{(j)})$ and $(y_i^{(\ell)}, s_i^{(\ell)})$ we count

0 edges	if $p = q'$ and $q = p'$;
1 edge	if ($p = q'$ and $q \neq p'$) or ($p \neq q'$ and $q = p'$);
1 edge	if $p \neq q'$, $q \neq p'$, and $p \neq p'$;
2 edges	if $p = p'$.

Given $p, q \in \{1, \dots, d_i\}$, let $X_{p,q}$ denote the number of pairs with label (p, q). Thus there are at least

$$\binom{d_i}{2} + \sum_{(p,q)} \binom{X_{p,q}}{2} - \sum_{p<q} X_{p,q} X_{q,p} = \binom{d_i}{2} - \frac{1}{2} d_i + \frac{1}{2} \sum_{p<q} (X_{p,q} - X_{q,p})^2$$
$$\geq \frac{1}{2} d_i (d_i - 2)$$

edges incident to $\{y_i^{(1)}, \dots, y_i^{(d_i)}\}$ that do not have both endpoints in $Y_i(w)$ for some w. Consequently there are at least

$$\tfrac{1}{2} d_i (d_i - 2) + \sum_w \tfrac{1}{2} |Y_i(w)|(|Y_i(w)| - 2)$$

edges incident to Y_i, none of which have endpoints in Y_j for $j \neq i$.

Summing up over all z_i, we get

$$|E(X)| \geq \sum_{i=1}^{|Z|} \left(\tfrac{1}{2} d_i (d_i - 2) + \sum_w \tfrac{1}{2} |Y_i(w)|(|Y_i(w)| - 2) \right).$$

This bound is minimized for fixed $|X|$ when all of the $|N(S_i)|$ and all of the $|Y_i(w)|$ are as equal as possible, and $|Z|$ is maximized, i.e. $|Z| = 4$. Further, we may modify W as follows so that there is no $s \in S_i$ with $N(s) \cap W = \{z_i\}$. If such a vertex has neighborhood $N(s) = \{z_i, y, y'\}$, then add y to W, so that $Y_i(y) = \{y'\}$. If such a vertex has neighborhood $N(s) = \{z_i, y, y', y''\}$, then add y to W, so that $Y_i(y) = \{y', y''\}$. In either case, note that d_i is unchanged and we add a term of at most zero to the sum, so the bound will not increase. Relaxing all integrality constraints and setting $|N(S_i)| = \frac{1}{4}|X|$ and $|Y_i(w)| = \frac{1}{d}(\frac{1}{4}|X| - d - 1)$, the bound only depends on $|X|$ and $d = d_i$ (note that $|X| = |Y| + 4(d + 1)$). We get

$$|E(X)| \geq 2d(d - 2) + \tfrac{2}{d}(\tfrac{|X|}{4} - d - 1)^2 - |X| + 4d + 4,$$

and thus

$$|E(X)| - 3|X| \geq 2d(d - 2) + \tfrac{2}{d}(\tfrac{|X|}{4} - d - 1)^2 - 4|X| + 4d + 4$$
$$= 8 + 2d + 2d^2 + \frac{(|X| - 4)^2}{8d} - 5|X|.$$

Given $d > 0$ and $|X| > 0$, the right side is minimized for $d = 12$ and $|X| = 244$, and thus

$$|E(X)| - 3(|X| - n) \geq 3n - 300 \geq 0,$$

a contradiction to Claim 3. This completes the proof of Lemma 3. □

Note that we are very generous with our bound on $|E(X)|$. We heavily undercount the edges between $Y_i(w) \cup w$ and $Y_i(w') \cup w'$, and we do not count the edges between Y_i and Y_j at all. Further note that for the case that $S \subseteq V_i$, the bound can easily be improved to

$$|E(X)| \geq \sum_{i=1}^{|Z|} \left(d_i(d_i - 1) + \sum_{w} \tfrac{1}{2}|Y_i(w)|(|Y_i(w)| - 1) \right).$$

For the case that S contains vertices in both V_i and V_j, it is not hard to see that $|Z| \leq 3$. This can be further lowered to $|Z| = 1$ if one treats a few exceptional cases. All these arguments can be used to lower the bound on n in the lemma, but the technicalities involved are too great to justify their exposition here, especially as one would still need to require $n \geq 20$ or so.

As there cannot be a vertex of degree less than 2 in a K_3-saturated subgraph of K_k^n, it only remains to consider the case where $\delta(G) = 2$ in order to complete the proof of Theorem 2.

Lemma 4. *If $n \geq 8$, $k \geq 3$, and G is a K_3-saturated subgraph of K_k^n of minimum size with minimum degree 2, then G is one of the graphs from Constructions 1 and 2, and in particular*

$$|E(G)| = \min\{2kn + n^2 - 4n - 1, 3kn - 3n - 6\}.$$

Proof. Without loss of generality, assume that $N(v_3^1) = \{v_1^1, v_2^1\}$. Partition the vertices as follows:

$$\begin{aligned}
A_i &= \{u \in V_i : v_1^1, v_2^1 \notin N(u)\},\\
B_i &= \{u \in V_i : v_1^1 \in N(u), v_2^1 \notin N(u)\},\\
C_i &= \{u \in V_i : v_1^1 \notin N(u), v_2^1 \in N(u)\}, \text{ and}\\
D_i &= \{u \in V_i : v_1^1, v_2^1 \in N(u)\}.
\end{aligned}$$

Note that $B_1 = D_1 = C_2 = D_2 = \emptyset$. Also, $A_1 = \{v_1^1\}$, $A_2 = \{v_2^1\}$ and $A_\ell = \emptyset$ for $\ell \geq 4$, as G is K_3-saturated and $N(v_i^j) \cap N(v_3^1) \neq \emptyset$ for $1 \leq i \leq 2$ and $2 \leq j \leq n$, and for $4 \leq i \leq k$ and $1 \leq j \leq n$.

Let $A = \bigcup A_i$, $B = \bigcup B_i$, $C = \bigcup C_i$, and $D = \bigcup D_i$. Note that $B \cup D$ and $C \cup D$ are independent sets, lest G contain a triangle. Thus, in particular, $N(B) \subseteq (A \setminus \{v_2^1\}) \cup C$ and $N(C) \subseteq (A \setminus \{v_1^1\}) \cup B$.

First, consider the case that $A_3 = \emptyset$. Then, for every $u \in D$, $N(u) = \{v_1^1, v_2^1\}$. Further, the sets C_i and B_j induce a complete bipartite graph for any $i \neq j$ as the intersection of their neighborhoods is empty. Thus, once given the sizes of the B_i and C_i, G is completely determined. Note that every vertex in $B \cup C$ has degree at least n, whereas vertices in D have degree 2. Thus, $|E(G)|$ is minimized if $|B_i| = |C_i| = 0$ for $3 \leq i \leq k$, which yields the graph in Construction 1.

Now suppose that $|A_3| = 1$, say $A_3 = \{v_3^2\}$. Further suppose that $u \in B \setminus B_2$. If $uv_3^2 \notin E(G)$, then $C_1 \subseteq N(u)$. If, on the other hand, $uv_3^2 \in E(G)$, then $N(u)$ contains a vertex in $C \setminus N(v_2^2)$, as otherwise there is no path of length at most 2 from u to v_2^1. Analogous statements hold for vertices $w \in C \setminus C_1$. Further, $D \setminus D_3 \subseteq N(v_3^2)$. This implies that

$$\begin{aligned}
|E(G)| \geq\ & 3|D \setminus D_3| + n|B \setminus (B_2 \cup N(v_3^2))| + n|C \setminus (C_1 \cup N(v_3^2))| \\
& + \tfrac{n}{2}|(C_1 \cup B_2) \setminus N(v_3^2)| + 3|B \cap N(v_3^2) \setminus B_2| \\
& + 3|C \cap N(v_3^2) \setminus C_1| + 2|((C_1 \cup B_2) \cap N(v_3^2)) \cup D_3| \\
\geq\ & 3|D \setminus D_3| + 3|B \setminus B_2| + 3|C \setminus C_1| + 2|C_1 \cup B_2 \cup D_3| \\
=\ & 3(|V(G)| - |A|) - |C_1 \cup B_2 \cup D_3| \\
=\ & 3(kn - 3) - (|C_1| + |B_2| + |D_3|) \\
\geq\ & 3kn - 3n - 6.
\end{aligned}$$

Note that equality holds only if $B \subset N(v_3^2)$ and $C \subset N(v_3^2)$, which then implies that in fact $B = B_2$ and $C = C_1$. It then follows that G is the graph from Construction 2.

Finally suppose that $|A_3| \geq 2$. To count the edges, we assign a charge of 1 to each edge uw and distribute the charge onto u and w as follows in this order, taking symmetry into account:

$$\begin{array}{rl}
u \in A & 0 \to u,\ 1 \to w \\
|N(u) \cap A| \geq 3 \text{ and } |N(w) \cap A| \leq 2 & 0 \to u,\ 1 \to w \\
\text{otherwise} & .5 \to u, .5 \to w
\end{array}$$

If every vertex in $B \cup C$ receives a total charge of at least 3, then

$$|E(G)| \geq 2|D| + |A_3||D \setminus D_3| + 3|B| + 3|C|$$

$$\begin{aligned}&\geq 3kn - 3|A| - |D_3|\\&= 3kn - 6 - 3|A_3| - |D_3|\\&\geq 3kn - 6 - 3n + 2,\end{aligned}$$

so we suppose that there exists $b \in B_i$ with total charge at most 2.5. If $N(b) \cap A_3 = \emptyset$, then $C_1 \subset N(b)$, and b has charge at least $(n+1)/2$, so this is not the case. Thus, $|N(b) \cap A_3| = 1$. Let $N(b) \cap A_3 = \{a\}$, and let $a' \in A \setminus a$. Let $c \in N(b) \cap C_j$ be b's only neighbor in C. Such a vertex must exist as there is a path of length 2 from b to a'. As b has charge only 2.5, $N(c) \cap A_3 = \{a'\}$. Note that this argument also implies that $A_3 = \{a, a'\}$.

We complete this case in a manner similar to when $|A_3| = 1$. Let $u \in B_i$ (the case for $w \in C_i$ is symmetric). If $N(u) \cap A_3 = \emptyset$, then $C_1 \subseteq N(u)$, so u has charge at least $(n+1)/2$. If $N(u) \cap A_3 = A_3$,then u has charge at least 3 (in fact, exactly 3). If $N(u) \cap A_3 = \{a\}$, then $\emptyset \subsetneq C \setminus (N(a) \cup C_i) \subset N(u)$, so u has charge at least $2 + |C \setminus (N(a) \cup C_i)|/2$. The only way that u has charge less than 3 in this case is if there exists $w \in C_j$ with $j \neq i$, such that $C \setminus (N(a) \cup C_i) = \{w\}$. Note that in this case $N(w) \cap A_3 = \{a'\}$. The charge of w is at least 2.5, so the combined charge of u and w is at least 5. Now let U be the set of vertices $u' \in B \setminus B_j$ with weight 2.5 satisfying $N(u') \cap A_3 = \{a\}$. Thus $u'w \in E(G)$, and the combined charge of U and w is at least $3|U| + 2$.

Now consider U', the set of vertices $u'' \in B_j$ with charge 2.5 satisfying $N(u'') \cap A_3 = \{a\}$. Thus $N(U') \cap C = \{w'\}$, and as $N(U) \cap C = \{w\}$, it follows that $w \in C_i$. Further, the total charge of U' and w' is at least $3|U'| + 2$. Very similar conclusions hold for the case of $N(u) \cap A_3 = \{a'\}$. In conclusion, the total charge of $B \cup C$ is at least $3|B \cup C| - 4$. Thus,

$$\begin{aligned}|E(G)| &\geq 2|D| + |A_3||D \setminus D_3| + 3|B| + 3|C| - 4\\&\geq 3kn - 3|A| - |D_3| - 4\\&= 3kn - |D_3| - 16\\&\geq 3kn - n - 14\\&> 3kn - 3n.\end{aligned}$$ □

Together, Lemmas 3 and 4 provide the proof of Theorem 2. When $k = 3$, it is relatively straightforward to determine $\mathrm{sat}(K_3, K_k^n)$ for all values of n.

Theorem 5. *For $n \geq 2$, $\mathrm{sat}(K_3, K_3^n) = 6n - 6$.*

Proof. Observe that $n - 1 + 5/n > 3$ for all $n \geq 2$. Thus $\mathrm{sat}(K_3, K_3^n) \leq 6n - 6$ by Lemma 1. Through the remainder of the proof we perform all arithmetic modulo 3.

Let G be a K_3-saturated subgraph of K_3^n. Let δ_i denote the minimum degree in G among the vertices in V_i. Assume that $\delta_1 \leq \delta_2 \leq \delta_3$. Each vertex in V_i either has a neighbor in both V_{i+1} and V_{i+2} or is completely joined to V_{i+1} or V_{i+2}; thus $\delta(G) \geq 2$.

Let v_i^1 be a vertex in V_i with degree δ_i. Every vertex in $V_{i+1} \cup V_{i+2}$ that is not adjacent to v_i^1 has at least one neighbor among the δ_i neighbors of v_i^1. Thus there are at least $2n - \delta_i$ edges joining V_{i+1} and V_{i+2}. Furthermore, there are at least $\delta_i n$ edges incident to the vertices in V_i. If $\delta_i \geq 4$, then $E(G) \geq 4n + 2n - 4 = 6n - 4$. Thus we may assume that $\delta_i \leq 3$ for all $i \in \{1, 2, 3\}$.

If $\delta_1 = \delta_2 = \delta_3 = 2$, then there are at least $2n - 2$ edges joining each pair of V_1, V_2, and V_3. Thus $|E(G)| \geq 6n - 6$.

Now suppose that $\delta_1 = 2$ and $\delta_3 = 3$. Every vertex of degree 2 in V_1 is adjacent to a vertex of degree at least n in V_3. Therefore, there are at least $2n-3$ edges joining V_1 and V_2, and V_3 has degree sum at least $3(n-1)+n$. Thus $|E(G)| \geq 2n - 3 + 3(n-1) + n = 6n - 6$.

Finally assume that $\delta_1 = \delta_2 = \delta_3 = 3$. A vertex of degree 3 in V_i has a neighbor that is incident to $n - 2$ edges joining V_{i+1} and V_{i+2}. Thus for each $j, l \in \{1, 2, 3\}$, $j < l$, there is a vertex $x_{j,l}$ that is incident to $n - 2$ edges joining V_j and V_l. If $x_{1,2}$, $x_{1,3}$ and $x_{2,3}$ are distinct, then G contains three vertices of degree at least $n - 1$. It follows that $|E(G)| \geq \frac{1}{2}(3(n-1)+3(3n-3)) = 6n-6$. If, without loss of generality, $x_{1,2} = x_{1,3}$, then $d(x_{1,2}) \geq 2n-4$ and $d(x_{2,3}) \geq n-1$. Thus $|E(G)| \geq \frac{1}{2}(3n-5+3(3n-2)) = \frac{1}{2}(12n - 11) > 6n - 6$. □

3. K_t-saturated subgraphs for $t \geq 4$

In this section we provide constructions of K_t-saturated subgraphs of K_k^n of small size for $t \geq 4$. We start with natural generalizations of Constructions 1 and 2.

Construction 3. Let $k \geq 2t - 4$, and let $S = \{v_1^1, \ldots, v_{2t-4}^1\}$. To construct $G_{k,n,t}$, place a complete graph on S and remove the $t-2$-edge matching $\{v_1^1 v_2^1, v_3^1 v_4^1, \ldots, v_{2t-5}^1 v_{2t-4}^1\}$. Now, for $r \in \{1, 3, \ldots, 2t-5\}$ completely join $V_r - v_r^1$ and $V_{r+1} - v_{r+1}^1$. Finally, add all edges from K_k^n joining S and $\overline{S}$. That is,

$$E(G_{k,n,t}) \\ = \left[\{v_r^1 v_s^1 : r \leq 2t-4, s \leq 2t-4, r \neq s\} \setminus \{v_1^1 v_2^1, v_3^1 v_4^1, \ldots, v_{2t-5}^1 v_{2t-4}^1\}\right]$$

$$\cup \{v_r^i v_{r+1}^j : i \geq 2, j \geq 2, r \in \{1, 3, \ldots, 2t-5\}\}$$
$$\cup \{v_r^i v_s^1 : i \geq 2, r \leq 2t-4, s \leq 2t-4, r \neq s\}$$
$$\cup \{v_r^i v_s^1 : i \leq n, r > 2t-4, s \leq 2t-4\}.$$

The number of edges in $G_{k,n,t}$ is

$$\begin{aligned}|E(G_{k,n,t})| &= \binom{2t-4}{2} - (t-2) + (t-2)(n-1)^2 \\ &\quad + (2t-4)(2t-5)(n-1) + (2t-4)(k-2t+4)n \\ &= (t-2)n^2 + (2t-4)kn - 2(2t-4)n - \binom{2t-4}{2}.\end{aligned}$$

Construction 4. Let $k \geq 2t-3$, and let $S = \{v_1^1, \ldots, v_1^{2t-3}\}$. To construct $H_{k,n,t}$, begin by placing a complete graph on S and removing the $2t-3$-cycle with edges $\{v_1^r v_1^s : |r-s| \in \{t-2, t-1\}\}$. Finally, add all edges from K_k^n joining S and $\overline{S}$. That is,

$$\begin{aligned}&E(H_{k,n,t}) \\ &= \left[\{v_1^r v_1^s : r \leq 2t-3, s \leq 2t-3\, r \neq s\} \setminus \{v_1^r v_1^s : |r-s| \in \{t-2, t-1\}\}\right] \\ &\quad \cup \{v_i^r v_1^s : i \geq 2, r \leq 2t-3, s \leq 2t-3, r \neq s\} \\ &\quad \cup \{v_i^r v_1^s : i \in [n], r > 2t-3, s \leq 2t-3\}.\end{aligned}$$

The number of edges in $H_{k,n,t}$ is

$$\begin{aligned}|E(H_{k,n,t})| &= (2t-3)(t-3) + (2t-3)(2t-4)(n-1) \\ &\quad + (k-2t+3)(2t-3)n \\ &= (2t-3)kn - (2t-3)n - (2t-3)(t-1).\end{aligned}$$

It is tedious but straightforward to verify that both $G_{k,n,t}$ and $H_{k,n,t}$ are K_t-saturated subgraphs of K_k^n for $k \geq 2t-4$ and $k \geq 2t-3$, respectively. Consequently, we have the following bound on $\mathrm{sat}(K_k^n, K_t)$ for $t \geq 3$ and $k \geq 2t-3$.

Theorem 6. *If $t \geq 3$ and $k \geq 2t-3$, then*

$$\mathrm{sat}(K_k^n, K_t) \leq \min \left\{ \begin{array}{l} (2t-4)kn + (t-2)n^2 - 2(2t-4)n - \binom{2t-4}{2}, \\ (2t-3)kn - (2t-3)n - (2t-3)(t-1) \end{array} \right\}.$$

As $G_{k,n,t}$ and $H_{k,n,t}$ are structurally similar to the unique minimal saturated graphs from Theorem 2, we conjecture that the bound in Theorem 6 is sharp when k is sufficiently large relative to t and $n \geq 2$.

The remaining constructions in this section follow the same general approach to building a K_t-saturated subgraph of K_k^n. First we select a small set of vertices S and construct on S a K_t-free graph that, for each choice of a two partite sets in K_k^n, contains a copy of K_{t-2} on $t-2$ vertices not lying in the two selected partite sets. We then add all edges in K_k^n joining S and $\overline{S}$. Finally, if necessary, iteratively add edges joining vertices in S provided that these edges do not complete any t-cliques. The resulting graph is a K_t-saturated subgraph of K_k^n and the number of edges is on the order of $|S|nk$.

We now turn our attention to K_t-saturated subgraphs of K_k^n for $k \in \{t, \ldots, 2t-5\}$. In this case. it seems that there may be a rich structure to the family of minimal K_t-saturated subgraphs of K_k^n. We present two additional constructions. The first applies to all values of t, k, and n, while the second applies only when t is even and $k \geq \frac{3}{2}(t-2)$.

Construction 5. Let $k \geq t$ and construct the graph $F_{k,n,t}$ as follows. First, list all $t-2$-element subsets of $[t]$ in lexicographic order. Thus for any $R \in \binom{[t]}{t-2} \setminus \{\{1, \ldots, t-2\}\}$, there is a $t-2$-set R' preceding R that contains the $t-3$ lowest elements of R. Begin by letting S contain one vertex from each of $V_1 \ldots, V_{t-2}$ and constructing a $t-2$-clique on those vertices. For each subsequent set R in the ordering of $\binom{[t]}{t-2}$, add a vertex from $V_{\max(R)}$ to S and join it to a $t-3$-clique in S whose vertices lie in the sets indexed by $R - \max(R)$. Thus for each set $R \in \binom{[t]}{t-2}$ there is a $t-2$-clique whose vertices lie in the partite sets indexed by R. Next, add all edges from K_k^n joining S and $\overline{S}$. Finally, iteratively add edges from K_k^n joining vertices in S provided that those edges do not complete any t-cliques.

Construction 6. For $t = 2m$ with $m \geq 3$, and $k \geq \frac{3}{2}(t-2)$ we construct the graph $I_{k,n,t}$ as follows. Let

$$S = \{v_1^1, v_1^2, v_2^1, v_2^2, \ldots, v_{3(t-2)/2}^1, v_{3(t-2)/2}^2\}$$

and start with the induced subgraph of K_k^n on S. If $t \equiv 2 \pmod 4$ (see Figure 2), then for $i \in \{0, \ldots, \frac{t-2}{4} - 1\}$, delete the edges of the following triangles:

$$\{v_{6i+1}^1, v_{6i+2}^1, v_{6i+3}^1\}, \{v_{6i+4}^1, v_{6i+5}^1, v_{6i+6}^1\},$$
$$\{v_{6i+1}^2, v_{6i+3}^2, v_{6i+5}^2\}, \{v_{6i+2}^2, v_{6i+4}^2, v_{6i+6}^2\}.$$

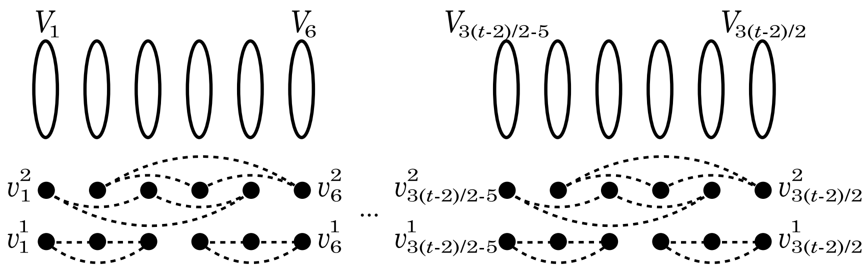

Figure 2: Constructing $I_{k,n,t}$: Nonedges in S when $k \equiv 2 \pmod 4$.

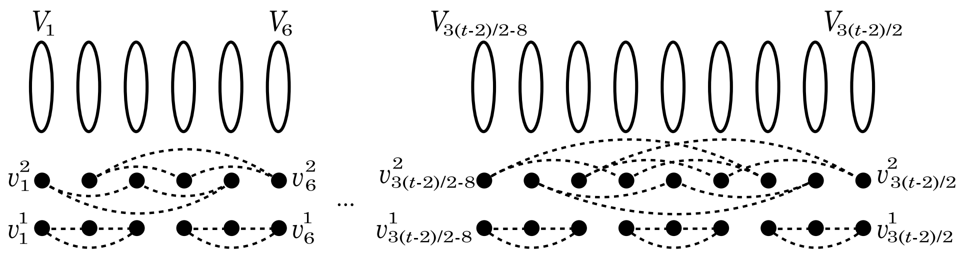

Figure 3: Constructing $I_{k,n,t}$: Nonedges in S when $k \equiv 0 \pmod 4$.

If $t \equiv 0 \pmod 4$ (see Figure 3), then for $i \in \{0, \ldots, \frac{t-4}{4} - 1\}$, delete the edges of the triangles

$$\{v_{6i+1}^1, v_{6i+2}^1, v_{6i+3}^1\}, \{v_{6i+4}^1, v_{6i+5}^1, v_{6i+6}^1\},$$
$$\{v_{6i+1}^2, v_{6i+3}^2, v_{6i+5}^2\}, \{v_{6i+2}^2, v_{6i+4}^2, v_{6i+6}^2\}$$

and also delete the edges of the triangles

$$\{v^1_{\frac{3}{2}(t-2)-8}, v^1_{\frac{3}{2}(t-2)-7}, v^1_{\frac{3}{2}(t-2)-6}\}, \{v^1_{\frac{3}{2}(t-2)-5}, v^1_{\frac{3}{2}(t-2)-4}, v^1_{\frac{3}{2}(t-2)-3}\},$$
$$\{v^1_{\frac{3}{2}(t-2)-2}, v^1_{\frac{3}{2}(t-2)-1}, v^1_{\frac{3}{2}(t-2)}\},$$
$$\{v^2_{\frac{3}{2}(t-2)-8}, v^2_{\frac{3}{2}(t-2)-5}, v^2_{\frac{3}{2}(t-2)-2}\}, \{v^2_{\frac{3}{2}(t-2)-7}, v^2_{\frac{3}{2}(t-2)-4}, v^2_{\frac{3}{2}(t-2)-1}\},$$
$$\{v^2_{\frac{3}{2}(t-2)-6}, v^2_{\frac{3}{2}(t-2)-3}, v^2_{\frac{3}{2}(t-2)}\}.$$

To complete the construction, add all edges in K_k^n joining vertices in S to vertices in $\overline{S}$.

Recall that both $F_{k,n,t}$ and $I_{k,n,t}$ have size on the order of $|S|nk$, for those sets S given in their respective constructions. This yields the following theorem, the details of which are again tedious but straightforward, and hence left to the reader.

Theorem 7. *1. For $k \geq t \geq 4$ and $n \geq 2$, $F_{k,n,t}$ is a K_t-saturated subgraph of K_k^n, so*

$$\mathrm{sat}(K_k^n, K_t) \leq 3(t-2)nk + o(nk).$$

2. For even $t \geq 6$ and $k \geq 3(t-2)$, $I_{k,n,t}$ is a K_t-saturated subgraph of K_k^n, so

$$\mathrm{sat}(K_k^n, K_t) \leq \frac{1}{2}(t^2 + t - 6)nk + o(nk).$$

We end with the following question, motivated by the differing number of edges in $F_{k,n,t}$ and $I_{k,n,t}$.

Question 1. Is there a linear function $f(t)$ such that for all $k \geq t \geq 3$ and n sufficiently large, $\mathrm{sat}(K_k^n, K_t) \leq f(t)kn$?

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