Journal of Combinatorics
Volume 6, Number 1–2, 235–248, 2015

Permutation statistics and multiple pattern avoidance

Wuttisak Trongsiriwat

For a set of patterns Π, let $F_n^{\mathrm{st}}(\Pi; q)$ be the st-polynomial of permutations avoiding all patterns in Π. Suppose $312 \in \Pi$. For some permutation statistic st, we give a formula that expresses $F_n^{\mathrm{st}}(\Pi; q)$ in terms of these st-polynomials where we take some subblocks of the patterns in Π. Using this formula, we construct examples of nontrivial st-Wilf equivalences. In particular, this disproves a conjecture by Dokos, Dwyer, Johnson, Sagan, and Selsor that all inv-Wilf equivalences are trivial.

1. Introduction

Let $\mathfrak{S}_n$ be the set of permutations of $[n] := \{1, 2, \ldots, n\}$ and let $\mathfrak{S} = \bigcup_{n\geq 0} \mathfrak{S}_n$, where $\mathfrak{S}_0$ contains only one element ϵ – the empty permutation. For permutations $\pi, \sigma \in \mathfrak{S}$ we say that the permutation σ *contains* π if there is a subsequence of σ having the same relative order as π. In particular, every permutation contains ϵ, and every permutation except ϵ contains $1 \in \mathfrak{S}_1$. For consistency, we will use the letter σ to represent a permutation and π to represent a pattern. We say that σ *avoids* π (or σ is π-*avoiding*) if σ does not contain π. For example, the permutation 46127538 contains 3142 while the permutation 46123578 avoids 3142. We denote by $\mathfrak{S}_n(\pi)$, where $\pi \in \mathfrak{S}$, the set of permutations $\sigma \in \mathfrak{S}_n$ avoiding π. More generally we denote by $\mathfrak{S}_n(\Pi)$, where $\Pi \subseteq \mathfrak{S}$, the set of permutations avoiding each pattern $\pi \in \Pi$ simultaneously, i.e. $\mathfrak{S}_n(\Pi) = \bigcap_{\pi\in\Pi} \mathfrak{S}_n(\pi)$. Two sets of patterns Π and Π' are called *Wilf equivalent*, written $\Pi \equiv \Pi'$, if $|\mathfrak{S}_n(\Pi)| = |\mathfrak{S}_n(\Pi')|$ for all integers $n \geq 0$.

Now we define q-analogues of pattern avoidance using permutation statistics. A *permutation statistic* (or sometimes just *statistic*) is a function st : $\mathfrak{S} \to \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers. Given a permutation statistic st, we define the *st-polynomial* of Π-avoiding permutations to be

$$F_n^{\mathrm{st}}(\Pi) = F_n^{\mathrm{st}}(\Pi; q) := \sum_{\sigma\in\mathfrak{S}_n(\Pi)} q^{\mathrm{st}(\sigma)}.$$

arXiv: 1309.3028

We may drop the q if it is clear from the context. The sets of patterns Π and Π' are said to be st-*Wilf equivalent*, written $\Pi \overset{\text{st}}{\equiv} \Pi'$, if $F_n^{\text{st}}(\Pi; q) = F_n^{\text{st}}(\Pi'; q)$ for all $n \geq 0$.

The study of q-analogues of pattern avoidance using permutation statistics and the st-Wilf equivalences began in 2002, as initiated by Robertson, Saracino, and Zeilberger [6], with the emphasis on the number of fixed points. Elizalde subsequently refined results of Robertson et al. by considering the excedance statistic [2] and later extended the study to cases of multiple patterns [3]. A bijective proof was later given by Elizalde and Pak [4]. Dokos et al. [1] studied pattern avoidance on the inversion and major statistics, as remarked by Savage and Sagan in their study of Mahonian pairs [7].

In this paper, we study multiple pattern avoidance on a class of permutation statistics which includes the inversion and descent statistics. The *inversion number* of $\sigma \in \mathfrak{S}_n$ is

$$\operatorname{inv}(\sigma) = \#\{(i,j) \in [n]^2 : i < j \text{ and } \sigma(i) > \sigma(j)\}.$$

The *descent number* of $\sigma \in \mathfrak{S}_n$ is

$$\operatorname{des}(\sigma) = \#\{i \in [n-1] : \sigma(i) > \sigma(i+1)\}.$$

For example $\operatorname{inv}(3142) = \#\{(1,2),(1,4),(3,4)\} = 3$ and $\operatorname{des}(3142) = \#\{1,3\} = 2$.

In [1], Dokos et al. conjectured that there are only essentially trivial inv-Wilf equivalences, obtained by rotations and reflections of permutation matrices. Let us describe these more precisely. The notations used below are mostly taken from [1].

Given a permutation $\sigma \in \mathfrak{S}_n$, we represent it geometrically using the squares $(1, \sigma(1)), (2,\ \sigma(2)), \ldots, (n, \sigma(n))$ of the n-by-n grid, which is coordinated according to the xy-plane. This will be referred as the *permutation matrix* of σ. The diagram to the left in Figure 1 is the permutation matrix of 46127538. In the diagram to the right, the red squares correspond to the subsequence 4173, which is an occurrence of the pattern 3142.

By representing each $\sigma \in \mathfrak{S}$ as a permutation matrix, we have an action of the dihedral group of square D_4 on $\mathfrak{S}$ by the corresponding action on the permutation matrices. We denote the elements of D_4 by

$$D_4 = \{R_0, R_{90}, R_{180}, R_{270}, r_{-1}, r_0, r_1, r_\infty\},$$

where R_θ is the counterclockwise rotation by θ degrees and r_m is the reflection in a line of slope m. We will sometimes write Π^t for $r_{-1}(\Pi)$. Note that

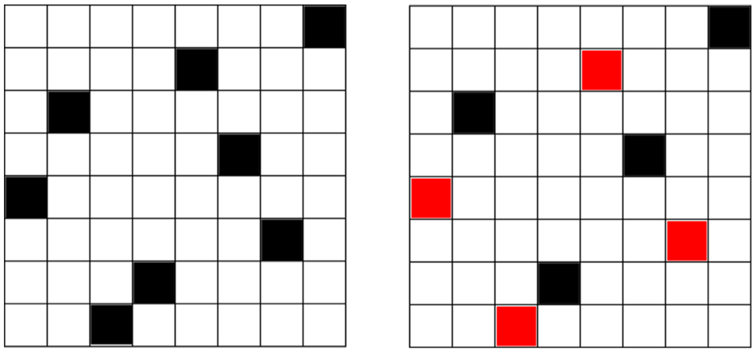

Figure 1: The permutation matrix of 46127538 (left) with an occurrence of 3142 colored (right).

R_0, R_{180}, r_{-1}, and r_1 preserve the inversion statistic while the others reverse it, i.e.

$$\operatorname{inv}(f(\sigma)) = \begin{cases} \operatorname{inv}(\sigma) & \text{if } f \in \{R_0, R_{180}, r_{-1}, r_1\}, \\ \binom{n}{2} - \operatorname{inv}(\sigma) & \text{if } f \in \{R_{90}, R_{270}, r_0, r_\infty\}. \end{cases}$$

It follows that Π and $f(\Pi)$ are inv-Wilf equivalent for all $\Pi \subseteq \mathfrak{S}$ and $f \in \{R_0, R_{180}, r_{-1}, r_1\}$. We call these equivalences trivial. With these notations, the conjecture by Dokos et al. can be stated as the following.

Conjecture 1.1 ([1], conj. 2.4)**.** Π *and* Π' *are inv-Wilf equivalent iff* $\Pi = f(\Pi')$ *for some* $f \in \{R_0, R_{180}, r_{-1}, r_1\}$.

Given permutations $\pi = a_1a_2\dots a_k \in \mathfrak{S}_k$ and $\sigma_1, \dots, \sigma_k \in \mathfrak{S}$, the *inflation* $\pi[\sigma_1, \dots, \sigma_k]$ of π by the σ_i is the permutation whose permutation matrix is obtained by putting the permutation matrices of σ_i in the relative order of π; for instance, 213[123,1,21] = 234165 as illustrated in Figure 2.

For convenience, we write

$$\pi_* := 21[\pi, 1].$$

In other words, π_* is the permutation whose permutation matrix is obtained by adding a box to the lower right corner of the permutation matrix of π.

The next proposition is one of the main results of this paper, which disproves the conjecture above. This is a special case of the corollary of Theorem 2.4 in the next section.

Proposition 1.2. *Let* ι_r *be the permutation* $12\dots r \in \mathfrak{S}_r$. *Let* $\pi_1, \dots, \pi_r, \pi'_1, \dots, \pi'_r$ *be permutations such that* $\{312, \pi_i\} \overset{inv}{\equiv} \{312, \pi'_i\}$ *for all* i. *Set* $\pi =$

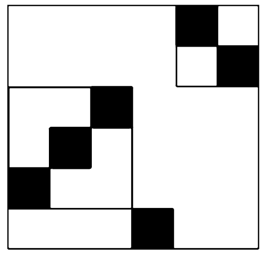

Figure 2: The permutation 213[123,1,21].

$\iota_r[\pi_{1*}, \ldots, \pi_{r*}]$ and $\pi' = \iota_r[\pi'_{1*}, \ldots, \pi'_{r*}]$. Then $\{312, \pi\}$ and $\{312, \pi'\}$ are inv-Wilf equivalent, i.e. $F_n^{inv}(312, \pi) = F_n^{inv}(312, \pi')$ for all n.

In particular, if we set each π'_i to be either π_i or π_i^t, then the conditions $\{312, \pi_i\} \overset{\text{inv}}{\equiv} \{312, \pi'_i\}$ are satisfied. By this construction Π' is generally not of the form $f(\Pi)$ for any $f \in \{R_0, R_{180}, r_{-1}, r_1\}$. For example, the pair $\Pi = \{312, 32415\}$ and $\Pi' = \{312, 24315\}$ is an example of smallest size of nontrivial inv-Wilf equivalences constructed this way.

2. Avoiding two patterns

In this section, we study the st-polynomials in the case when Π consists of 312 and another permutation π. For this set of patterns Π, Mansour and Vainshtein [5] gave a recursive formula for $|\mathfrak{S}_n(\Pi)|$. Here, we give a recursive formula for the st-polynomials $F_n^{\text{st}}(\Pi)$, which generalizes the result of Mansour and Vainshtein. Then we present its corollary, which gives a construction of nontrivial st-Wilf equivalences. We note that Proposition 2.1 and Lemma 2.2 appear in [5] as small observations.

Suppose $\sigma \in \mathfrak{S}_{n+1}(312)$ with $\sigma(k+1) = 1$. Then, for every pair of indices (i, j) with $i < k+1 < j$, we must have $\sigma(i) < \sigma(j)$; otherwise $\sigma(i)\sigma(k+1)\sigma(j)$ is an occurrence of the pattern 312 in σ. So σ can be written as $\sigma = 213[\sigma_1, 1, \sigma_2]$ with $\sigma_1 \in \mathfrak{S}_k$ and $\sigma_2 \in \mathfrak{S}_{n-k}$. For the rest of the paper, we will always consider σ in this inflation form.

We also assume that the permutation statistic $\text{st} : \mathfrak{S}_n \to \mathbb{N}$ satisfies

$$(\dagger) \qquad \text{st}(\sigma) = f(k, n-k) + \text{st}(\sigma_1) + \text{st}(\sigma_2)$$

for some function $f : \mathbb{N}^2 \to \mathbb{N}$ which does not depend on σ. Some examples of such statistics are the inversion number, the descent number, and the number of occurrences of the consecutive pattern 213:

$$\underline{213}(\sigma) = \#\{i \in [n-2] : \sigma(i+1) < \sigma(i) < \sigma(i+2)\}.$$

For these statistics, we have

$$\begin{aligned}
\operatorname{inv}(\sigma) &= k + \operatorname{inv}(\sigma_1) + \operatorname{inv}(\sigma_2), \\
\operatorname{des}(\sigma) &= 1 - \delta_{0,k} + \operatorname{des}(\sigma_1) + \operatorname{des}(\sigma_2), \\
\underline{213}(\sigma) &= (1 - \delta_{0,k})(1 - \delta_{k,n}) + \underline{213}(\sigma_1) + \underline{213}(\sigma_2),
\end{aligned}$$

where δ is the Kronecker delta function.

It will be more beneficial to consider the permutation patterns in their *block decomposition* form as in the following proposition.

Proposition 2.1. *Every 312-avoiding permutation $\pi \in \mathfrak{S}_n(312)$ can be written uniquely as*

$$\pi = \iota_r[\pi_{1*}, \ldots, \pi_{r*}]$$

where $r \geq 0$ and $\pi_i \in \mathfrak{S}(312)$.

Proof. The uniqueness part is trivial. The proof of existence of $\pi_1, \ldots, \pi_r$ is by induction on n. If $n = 0$, there is nothing to prove. Suppose the result holds for n. Suppose that $\pi(k+1) = 1$. Then $\pi = 213[\pi_1, 1, \pi'] = 12[\pi_{1*}, \pi']$ where $\pi_1 \in \mathfrak{S}_k(312)$ and $\pi' \in \mathfrak{S}_{n-k}(312)$. Applying the inductive hypothesis on π', we are done. ☐

Suppose that $\pi \in \mathfrak{S}_n(312)$ has the block decomposition $\pi = \iota_r[\pi_{1*}, \ldots, \pi_{r*}]$. For $1 \leq i \leq r$, we define $\underline{\pi}(i)$ and $\overline{\pi}(i)$ to be

$$\underline{\pi}(i) = \begin{cases} \pi_1 & \text{if } i = 1, \\ \iota_i[\pi_{1*}, \ldots, \pi_{i*}] & \text{otherwise,} \end{cases}$$

and

$$\overline{\pi}(i) = \iota_{r-i+1}[\pi_{i_*}, \ldots, \pi_{r_*}].$$

Let Π be a set of patterns containing 312. If $\pi \in \Pi \setminus \{312\}$ contains the pattern 312, then every permutation avoiding 312 will automatically avoid π, which means $F_n^{\mathrm{st}}(\Pi) = F_n^{\mathrm{st}}(\Pi \setminus \{\pi\})$. So we may assume that each pattern besides 312 in Π avoids 312. The following lemma gives a recursive condition for a permutation $\sigma = 213[\sigma_1, 1, \sigma_2] \in \mathfrak{S}(312)$ to avoid π, in terms of σ_1, σ_2, and the blocks π_{i*} of π.

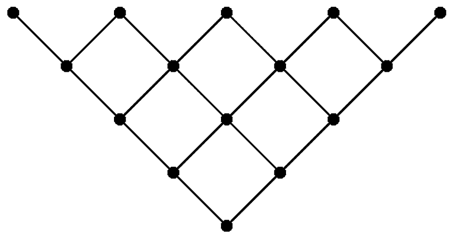

Figure 3: The poset L_5.

Lemma 2.2. *Let* $\sigma = 213[\sigma_1, 1, \sigma_2], \pi = \iota_r[\pi_{1*}, \ldots, \pi_{r*}] \in \mathfrak{S}(312)$*. Then* σ *avoids* π *if and only if the condition*

(C_i) : σ_1 *avoids* $\underline{\pi}(i)$ *and* σ_2 *avoids* $\overline{\pi}(i)$

holds for some $i \in [r]$*.*

Proof. First, suppose that σ contains π. Let j be the largest number for which σ_1 contains $\underline{\pi}(j)$. Then σ_2 must contain $\overline{\pi}(j+1)$. So σ_1 contains $\underline{\pi}(i)$ for all $i \leq j$, and σ_2 contains $\overline{\pi}(i)$ for all $i > j$. Thus none of the C_i holds.

On the other hand, suppose that there is a permutation $\sigma \in \mathfrak{S}(312)$ that avoids π but does not satisfy any C_i. This means, for every i, either σ_1 contains $\underline{\pi}(i)$ or σ_2 contains $\overline{\pi}(i)$. Let j be the smallest number such that σ_1 does not contain $\underline{\pi}(j)$. Note that j exists and $j > 1$ since $j = 1$ implies σ_2 contains $\overline{\pi}(1) = \pi$, a contradiction. Since σ_1 does not contain $\underline{\pi}(j)$, σ_2 must contain $\overline{\pi}(j)$ (by C_j). But since σ_1 contains $\underline{\pi}(j-1)$ by minimality of j, we have found a copy of π in σ with $\underline{\pi}(j-1)$ from σ_1 and $\overline{\pi}(j)$ from σ_2, a contradiction. (For $j = 2$, the number 1 in σ together with π_1 in σ_1 give π_{1*}.) □

Before presenting the main result, we state a technical lemma regarding the Möbius function of certain posets. See, for example, Chapter 3 of [8] for definitions and terminologies about posets and the general treatment of the subject.

Let $\mathbf{r}$ be the chain of r elements $0 < 1 < \cdots < r-1$. Let L_r be the poset obtained by taking the elements of $\mathbf{r} \times \mathbf{r}$ of rank 0 to $r-1$, i.e. the elements of L_r are the lattice points (a, b) where $a, b \geq 0$ and $a + b < r$. For instance, L_5 is the poset shown in Figure 3. We denote its unique minimal element $(0, 0)$ by $\hat{0}$. Let $\hat{L}_r$ be the poset L_r with the unique maximum element $\hat{1}$ adjoined.

For a poset P, we denote the Möbius function of P by μ_P. Note that for every element $a \in \hat{L}_r$ the up-set $U(a) := \{x \in \hat{L}_r : x \geq a\}$ of a is isomorphic to $\hat{L}_{r-l(a)}$ where $l(a)$ is the rank of a in L_r. Therefore, the problem of computing $\mu_{\hat{L}_r}(x,\hat{1})$ for every r is equivalent to computing $\mu_{\hat{L}_r}(\hat{0},\hat{1})$ for every r, which is given by the following lemma. The proof is omitted since it is by a straightforward calculation.

Lemma 2.3. *We have*

$$\mu_{\hat{L}_r}(\hat{0},\hat{1}) = \begin{cases} (-1)^r, & \text{if } r = 1,2,\\ 0, & \text{otherwise.}\end{cases}$$

We now present the main theorem of this section.

Theorem 2.4. *Let $\Pi = \{312, \pi\}$. Suppose that the statistic $st : \mathfrak{S} \to \mathbb{N}$ satisfies the condition (†). Then $F_n^{st}(\Pi; q)$ satisfies*

$$(*)\quad F_{n+1}^{st}(\Pi;q) = \sum_{k=0}^{n} q^{f(k,n-k)}\left[\sum_{i=1}^{r} F_k^{st}(312,\underline{\pi}(i))\cdot F_{n-k}^{st}(312,\overline{\pi}(i)) - \sum_{i=1}^{r-1} F_k^{st}(312,\underline{\pi}(i))\cdot F_{n-k}^{st}(312,\overline{\pi}(i+1))\right],$$

for all $n \ge 0$, where $F_0^{st}(\Pi;q) = 0$ if $\pi = \epsilon$, and 1 otherwise.

Proof. For $k \in \{0,1,\dots,n\}$ and $\Sigma \subset \mathfrak{S}$, we write $\mathfrak{S}_{n+1}^k(\Sigma)$ to denote the set of permutations $\sigma \in \mathfrak{S}_{n+1}(\Sigma)$ such that $\sigma(k+1) = 1$. In particular,

$$\mathfrak{S}_{n+1}^k(312) = \{\sigma = 213[\sigma_1, 1, \sigma_2] : \sigma_1 \in \mathfrak{S}_k(312) \text{ and } \sigma_2 \in \mathfrak{S}_{n-k}(312)\}.$$

Fix k, and let $A_i (i \in [r])$ be the set of permutations in $\mathfrak{S}_{n+1}^k(312)$ satisfying the condition C_i. So $\mathfrak{S}_{n+1}^k(\Pi) = A_1 \cup A_2 \cup \cdots \cup A_r =: A$ by Lemma 2.2. Observe that if $i_1 < \cdots < i_k$ then

$$A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k} = A_{i_1} \cap A_{i_k} =: A_{i_1,i_k}$$

since satisfying the conditions $C_{i_1}, \dots, C_{i_k}$ is equivalent to satisfying the conditions C_{i_1} and C_{i_k}.

Let P be the intersection poset of $A_1, \dots, A_r$, where the order is given by $A \le B$ if $A \subseteq B$. The elements of P are A, A_i $(1 \le i \le r)$, and $A_{i,j}$ $(1 \le i < j \le r)$. We see that P is isomorphic to the set $\hat{L}_r$, so the Möbius function $\mu_P(T,A)$ for $T \in P$ is given by

$$\mu_P(T,A) = \begin{cases} 1 & \text{if } T = A \text{ or } A_{i,i+1} \text{ for some } i, \\ -1 & \text{if } T = A_i \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

For $T \in P$, we define $g : P \to \mathbb{C}(x : x \in A)$ by

$$g(T) = \sum_{x \in T} x.$$

The Möbius inversion formula ([8], Section 3.7) implies that

$$\begin{aligned} g(A) &= -\sum_{T<A} \mu_P(T,A) g(T) \\ &= \sum_{i=1}^{r} g(A_i) - \sum_{i=1}^{r-1} g(A_i \cap A_{i+1}). \end{aligned}$$

By mapping $\sigma \mapsto q^{\mathrm{st}(\sigma)}$ for all $\sigma \in A$, $g(A)$ is sent to $F^{\mathrm{st}}_{n+1,k}(\Pi;q) := \sum_{\sigma \in \mathfrak{S}^k_{n+1}(\Pi)} q^{\mathrm{st}(\sigma)}$. Hence,

$$\begin{aligned} F^{\mathrm{st}}_{n+1,k}(\Pi;q) &= \sum_{i=1}^{r} \sum_{\sigma \in A_i} q^{\mathrm{st}(\sigma)} - \sum_{i=1}^{r-1} \sum_{\sigma \in A_i \cap A_{i+1}} q^{\mathrm{st}(\sigma)} \\ &= q^{f(k,n-k)} \left[\sum_{i=1}^{r} \sum_{\sigma \in A_i} q^{\mathrm{st}(\sigma_1)+\mathrm{st}(\sigma_2)} - \sum_{i=1}^{r-1} \sum_{\sigma \in A_i \cap A_{i+1}} q^{\mathrm{st}(\sigma_1)+\mathrm{st}(\sigma_2)} \right], \end{aligned}$$

where the second equality is obtained from the condition (†).

Note that $\sigma \in A_i$ iff σ_1 avoids $\underline{\pi}(i)$ and σ_2 avoids $\overline{\pi}(i)$, and $\sigma \in A_i \cap A_{i+1}$ iff σ_1 avoids $\underline{\pi}(i)$ and σ_2 avoids $\overline{\pi}(i+1)$. Thus

$$\sum_{\sigma \in A_i} q^{\mathrm{st}(\sigma_1)+\mathrm{st}(\sigma_2)} = F^{\mathrm{st}}_k(312, \underline{\pi}(i)) \cdot F^{\mathrm{st}}_{n-k}(312, \overline{\pi}(i))$$

and

$$\sum_{\sigma \in A_i \cap A_{i+1}} q^{\mathrm{st}(\sigma_1)+\mathrm{st}(\sigma_2)} = F^{\mathrm{st}}_k(312, \underline{\pi}(i)) \cdot F^{\mathrm{st}}_{n-k}(312, \overline{\pi}(i+1)).$$

Therefore

$$F^{\text{st}}_{n+1,k}(\Pi;q) = q^{f(k,n-k)}\left[\sum_{i=1}^{r} F^{\text{st}}_{k}(312,\underline{\pi}(i))\cdot F^{\text{st}}_{n-k}(312,\overline{\pi}(i)) - \sum_{i=1}^{r-1} F^{\text{st}}_{k}(312,\underline{\pi}(i))\cdot F^{\text{st}}_{n-k}(312,\overline{\pi}(i+1))\right].$$

We get the stated result by summing the preceding equation from $k = 0$ to n. □

Example 2.5 (q-analogues of odd Fibonacci numbers)**.** It is well known that the permutations avoiding 312 and 1432 are counted by the Fibonacci numbers F_{2n+1}, assuming $F_1 = F_2 = 1$ (see [9] for example). Let $A_n = F_{2n+1}$. It can be shown that the A_n satisfy

$$A_{n+1} = A_n + \sum_{k=0}^{n-1} 2^{n-k-1} A_k.$$

Theorem 2.4 gives q-analogues of this relation. Here, we consider the inversion statistic.

Let $\pi = 1432 = 12[\epsilon_*, 21_*]$ and $\Pi = \{312, \pi\}$. Since $\underline{\pi}(1) = \epsilon$ and $F^{\text{inv}}_n(312,\epsilon) = 0$ for all n, Theorem 2.4 implies

$$\begin{aligned} F^{\text{inv}}_{n+1}(\Pi) &= \sum_{k=0}^{n} F^{\text{inv}}_{k}(\Pi) F^{\text{inv}}_{n-k}(312, 321) \\ &= q^n F^{\text{inv}}_n(\Pi) + \sum_{k=0}^{n-1} q^k (1+q)^{n-k-1} F^{\text{inv}}_k(\Pi), \end{aligned}$$

where the last equality is by [1], Proposition 4.2.

Corollary 2.6. *Let st be a statistic satisfying* (†)*. Let* $\pi_1, \ldots, \pi_r, \pi'_1, \ldots, \pi'_r$ *be permutations such that* $\{312, \pi_i\} \overset{st}{\equiv} \{312, \pi'_i\}$ *for all* i*. Set* $\pi = \iota_r[\pi_{1*}, \ldots, \pi_{r*}]$ *and* $\pi' = \iota_r[\pi'_{1*}, \ldots, \pi'_{r*}]$*. Then* $\{312, \pi\}$ *and* $\{312, \pi'\}$ *are also st-Wilf equivalent, i.e.* $F^{st}_n(312, \pi) = F^{st}_n(312, \pi')$ *for all* n*.*

Proof. The proof is by induction on n. If $n = 0$, then the statement trivially holds. Now suppose the statement holds up to n. Then for $0 \le k \le n$ and $1 \le i \le r$, we have $F^{\text{st}}_k(312, \underline{\pi}(i)) = F^{\text{st}}_k(312, \underline{\pi'}(i))$ and $F^{\text{st}}_{n-k}(312, \overline{\pi}(i)) = F^{\text{st}}_{n-k}(312, \overline{\pi'}(i))$. Hence $F^{\text{st}}_{n+1}(312, \pi) = F^{\text{st}}_{n+1}(312, \pi')$ by comparing the terms on the right-hand side of (*). □

As mentioned at the end of Section 1, for the inversion statistic we can choose each π_i' to be either π_i or π_i^t. Of course, this construction works for every statistic st satisfying (†) and that $\mathrm{st}(\sigma) = \mathrm{st}(\sigma^t)$ for all $\sigma \in \mathfrak{S}(312)$. Besides the inversion statistic, the descent statistic for example also possesses this property. To justify this fact, we write $\sigma = 213[\sigma_1, 1, \sigma_2] \in \mathfrak{S}(312)$ where $\sigma_1, \sigma_2 \in \mathfrak{S}$. Observe that $\sigma^t = 132[\sigma_2^t, \sigma_1^t, 1]$ and

$$\mathrm{des}(\sigma^t) = \mathrm{des}(\sigma_2^t) + \mathrm{des}(\sigma_1^t) + (1 - \delta_{0,k})$$

where $k = |\sigma_1^t| = |\sigma_1|$. The proof then proceeds by induction on $n = |\sigma|$. It is, however, not true in general that the matrix transposition preserves the descent number. For instance, if $\sigma = 2413$, then $\mathrm{des}(\sigma) = 1$ while $\mathrm{des}(\sigma^t) = 2$.

3. Generalization

In this section, we generalize the results from Section 2 to the case when Π consists of 312 and other patterns. We again begin with a lemma regarding the Möbius function.

Lemma 3.1. *Let L be the poset $L_{r_1} \times \cdots \times L_{r_m}$ and $\hat{L}$ the poset $L \cup \{\hat{1}\}$. Let $\mu = \mu_{\hat{L}}$ be the Möbius function on $\hat{L}$. Then $\mu(\hat{0}, \hat{1}) = 0$ unless each $r_i \in \{1, 2\}$, in which case $\mu(\hat{0}, \hat{1}) = (-1)^{|S|+1}$, where $S = \{i : r_i = 2\}$.*

Proof. Let $a = (a_1, \ldots, a_m) \in L$. Then $\mu(\hat{0}, a) = \prod_{i=1}^m \mu_i(\hat{0}, a_i)$, where μ_i is the Möbius function of L_{r_i}. So

$$\mu(\hat{0}, \hat{1}) = -\sum_{a \in L} \mu(\hat{0}, a) = -\prod_{i=1}^{m} \left(\sum_{a_i \in L_{r_i}} \mu_i(\hat{0}, a_i) \right).$$

Note that if $r \geq 3$, the Möbius function $\mu_{L_r}(\hat{0}, a)$ vanishes unless $a \in \{(0,0), (1,0), (0,1), (1,1)\}$, in which cases the value of $\mu_{L_r}(\hat{0}, a)$ is 1, -1, -1, 1, respectively. So $\sum_{a \in L_r} \mu_{L_r}(\hat{0}, a) = 0$ unless $r = 1, 2$. For $r = 1, 2$, it can easily be checked that $\sum_{a \in L_r} \mu_{L_r}(\hat{0}, a) = 1$ if $r = 1$ and -1 if $r = 2$. So if $r_i \geq 3$ for some i, then $\mu(\hat{0}, \hat{1}) = 0$. If each $r_i \in \{1, 2\}$, then each index i for which $r_i = 2$ contributes a -1 to the product on the right-hand side of the previous equation. Thus $\mu(\hat{0}, \hat{1}) = (-1)^{|S|+1}$. □

For convenience, we introduce the following notations. Let $\Pi = \{312, \pi^{(1)}, \ldots, \pi^{(m)}\}$ where $\pi^{(j)} = \iota_{r_j}[(\pi_1^{(j)})_*, \ldots, (\pi_{r_j}^{(j)})_*]$. For $I = (i_1, \ldots, i_m)$, we define

$$\underline{\Pi}_I = \{312, \underline{\pi^{(1)}}(i_1), \ldots, \underline{\pi^{(m)}}(i_m)\}$$

and

$$\overline{\Pi}_I = \{312, \overline{\pi^{(1)}}(i_1), \ldots, \overline{\pi^{(m)}}(i_m)\}.$$

A generalization of Theorem 2.4 can be stated as the following.

Theorem 3.2. *Suppose that the statistic* $st : \mathfrak{S} \to \mathbb{N}$ *satisfies the condition* (†). *Let* $\Pi = \{312, \pi^{(1)}, \ldots, \pi^{(m)}\}$ *where* $\pi^{(i)} = \iota_{r_i}[(\pi_1^{(i)})_*, \ldots, (\pi_{r_i}^{(i)})_*]$. *Then* $F_0^{st}(\Pi) = 0$ *if* $\pi_i = \epsilon$ *for some* i *and* 1 *otherwise, and for* $n \geq 1$ *the st-polynomial* $F_n^{st}(\Pi; q)$ *satisfies*

$$F_{n+1}^{st}(\Pi; q) = \sum_{k=0}^{n} q^{f(k,n-k)} \left[\sum_{S \subseteq [m]} (-1)^{|S|} \sum_{\substack{I=(i_1,\ldots,i_m): \\ 1 \leq i_j \leq r_j - \delta_j}} F_k^{st}(\underline{\Pi}_I) \cdot F_{n-k}^{st}(\overline{\Pi}_{I+\delta}) \right],$$

where $\delta = (\delta_1, \ldots, \delta_m)$ *with* $\delta_j = 1$ *if* $j \in S$ *and* 0 *if* $j \notin S$.

Proof. Recall that by Lemma 2.2 a permutation $\sigma = 213[\sigma_1, 1, \sigma_2] \in \mathfrak{S}(312)$ avoids $\pi^{(j)}$ iff σ satisfies the condition

(C_i^j) : σ_1 avoids $\underline{\pi^{(j)}}(i)$ and σ_2 avoids $\overline{\pi^{(j)}}(i)$

for some $i \in [r_j]$. So $\sigma \in \mathfrak{S}(312)$ belongs to $\mathfrak{S}(\Pi)$ iff, for every j, there is an $i \in [r_j]$ for which σ satisfies (C_i^j). Fix k and let $\mathfrak{S}_{n+1}^k(312)$ be as in the proof of Theorem 2.4. Let A_i^j be the set of $\pi^{(j)}$-avoiding permutations in $\mathfrak{S}_{n+1}^k(312)$ satisfying the condition (C_i^j). For $I = (i_1, \ldots, i_m) \in [r_1] \times [r_2] \times \cdots \times [r_m]$, we define the set A_I to be

$$A_I = A_{i_1,i_2,\ldots,i_m} := A_{i_1}^1 \cap A_{i_2}^2 \cap A_{i_m}^m.$$

So $\mathfrak{S}_{n+1}^k(\Pi)$ is the union

$$\mathfrak{S}_{n+1}^k(\Pi) = \bigcup_{i_1,\ldots,i_m} A_{i_1,i_2,\ldots,i_m},$$

where the union is taken over all m-tuples $I = (i_1, \ldots, i_m)$ in $[r_1] \times [r_2] \times \cdots \times [r_m]$. Let $\hat{P}_j$ be the intersection poset of $A_1^j, \ldots, A_{r_j}^j$, and let P_j be the poset $\hat{P}_j \setminus \{A^j\}$, where $A^j = A_1^j \cup \cdots \cup A_{r_j}^j$ is the unique maximum element of $\hat{P}_j$. Recall that P_j is isomorphic to L_{r_j}. Let P be the intersection poset of the A_I. The elements of P are the unique maximal element $A = \mathfrak{S}_{n+1}^k(\Pi)$ and

$$T = T^1 \cap T^2 \cap \cdots \cap T^m,$$

where each T^j is an element of P_j. Thus P is isomorphic to $L_{r_1} \times \cdots \times L_{r_m} \cup \{\hat{1}\}$. For $S \subseteq [n]$, we say that an element $T \in P$ has type S if

$T^j = A^j_i$ for some i when $j \notin S$ and $T^j = A^j_i \cap A^j_{i+1}$ for some i when $j \in S$. Using Lemma 3.1, we know that the value of $\mu_P(T, A)$ where $T = T^1 \cap T^2 \cap \cdots \cap T^m \neq A$ is

$$\mu_P(T,A) = \begin{cases} (-1)^{|S|+1}, & \text{if } T \text{ has type } S, \\ 0, & \text{otherwise.} \end{cases}$$

For $T \in P$, we define $g : P \to \mathbb{C}(x : x \in A)$ by $g(T) = \sum_{x \in T} x$, so that

$$g(A) = \sum_{S \subseteq [n]} (-1)^{|S|} \sum_{T \text{ has type } S} g(T)$$

by the Möbius inversion formula.

Now, by the definition of type S, we have

$$\sum_{T \text{ has type } S} g(T) = \sum_{\substack{i_1,\dots,i_m: \\ 1 \le i_j \le r_j - \delta_j}} g\left(\bigcap_{j \notin S} A^j_{i_j} \cap \bigcap_{j \in S} (A^j_{i_j} \cap A^j_{i_j+1}) \right),$$

where $\delta_j = 1$ if $j \in S$ and 0 if $j \notin S$. Recall that $\sigma \in A^j_{i_j}$ iff σ_1 avoids $\underline{\pi^{(j)}}(i_j)$ and σ_2 avoids $\overline{\pi^{(j)}}(i_j)$, and $\sigma \in A^j_{i_j} \cap A^j_{i_j+1}$ iff σ_1 avoids $\underline{\pi^{(j)}}(i_j)$ and σ_2 avoids $\overline{\pi^{(j)}}(i_j + 1)$. Therefore, by mapping $\sigma \mapsto q^{\mathrm{st}(\sigma)}$, we have

$$g\left(\bigcap_{j \notin S} A^j_{i_j} \cap \bigcap_{j \in S} (A^j_{i_j} \cap A^j_{i_j+1}) \right) \mapsto q^{f(k,n-k)} F^{\mathrm{st}}_k(312, \underline{\pi^{(1)}}(i_1), \dots, \underline{\pi^{(m)}}(i_m)) \cdot F^{\mathrm{st}}_{n-k}(312, \overline{\pi^{(1)}}(i_1 + \delta_1), \dots, \overline{\pi^{(m)}}(i_m + \delta_m)).$$

Therefore,

$$F^{\mathrm{st}}_{n+1,k}(\Pi; q) = q^{f(k,n-k)} \left[\sum_{S \subseteq [m]} (-1)^{|S|} \sum_{\substack{i_1,\dots,i_m: \\ 1 \le i_j \le r_j - \delta_j}} F^{\mathrm{st}}_k(\underline{\Pi}_I) \cdot F^{\mathrm{st}}_{n-k}(\overline{\Pi}_{I+\delta}) \right],$$

and we are done. □

Example 3.3. Let $\Pi = \{312, \pi^{(1)}, \pi^{(2)}\}$ where $\pi^{(1)} = 2314 = 12[12_*, \epsilon_*]$ and $\pi^{(2)} = 2143 = 12[1_*, 1_*]$. We want to compute $a_n = F^{\mathrm{inv}}_n(\Pi)$ by using Theorem 3.2. There are four possibilities of $S \subseteq \{1, 2\}$, and for each possibility the following table shows the appearing terms, where δ is again the Kronecker delta function.

$S = \emptyset$:	$F_k^{\text{inv}}(312, 12, 1) \cdot F_{n-k}^{\text{inv}}(\Pi)$ $F_k^{\text{inv}}(312, 2314, 1) \cdot F_{n-k}^{\text{inv}}(312, 1, 2143)$ $F_k^{\text{inv}}(312, 12, 2143) \cdot F_{n-k}^{\text{inv}}(312, 2314, 21)$ $F_k^{\text{inv}}(\Pi) \cdot F_{n-k}^{\text{inv}}(312, 1, 21)$	$= \delta_{0,k} \cdot a_{n-k}$ $= \delta_{0,k} \cdot \delta_{0,n-k}$ $= 1$ $= \delta_{0,n-k} \cdot a_k$
$S = \{1\}$:	$F_k^{\text{inv}}(312, 12, 1) \cdot F_{n-k}^{\text{inv}}(312, 1, 2143)$ $F_k^{\text{inv}}(312, 12, 2143) \cdot F_{n-k}^{\text{inv}}(312, 1, 21)$	$= \delta_{0,k} \cdot \delta_{0,n-k}$ $= \delta_{0,n-k}$
$S = \{2\}$:	$F_k^{\text{inv}}(312, 12, 1) \cdot F_{n-k}^{\text{inv}}(312, 2314, 21)$ $F_k^{\text{inv}}(312, 2314, 1) \cdot F_{n-k}^{\text{inv}}(312, 1, 21)$	$= \delta_{0,k}$ $= \delta_{0,k} \cdot \delta_{0,n-k}$
$S = \{1, 2\}$:	$F_k^{\text{inv}}(312, 21, 1) \cdot F_{n-k}^{\text{inv}}(312, 1, 21)$	$= \delta_{0,k} \cdot \delta_{0,n-k}$

Hence the a_n satisfy

$$\begin{aligned} a_{n+1} &= \sum_{q=0}^{n} q^k \left[\delta_{0,k} a_{n-k} + \delta_{0,n-k} \cdot a_k + 1 - \delta_{0,k} - \delta_{0,n-k}\right] \\ &= (1 + q^n) a_n + \frac{1 - q^{n+1}}{1 - q} - (1 + q^n) \\ &= (1 + q^n) a_n + q \left(\frac{1 - q^{n-1}}{1 - q}\right). \end{aligned}$$

In particular, by setting $q = 1$ we get $a_{n+1} = 2a_n + n - 1$ with $a_0 = a_1 = 1$. Thus

$$|\mathfrak{S}_n(312, 2314, 2143)| = 2^n - n.$$

The following construction of st-Wilf equivalences can be extracted from Theorem 3.2. A proof of this corollary uses a similar argument to that of Corollary 2.6 and is omitted here.

Corollary 3.4. *Let st be a statistic satisfying (†). Let $\pi_i^{(j)}, \pi_i'^{(j)}, 1 \le j \le m, 1 \le i \le r_m$, be permutations such that*

$$\{312, \pi_{i_1}^{(1)}, \ldots, \pi_{i_m}^{(m)}\} \stackrel{st}{\equiv} \{312, \pi_{i_1}'^{(1)}, \ldots, \pi_{i_m}'^{(m)}\}$$

for all m-tuples $I = (i_1, \ldots, i_m) \in [r_1] \times \cdots \times [r_m]$. Set $\pi^{(j)} = \iota_r[\pi_{1*}^{(j)}, \ldots, \pi_{r_j*}^{(j)}]$ and $\pi'^{(j)} = \iota_r[\pi_{1*}'^{(j)}, \ldots, \pi_{r_j*}'^{(j)}]$. Then $\Pi = \{312, \pi^{(1)}, \ldots, \pi^{(m)}\}$ and $\Pi' = \{312, \pi'^{(1)}, \ldots, \pi'^{(m)}\}$ are st-Wilf equivalent.

References

[1] T. Dokos, T. Dwyer, B. P. Johnson, B. E. Sagan, and K. Selsor (2012). Permutation patterns and statistics. *Discrete Mathematics* **312**(8) 2760–2775. MR2945168

[2] S. Elizalde (2011). Fixed points and excedances in restricted permutations. *Electron. J. Combin.* **18**(2). MR2880679

[3] S. Elizalde (2004). Multiple pattern-avoidance with respect to fixed points and excedances. *Electron. J. Combin.* **11**(1). MR2097317

[4] S. Elizalde and I. Pak (2004). Bijections for refined restricted permutations. *J. Combin. Theory A* **105**(2) 207–219. MR2046080

[5] T. Mansour and A. Vainshtein (2001). Restricted 132-Avoiding Permutations. *Adv. in Appl. Math.* **26**(3) 258–269. MR1818747

[6] A. Robertson, D. Saracino, and D. Zeilberger (2003). Refined Restricted Permutations. *Annals of Combin.* **6**(3) 427–444. MR1980351

[7] B. E. Sagan and C. D. Savage (2012). Mahonian Pairs. *J. Combin. Theory Ser. A* **119**(3) 526–545. MR2871748

[8] R. P. Stanley (2012). *Enumerative Combinatorics Vol. 1*, 2nd ed. Cambridge University Press, Cambridge. MR2868112

[9] J. West (1996). Generating trees and forbidden subsequences. *Discrete Mathematics* **157**(1–3) 363–374. MR1417303

Wuttisak Trongsiriwat
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139
USA
E-mail address: wuttisak@mit.edu

Received 29 January 2014