Signed arc permutations

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Arc permutations, which were originally introduced in the study of triangulations and characters, have recently been shown to have interesting combinatorial properties. The first part of this paper continues their study by providing signed enumeration formulas with respect to their descent set and major index. Next, we generalize the notion of arc permutations to the hyperoctahedral group in two different directions. We show that these extensions to type B carry interesting analogues of the properties of type A arc permutations, such as characterizations by pattern avoidance, and elegant unsigned and signed enumeration formulas with respect to the flag-major index.

1. Introduction

The enumeration of permutations taking into account their sign, usually referred to as *signed enumeration*, was studied for subsets of the symmetric group $\mathcal{S}_n$ in the seminal paper of Simion and Schmidt on pattern-avoiding permutations [24]. Among many other instances of sign enumeration in the literature, we highlight an elegant formula for the signed descent number enumerator conjectured by Loday [20] and proved by Désarmenien and Foata [15] and by Wachs [25]. Type B analogues were given later by Reiner [22].

In analogy to MacMahon's well-known product formula enumerating permutations in $\mathcal{S}_n$ with respect to the major index, a factorial-type formula for the signed major index enumerator on $\mathcal{S}_n$ was given by Gessel and Simion [25, Cor. 2]. For generalizations to other groups, see [3, 7, 13, 10, 9, 12]. In this paper we study signed major index enumerators and other

related polynomials for arc permutations, both in the symmetric group $\mathcal{S}_n$ and in the hyperoctahedral group B_n.

Arc permutations were introduced in [5] as a subset of the symmetric group. These permutations play an important role in the study of flip graphs of polygon triangulations and associated affine Weyl group actions. It was shown in [16] that arc permutations can be characterized in terms of pattern avoidance. A descent-set-preserving map from arc permutations to Young tableaux was constructed in [16] to deduce a conjectured character formula of Regev.

In this paper we propose two different generalizations of the notion of arc permutations to the hyperoctahedral group. These generalizations, which we call *signed arc permutations* and *B-arc permutations*, carry known properties of unsigned arc permutations and reveal new ones. In particular, we give characterizations of both generalizations by forbidden patterns (see Theorems 4.4 and 5.4), in analogy to the results from [16] in the unsigned case. Additionally, we show in Subsection 5.3 that both unsigned arc permutations and B-arc permutations may be characterized by their canonical expressions. This characterization will be used to derive the signed and unsigned flag-major index enumerators for B-arc permutations. In the case of signed arc permutations, different tools are used to derive similar formulas in Section 4.

For both generalizations of arc permutations to type B, we obtain nice product formulas for their descent set enumerators (see Theorems 4.5 and 5.10). Even though the descent set has a different distribution on these two definitions, it turns out that they both carry the same unsigned and signed flag-major index enumerators (see Corollary 6.1). This surprising phenomenon deserves further study.

2. Arc permutations in the symmetric group

2.1. Definition and basic properties

We start by reviewing two definitions and a result from [16]. Recall that an interval of $\mathbb{Z}_n$ is a set of the form $\{a, a+1, \ldots, b\}$ or $\{b, b+1, \ldots, n, 1, 2, \ldots, a\}$ where $1 \leq a \leq b \leq n$.

Definition 2.1. *A permutation $\pi \in \mathcal{S}_n$ is an* arc permutation *if, for every $1 \leq j \leq n$, the first j letters in π form an interval in $\mathbb{Z}_n$. Denote by $\mathcal{A}_n$ the set of arc permutations in $\mathcal{S}_n$.*

A permutation $\pi \in \mathcal{A}_n$ is left-unimodal *if, for every $1 \leq j \leq n$, the first j letters in π form an interval in $\mathbb{Z}$. Denote by $\mathcal{L}_n$ the set of left-unimodal permutations in $\mathcal{S}_n$.*

Example 1. We have that $12543 \in \mathcal{A}_5$, but $125436 \notin \mathcal{A}_6$, since $\{1, 2, 5\}$ is an interval in $\mathbb{Z}_5$ but not in $\mathbb{Z}_6$.

It is easy to show that $|\mathcal{A}_n| = n2^{n-2}$ for $n \geq 2$ [16]. Arc permutations can be characterized in terms of pattern avoidance, as those permutations avoiding the eight patterns $\tau \in \mathcal{S}_4$ with $|\tau(1) - \tau(2)| = 2$.

Theorem 2.2 ([16]).

$$\mathcal{A}_n = \mathcal{S}_n(1324, 1342, 2413, 2431, 3124, 3142, 4213, 4231).$$

2.2. Enumeration

For a permutation $\pi \in \mathcal{S}_n$, recall the definition of its descent set

$$\mathrm{Des}(\pi) := \{i : \pi(i) > \pi(i + 1)\},$$

its major index

$$\mathrm{maj}(\pi) := \sum_{i \in \mathrm{Des}(\pi)} i,$$

and its inversion number

$$\mathrm{inv}(\pi) := \#\{i < j : \pi(i) > \pi(j)\}.$$

For a set $D = \{i_1, \ldots, i_k\}$ denote $\mathbf{x}^D = x_{i_1} \cdots x_{i_k}$.

Theorem 2.3. *For every $n \geq 2$,*

$$\sum_{\pi \in \mathcal{A}_n} t^{\mathrm{inv}(\pi)} \mathbf{x}^{\mathrm{Des}(\pi)}$$

$$= \prod_{i=1}^{n-1} (1 + t^i x_i) + \sum_{j=1}^{n-2} \left((t^{j(n-j)} x_j + t^{n-j-1} x_{j+1}) \right.$$

$$\left. \times \prod_{i=1}^{j-1} (1 + t^i x_i) \prod_{i=j+2}^{n-1} (1 + t^{n-i} x_i) \right).$$

Proof. We separate permutations $\pi \in \mathcal{A}_n$ into those that are left-unimodal and those that are not.

Left-unimodal permutations are in bijection with subsets of $[n - 1]$, the bijection given by taking their descent set. Thus, such permutations are determined by choosing, for each $1 \leq i \leq n - 1$, whether $\pi(i) > \pi(i + 1)$

or $\pi(i) < \pi(i+1)$. In the first case, we introduce a descent in position i, and inversions between π_{i+1} and all the preceding entries of π, contributing $t^i x_i$ to the generating function, while in the second case no descents or inversions are created. It follows that left-unimodal permutations contribute $\prod_{i=1}^{n-1}(1 + t^i x_i)$ to the generating function.

If π is not left-unimodal, let j be the largest such that $\{\pi(1), \ldots, \pi(j)\}$ is an interval in $\mathbb{Z}$. Note that $1 \le j \le n-2$, and that $\pi(j+1) \in \{1, n\}$.

If $\pi(j+1) = 1$, then the first j entries in π are larger than the last $n-j$ entries, creating $j(n-j)$ inversions and a descent in position j. For each i with $1 \le i \le j-1$ or $j+2 \le i \le n-1$, we have the choice of whether $\pi(i) > \pi(i+1)$ or $\pi(i) < \pi(i+1)$. For $1 \le i \le j-1$, the first option introduces a descent in position i and inversions between π_{i+1} and the entries to its left, contributing $t^i x_i$. Similarly, for $j+2 \le i \le n-1$, the choice $\pi(i) > \pi(i+1)$ introduces inversions between π_i and the entries to its right, contributing $t^{n-i} x_i$. In total, the contribution of non-left-unimodal permutations with $\pi(j+1) = 1$ is

$$t^{j(n-j)} x_j \prod_{i=1}^{j-1}(1 + t^i x_i) \prod_{i=j+2}^{n-1}(1 + t^{n-i} x_i).$$

If $\pi(j+1) = n$, the argument is similar, except that instead of a descent in position j there is a descent in position $j+1$, and there are inversions between $\pi(j+1)$ and the entries to its right, so the contribution in this case is

$$t^{n-j-1} x_{j+1} \prod_{i=1}^{j-1}(1 + t^i x_i) \prod_{i=j+2}^{n-1}(1 + t^{n-i} x_i). \qquad \square$$

Substituting $t = 1$ in Theorem 2.3 we recover the following formula from [16]:

$$(1) \qquad \sum_{\pi \in \mathcal{A}_n} \mathbf{x}^{\mathrm{Des}(\pi)} = \prod_{i=1}^{n-1}(1 + x_i)\left(1 + \sum_{j=1}^{n-2} \frac{x_j + x_{j+1}}{(1 + x_j)(1 + x_{j+1})}\right)$$

for every $n \ge 2$. It is now easy to obtain the (des, maj)-enumerator for arc permutations. Recall the notation $[n]_q = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$.

Corollary 2.4. *For every* $n \ge 2$,

$$\sum_{\pi \in \mathcal{A}_n} t^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)} = \prod_{i=2}^{n-2}(1 + tq^i)\left(1 + 2tq[n-1]_q + t^2 q^n\right).$$

In particular,

$$\sum_{\pi \in \mathcal{A}_n} t^{\text{des}(\pi)} = (1+t)^{n-3}\left(1 + 2(n-1)t + t^2\right).$$

Proof. Substituting $x_i = tq^i$ for $1 \le i \le n-1$ in Equation (1), we get

$$\sum_{\pi \in \mathcal{A}_n} t^{\text{des}(\pi)} q^{\text{maj}(\pi)} = \prod_{i=1}^{n-1}(1 + tq^i)\left(1 + \sum_{j=1}^{n-2} \frac{tq^j(1+q)}{(1+tq^j)(1+tq^{j+1})}\right).$$

Using that

$$\frac{tq^j}{(1+tq^j)(1+tq^{j+1})} = \frac{1}{1-q}\left(\frac{1}{1+tq^{j+1}} - \frac{1}{1+tq^j}\right),$$

the summation on the right-hand side becomes a telescopic sum that simplifies to

$$\frac{(1+q)tq[n-2]_q}{(1+tq)(1+tq^{n-1})},$$

from where the first formula in the statement follows. The second formula is obtained by substituting $q = 1$. $\square$

Now we turn to signed enumeration of arc permutations. Recall that $\text{sign}(\pi) = (-1)^{\text{inv}(\pi)}$. Setting $t = -1$ in Theorem 2.3, we get

$$(2) \quad \sum_{\pi \in \mathcal{A}_n} \text{sign}(\pi)\mathbf{x}^{\text{Des}(\pi)}$$

$$= \prod_{i=1}^{n-1}(1 + (-1)^i x_i) + \sum_{j=1}^{n-2}\left(((-1)^{j(n-j)}x_j + (-1)^{n-j-1}x_{j+1})\right.$$

$$\left.\times \prod_{i=1}^{j-1}(1 + (-1)^i x_i) \prod_{i=j+2}^{n-1}(1 + (-1)^{n-i}x_i)\right).$$

When n is even, this formula simplifies to

$$(3) \quad \sum_{\pi \in \mathcal{A}_n} \text{sign}(\pi)\mathbf{x}^{\text{Des}(\pi)}$$

$$= \prod_{i=1}^{n-1}(1 + (-1)^i x_i)\left(1 + \sum_{j=1}^{n-2} \frac{(-1)^j(x_j - x_{j+1})}{(1+(-1)^j x_j)(1+(-1)^{j+1}x_{j+1})}\right).$$

Theorem 2.5. *For every $n \geq 2$*

$$\sum_{\pi \in \mathcal{A}_n} q^{\mathrm{maj}(\pi)} = [n]_q \prod_{i=1}^{n-2}(1 + q^i),$$

$$\sum_{\pi \in \mathcal{A}_n} \mathrm{sign}(\pi)q^{\mathrm{maj}(\pi)} = [n]_{(-1)^{n-1}q} \prod_{i=1}^{n-2}(1 + (-q)^i).$$

Proof. Substituting $t = 1$ in Corollary 2.4 gives the first formula, which already appears in [16, Cor. 7]. To prove the second formula, we consider two cases depending on the parity of n.

If n is even, substituting $x_i = q^i$ for $1 \leq i \leq n - 1$ in Equation (3) gives

$$\sum_{\pi \in \mathcal{A}_n} \mathrm{sign}(\pi)q^{\mathrm{maj}(\pi)} = \prod_{i=1}^{n-1}(1 + (-q)^i)\left(1 + \sum_{j=1}^{n-2} \frac{(-1)^j(q^j - q^{j+1})}{(1 + (-q)^j)(1 + (-q)^{j+1})}\right).$$

Letting $z = -q$, the formula becomes

$$(4) \qquad \prod_{i=1}^{n-1}(1 + z^i)\left(1 + \sum_{j=1}^{n-2} \frac{z^j + z^{j+1}}{(1 + z^j)(1 + z^{j+1})}\right),$$

where the sum can be simplified as

$$\frac{1+z}{1-z}\sum_{j=1}^{n-2}\left(\frac{1}{1 + z^{j+1}} - \frac{1}{1 + z^j}\right)$$

$$= \frac{1+z}{1-z}\left(\frac{1}{1 + z^{n-1}} - \frac{1}{1 + z}\right) = \frac{z - z^{n-1}}{(1 - z)(1 + z^{n-1})},$$

and so Equation (4) equals

$$\prod_{i=1}^{n-1}(1+z^i)\left(1 + \frac{z - z^{n-1}}{(1 - z)(1 + z^{n-1})}\right) = \prod_{i=1}^{n-2}(1+z^i)[n]_z = [n]_{-q}\prod_{i=1}^{n-2}(1+(-q)^i).$$

If n is odd, substituting $x_j = q^j$ in Equation (2) gives

(5) $\displaystyle\sum_{\pi\in\mathcal{A}_n}\text{sign}(\pi)q^{\text{maj}(\pi)}$

$$= \prod_{i=1}^{n-1}(1+(-q)^i) + \sum_{j=1}^{n-2}\left((q^j+(-1)^jq^{j+1})\right.$$

$$\left.\times\prod_{i=1}^{j-1}(1+(-q)^i)\prod_{i=j+2}^{n-1}(1-(-q)^i)\right)$$

$$= \prod_{i=1}^{n-1}(1+z^i) + \sum_{j=1}^{n-2}\left(((-1)^jz^j-z^{j+1})\prod_{i=1}^{j-1}(1+z^i)\prod_{i=j+2}^{n-1}(1-z^i)\right),$$

letting $z=-q$ again. Writing $(-1)^jz^j-z^{j+1}=(1-z^{j+1})-(1-(-1)^jz^j)$, the summation on the right-hand side of Equation (5) becomes a telescopic sum

(6)
$$\sum_{j=1}^{n-2}\left(\prod_{i=1}^{j-1}(1+z^i)\prod_{i=j+1}^{n-1}(1-z^i)-(1-(-1)^jz^j)\prod_{i=1}^{j-1}(1+z^i)\prod_{i=j+2}^{n-1}(1-z^i)\right)$$

$$= \prod_{i=2}^{n-1}(1-z^i)+\sum_{\substack{j=2\\j\,\text{even}}}^{n-3}\left(2z^j\prod_{i=1}^{j-1}(1+z^i)\prod_{i=j+2}^{n-1}(1-z^i)\right)-\prod_{i=1}^{n-2}(1+z^i),$$

noting that $-(1-(-1)^jz^j)+(1+z^j)=2z^j$ when j is even and 0 otherwise. Now, writing

$$2z^j=\frac{(1+z^j)(1+z^{j+1})-(1-z^j)(1-z^{j+1})}{1+z},$$

the summation in the middle of Equation (6) also becomes a telescopic sum

$$\frac{1}{1+z}\sum_{\substack{j=2\\j\,\text{even}}}^{n-3}\left(\prod_{i=1}^{j+1}(1+z^i)\prod_{i=j+2}^{n-1}(1-z^i)-\prod_{i=1}^{j-1}(1+z^i)\prod_{i=j}^{n-1}(1-z^i)\right)$$

$$= \frac{1}{1+z}\left(-(1+z)\prod_{i=2}^{n-1}(1-z^i)+(1-z^{n-1})\prod_{i=1}^{n-2}(1+z^i)\right)$$

$$= -\prod_{i=2}^{n-1}(1-z^i)+(1-z^{n-1})\prod_{i=2}^{n-2}(1+z^i).$$

With these simplifications, Equation (5) equals

$$\prod_{i=1}^{n-1}(1+z^i) + \prod_{i=2}^{n-1}(1-z^i) - \prod_{i=2}^{n-1}(1-z^i) + (1-z^{n-1})\prod_{i=2}^{n-2}(1+z^i) - \prod_{i=1}^{n-2}(1+z^i)$$

$$= \prod_{i=1}^{n-2}(1+z^i)\left(1+z^{n-1}+\frac{1-z^{n-1}}{1+z}-1\right)$$

$$= [n]_{-z}\prod_{i=1}^{n-2}(1+z^i) = [n]_q\prod_{i=1}^{n-2}(1+(-q)^i). \qquad \square$$

A different approach to prove Theorem 2.5 will be described in Section 5.3.

3. The hyperoctahedral group: preliminaries and notation

The hyperoctahedral group B_n may be realized as a group of signed permutations as follows. We denote by B_n the group of all bijections π of the set $[\pm n] = \{-1, -2, \ldots, -n, 1, 2, \ldots, n\}$ onto itself such that

$$\pi(-a) = -\pi(a)$$

for every $1 \le a \le n$, with composition as the group operation. This group is usually known as the group of *signed permutations* on $\{1, 2, \ldots, n\}$, or as the *hyperoctahedral group* of rank n. We identify $\mathcal{S}_n$ as a subgroup of B_n, and B_n as a subgroup of $\mathcal{S}_{2n}$ in the natural ways.

If $\pi \in B_n$, we write $\pi = [a_1, \ldots, a_n]$ to mean that $\pi(i) = a_i$ for $1 \le i \le n$. The Coxeter generating set of B_n is $\mathsf{S} = \{\sigma_i : 0 \le i < n\}$, where $\sigma_0 = [-1, 2, 3, 4, \ldots, n]$ and, for $1 \le i < n$, σ_i is the adjacent transposition $(i, i+1)$.

We recall some statistics on B_n. For $\pi \in B_n$, we say that i is a descent in π if $\pi(i) > \pi(i+1)$ with respect to the order $-1 < -2 < \cdots < -n < 1 < 2 < \cdots < n$. We use the following standard notation:

$$\mathrm{Des}(\pi) := \{1 \le i \le n-1 : \ \pi(i) > \pi(i+1)\},$$
$$\mathrm{des}(\pi) := |\mathrm{Des}(\pi)|,$$
$$\mathrm{maj}(\pi) := \sum_{i \in \mathrm{Des}(\pi)} i,$$
$$\mathrm{Neg}(\pi) := \{1 \le i \le n : \ \pi(i) < 0\},$$
$$\mathrm{neg}(\pi) := |\mathrm{Neg}(\pi)|.$$

Two more statistics, defined in [4] and [1], respectively, are the *flag-major index*

$$\mathrm{fmaj}(\pi) := 2 \cdot \mathrm{maj}(\pi) + \mathrm{neg}(\pi),$$

and *flag-descent number*

$$\mathrm{fdes}(\pi) := 2 \cdot \mathrm{des}(\pi) + \delta(\pi(1) < 0),$$

where $\delta(a) := 1$ if the event a occurs and zero otherwise.

The statistics fmaj and fdes have been shown to play a significant role in the study of B_n, which is analogous to the role of the classical descent statistics on $\mathcal{S}_n$. Some examples in the literature are [2, 3, 7, 8, 14, 17, 18, 19].

4. Signed arc permutations

4.1. Definition and basic properties

In this section we introduce our first generalization of arc permutations to type B.

Definition 4.1. *A permutation $\pi = [\pi(1), \ldots, \pi(n)] \in B_n$ is a* signed arc permutation *if, for every $1 < i < n$,*

- *the prefix $\{|\pi(1)|, \ldots, |\pi(i)|\}$ forms an interval in $\mathbb{Z}_n$; and*
- *the sign of $\pi(i)$ is positive if $|\pi(i)| - 1 \in \{|\pi(1)|, \ldots, |\pi(i-1)|\}$ and negative if $|\pi(i)| + 1 \in \{|\pi(1)|, \ldots, |\pi(i-1)|\}$ (with addition in $\mathbb{Z}_n$).*

Denote by $\mathcal{A}_n^s$ the set of signed arc permutations in B_n.

Note that there is no restriction on the signs of $\pi(1)$ and $\pi(n)$.

Example 2. We have that $[2, -1, 3] \in \mathcal{A}_3^s$ and $[-3, -2, 4, 1] \in \mathcal{A}_4^s$, but $[-2, 1, 3] \notin \mathcal{A}_3^s$.

For $\pi \in B_n$, let $|\pi| = |\pi(1)||\pi(2)| \ldots |\pi(n)| \in \mathcal{S}_n$. Note that if $\pi \in \mathcal{A}_n^s$, then $|\pi| \in \mathcal{A}_n$.

Remark 4.2. The apparent ad-hoc determination of the signs in the second part of Definition 4.1 surprisingly results in a coherent combinatorial structure to be described below, which further leads to interesting quasi-symmetric functions of type B to be discussed in a forthcoming paper.

Claim 4.3. *For $n \geq 1$, $|\mathcal{A}_n^s| = n2^n$.*

Proof. The equality is trivial for $n = 1$, so we may assume that $n \geq 2$. From every $\sigma \in \mathcal{A}_n$, there are four permutations $\pi \in \mathcal{A}_n^s$ such that $|\pi| = \sigma$, since all the signs but those of the first and the last entry are determined. It follows that $|\mathcal{A}_n^s| = 4|\mathcal{A}_n| = n2^n$. $\qquad\square$

4.2. Characterization by pattern avoidance

Let us recall the standard definition of pattern avoidance in the hyperoctahedral group. Given $\pi = [\pi(1), \ldots, \pi(n)] \in B_n$ and $\sigma = [\sigma(1), \ldots, \sigma(k)] \in B_k$, we say that π contains the pattern σ if there exist indices $1 \leq i_1 < \cdots < i_k \leq n$ such that

- $\pi(i_j)$ and $\sigma(j)$ have the same sign for all $1 \leq j \leq k$, and
- $|\pi(i_1)||\pi(i_2)| \ldots |\pi(i_k)|$ is in the same relative order as $|\sigma(1)||\sigma(2)| \ldots |\sigma(k)|$.

In this case, $\pi(i_1)\pi(i_2) \ldots \pi(i_k)$ is called an *occurrence* of σ. Otherwise, we say that π *avoids* σ. For example, $[-3, 2, 5, -1, 4]$ contains the pattern $[-2, -1, 3]$, because the subsequence $-3, -1, 4$ is an occurrence of this pattern, but it avoids the pattern $[2, 1, 3]$.

In analogy with Theorem 2.2 for arc permutations in $\mathcal{S}_n$, we can characterize signed arc permutations in terms of pattern avoidance.

Theorem 4.4. *A permutation $\pi \in B_n$ is a signed arc permutation if and only if it avoids the following* 24 *patterns:*

$$[\pm 1, -2, \pm 3], [\pm 1, 3, \pm 2], [\pm 2, -3, \pm 1], [\pm 2, 1, \pm 3], [\pm 3, -1, \pm 2], [\pm 3, 2, \pm 1].$$

We say that a triple (a, b, c) of different integers in $\{1, 2, \ldots, n\}$ is a *clockwise* triple if either $a < b < c$, $b < c < a$ or $c < a < b$. Otherwise, we say that it is a *counterclockwise* triple. The name comes from the direction determined by the triple (a, b, c) in the circle where the entries $1, 2, \ldots, n$ have been written in clockwise order.

Note that the patterns listed in Theorem 4.4 are precisely those permutations in B_3 of the form $[\pm a, -b, \pm c]$ where (a, b, c) is a clockwise triple, and $[\pm a, b, \pm c]$ where (a, b, c) is a counterclockwise triple.

Proof of Theorem 4.4. In this proof, addition and subtraction are in $\mathbb{Z}_n$, and so are intervals.

Let $\pi \in B_n$ contain an occurrence $\pi(i_1)\pi(i_2)\pi(i_3)$ of one of the 24 listed patterns. Suppose for contradiction that $\pi \in \mathcal{A}_n^s$.

If $\pi(i_2) > 0$, then $(|\pi(i_1)|, |\pi(i_2)|, |\pi(i_3)|)$ is a counterclockwise triple. Since $\pi \in \mathcal{A}_n^s$ and $\pi(i_2)$ is positive, the interval $\{|\pi(1)|, \ldots, |\pi(i_2 - 1)|\}$ contains $|\pi(i_2)| - 1$ and $|\pi(i_1)|$, but not $|\pi(i_2)|$. Thus, it must also contain $|\pi(i_3)|$, which is a contradiction.

Similarly, if $\pi(i_2) < 0$, then $(|\pi(i_1)|, |\pi(i_2)|, |\pi(i_3)|)$ is a clockwise triple, and the interval $\{|\pi(1)|, \ldots, |\pi(i_2 - 1)|\}$ contains $|\pi(i_2)| + 1$ and $|\pi(i_1)|$, but not $|\pi(i_2)|$. Thus, it must also contain $|\pi(i_3)|$, again a contradiction.

To prove the converse, suppose now that $\pi \in B_n$ is not a signed arc permutation. Let i be the smallest index where the conditions from Definition 4.1 fail. This means that either $\{|\pi(1)|, \ldots, |\pi(i)|\}$ is not an interval in $\mathbb{Z}_n$, or $\pi(i)$ has the wrong sign. In the first case, neither of the values $|\pi(i)| \pm 1$ is in the interval $\{|\pi(1)|, \ldots, |\pi(i-1)|\}$. In the second case, either $\pi(i) > 0$ but $|\pi(i)| - 1 \notin \{|\pi(1)|, \ldots, |\pi(i-1)|\}$, or $\pi(i) < 0$ but $|\pi(i)| + 1 \notin \{|\pi(1)|, \ldots, |\pi(i-1)|\}$.

If $\pi(i) > 0$ (respectively, $\pi(i) < 0$), let $j > i$ be such that $|\pi(j)| = |\pi(i)| - 1$ (respectively, $|\pi(j)| = |\pi(i)| + 1$). Then $(|\pi(1)|, |\pi(i)|, |\pi(j)|)$ is a counterclockwise (respectively, clockwise) triple, so $\pi(1)\pi(i)\pi(j)$ is an occurrence of one of the 24 listed patterns. $\qquad\square$

4.3. Descent set enumerators

Next we describe the joint distribution of the descent set and the set of negative entries on signed arc permutations.

Theorem 4.5. *For every $n \geq 1$,*

(7)
$$\sum_{\pi \in \mathcal{A}_n^s} \mathbf{x}^{\mathrm{Des}(\pi)} \mathbf{y}^{\mathrm{Neg}(\pi)} = \prod_{i=1}^{n}(1 + x_{i-1}y_i)\left(1 + \sum_{j=1}^{n-1} \frac{(x_j + x_{j-1}y_j)(1 + y_{j+1})}{(1 + x_{j-1}y_j)(1 + x_j y_{j+1})}\right),$$

and

$$(8) \quad \sum_{\pi \in \mathcal{A}_n^s} t^{\mathrm{inv}(|\pi|)} \mathbf{x}^{\mathrm{Des}(\pi)} \mathbf{y}^{\mathrm{Neg}(\pi)}$$

$$= \prod_{i=1}^{n}(1 + t^{i-1}x_{i-1}y_i) + \sum_{j=1}^{n-1}\left((x_j + t^{j-1}x_{j-1}y_j)(t^{j(n-j)} + t^{n-j-1}y_{j+1})\right.$$

$$\left. \times \prod_{i=1}^{j-1}(1 + t^{i-1}x_{i-1}y_i)\prod_{i=j+2}^{n}(1 + t^{n-i}x_{i-1}y_i)\right),$$

with the convention that $x_0 := 1$.

Proof. Since Equation (7) follows from Equation (8) by setting $t = 1$ and simplifying, it suffices to prove Equation (8).

If $\pi \in \mathcal{A}_n^s$ is such that $|\pi|$ is left-unimodal, then $|\pi(n)| \in \{1, n\}$. Let us first consider signed arc permutations where $|\pi|$ is left-unimodal and $\pi(n) \in \{-1, n\}$. The contribution of such permutations to the generating

function is

$$\prod_{i=1}^{n}(1 + t^{i-1}x_{i-1}y_i).$$

Indeed, such permutations are uniquely determined by a choice of sign of $\pi(i)$ for $1 \le i \le n$. If $\pi(i)$ is negative, it creates a descent with $\pi(i-1)$ (for $i > 1$) and inversions in $|\pi|$ with all the preceding entries, contributing a factor $t^{i-1}x_{i-1}y_i$. If $\pi(i)$ is positive, then no descent or inversions with preceding entries are created.

   Let us now consider the remaining permutations $\pi \in \mathcal{A}_n^s$, and let $j+1$ be the first index where π fails to be in the set considered above. In other words, if $|\pi|$ is not left-unimodal, j is the largest such that $\{|\pi(1)|, \ldots, |\pi(j)|\}$ is an interval in $\mathbb{Z}$, and note that $1 \le j \le n-2$ in this case. On the other hand, if $|\pi|$ is left-unimodal but $\pi(n) \in \{1, -n\}$, then $j = n-1$. Consider two cases depending on the sign of $\pi(j+1)$.

- If $\pi(j+1)$ is positive, we must have $\pi(j+1) = 1$. In this case, the first j entries in $|\pi|$ are larger than the last $n-j$ entries, creating $j(n-j)$ inversions. The contribution of permutations where $\pi(j)$ is positive as well (and so $\pi(j) = n$) is then

  $$(9) \qquad t^{j(n-j)}x_j \cdot \prod_{i=1}^{j-1}(1 + t^{i-1}x_{i-1}y_i) \prod_{i=j+2}^{n}(1 + t^{n-i}x_{i-1}y_i).$$

  To see this, first notice that the factor x_j records the descent in position j. For $1 \le i \le j-1$, each negative entry $\pi(i)$ creates a descent with $\pi(i-1)$ and inversions with all the preceding entries in $|\pi|$, contributing $t^{i-1}x_{i-1}y_i$. For $j+2 \le i \le n$, each negative entry $\pi(i)$ creates a descent with $\pi(i-1)$ and inversions with all the following entries in $|\pi|$, contributing $t^{n-i}x_{i-1}y_i$. In both cases, positive entries $\pi(i)$ just contribute a factor of 1.

  The contribution of permutations where $\pi(j)$ is negative is given by replacing $t^{j(n-j)}x_j$ with $t^{j(n-j)}t^{j-1}x_{j-1}y_j$ in Equation (9), since now π has a descent in position $j-1$, and $|\pi(j)|$ creates inversions with all the preceding entries in $|\pi|$.

- If $\pi(j+1)$ is negative, we must have $\pi(j+1) = -n$. In this case, the contribution of permutations where $\pi(j)$ is negative (and so $\pi(j) = -1$) is given by replacing $t^{j(n-j)}x_j$ with $t^{j-1}t^{n-j-1}x_{j-1}y_jy_{j+1}$ in Equation (9). Indeed, π has a descent in position $j-1$, and there are inversions in $|\pi|$ between $|\pi(j)|$ and all the preceding entries, and between $|\pi(j+1)|$ and all the following entries.

Similarly, the contribution of permutations where $\pi(j)$ is positive is obtained by replacing $t^{j(n-j)}x_j$ with $t^{n-j-1}x_j y_{j+1}$ in Equation (9), since π has a descent in position $j-1$, and there are inversions in $|\pi|$ between $|\pi(j)|$ and all the preceding entries, and between $|\pi(j+1)|$ and all the following entries.

Adding all of the above contributions we obtain the stated formula. $\square$

4.4. The (fdes, fmaj)-enumerator

Corollary 4.6. *For every* $n \geq 2$,

$$\sum_{\pi \in \mathcal{A}_n^s} t^{\mathrm{fdes}(\pi)} q^{\mathrm{fmaj}(\pi)}$$

$$= (1+tq)\left(1 + tq(1+q) + 2t^2q^3[2n-3]_q + t^3q^{2n}(1+q) + t^4 q^{2n+2}\right)$$

$$\times \prod_{i=3}^{n-1}(1 + t^2 q^{2i-1}).$$

In particular,

$$\sum_{\pi \in \mathcal{A}_n^s} t^{\mathrm{fdes}(\pi)} = (1+t)(1+t^2)^{n-3}(1 + 2t + (4n-6)t^2 + 2t^3 + t^4).$$

Proof. Substituting $y_1 = tq$, $y_i = q$ for $2 \leq i \leq n$, and $x_i = t^2 q^{2i}$ for $1 \leq i \leq n-1$ in Equation (7), we get

$$\sum_{\pi \in \mathcal{A}_n^s} t^{\mathrm{fdes}(\pi)} q^{\mathrm{fmaj}(\pi)}$$

$$= (1+tq)\prod_{i=2}^{n}(1 + t^2 q^{2i-1})$$

$$\times \left(1 + \frac{tq(1+q)}{1+t^2 q^3} + \sum_{j=2}^{n-1} \frac{(1+q^2)t^2 q^{2j-1}}{(1+t^2 q^{2j-1})(1+t^2 q^{2j+1})}\right).$$

Using that

$$\frac{t^2 q^{2j-1}}{(1+t^2 q^{2j-1})(1+t^2 q^{2j+1})} = \frac{1}{1-q^2}\left(\frac{1}{1+t^2 q^{2j+1}} - \frac{1}{1+t^2 q^{2j-1}}\right),$$

the summation on the right-hand side becomes a telescopic sum that simplifies to

$$\frac{(1 + q)t^2q^3[2n - 4]_q}{(1 + t^2q^3)(1 + t^2q^{2n-1})}.$$

The first formula in the statement follows now from straightforward simplifications, and the second formula is obtained by substituting $q = 1$. $\square$

4.5. The signed fmaj-enumerator

Recall that B_n has four one-dimensional characters: the trivial character; the sign character $\text{sign}(\pi)$; $(-1)^{\text{neg}(\pi)}$; and the sign of $|\pi| \in \mathcal{S}_n$, denoted $\text{sign}(|\pi|)$. Let us now compute the enumerators for signed arc permutations with respect to fmaj and each one of these characters.

Corollary 4.7. *For every $n \geq 1$,*

$$(10) \qquad \sum_{\pi \in \mathcal{A}_n^s} q^{\text{fmaj}(\pi)} = [2n]_q \prod_{i=1}^{n-1}(1 + q^{2i-1}),$$

(11)

$$\sum_{\pi \in \mathcal{A}_n^s} \text{sign}(\pi)q^{\text{fmaj}(\pi)} = \begin{cases} (1 - q)[n]_{-q^2} \displaystyle\prod_{i=1}^{n-1}(1 + (-1)^i q^{2i-1}) & \text{if } n \text{ is odd,} \\ [2n]_q \displaystyle\prod_{i=1}^{n-1}(1 + (-1)^i q^{2i-1}) & \text{if } n \text{ is even,} \end{cases}$$

$$\sum_{\pi \in \mathcal{A}_n^s} (-1)^{\text{neg}(\pi)}q^{\text{fmaj}(\pi)} = [2n]_{-q} \prod_{i=1}^{n-1}(1 - q^{2i-1}),$$

$$\sum_{\pi \in \mathcal{A}_n^s} \text{sign}(|\pi|)q^{\text{fmaj}(\pi)} = \begin{cases} (1 + q)[n]_{-q^2} \displaystyle\prod_{i=1}^{n-1}(1 + (-1)^{i-1} q^{2i-1}) & \text{if } n \text{ is odd,} \\ [2n]_{-q} \displaystyle\prod_{i=1}^{n-1}(1 + (-1)^{i-1} q^{2i-1}) & \text{if } n \text{ is even.} \end{cases}$$

Proof. Equation (10) for $n \geq 2$ is obtained from Corollary 4.6 by substituting $t = 1$, and it is trivial for $n = 1$.

To prove Equation (11), we use that the sign of $\pi \in B_n$ can be expressed as $\text{sign}(\pi) = (-1)^{\text{neg}(\pi)}\text{sign}(|\pi|) = (-1)^{\text{inv}(|\pi|)+\text{neg}(\pi)}$. Substituting $t = -1$, $y_i = -q$ and $x_i = q^{2i}$ for all i in Theorem 4.5, we obtain

(12)

$$\sum_{\pi \in \mathcal{A}_n^s} \text{sign}(\pi) q^{\text{fmaj}(\pi)} = \sum_{\pi \in \mathcal{A}_n^s} (-1)^{\text{inv}(|\pi|) + \text{neg}(\pi)} q^{\text{fmaj}(\pi)}$$

$$= \prod_{i=1}^{n} (1 + (-1)^i q^{2i-1}) + \sum_{j=1}^{n-1} \left(q^{2j-1} (q + (-1)^j)((-1)^{j(n-j)} + (-1)^{n-j} q) \right.$$

$$\left. \times \prod_{i=1}^{j-1} (1 + (-1)^i q^{2i-1}) \prod_{i=j+2}^{n} (1 + (-1)^{n-i+1} q^{2i-1}) \right).$$

When n is odd, the right-hand side of Equation (12) simplifies to

$$\prod_{i=1}^{n} (1 + (-1)^i q^{2i-1}) \left(1 + \sum_{j=1}^{n-1} \frac{(-1)^j q^{2j-1}(1 - q^2)}{(1 + (-1)^j q^{2j-1})(1 + (-1)^{j+1} q^{2j+1})} \right).$$

The summation in the above formula can be written as a telescopic sum

$$\frac{1 - q^2}{1 + q^2} \sum_{j=1}^{n-1} \left(\frac{1}{1 + (-1)^{j+1} q^{2j+1}} - \frac{1}{1 + (-1)^j q^{2j-1}} \right)$$

$$= \frac{q(1 + q)((-1)^{n-1} q^{2n-2} - 1)}{(1 + q^2)(1 + (-1)^n q^{2n-1})},$$

from where we obtain the expression in the statement.

When n is even, using the shorthand $a_j = \prod_{i=1}^{j}(1 + (-1)^i q^{2i-1})$ and $b_j = \prod_{i=j}^{n}(1 + (-1)^{i-1} q^{2i-1})$, we can write the right-hand side of Equation (12) as

(13) $a_n + \sum_{j=1}^{n-1} q^{2j-1}(1 + (-1)^j q^2) a_{j-1} b_{j+2} + \sum_{j=1}^{n-1} q^{2j-1}(1 + (-1)^j q) a_{j-1} b_{j+2}.$

Using that $q^{2j-1}(1 + (-1)^j q^2) = (1 + (-1)^j q^{2j+1}) - (1 - q^{2j-1})$, the first summation in Equation (13) simplifies as a telescopic sum

$$\sum_{j=1}^{n-1} \left(a_{j-1} b_{j+1} - (1 - q^{2j-1}) a_{j-1} b_{j+2} \right) = b_2 - a_{n-1} + \sum_{\substack{j=2 \\ j \text{ even}}}^{n-2} 2q^{2j-1} a_{j-1} b_{j+2}.$$

Combining this expression with the fact that the second summation in Equa-

tion (13) can be written as

$$\sum_{\substack{j=2 \\ j\,\text{even}}}^{n-2} 2q^{2j} a_{j-1} b_{j+2},$$

Equation (13) equals

$$(14) \qquad a_n + b_2 - a_{n-1} + (1+q) \sum_{\substack{j=2 \\ j\,\text{even}}}^{n-2} 2q^{2j-1} a_{j-1} b_{j+2}.$$

Now, using that

$$2q^{2j-1} = \frac{(1-q^{2j+1})(1+q^{2j-1}) - (1+q^{2j+1})(1-q^{2j-1})}{1-q^2},$$

Equation (14) simplifies to

$$a_n + b_2 - a_{n-1} + \frac{1}{1-q} \sum_{\substack{j=2 \\ j\,\text{even}}}^{n-2} (a_{j+1} b_{j+2} - a_{j-1} b_j)$$

$$= a_n + b_2 - a_{n-1} + \frac{a_{n-1} b_n - a_1 b_2}{1-q} = a_{n-1}\left((1+q^{2n-1}) - 1 + \frac{b_n}{1-q}\right)$$

$$+ b_2\left(1 - \frac{a_1}{1-q}\right) = [2n]_q\, a_{n-1} = [2n]_q \prod_{i=1}^{n-1} (1+(-1)^i q^{2i-1}),$$

as claimed.

Finally, the generating functions $\sum_{\pi \in \mathcal{A}_n^s} (-1)^{\text{neg}(\pi)} q^{\text{fmaj}(\pi)}$ and $\sum_{\pi \in \mathcal{A}_n^s} \text{sign}(|\pi|) q^{\text{fmaj}(\pi)}$ are easily obtained by replacing q with $-q$ in Equation (10) and in Equation (11), respectively. $\qquad \square$

5. B-arc permutations

5.1. Definition and basic properties

In this section we introduce a different generalization of arc permutations to type B.

Let $\mathcal{O}_n$ be a circle with $2n$ points labeled $-1, -2, \dots, -n, 1, 2, \dots, n$ in clockwise order, as shown in Figure 1. One can think of these points as the

elements of $\mathbb{Z}_{2n}$, where for every $1 \leq j \leq n$, the letter $-j$ is identified with $n + j \in \mathbb{Z}_{2n}$.
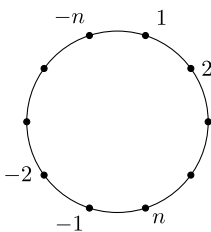

Figure 1: The circle $\mathcal{O}_n$.

Definition 5.1. *A permutation* $\pi = [\pi(1), \ldots, \pi(n)] \in B_n$ *is a B-arc permutation if, for every* $1 \leq j \leq n$, *the suffix* $\{\pi(j), \pi(j+1), \ldots, \pi(n)\}$ *forms an interval in* $\mathcal{O}_n$. *Denote by* $\mathcal{A}_n^B$ *the set of B-arc permutations in* B_n.

Example 3. We have that $[-2, 3, -1] \in \mathcal{A}_3^B$ and $[2, -1, 4, 3] \in \mathcal{A}_4^B$, but $[-3, -1, 2] \notin \mathcal{A}_3^B$ and $[5, 2, -1, 4, 3] \notin \mathcal{A}_5^B$.

Remark 5.2. While for permutations in $\mathcal{S}_n$ the suffix is an interval if and only if the prefix is an interval, this is not the case in B_n. If we replaced *suffix* with *prefix* in Definition 5.1, the corresponding formulas in Sections 5.3.2 and 5.4 would be less elegant.

Claim 5.3. *For* $n \geq 1$, $|\mathcal{A}_n^B| = n2^n$.

Proof. Writing the entries of $\pi \in \mathcal{A}_n^B$ from right to left, there are $2n$ choices for $\pi(n)$, and 2 choices for each entry thereafter, since every suffix has to be be an interval in $\mathcal{O}_n$. $\square$

5.2. Characterization by pattern avoidance

Paralleling our results for signed arc permutations, we can characterize B-arc permutations in terms of pattern avoidance.

Theorem 5.4. *A permutation* $\pi \in B_n$ *is a B-arc permutation if and only if it avoids the following* 24 *patterns:*

$$[\pm 2, 1, 3], [\pm 2, 3, 1], [\pm 3, 1, -2], [\pm 3, -2, 1], [\pm 1, 2, -3], [\pm 1, -3, 2],$$
$$[\pm 2, -1, -3], [\pm 2, -3, -1], [\pm 3, -1, 2], [\pm 3, 2, -1], [\pm 1, -2, 3], [\pm 1, 3, -2].$$

Note that the patterns listed in the above theorem are precisely those of the form $[a, b, c] \in B_3$ where b and c are at distance at least 2 in the circle $\mathcal{O}_3$.

Proof. If $\pi \in B_n$ contains an occurrence $\pi(i_1)\pi(i_2)\pi(i_3)$ of a pattern $[a, b, c] \in B_3$, where b and c are at distance at least 2 in $\mathcal{O}_3$, then the suffix $\{\pi(i_2), \pi(i_2 + 1), \ldots, \pi(n)\}$ is not an interval in $\mathcal{O}_n$, since it contains the letters $\pi(i_2)$ and $\pi(i_3)$, but neither of the letters $\pm\pi(i_1)$. Thus, $\pi \notin \mathcal{A}_n^B$.

For the converse, suppose now that $\pi \in B_n$ is not a B-arc permutation. Take the largest j such that $\{\pi(j), \pi(j + 1), \ldots, \pi(n)\}$ is not an interval in $\mathcal{O}_n$. Then $\pi(j)$ is at distance at least 2 from the interval $\{\pi(j + 1), \pi(j + 2), \ldots, \pi(n)\}$ in the circle $\mathcal{O}_n$. It follows that there is some value $1 \leq k \leq n$ such that $\pm k \notin \{\pi(j), \pi(j + 1), \ldots, \pi(n)\}$, but any interval containing $\pi(j)$ and $\pi(j+1)$ must also contain either k or $-k$. Let i be such that $\pi(i) = \pm k$, and note that $1 \leq i < j$. We claim that the subsequence $\pi(i)\pi(j)\pi(j + 1)$ is an occurrence of one of the patterns in the statement. Noticing that $\mathcal{A}_n^B$ is invariant under left multiplication by $[n, n - 1, \ldots, 1]$, we can assume without loss of generality that $|\pi(j)| < |\pi(j+1)|$. Additionally, by symmetry (reversing the signs if necessary), we can assume that $\pi(j) > 0$. Now these are the possibilities:

- if $0 < \pi(j + 1)$ then $0 < \pi(j) < k < \pi(j + 1)$, so $\pi(i)\pi(j)\pi(j + 1)$ is an occurrence of $[\pm 2, 1, 3]$;
- if $-k < \pi(j+1) < 0$ then $\pi(j) < k$, so $\pi(i)\pi(j)\pi(j+1)$ is an occurrence of $[\pm 3, 1, -2]$;
- if $\pi(j + 1) < -k$ then $\pi(j) > k$, so $\pi(i)\pi(j)\pi(j + 1)$ is an occurrence of $[\pm 1, 2, -3]$. $\qquad\square$

5.3. Canonical expressions and signed enumeration

In this subsection we characterize arc permutations and B-arc permutations in terms of their canonical expressions. This characterization is then applied to derive the unsigned and signed flag-major index enumerators.

5.3.1. The type A case For a positive integer $1 \leq m < n$ let $c_m := \sigma_m \sigma_{m-1} \cdots \sigma_1 = (m + 1, m, \ldots, 2, 1)$, in cycle notation. Every permutation $\pi \in \mathcal{S}_n$ has a unique expression

$$\pi = c_{n-1}^{k_{n-1}} c_{n-2}^{k_{n-2}} \cdots c_1^{k_1},$$

with $0 \leq k_i \leq i$ for all $1 \leq i < n - 1$. Recall from [4] that

$$(15) \qquad\qquad \text{maj}(\pi) = \sum_{i=1}^{n-1} k_i.$$

Indeed, in the above expression for π, each multiplication by c_m from the left rotates the values $1, 2, \ldots, m + 1$ cyclically. Changing the value 1 to $m + 1$ has the effect of moving a descent one position to the right, while the other descents remain unchanged.

Proposition 5.5. *A permutation $\pi \in \mathcal{S}_n$ is an arc permutation if and only if*

$$\pi = c_{n-1}^{k_{n-1}} c_{n-2}^{k_{n-2}} \cdots c_1^{k_1},$$

with $0 \leq k_{n-1} \leq n - 1$ and $k_i \in \{0, i\}$ for all $1 \leq i \leq n - 2$.

Proof. First, notice that a permutation in $\mathcal{S}_n$ is an arc permutation if and only if it may obtained by rotation of the values of a left-unimodal permutation, namely, $\pi = c_{n-1}^k u$ for some $u \in \mathcal{L}_n$. Next, a permutation $u \in \mathcal{S}_n$ is left-unimodal if and only if its inverse has descent set $\{1, 2, \ldots, j\}$ for some $1 \leq j \leq n$. Equivalently, its inverse may be obtained from a permutation whose inverse is in $\mathcal{L}_{n-1}$ by inserting the letter n at the beginning or at the end. Hence, by induction on n, we have that $u \in \mathcal{L}_n$ if and only if it has the form

$$u = c_{n-2}^{k_{n-2}} c_{n-2}^{k_{n-2}} \cdots c_1^{k_1},$$

with $k_i \in \{0, i\}$ for all $1 \leq i \leq n - 2$. $\qquad\qquad\qquad\qquad\qquad\quad$ □

The above characterization can be used to give a short algebraic proof of Theorem 2.5.

Alternate proof of Theorem 2.5. Let χ be a one-dimensional character of the symmetric group $\mathcal{S}_n$. Let $K_n := \{\mathbf{k} = (k_1, \ldots, k_{n-1}) : 0 \leq k_{n-1} \leq n - 1, \ k_i \in \{0, i\}$ for $1 \leq i \leq n - 2\}$.

By Proposition 5.5 and Equation (15),

$$\sum_{\pi \in \mathcal{A}_n} \chi(\pi) q^{\text{maj}(\pi)} = \sum_{\mathbf{k} \in K_n} \chi(c_{n-1}^{k_{n-1}} \cdots c_1^{k_1}) q^{\text{maj}(c_{n-1}^{k_{n-1}} \cdots c_1^{k_1})}$$

$$= \sum_{\mathbf{k} \in K_n} \chi(c_{n-1}^{k_{n-1}} \cdots c_1^{k_1}) q^{\sum k_i} = \sum_{\mathbf{k} \in K_n} \prod_{i=1}^{n-1} \chi(c_i)^{k_i} \, q^{\sum k_i}$$

$$= \sum_{\mathbf{k} \in K_n} \prod_{i=1}^{n-1} (\chi(c_i)q)^{k_i} = \sum_{k_{n-1}=0}^{n-1} (\chi(c_{n-1})q)^{k_{n-1}} \prod_{i=1}^{n-2} (1 + \chi(c_i)^i q^i)$$

$$= \begin{cases} [n]_q \displaystyle\prod_{i=1}^{n-1}(1 + q^i) & \text{if } \chi \text{ is the trivial character,} \\ [n]_{(-1)^{n-1}q} \displaystyle\prod_{i=1}^{n-2}(1 + (-q)^i) & \text{if } \chi \text{ is the sign character.} \end{cases} \qquad \square$$

5.3.2. The type B case In analogy with the formulas in Corollary 4.7 for signed arc permutations, in this section we give formulas enumerating B-arc permutations with respect to fmaj and each one of the four one-dimensional characters in type B.

We will use the following characterization of B-arc permutations, analogous to the characterization of arc permutations given in Proposition 5.5.

For a positive integer $0 \le m < n$ let now

$$c_m := \sigma_m \sigma_{m-1} \cdots \sigma_1 \sigma_0 = [-(m+1), 1, 2, \ldots, m, m+2, \ldots, n].$$

Note that $c_m = (m+1, m, \ldots, 1, -(m+1), -m, \ldots, -1)$ in cycle notation, and it has order $2m + 2$. Every $\pi \in B_n$ has a unique expression

$$\pi = c_{n-1}^{k_{n-1}} c_{n-2}^{k_{n-2}} \cdots c_1^{k_1} c_0^{k_0},$$

with $0 \le k_i \le 2i + 1$ for all $0 \le i < n$. Recall from [4] that

(16) $$\operatorname{fmaj}(\pi) = \sum_{i=0}^{n-1} k_i.$$

Proposition 5.6. *A permutation $\pi \in B_n$ is a B-arc permutation if and only if*

$$\pi = c_{n-1}^{k_{n-1}} c_{n-2}^{k_{n-2}} \cdots c_1^{k_1} c_0^{k_0},$$

with $0 \le k_{n-1} \le 2n - 1$ and $k_i \in \{0, 2i + 1\}$ for all $0 \le i \le n - 2$.

Proof. For every $0 \le i < n$, if $\pi \in \mathcal{A}_n^B$ is such that $\pi(j) = j$ for all $j > i$, the permutation $c_i^{2i+1}\pi = c_i^{-1}\pi$ is also a B-arc permutation. It follows by induction that $c_{n-2}^{k_{n-2}} \cdots c_0^{k_0} \in \mathcal{A}_n^B$ for all choices of $k_i \in \{0, 2i + 1\}$. Next, notice that $\mathcal{A}_n^B$ is invariant under left multiplication by c_{n-1}, since this

operation is a counterclockwise rotation of the letters in $\mathcal{O}_n$. One concludes that

$$\{c_{n-1}^{k_{n-1}} c_{n-2}^{k_{n-2}} \cdots c_0^{k_0} : 0 \le k_n \le 2n - 1 \text{ and } k_i \in \{0, 2i + 1\}$$
$$\text{for all } 0 \le i < n \} \subseteq \mathcal{A}_n^B.$$

Finally, we prove that these two sets are equal by showing that they have the same cardinality. The set on the left-hand side has size $n2^n$, because each choice of the k_i yields a different element of B_n. By Claim 5.3, this coincides with the cardinality of $\mathcal{A}_n^B$. $\square$

Product formulas for unsigned, signed and other one-dimensional character enumerators for the flag-major index follow.

Theorem 5.7. *For every $n \ge 1$,*

(17)
$$\sum_{\pi \in \mathcal{A}_n^B} q^{\mathrm{fmaj}(\pi)} = [2n]_q \prod_{i=1}^{n-1} (1 + q^{2i-1}),$$

$$\sum_{\pi \in \mathcal{A}_n^B} \mathrm{sign}(\pi) q^{\mathrm{fmaj}(\pi)} = [2n]_{(-1)^n q} \prod_{i=1}^{n-1} (1 + (-1)^i q^{2i-1}),$$

$$\sum_{\pi \in \mathcal{A}_n^B} (-1)^{\mathrm{neg}(\pi)} q^{\mathrm{fmaj}(\pi)} = [2n]_{-q} \prod_{i=1}^{n-1} (1 - q^{2i-1}),$$

$$\sum_{\pi \in \mathcal{A}_n^B} \mathrm{sign}(|\pi|) q^{\mathrm{fmaj}(\pi)} = [2n]_{(-1)^{n-1} q} \prod_{i=1}^{n-1} (1 + (-1)^{i-1} q^{2i-1}).$$

Proof. Let χ be a one-dimensional character of B_n. Let $K_n' := \{\mathbf{k} = (k_0, k_1, \ldots, k_{n-1}) : 0 \le k_{n-1} \le 2n - 1, \ k_i \in \{0, 2i + 1\}$ for $0 \le i \le n - 2\}$.
By Proposition 5.6 and Equation (16),

$$\sum_{\pi \in \mathcal{A}_n^B} \chi(\pi) q^{\mathrm{fmaj}(\pi)} = \sum_{\mathbf{k} \in K_n'} \chi(c_{n-1}^{k_{n-1}} \cdots c_1^{k_1}) q^{\mathrm{fmaj}(c_{n-1}^{k_{n-1}} \cdots c_0^{k_0})}$$

$$= \sum_{\mathbf{k} \in K_n'} \chi(c_{n-1}^{k_{n-1}} \cdots c_0^{k_0}) q^{\sum k_i} = \sum_{\mathbf{k} \in K_n'} \prod_{i=0}^{n-1} \chi(c_i)^{k_i} q^{\sum k_i}$$

$$= \sum_{\mathbf{k} \in K_n'} \prod_{i=0}^{n-1} (\chi(c_i) q)^{k_i} = \sum_{k_{n-1}=0}^{2n-1} (\chi(c_{n-1}) q)^{k_{n-1}} \prod_{i=0}^{n-2} (1 + \chi(c_i) q^{2i+1}). \quad \square$$

Remark 5.8. A characterization similar to Proposition 5.6 holds for signed arc permutations. Letting t_n be the reflection $[1, 2, \ldots, n-1, -n]$, one can show that a permutation $\pi \in B_n$ is a signed arc permutation if and only if

$$\pi = (c_{n-1}c_0)^{k_n} t_n^{k_{n-1}} c_{n-2}^{k_{n-2}} \cdots c_1^{k_1} c_0^{k_0},$$

with $0 \leq k_n \leq n-1$ and $k_i \in \{0, -1\}$ for all $0 \leq i \leq n-1$. Unlike in the case of B-arc permutations, we did not find this characterization helpful in computing enumerators.

5.4. The $(\mathrm{fdes}, \mathrm{fmaj})$-enumerator

Next we apply a coset analysis to calculate the bivariate $(\mathrm{fdes}, \mathrm{fmaj})$-enumerator on B-arc permutations.

Theorem 5.9. *For every $n \geq 2$,*

$$(18) \qquad \sum_{\pi \in \mathcal{A}_n^B} t^{\mathrm{fdes}(\pi)} q^{\mathrm{fmaj}(\pi)} = \frac{(1+tq)(1+tq^n)}{1-q}$$

$$\times \left((1 - tq^n) \prod_{i=1}^{n-2} (1 + t^2 q^{2i+1}) - (1-t)q \prod_{i=1}^{n-2} (1 + t^2 q^{2i+2}) \right),$$

$$(19) \qquad \sum_{\pi \in \mathcal{A}_n^B} t^{\mathrm{fdes}(\pi)} = (1+t)^3 (1+t^2)^{n-3} (1 + (n-2)t + t^2).$$

Proof. Fix n, and let $c := c_{n-1} = [-n, 1, 2, \ldots, n-1] = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_0$, which is a Coxeter element in B_n, and it has order $2n$. Recall that $\mathcal{A}_n^B$ is closed under left multiplication by c, which corresponds to shifting the values of π one position counterclockwise in $\mathcal{O}_n$. A collection of representatives of the distinct left cosets of the cyclic subgroup generated by c is given by $\{\pi \in \mathcal{A}_n^B : \pi(n) = n\}$. Denoting this set by $\widetilde{\mathcal{A}_n^B}$, we can write $\mathcal{A}_n^B$ as a disjoint union

$$\mathcal{A}_n^B = \bigcup_{j=0}^{2n-1} \{c^j \pi : \pi \in \widetilde{\mathcal{A}_n^B}\}.$$

Before proving Equation (18), we start with the case $t = 1$ to illustrate our technique. This is Equation (17), which we proved above using a different method. We first show that

$$\sum_{\pi \in \widetilde{\mathcal{A}_n^B}} q^{\text{fmaj}(\pi)} = \prod_{i=1}^{n-1} (1 + q^{2i-1}).$$

Indeed, for every $1 \le i < n$, given a suffix of $n-i$ letters, which is an interval containing n, there are two choices for the preceding letter $\pi(i)$: positive and maximal among the remaining letters, or negative and minimal. In the first case, $\pi(i-1)$ must be smaller than $\pi(i)$ and the contribution to the flag-major index is zero. In the second case, since $\pi(i)$ is negative and minimal among the remaining letters, $i-1$ must be a descent, and the contribution to the flag-major index is $2(i-1)+1$.

It is easy to verify that for every $\pi \in \mathcal{A}_n^B$ and $0 \le j < 2n$,

(20)
$$\text{fmaj}(c^j \pi) = \text{fmaj}(\pi) + j.$$

One concludes that

$$\sum_{\pi \in \mathcal{A}_n^B} q^{\text{fmaj}(\pi)} = [2n]_q \sum_{\pi \in \widetilde{\mathcal{A}_n^B}} q^{\text{fmaj}(\pi)},$$

which implies Equation (17).

Refining the above argument, we can enumerate permutations $\pi \in \widetilde{\mathcal{A}_n^B}$ with $\pi(1) > 0$ according to the descent set, the value of $\pi(1)$, and $\text{neg}(\pi)$ as follows:

$$\sum_{\{\pi \in \widetilde{\mathcal{A}_n^B} \colon \pi(1)>0\}} \mathbf{x}^{\text{Des}(\pi)} y^{\pi(1)} z^{\text{neg}(\pi)}$$

$$= y^{n-1} \left(x_{n-2}z + \frac{1}{y} \right) \left(x_{n-3}z + \frac{1}{y} \right) \dots \left(x_1 z + \frac{1}{y} \right) = y \prod_{i=1}^{n-2} (1 + x_i y z).$$

To see this, let $2 \le i < n$, and suppose that the entries $\pi(i+1), \pi(i+2), \dots, \pi(n)$ have been chosen, forming an interval in $\mathcal{O}_n$ containing n. Suppose that this interval is bounded by $-k < 0$ and $m > 0$. There are two choices for the entry $\pi(i)$, namely $-k-1$ and $m-1$. If $\pi(i) = -k-1$, then $\pi(i-1)\pi(i)$ will be a descent, regardless of how $\pi(i-1)$ is chosen, and additionally $\pi(i)$ contributes to $\text{neg}(\pi)$. On the other hand, if $\pi(i) = m-1$, then $\pi(i-1)\pi(i)$ will not be a descent. Finally, there is only one choice for $\pi(1)$ once $\pi(2), \pi(3), \dots, \pi(n)$ have been chosen, since $\pi(1) > 0$, and its value will be $n-1$ minus the number of indices $2 \le i < n$ for which the positive choice for $\pi(i)$ has been made.

Similarly, for permutations $\pi \in \widetilde{\mathcal{A}_n^B}$ with $\pi(1) < 0$, we get

$$\sum_{\{\pi \in \widetilde{\mathcal{A}_n^B}:\, \pi(1)<0\}} \mathbf{x}^{\mathrm{Des}(\pi)} y^{|\pi(1)|} z^{\mathrm{neg}(\pi)} = yz \prod_{i=1}^{n-2}(1 + x_i yz),$$

and so

$$\sum_{\pi \in \widetilde{\mathcal{A}_n^B}} \mathbf{x}^{\mathrm{Des}(\pi)} y^{|\pi(1)|} z^{\mathrm{neg}(\pi)} u^{\delta(\pi(1)<0)} = y(1 + uz) \prod_{i=1}^{n-2}(1 + x_i yz).$$

Making the substitutions $x_i = t^2 q^{2i}$, $z = q$ and $u = t$, we obtain

$$P(t,q,y) := \sum_{\pi \in \widetilde{\mathcal{A}_n^B}} t^{\mathrm{fdes}(\pi)} q^{\mathrm{fmaj}(\pi)} y^{|\pi(1)|} = y(1 + tq) \prod_{i=1}^{n-2}(1 + yt^2 q^{2i+1}).$$

Given $\pi \in \widetilde{\mathcal{A}_n^B}$ with $\pi(1) = a > 0$, let us analyze the values of fdes on the coset $\{c^j \pi : 0 \le j < 2n\}$. To see how fdes changes when multiplying by c, note that $\mathrm{des}(c\sigma) = \mathrm{des}(\sigma)$ unless $\sigma(1) = -1$, in which case $\mathrm{des}(c\sigma) = \mathrm{des}(\sigma) + 1$, or $\sigma(n) = -1$, in which case $\mathrm{des}(c\sigma) = \mathrm{des}(\sigma) - 1$. Thus,

$$\mathrm{des}(c^j \pi) = \begin{cases} \mathrm{des}(\pi) & \text{if } 0 \le j < n + a, \\ \mathrm{des}(\pi) + 1 & \text{if } n + a \le j < 2n. \end{cases}$$

Since $c^j \pi(1) < 0$ precisely for $a \le j < n + a$, it follows that

$$(21) \qquad \mathrm{fdes}(c^j \pi) = \begin{cases} \mathrm{fdes}(\pi) & \text{if } 0 \le j < a, \\ \mathrm{fdes}(\pi) + 1 & \text{if } a \le j < n + a, \\ \mathrm{fdes}(\pi) + 2 & \text{if } n + a \le j < 2n. \end{cases}$$

Similarly, given $\pi \in \widetilde{\mathcal{A}_n^B}$ with $\pi(1) = a < 0$, we have

$$\mathrm{des}(c^j \pi) = \begin{cases} \mathrm{des}(\pi) & \text{if } 0 \le j < a, \\ \mathrm{des}(\pi) + 1 & \text{if } a \le j < n + a, \\ \mathrm{des}(\pi) & \text{if } n + a \le j < 2n, \end{cases}$$

and since $c^j \pi(1) < 0$ precisely when $0 \le j < a$ or $n + a \le j < 2n$, the same Formula (21) for $\mathrm{fdes}(c^j \pi)$ holds.

Using Equations (21) and (20), we see that if the contribution of $\pi \in \widetilde{\mathcal{A}_n^B}$ to the generating function $P(t, q, y)$ is $t^{\mathrm{fdes}(\pi)} q^{\mathrm{fmaj}(\pi)} y^{|\pi(1)|} = t^d q^m y^a$, then the contribution of the coset $\{c^j \pi : 0 \leq j < 2n\}$ to the generating function $\sum_{\pi \in \mathcal{A}_n^B} t^{\mathrm{fdes}(\pi)} q^{\mathrm{fmaj}(\pi)}$ is

$$t^d q^m (1 + q + \cdots + q^{a-1} + tq^a + tq^{a+1} + \cdots$$
$$+ tq^{n+a-1} + t^2 q^{n+a} + t^2 q^{n+a+1} + \cdots + t^2 q^{2n-1})$$
$$= t^d q^m ([a]_q + tq^a [n]_q + t^2 q^{n+a} [n-a]_q)$$
$$= t^d q^m \frac{1 - t^2 q^{2n} - (1-t)(1 + tq^n) q^a}{1 - q}.$$

It follows that

$$\sum_{\pi \in \mathcal{A}_n^B} t^{\mathrm{fdes}(\pi)} q^{\mathrm{fmaj}(\pi)}$$
$$= \frac{(1 - t^2 q^{2n}) P(t, q, 1) - (1-t)(1 + tq^n) P(t, q, q)}{1 - q} = \frac{(1 + tq)(1 + tq^n)}{1 - q}$$
$$\times \left((1 - tq^n) \prod_{i=1}^{n-2} (1 + t^2 q^{2i+1}) - (1-t) q \prod_{i=1}^{n-2} (1 + t^2 q^{2i+2}) \right),$$

proving Equation (18).

When $q = 1$, it is easy to realize that if the contribution of a permutation $\pi \in \widetilde{\mathcal{A}_n^B}$ to $P(t, 1, y)$ is $t^d y^a$, then the contribution of the coset $\{c^j \pi : 0 \leq j < 2n\}$ to $\sum_{\pi \in \mathcal{A}_n^B} t^{\mathrm{fdes}(\pi)}$ is

$$t^d (a + tn + t^2 (n - a)) = t^d (a(1 - t^2) + nt(1 + t)).$$

It follows that

$$\sum_{\pi \in \mathcal{A}_n^B} t^{\mathrm{fdes}(\pi)} = (1 - t^2) \frac{\partial}{\partial y} P(t, 1, y) \bigg|_{y=1} + nt(1 + t) P(t, 1, 1)$$
$$= (1 - t^2)(1 + t)(1 + t^2)^{n-3} (1 + (n-1)t^2)$$
$$+ nt(1 + t)^2 (1 + t^2)^{n-2}$$
$$= (1 + t)^2 (1 + t^2)^{n-3} \left((1 - t)(1 + (n-1)t^2) + nt(1 + t^2) \right)$$
$$= (1 + t)^3 (1 + t^2)^{n-3} (1 + (n-2)t + t^2),$$

proving Equation (19). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

5.5. The descent set enumerator

In this subsection we apply a descent-set-preserving map to reduce the calculation of the descent set enumerator on B-arc permutations to the type A case.

Theorem 5.10. *For every $n \geq 2$,*

$$(22) \qquad \sum_{\pi \in \mathcal{A}_n^B} \mathbf{x}^{\mathrm{Des}(\pi)} = \prod_{i=1}^{n-1}(1+x_i)\left(2+n+2\sum_{i=1}^{n-2}\frac{x_i+x_{i+1}}{(1+x_i)(1+x_{i+1})}\right).$$

Proof. We show that there exists an n-to-1 descent-set-preserving map from the subset of permutations in $\mathcal{A}_n^B$ which contain the letter 1 to $\mathcal{L}_n$, and a 2-to-1 descent-set-preserving map from permutations in $\mathcal{A}_n^B$ which contain the letter -1 to $\mathcal{A}_n$.

For $1 \leq k \leq n$, denote by $B_{n,k}$ the set of permutations in $\mathcal{A}_n^B$ whose support is $-k+1, -k+2\ldots, -n, 1, \ldots, k$. Note that $\bigcup_{1 \leq k \leq n} B_{n,k}$ is the subset of permutations in $\mathcal{A}_n^B$ which contain the entry 1. Permutations $\pi \in B_{n,k}$ are determined by choosing, for $1 \leq i \leq n-1$, whether $\pi(i)$ is the largest or the smallest of the remaining entries. Clearly, $\pi(i)$ creates a descent with $\pi(i+1)$ only in the first case. It follows that

$$(23) \qquad \sum_{\pi \in B_{n,k}} \mathbf{x}^{\mathrm{Des}(\pi)} = \prod_{i=1}^{n-1}(1+x_i),$$

which, as shown in the proof of Theorem 2.3, coincides with the descent set enumerator on $\mathcal{L}_n$. In fact, this construction gives a natural descent-set-preserving bijection from $B_{n,k}$ to $\mathcal{L}_n$, and thus an n-to-1 descent-set-preserving map from permutations in $\mathcal{A}_n^B$ which contain 1 to $\mathcal{L}_n$.

Next we describe a 2-to-1 descent-set-preserving map from permutations in $\mathcal{A}_n^B$ which contain -1 to $\mathcal{A}_n$. The image of π is simply defined to be $|\pi|$, that is, the permutation obtained by forgetting the signs. It is easy to check that $|\pi| \in \mathcal{A}_n$ and that this map preserves the descent set.

To see that it is a 2-to-1 map, we show that each permutation $[a_1, a_2, \ldots, a_n] \in \mathcal{A}_n$ has exactly two preimages. If $a_1 \neq 1$, the preimages are $[a_1, a_2', a_3', \ldots, a_n']$ and $[-a_1, a_2', a_3', \ldots, a_n']$, where

$$a_i' = \begin{cases} a_i & \text{if } a_i > a_1, \\ -a_i & \text{otherwise.} \end{cases}$$

If $a_1 = 1$, the preimages are $[-1, a_2, \ldots, a_n]$ and $[-1, -a_2, \ldots, -a_n]$.

Combining Equation (1), which gives the distribution of the descent set on $\mathcal{A}_n$, with Equation (23), we conclude that

$$\sum_{\pi \in \mathcal{A}_n^B} \mathbf{x}^{\mathrm{Des}(\pi)} = n \sum_{\pi \in \mathcal{L}_n} \mathbf{x}^{\mathrm{Des}(\pi)} + 2 \sum_{\pi \in \mathcal{A}_n} \mathbf{x}^{\mathrm{Des}(\pi)}$$

$$= n \prod_{i=1}^{n-1} (1 + x_i) + 2 \prod_{i=1}^{n-1} (1 + x_i) \left(1 + \sum_{i=1}^{n-2} \frac{x_i + x_{i+1}}{(1 + x_i)(1 + x_{i+1})} \right),$$

which equals the right-hand side of Equation (22). $\square$

6. Final remarks and open problems

Comparing Theorem 4.5 with Theorem 5.10, we see that the descent set has different distributions on $\mathcal{A}_n^s$ and $\mathcal{A}_n^B$. However, combining Theorem 5.7 with Corollary 4.7, we obtain the following equidistribution phenomena. It would be natural to look for bijective proofs.

Corollary 6.1.   *1. For every $n \geq 1$,*

$$\sum_{\pi \in \mathcal{A}_n^B} q^{\mathrm{fmaj}(\pi)} = \sum_{\pi \in \mathcal{A}_n^s} q^{\mathrm{fmaj}(\pi)}.$$

2. For every even $n \geq 2$,

$$\sum_{\pi \in \mathcal{A}_n^B} \mathrm{sign}(\pi) q^{\mathrm{fmaj}(\pi)} = \sum_{\pi \in \mathcal{A}_n^s} \mathrm{sign}(\pi) q^{\mathrm{fmaj}(\pi)}.$$

Signed arc permutations and B-arc permutations have further properties analogous to those of unsigned arc permutations. In particular, both sets carry affine Weyl group actions, interesting underlying graph structures, and descent-set-preserving maps to standard Young tableaux. Whereas the definition of B-arc permutations is more natural and gives rise to a nicer underlying graph structure, signed arc permutations have a finer joint distribution of the descent set and the set of negative entries, which leads to interesting quasi-symmetric functions of type B to be discussed in a forthcoming paper.

We conclude by mentioning a natural direction in which our work could be extended. The flag-major index and flag-descent number have been generalized to classical complex reflection groups in [11, 9, 6, 23].

Problem 6.2. *Generalize the concept of arc permutations to the complex reflection group $G(r, p, n)$.*

Finding elegant descent enumerators on these generalized arc permutations may serve as an indicator of a "correct" generalization. It should be noted that natural analogues of Equation (15) and Proposition 5.5 hold on wreath products $G(r,1,n) = \mathbb{Z}_r \wr \mathcal{S}_n$. Thus, enumerators on B-arc permutations could be generalized to all one-dimensional character enumerators for the flag-major index on these sets.

A more challenging task is to find a unified abstract generalization of arc permutations to all Coxeter groups, including affine as well as exceptional types.

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