Journal of Combinatorics
Volume 5, Number 3, 355–378, 2014

2-adic partial Stirling functions and their zeros

Donald M. Davis

Let $P_n(x) = \frac{1}{n!}\sum\binom{n}{2i+1}(2i+1)^x$. This extends to a continuous function on the 2-adic integers, the nth 2-adic partial Stirling function. We show that $(-1)^{n+1}P_n$ is the only 2-adically continuous approximation to $S(x,n)$, the Stirling number of the second kind. We present extensive information about the zeros of P_n, for which there are many interesting patterns. We prove that if $e \geq 2$ and $2^e+1 \leq n \leq 2^e+4$, then P_n has exactly 2^{e-1} zeros, one in each mod 2^{e-1} congruence. We study the relationship between the zeros of $P_{2^e+\Delta}$ and P_Δ, for $1 \leq \Delta \leq 2^e$, and the convergence of $P_{2^e+\Delta}(x)$ as $e \to \infty$.

AMS 2000 subject classifications: Primary 11B73, 05A99.
Keywords and phrases: Stirling number, 2-adic integers.

1. Introduction

The numbers

$$T_n(x) := \sum_{j \text{ odd}} \binom{n}{j} j^x$$

were called partial Stirling numbers in [12], and this terminology (with varying notation) was continued in [3], [6], [7], and [15]. Although our results can no doubt be adapted to odd-primary results, we focus entirely on the prime 2 for simplicity. The 2-exponents $\nu(T_n(x))$ are important in algebraic topology ([1], [4], [9], [10], [13]). Here and throughout, $\nu(-)$ denotes the exponent of 2 in an integer or rational number or 2-adic integer.

Since the Stirling numbers of the second kind satisfy

$$S(x,n) = \tfrac{1}{n!}\sum(-1)^{n-j}\binom{n}{j}j^x, \quad x \geq 0,$$

it would seem more reasonable to call

$$P_n(x) := \tfrac{1}{n!}\sum_{j \text{ odd}} \binom{n}{j} j^x \tag{1.1}$$

arXiv: 1402.0433

the partial Stirling numbers, defined for any integer x. Of course, information about either $T_n(x)$ or $P_n(x)$ is easily transformed into information about the other. We prefer to work with $P_n(x)$ because of its closer relationship with the Stirling numbers and because of

Proposition 1.2. *For any integer x, $\nu(P_n(x)) \geq 0$ with equality iff $\binom{2x-n-1}{n-1}$ is odd.*

This implies, of course, that $\nu(T_n(x)) \geq \nu(n!)$, which is fine, but less elegant. Proposition 1.2 follows easily from the known similar result for $S(x,n)$ when $x \geq n$, that $\nu((-1)^{n+1}P_n(x) - S(x,n)) \geq x - \nu(n!)$, and periodicity of P_n given in the next proposition, which we will prove in Section 4.

Proposition 1.3. *Let $\lg(n) = [\log_2(n)]$. For all integers x,*

$$P_n(x + 2^t) \equiv P_n(x) \mod 2^{t+1-\lg(n)}.$$

An immediate consequence is

Corollary 1.4. *P_n extends to a continuous function $\mathbf{Z}_2 \to \mathbf{Z}_2$, where $\mathbf{Z}_2$ denotes the 2-adic integers, with the usual 2-adic metric $d(x,y) = 1/2^{\nu(x-y)}$.*

This was pointed out by Clarke in [3], where he also noted that the function P_n is analytic on $2\mathbf{Z}_2 + \varepsilon$, $\varepsilon \in \{0,1\}$. We call P_n a *partial Stirling function*.

In [5], the author proved that there exist 2-adic integers x_0 and x_1 such that $\nu(P_5(2x)) = \nu(x - x_0)$ and $\nu(P_5(2x+1)) = \nu(x - x_1)$ for all $x \in \mathbf{Z}_2$, and in [3], Clarke noted that $2x_0$ and $2x_1+1$ should be thought of as 2-adic zeros of the function P_5, and these are the only two zeros of P_5 on $\mathbf{Z}_2$. Recently, in [7], the author showed that this sort of behavior occurs frequently for the functions P_n restricted to certain congruence classes. In this paper, we will continue this investigation of the zeros of P_n. Related to this, we will also discuss $\lim_{e\to\infty} P_{2^e+\Delta}(x)$ for fixed $\Delta > 0$.

Next we compare with similar notions for the actual Stirling numbers of the second kind. There are results ([2], [11]) somewhat similar to our Proposition 1.3 saying

$$S(x + 2^t, n) \equiv S(x,n) \mod 2^{\min(t+1-\lg(n), x-\nu(n!))}$$

if $x \geq n$. Since, if $n \ll x \ll t$,

$$\begin{aligned} \nu(S(x+2^t,n) - S(x,n)) &= \nu\left(\tfrac{1}{n!}\sum(-1)^{n-j}\tbinom{n}{j}j^x(j^{2^t}-1)\right) \\ &= \nu\left(\tfrac{1}{n!}\tbinom{n}{2}2^x\right) = x - 1 - \nu((n-2)!), \end{aligned}$$

we conclude that $x \mapsto S(x,n)$ is not continuous in the 2-adic metric on any domain containing arbitrarily large x. Our partial Stirling function $(-1)^{n+1}P_n$ is the only 2-adically continuous approximation to $S(-,n)$, which is made precise in the following result.

Proposition 1.5. *For all $x \geq n \geq 1$, $(-1)^{n+1}P_n(x) \equiv S(x,n) \bmod 2^{x-\nu(n!)}$. Moreover $(-1)^{n+1}P_n$ is the only continuous function $f : \mathbf{Z}_2 \to \mathbf{Z}_2$ for which there exists an integer c satisfying that for all $x \geq n$, $f(x) \equiv S(x,n)$ mod 2^{x-c}.*

Proof. The first part is true since

$$(-1)^{n+1}P_n(x) - S(x,n) = (-1)^{n+1}\sum \tbinom{n}{2j}(2j)^x/n!.$$

For the second part, we have, for any positive integers x and L with L sufficiently large,

$$\begin{aligned} & d(f(x+2^L), (-1)^{n+1}P_n(x+2^L)) \\ \leq\ & d(f(x+2^L), S(x+2^L,n)) + d(S(x+2^L,n), (-1)^{n+1}P_n(x+2^L)) \\ \leq\ & 1/2^{x+2^L-c} + 1/2^{x+2^L-\nu(n!)}, \end{aligned}$$

which approaches 0 as $L \to \infty$. Thus $f(x) = (-1)^{n+1}P_n(x)$ since both functions are continuous. Since positive integers are dense in $\mathbf{Z}_2$, $f = (-1)^{n+1}P_n$ on $\mathbf{Z}_2$. $\square$

Clarke ([3]) conjectured that if, as is often the case, $\nu(P_n(x)) = \nu(x - x_0) + c_0$ for some $x_0 \in \mathbf{Z}_2$, $c_0 \in \mathbb{Z}$, and all x in a congruence class, then $\nu(S(x,n)) = \nu(x-x_0)+c_0$ on the same congruence class, provided $x \geq n$, and that moreover $\nu(S(x,n)) = \nu(P_n(x))$ for all integers $x \geq n$. He pointed out the difficulty of proving this, which can be thought of as the possibility that x_0 might contain extraordinarily long strings of zeros in its binary expansion. This will be discussed in more detail after (2.11).

2. Main theorems

In [7], we showed that for $e \geq 2$, the functions P_{2^e+1} and P_{2^e+2} have exactly 2^{e-1} zeros, one in each mod 2^{e-1} congruence class. One of our main new results is to extend this to P_{2^e+3} and P_{2^e+4}. We will prove the following result in Section 4.

Theorem 2.1. *Let $1 \le \Delta \le 4$, $e \ge 2$, $0 \le p < 2^{e-1}$, and p_2 the mod-2 reduction of p. There exists $x_{e,\Delta,p} \in \mathbf{Z}_2$ such that for all integers x*
(2.2)
$$\nu(P_{2^e+\Delta}(2^{e-1}x+p)) = \nu(x-x_{e,\Delta,p}) + \begin{cases} 2 & \text{if } (\Delta,p_2)=(3,0) \text{ or } (4,1),\ e=2 \\ 1 & \text{if } (\Delta,p_2)=(3,0) \text{ or } (4,1),\ e>2 \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 2.3. *If $1 \le \Delta \le 4$ and $e \ge 2$, the function $P_{2^e+\Delta}$ has exactly 2^{e-1} zeros on $\mathbf{Z}_2$, given by the 2-adic integers $2^{e-1}x_{e,\Delta,p}+p$ for $0 \le p < 2^{e-1}$.*

It is easy to see that, as noted in [5], P_n has no zeros if $1 \le n \le 4$. Corollary 2.3 says that P_n has 2 (resp. 4) zeros for $5 \le n \le 8$ (resp. $9 \le n \le 12$). In Section 3, we discuss patterns in the zeros of P_n, extending work in [5]. We have located all the zeros of P_n for $n \le 101$, and present the results for $n \le 64$ in Tables 3.4 and 3.7. The number of zeros of P_n appears to equal, with several exceptions,

$$2\left[\frac{n-1}{4}\right] + \begin{cases} -2 & n \equiv 13\ (16) \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

For $n \le 101$, the exceptions are that the number of zeros of P_n is two less than that given in (2.4) if $n = 21, 71$, or 90. This is a tantalizing aspect of this study—patterns appear, leading perhaps to conjectures, but then there are exceptions. The most striking example of this is that we were conjecturing that if $1 \le \Delta \le 2^e$, then $P_{2^e+\Delta}$ has exactly one zero in every mod 2^{e-1} congruence class that does not contain a zero of P_Δ. This fails only once for $2^e + \Delta \le 101$: for $x \equiv 4$ mod 16, P_{53} has three zeros, while P_{21} has none. The zeros of $P_{2^e+\Delta}$ in mod 2^{e-1} congruence classes in which P_Δ has zeros are somewhat more complicated, although usually $P_{2^e+\Delta}$ has two zeros in such mod 2^{e-1} classes. We will discuss this in Section 3.

Next we describe another approach related to the zeros of $P_{2^e+\Delta}$. We begin with a simple lemma, which was proved in [8]. Let $\mathrm{U}(n) = n/2^{\nu(n)}$ denote the odd part of n.

Lemma 2.5. *For all $e \ge 1$, $\mathrm{U}(2^{e-1}!) \equiv \mathrm{U}(2^e!) \mod 2^e$.*

Thus there is a well-defined element $\mathrm{U}(2^\infty!) := \lim U(2^e!)$ in $\mathbf{Z}_2$. Its backwards binary expansion begins $1101000101101\cdots$.

The following theorem will be proved in Section 5.

Theorem 2.6. *For $x \ge 0$ and $0 \le \Delta < 2^e$,*

$$P_{2^e+\Delta}(x) \equiv \tfrac{1}{\mathrm{U}(2^e!)}\tfrac{1}{\Delta!}\sum_{j=0}^{\Delta}\tbinom{\Delta}{j}j^x \mod 2^{e-\max(\lg(x-\Delta)+1,\lg(\Delta)-1)}. \tag{2.7}$$

Here we ignore $\lg(x-\Delta)$ if $x-\Delta \le 0$ (or call it $-\infty$).

This has as an immediate corollary that the 2-adic limit of (2.7), as $e \to \infty$, equals the RHS of the following:

$$P_{2^\infty+\Delta}(x) := \lim_{e\to\infty} P_{2^e+\Delta}(x) = \tfrac{1}{\mathrm{U}(2^\infty!)}\tfrac{1}{\Delta!}\sum_{j=0}^{\Delta}\tbinom{\Delta}{j}j^x, \quad x \ge 0. \tag{2.8}$$

The novelty here is that we have defined a function, at least for positive integers x, of the form $P_{2^\infty+\Delta}(x)$ and then related it to the finite sum $\sum_{j=0}^{\Delta}\binom{\Delta}{j}j^x$.

We now explain the relevance of (2.8) to the zeros of P_n. Note that the RHS of (2.8) is a sum over all j, not just odd j. Since $S(x,n)=0$ when $x<n$ (and $S(x,n)$ is the difference of the sum over odd j and the sum over even j), we have

$$\tfrac{1}{n!}\sum_{j=0}^{n}\tbinom{n}{j}j^x = \tfrac{2}{n!}\sum_{j \text{ odd}}\tbinom{n}{j}j^x \quad \text{if } 0 \le x < n. \tag{2.9}$$

On the other hand, if $x \ge n$, then

$$\tfrac{1}{n!}\sum_{j \text{ odd}}\tbinom{n}{j}j^x \equiv \tfrac{1}{n!}\sum_{j=0}^{n}\tbinom{n}{j}j^x \mod 2^{x-\nu(n!)}. \tag{2.10}$$

It is likely that, as a consequence of (2.10), we have

$$\nu\big(\tfrac{1}{\Delta!}\sum_{j=0}^{\Delta}\tbinom{\Delta}{j}j^x\big) = \nu\big(\tfrac{1}{\Delta!}\sum_{j \text{ odd}}\tbinom{\Delta}{j}j^x\big) = \nu(P_\Delta(x)) = \nu(x-x_0)+c_0 \quad \text{if } x \gg \Delta \tag{2.11}$$

for x in a congruence class for which the last equality holds for some $x_0 \in \mathbf{Z}_2$. That it is only "likely" is due to the possibility that it might conceivably happen that the zero x_0 of P_Δ satisfies that

$$\nu(x_0-A) \ge A \tag{2.12}$$

for some large integer A. This refers to a long string of zeros in the binary expansion of x_0 mentioned at the end of Section 1. Then the inequality

$$\nu(\tfrac{1}{\Delta!}\sum_{j=0}^{\Delta}\tbinom{\Delta}{j}j^A - P_\Delta(A)) \geq A - \nu(\Delta!)$$

implied by (2.10) would not be sufficient to deduce from $\nu(P_\Delta(A)) = \nu(A - x_0) + c_0$ that $\nu(\frac{1}{\Delta!}\sum_{j=0}^{\Delta}\binom{\Delta}{j}j^A) = \nu(A - x_0) + c_0$, as desired. The situation (2.12) would have to happen infinitely often in x_0 to create a real problem.

Assuming (2.11), it would follow from (2.8) that the zeros of $P_{2^\infty+\Delta}$ are exactly those of P_Δ. Unfortunately, this does not give information about the zeros of $P_{2^e+\Delta}$, since the convergence in (2.8) is not uniform. Nevertheless, it is interesting that for all positive integers x, the sequence $P_{2^e+\Delta}(x)$ converges in $\mathbf{Z}_2$ as $e \to \infty$. This leads one to wonder whether the same thing is true if x is in $\mathbf{Z}_2 - \mathbb{Z}^+$. Quite possibly, the answer is that a variant of this is true iff x is rational.

Our investigation of this has been focused primarily on the case $\Delta = 1$, but we anticipate similar results for any $\Delta > 0$. Our main conjecture here is as follows. Throughout the following, if $x \in \mathbf{Z}_2$, we let x_i denote the 2^i-bit of x; i.e. $x = \sum_{i\geq 0} x_i 2^i$ with $x_i \in \{0, 1\}$.

Conjecture 2.13. *If, for some $d \geq 2$ and $i_0 \geq 0$, $x \in \mathbf{Z}_2$ satisfies $x_{i+d} = x_i$ for all $i \geq i_0$, then for any e, $\lim_{j\to\infty} P_{2^{e+dj}+1}(x)$ exists in $\mathbf{Z}_2$.*

That is, if x is a 2-adic integer with eventual period d in its binary expansion, then the sequence of $P_{2^e+1}(x)$ as $e \to \infty$ splits into d convergent subsequences.

Table 2.15 illustrates this phenomenon. Here we deal with $z_n := 3 \cdot \sum_{i=0}^{n} 2^{3i}$ and tabulate the backwards binary expansion of $P_{2^e+1}(z_n)$ for $n \geq n_0$, as listed. We list $\nu(P_{2^e+1}(z_n) - P_{2^{e-3}+1}(z_n))$ for emphasis, although these values are clear from comparison of the 12-bit expansions.

We now state a more detailed conjecture which implies Conjecture 2.13.

Conjecture 2.14. *Suppose x is a finite element of $\mathbf{Z}_2$, and i_0 and d are positive integers such that $x_{i_0} = 0$ and $x_{i+d} = x_i$ for all $i \geq i_0$, provided $2^{i+d} \leq x$. Denote by $R(x) := \lg(x) + 1 - (i_0 + d)$ the number of repeating bits of x. Then*

$$\nu(P_{2^{e+d}+1}(x+1) - P_{2^e+1}(x+1)) \geq e - i_0,$$

provided $R(x) \geq 2(e - i_0) - 1$.

Table 2.15: $P_{2^e+1}(z_n)$ for $n \ge n_0$

e	$P_{2^e+1}(z_n)$	n_0	$\nu(P_{2^e+1}(z_n) - P_{2^{e-3}+1}(z_n))$
4	$011011101000\cdots$	4	
5	$001110101111\cdots$	4	
6	$111101001101\cdots$	5	
7	$011001110011\cdots$	5	4
8	$001110110100\cdots$	6	7
9	$111101100101\cdots$	6	6
10	$011001111001\cdots$	6	8
11	$001110110110\cdots$	7	10
12	$111101100011\cdots$	7	9
13	$011001111000\cdots$	7	11
14	$001110110110\cdots$	8	12
15	$111101100011\cdots$	8	12

Proof that Conjecture 2.14 implies Conjecture 2.13. Let x be as in Conjecture 2.13. Let $x[n] := \sum_{i\le n} x_i 2^i$, and let $Q_e := P_{2^e+1}$. We have

$$\begin{aligned} & \nu(Q_{e+d}(x) - Q_e(x)) \\ \ge\ & \min\bigl(\nu(Q_{e+d}(x) - Q_{e+d}(x[n])), \nu(Q_{e+d}(x[n]) - Q_e(x[n])), \\ & \quad \nu(Q_e(x[n]) - Q_e(x))\bigr) \\ \ge\ & \min(n+2-e-d, e-i_0, n+2-e), \end{aligned}$$

provided $n - d \ge 2e - i_0 - 2$, using Proposition 1.3 for the first and last parts. For any e, we can make this $\ge e - i_0$ by choosing n sufficiently large. Thus the sequence $\langle Q_{e+jd}(x)\rangle$ is Cauchy. $\square$

Conjecture 2.14 has been verified for $i_0 = 5$, $2 \le d \le 7$, $6 \le e \le 9$, and many values of x mod 2^{i_0}.

3. Zeros of P_n

In this section, we describe various facts about the zeros of the functions P_n. Most of these can be considered to be extensions of results of [5], but the emphasis here is on the zeros rather than divisibility.

We begin with a broad outline of our proofs, but defer most details to the following section. This outline is needed to understand certain aspects of our tabulated results.

One of our main tools is the following result, which is a slight refinement of [5, Theorem 1]. Here we use the notation that $\min'(a, b) = \min(a, b)$ if $a \ne b$, while $\min'(a, a) > a$.

Lemma 3.1 ([5])**.** *A function $f : \mathbb{Z} \to \mathbb{Z} \cup \{\infty\}$ satisfies that there exists $z_0 \in \mathbf{Z}_2$ such that $f(x) = \nu(x - z_0)$ for all integers x iff $f(0) \geq 0$ and for all $x \in \mathbb{Z}$ and all $d \geq 0$,*

$$f(x + 2^d) = \min{}'(f(x), d).$$

The difference between this and the result of [5] is that here we do not assume at the outset that $f(x) \geq 0$ for all x. As can be seen from the proof in [5], all that is required is $f(0) \geq 0$ since $z_0 = 2^{e_0} + 2^{e_1} + \cdots$ with $e_0 < e_1 < \cdots$ and $e_0 = f(0)$, $e_i = f(2^{e_0} + \cdots + 2^{e_{i-1}})$.

Corollary 3.2. *If $g : \mathbb{Z} \to \mathbb{Q} \cup \{\infty\}$ satisfies that there exists an integer c such that $\nu(g(0)) \geq c$ and, for all integers x and d with $d \geq 0$,*

$$\nu(g(x + 2^d) - g(x)) = d + c,$$

then there exists $z_0 \in \mathbf{Z}_2$ such that, for all $x \in \mathbb{Z}$,

$$\nu(g(x)) = \nu(x - z_0) + c.$$

Proof. The hypothesis implies that

$$\nu(g(x + 2^d)) = \min{}'(\nu(g(x)), d + c).$$

Apply the lemma to $f(x) = \nu(g(x)) - c$. $\square$

Let

$$\Phi_n(s) = \tfrac{1}{n!} \sum_i \tbinom{n}{2i+1} (2i)^s. \tag{3.3}$$

Since

$$\begin{aligned}
& P_n(2^{e-1}(x + 2^d) + p) - P_n(2^{e-1}x + p) \\
= \; & \tfrac{1}{n!} \sum_i \tbinom{n}{2i+1} (2i + 1)^{2^{e-1}x+p}((2i + 1)^{2^{e-1+d}} - 1) \\
= \; & \sum_{k \geq 0} \tbinom{2^{e-1}x+p}{k} \sum_{j>0} \tbinom{2^{e-1+d}}{j} \Phi_n(j + k),
\end{aligned}$$

Corollary 3.2 implies that to show

$$\nu(P_n(2^{e-1}x + p)) = \nu(x - x_0) + c$$

for some $x_0 \in \mathbf{Z}_2$, it suffices to prove that

Table 3.4: Zeros of P_n in (p mod 8), $17 \le n \le 32$

n \ p	0	1	2	3	4	5	6	7
17	○	○	○	○	○	○	○	○
18	○	○	○	○	○	○	○	○
19	○	○	•	○	○	○	•	○
20	○	•	○	•	○	•	○	•
21	○	○	○	•		○	○	• •
22	○	○	○	○	• •	• •	○	○
23	○	○	•	○	•	• •	• •	○
24	○	•	•	•	○	•	• •	• •
25	○ ○	○	○	•	○	• •	• •	• •
26	○ ○	○ ○	○	○	○	○	• •	• •
27	○ ○	○ ○	• •	○	○	○	•	• •
28	○ ○	• •	○ ○	• •	○	•	○	•
29	○ ○	○ ○	○ ○	•	• •	○	○	•
30	○ ○	○ ○	○ ○	○ ○	• •	• •	○	○
31	•	○ ○	• •	○ ○	• •	• •	• •	○
32	○	•	• •	• •	• •	• •	• •	• •

$$\nu\Big(\sum_{k\ge 0}\binom{2^{e-1}x+p}{k}\sum_{j>0}\tfrac{1}{2^d}\binom{2^{e-1+d}}{j}\Phi_n(j+k)\Big)=c \tag{3.5}$$

for all integers x and d with $d \ge 0$ (and that $\nu(P_n(p)) \ge c$). The study of (3.5) will occupy much of our effort.

Table 3.4 describes the location of the zeros of P_n for $17 \le n \le 32$. This information was given, in a different form, in [5, Table 1.3, 1.4].

We now explain the table. We temporarily refer to either a • or a ○ as a "dot." The dots in (n,p) represent the zeros of $P_n(z)$ for which $z \equiv p$ mod 8. A dot in the center of square (n,p) means that P_n has a zero of the form $8x_0 + p$ for some $x_0 \in \mathbf{Z}_2$, and that, moreover, there is an integer c such that

$$\nu(P_n(8x+p)) = \nu(x-x_0)+c \tag{3.6}$$

for all integers x. Two horizontally-displaced dots in a box mean that P_n has zeros of the form $16x_0+p$ and $16x_1+8+p$, and analogues of (3.6) hold for $\nu(P_n(16x+p))$ and $\nu(P_n(16x+8+p))$. Two vertically-displaced dots on the left side of a box mean that P_n has zeros of the form $32x_0+p$ and $32x_1+16+p$, with analogues of (3.6). The single dot on the right side of $(29,3)$ is a zero of the form $16x_0+11$.

Next we explain the difference between $\circ$ and $\bullet$ in the table. In order to prove (3.6), we would like to prove (3.5) with $e=4$. The cases indicated by a single $\circ$ are those in which, for all $k\ge 0$ and $j>0$

$$\nu\Big(\tbinom{8x+p}{k}\tfrac{1}{2^d}\tbinom{2^{d+3}}{j}\Phi_n(j+k)\Big)\ge c,$$

with equality for a unique pair (k,j). Cases with two horizontally-displaced $\circ$'s are analogous with $8x+p$ replaced by $16x+p$ and $16x+8+p$, except that here the minimum value will occur for a unique (j,k) for $16x+p$, and for three (j,k)'s for $16x+8+p$. The $\bullet$'s in the table are zeros of P_n in which some of the terms $\binom{8x+p}{k}\frac{1}{2^d}\binom{2^{d+3}}{j}\Phi_n(j+k)$ have 2-exponent smaller than that of their sum, and so more complicated combinations, involving odd factors of some terms, must be considered.

Next we present the analogue of Table 3.4 for $33\le n\le 64$. The main reason for including such a large table is to illustrate the great deal of regularity, marred by a few exceptions. After presenting the table, we will explain the aspects in which it differs from Table 3.4.

A box (n,p) in Table 3.7 with one dot on the left side and two vertically-placed dots on the right says that, in (p mod 16), P_n has zeros of the form $32x_0+p$, $64x_1+16+p$, and $64x_2+48+p$ with formulas analogous to (3.6) in each congruence class. If box (n,p) has a number 2 on its left side, P_n has two zeros in (p mod 16) of the form 2^tx_0+32+p and $2^tx_1+2^{t-1}+32+p$ for $t=9$, 7, 9, 8, if $n=41$, 45, 53, 57, resp.

In [7, Theorem 1.7], we proved a general result describing a large family of cases in which, if $e=\lg(n-1)$, $P_n(2^{e-1}x+p)$ has a single zero, due to $\nu(\binom{2^{e-1}x+p}{k}\binom{2^{e-1}}{j}\Phi_n(j+k))$ obtaining its minimum value for a unique (j,k). For $17\le n\le 64$, these are the cases where the box (n,p) in Table 3.4 or 3.7 has a single $\circ$ in the center of the box. We restate the result in a slightly simpler way here. Here we introduce the notation $\alpha(n)$ for the number of

Table 3.7: Zeros of P_n in (p mod 16), $33 \le n \le 64$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
33	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
34	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
35	○	○	•	○	○	○	•	○	○	○	•	○	○	○	•	○
36	○	•	○	•	○	•	○	•	○	•	○	•	○	•	○	•
37	○	○	○	•	•	○	○	•	○	○	○	•	• •	○	○	• •
38	○	○	○	○	• •	•	○	○	○	○	○	○	•	• •	○	○
39	○	○	•	○	•	•	•	○	○	○	•	○	•	• •	• •	○
40	○	•	•	•	○	•	•	•	○	•	•	•	○	•	• •	• •
41	○	○	○	•	○	• •	2	• •	• •	○	○	•	○	•	•	•
42	○	○	○	○	○	○	•	• •	• •	• •	○	○	○	○	• •	•
43	○	○	•	○	○	○	•	• •	• •	• •	• •	○	○	○	•	•
44	○	•	○	•	○	•	○	•	• •	• •	• •	• •	○	•	○	•
45	○	○	○	•	•	○	○	•	2	• •	• •	•	• •	○	○	•
46	○	○	○	○	•	•	○	○	• •	• •	• •	• •	• •	• •	○	○
47	○	○	•	○	•	•	•	○	•	• •	• •	• •	• •	• •	• •	○
48	○	•	•	•	•	•	•	•	○	•	• •	• •	• •	• •	• •	• •
49	○ ○	○	○	•	○	•	•	•	○	• •	• •	• •	• •	• •	• •	• •
50	○ ○	○ ○	○	○	○	○	•	•	○	○	• •	• •	• •	• •	• •	• •
51	○ ○	○ ○	• •	○	○	○	•	•	○	○	•	• •	• •	• •	• •	• •
52	○ ○	• •	○ ○	• •	○	•	○	•	○	•	○	•	• •	• •	• •	• •
53	○ ○	○ ○	○ ○	• •	• • •	○	○	2	○	○	○	•	•	• •	• •	• •
54	○ ○	○ ○	○ ○	○ ○	• •	• • •	○	○	○	○	○	○	• •	•	• •	• •
55	○ ○	○ ○	• •	○ ○	• •	• • •	• • •	○	○	○	•	○	•	•	•	• •
56	○ ○	• •	• •	• •	○ ○	• •	• • •	• • •	○	•	•	•	○	•	•	•
57	○ ○	○ ○	○ ○	• •	○ ○	• •	• •	• •	• •	○	○	•	○	• •	2	• •
58	○ ○	○ ○	○ ○	○ ○	○ ○	○ ○	• • •	• •	• •	• •	○	○	○	○	•	• •
59	○ ○	○ ○	• •	○ ○	○ ○	○ ○	• •	• •	• •	• •	○ ○	○	○	○	•	• •
60	○ ○	• •	○ ○	• •	○ ○	• •	○ ○	• •	• •	• •	• •	• •	○	•	○	•
61	○ ○	○ ○	○ ○	•	• •	○ ○	○ ○	• •	• •	• •	• •	• •	• •	○	○	•
62	○ ○	○ ○	○ ○	○ ○	• •	• •	○ ○	○ ○	• •	• •	• •	• •	• •	○ ○	○	○
63	•	○ ○	• •	○ ○	• •	• •	• •	○ ○	• •	• •	• •	• •	• •	• •	• •	○
64	○	•	• •	• •	• •	• •	• •	• •	• •	• •	• •	• •	• •	• •	• •	• •

1's in the binary expansion of n. This notation will occur frequently in our proofs, mainly due to the well-known formulas

$$\nu(n!) = n - \alpha(n) \quad \text{and} \quad \nu\tbinom{m}{n} = \alpha(n) + \alpha(m-n) - \alpha(m),$$

which we will use without comment.

Theorem 3.8 ([7, 1.7])**.** *Let $e = \lg(n-1)$ and $t = \lg(n-2^e)$. Suppose $\max(0, n-2^e-2^{e-1}) \le p < 2^{e-1}$ and $\binom{n-1-p}{p}$ is odd, and let p_0 denote the mod 2^t reduction of p. Suppose*

$$q = p + \varepsilon \cdot 2^{\nu(n)-1} + b \cdot 2^{t+1}$$

for $\varepsilon \in \{0,1\}$ and $b \ge 0$, with $q < 2^{e-1}$. Then

$$\nu\left(\binom{2^{e-1}x+q}{k}\binom{2^{e-1}}{j}\Phi_n(j,k)\right) \ge \alpha(n) - 2 - \alpha(p_0)$$

with equality iff $(j,k) = (2^{e-1}, p_0)$.

Corollary 3.9. *If n, p_0, and q are as above, then there exists $x_0 \in \mathbf{Z}_2$ such that for all integers x*

$$\nu(P_n(2^{e-1}x+q)) = \nu(x-x_0) + \alpha(n) - 2 - \alpha(p_0).$$

Hence, P_n has a unique zero, $2^{e-1}x_0 + q$, in (q mod 2^{e-1}).

Remark 3.10. The boxes in Tables 3.4 and 3.7 with a single $\circ$ in the center are all the cases in this range in which, if $e = \lg(n-1)$, $P_n(2^{e-1}x+p)$ has a single zero, due to $\nu(\binom{2^{e-1}x+p}{k}\binom{2^{e-1}}{j}\Phi_n(j+k))$ obtaining its minimum value for a unique (j,k). All of these cases fit into families that work for all e. However, there are several of these which are not covered by Theorem 3.8. When $n = 2^e+1$ or 2^e+2, not all values of p are handled by Theorem 3.8, but they are handled by Theorem 2.1. Also the case $(n,p) = (2^e+2^e, 0)$ is not covered by Theorem 3.8, but it is easily proved, using Proposition 4.1, that if $j > 0$

$$\nu\big(\tbinom{2^{e-1}x}{k}\tbinom{2^{e-1}}{j}\Phi_{2^{e+1}}(j+k)\big) \ge 0 \text{ with equality iff} (j,k) = (2^{e-1}, 0),$$

and hence $P_{2^{e+1}}$ has a single zero in (0 mod 2^{e-1}). We conjecture that, for all e and n with $\lg(n-1) = e$, these families together provide all cases in which $P_n(2^{e-1}x+p)$ has a single zero due to a single (j,k).

A similar result describes cases in which (with $e = \lg(n-1)$) a mod 2^{e-1} class splits into two mod-2^e classes with each having a single zero of P_n, the first of the two determined by a unique (j,k) and the second by three (j,k)'s. These are represented in Tables 3.4 and 3.7 by boxes with two horizontally-displaced $\circ$'s.

Theorem 3.11. *Let $3\cdot 2^{e-1} < n < 2^{e+1}$. Suppose $0 \le p \le [(n-3\cdot 2^{e-1})/2]$, $(n,p) \ne (2^{e+1}-1,0)$, and $\binom{n-1-p}{p}$ is odd. Let $\ell = \lg(2^{e+1}-(n-p))$. If $q = p+\varepsilon\cdot 2^{\nu(n)-1}$ for $\varepsilon \in \{0,1\}$, then, if $\delta \in \{0,1\}$, $x \in \mathbb{Z}$, $k \ge 0$, and $j > 0$,*

$$\nu\left(\binom{2^e x + \delta 2^{e-1} + q}{k}\binom{2^e}{j}\Phi_n(j+k)\right) \ge \alpha(n) - 1 - \alpha(p) \tag{3.12}$$

with equality iff $(j,k) = (2^e - 2^\ell, p)$ or $\delta = 1$ and

$$(j,k) \in \{(2^{e-1}-2^\ell, 2^{e-1}+p),\ (2^{e-1}, 2^{e-1}+p-2^\ell)\}.$$

Corollary 3.13. *Let n, p, q and δ be as above. Then there exists a 2-adic integer x_δ such that for all integers x*

$$\nu(P_n(2^e x + 2^{e-1}\delta + q)) = \nu(x - x_\delta) + \alpha(n) - 1 - \alpha(p).$$

Hence P_n has a unique zero, $2^e x_\delta + 2^{e-1}\delta + q$, in $(2^{e-1}\delta + q \bmod 2^e)$.

There are two types of proofs which we require, both involving the verification of (3.5). One is to establish the zeros of P_n for a specific n and specific congruence class, as are presented in Tables 3.4 and 3.7. The other is to prove general results, such as Theorems 2.1 and 3.11, which apply to infinitely many values of n. The first of these types can be accomplished using `Maple`, using Proposition 3.14 to limit the set of values of (j,k) that need to be checked. This will be discussed in the remainder of this section. The second type involves using general results about $\nu(\Phi_n(s))$; this will be discussed in Section 4.

The following useful result is a restatement of [10, Theorem 3.4].

Proposition 3.14. *If n and s are nonnegative integers, then $\nu(\Phi_n(s)) \ge s - [n/2]$.*

The results in Tables 3.4 and 3.7 are discovered by having `Maple` compute the numbers $\nu(P_n(2^{e-1}x+p))$ for many values of x. We illustrate the process of discovery and proof with two examples.

Even though (almost) all the $\circ$'s in the tables are proved in Corollaries 3.9 and 3.13, we briefly sketch how one is discovered and proved. We consider the box $(n,p)=(29,2)$, which contains a double $\circ$. Indeed, P_{29} turns out to have two zeros in (2 mod 8), one in 2 mod 16, and one in 10 mod 16. The likelihood of this is seen when `Maple` computes $\nu(P_{29}(16x+2))$ for consecutive values of x, and the values are 2, 3, 2, 4, 2, 3, 2, 5, 2, ..., which is the pattern of $\nu(x-x_0)+2$, and a similar result is seen for $\nu(P_{29}(16x+10))$. To prove that $\nu(P_{29}(16x+10))=\nu(x-x_0)+2$ for some 2-adic integer x_0, it suffices to show, by the argument leading to (3.5), that for all $j>0$ and $k\ge 0$

$$\nu\tbinom{16x+10}{k}+4-\nu(j)+\nu(\Phi_{29}(j+k))\ge 2 \tag{3.15}$$

with equality occurring for an odd number of pairs (j,k). We use here and later that $\nu\tbinom{2^t}{j}=t-\nu(j)$ for $1\le j\le 2^t$.

Since Proposition 3.14 implies that $\nu(\Phi_{29}(j+k))\ge j+k-14$, strict inequality holds in (3.15) provided $j+k\ge 17$. Thus we are reduced to a finite number of verifications, and `Maple` shows that (3.15) holds with equality iff $(j,k)=(4,10)$, $(12,2)$, or $(8,6)$.

Most[1] of the $\bullet$'s in the tables are zeros of P_n in which some of the (j,k)-terms of (3.5) have 2-exponent smaller than that of their sum, and so more than just the 2-exponent of the terms must be considered. We illustrate with the proof for a typical such case, the left dot in $(31,2)$. The sequence of $\nu(P_{31}(16x+2))$ is 7, 8, 7, 9, 7, 8, 7, 10, 7, ..., and so we wish to prove

$$\nu\Big(\sum_{k\ge 0}\tbinom{16x+2}{k}\sum_{j>0}\tfrac{1}{2^d}\tbinom{2^{d+4}}{j}\Phi_{31}(j+k)\Big)=7 \tag{3.16}$$

for all $x\in\mathbb{Z}$ and all $d\ge 0$. `Maple` verifies that

$$\nu\tbinom{16x+2}{k}+4-\nu(j)+\nu(\Phi_{31}(j+k))\ge 4 \tag{3.17}$$

for all $j>0$, $k\ge 0$ with $j+k\le 19$, which by Proposition 3.14 are the only values of j and k that we need to consider, and for $0\le x\le 7$. There are many pairs (j,k) for which this value equals 4, 5, 6, and 7, and these combine in a complicated way to give 2-exponent 7 for the sum. `Maple` can easily enough check this value for the sum, but there are two things that must be considered in giving a proof valid for all integers x and d. For (j,k)-summands with $\nu=4$, the mod 16 value of the odd part of $\frac{1}{2^d}\binom{2^{4+d}}{j}$ for

[1]Several boxes in Table 3.7 with two horizontally-displaced $\bullet$'s have minimal $\nu(\binom{2^5x+p}{k}\binom{2^5}{j}\Phi_n(j+k))$ occurring an odd number of times but more than once; these do not seem to fit into an easily-proved general formula.

various values of d must be taken into account, and, similarly, changing x causes changes in $\binom{16x+2}{k}$, which are essential in proving that (3.16) holds for all x and d.

Similarly to Lemma 5.1, one easily proves

$$\nu\left(\frac{1}{2^{d+1}}\binom{2^{d+1+b}}{j}-\frac{1}{2^d}\binom{2^{d+b}}{j}\right)=2b+d-\lg(j-1)-\nu(j), \tag{3.18}$$

and so the odd factors for d and $d+1$ are congruent mod $2^{b+d-\lg(j-1)}$. As we have $b=4$ and $\lg(j-1)\le 4$, these odd factors will be congruent mod 16 provided $d\ge 4$. Thus the validity of (3.16) for $d\le 4$, which is checked by `Maple`, implies its validity for all d. A similar argument shows that changing x by a multiple of 8 changes (3.16) by a multiple of 2^8, and so verifying (3.16) for $0\le x\le 7$ implies it for all x.

A similar analysis proves the blanks in Tables 3.4 and 3.7, namely that there are no zeros in certain congruences. We illustrate with the case $n=23$, right side of column 6. `Maple` verifies that $\nu(P_{23}(16x+14))=4$ for $0\le x\le 3$. We will prove that

$$\nu(P_{23}(16x+64i+14)-P_{23}(16x+14))\ge 5. \tag{3.19}$$

These together imply that $\nu(P_{23}(16x+14))=4$ for all integers x, and hence P_{23} has no zeros in (14 mod 16).

We write

$$P_{23}(16x+14)=\sum \tbinom{16x+14}{k}\Phi_{23}(k).$$

Using Proposition 3.14, we easily see that terms with $k>14$ have 2-exponent ≥ 5. Using `Maple`, we see that all terms have 2-exponent ≥ 2. Similarly to the proof of 5.1, we can prove that for $0\le k\le 14$

$$\nu\left(\tbinom{16x+64i+14}{k}-\tbinom{16x+14}{k}\right)\ge 3+\nu(i)+\nu\tbinom{16x+14}{k},$$

implying (3.19).

Using the ideas discussed in the above examples, `Maple` can systematically find and prove all the results in Tables 3.4 and 3.7. In each case, `Maple` discovers the value of c by computing a sequence of values of $\nu(P_n(2^{e-1}x+p))$. The following remarkable formula, obtained by inspecting the c-values

obtained in all cases, gives the value of c in every case of Table 3.4 and 3.7.
(3.20)

$$\nu(P_n(z)) = \sum \nu(z-z_i) - \nu([\tfrac{n-1}{2}]!) + \begin{cases} \nu([\frac{n+1}{2}]) & n \equiv 0,3\ (4) \\ & \text{and } n+z \text{ even} \\ \min(15, 2\nu(z-148)) & n = 21 \\ \min(9, 2\nu(z-19)) & n = 29 \\ \min(8, 2\nu(z-11)) & n = 45 \\ \min(10, 2\nu(z-3) & n = 61 \\ 0 & \text{otherwise,} \end{cases}$$

where the sum is taken over all the zeros z_i of P_n.

We illustrate this formula with the example of $P_{29}(16x+10)$ considered above. Let $z = 16x+10$. From Table 3.4, we observe that P_{29} has five zeros z_i with z_i odd, four with $z_i \equiv 0 \bmod 4$, one with $z_i \equiv 6 \bmod 8$, one with $z_i \equiv 2 \bmod 16$, and one with $z_i = 16x_0 + 10$ for some $x_0 \in \mathbf{Z}_2$. The sum of $\nu(z-z_i)$ is $5 \cdot 0 + 4 \cdot 1 + 2 + 3 + \nu((16x+10) - (16x_0+10)) = 13 + \nu(x - x_0)$. Since $\nu([\frac{29-1}{2}]!) = 11$, (3.20) becomes $\nu(P_{29}(z)) = \nu(x-x_0) + 2$, consistent with the worked-out example above.

4. Some proofs

In this section, we prove Proposition 1.3, Theorem 2.1, and Theorem 3.11. We will need the following results related to $\nu(\Phi_n(k))$. One such result was stated earlier as Proposition 3.14.

The next result is an extension of [7, Prop. 2.4]. The proof of the new part, the condition for equality, will appear at the end of this section.

Proposition 4.1. *For any nonnegative integers n and k,*

$$\nu(\Phi_n(k)) \geq 0.$$

If $n = 2^e + \Delta$, with $0 \leq \Delta < 2^e$, then equality occurs here iff $\binom{2^{e-1}-1-[\Delta/2]}{k-\Delta}$ is odd.

The next result restates [7, Props 2.5, 2.6].

Proposition 4.2. *Let n and k be nonnegative integers with $n > k$. Then $\nu(\Phi_n(k)) \geq \alpha(n) - 1 - \alpha(k)$ with equality iff $\binom{n-1-k}{k}$ is odd. Mod 4,*

$$\sum_i \binom{n}{2i+1} i^k / (2^{n-1-2k} k!) \equiv \binom{n-1-k}{k} + \begin{cases} 2\binom{n-1-k}{k-2} & \text{if } n-1 \text{ and } k \text{ are even} \\ 0 & \text{otherwise.} \end{cases}$$

The last result is an extension of [7, Prop. 2.3]. The proof of the new part will appear near the end of this section.

Proposition 4.3. *Mod* 4

$$\frac{1}{n!}\sum_i \binom{2n+\varepsilon}{2i+b} i^k \equiv \begin{cases} S(k,n)+2nS(k,n-1) & \varepsilon=0,\ b=0 \\ (2n+1)S(k,n)+2(n+1)S(k,n-1) & \varepsilon=1,\ b=0 \\ S(k,n)+2(n+1)S(k,n-1) & \varepsilon=1,\ b=1. \end{cases}$$

Integrally

$$\frac{1}{n!}\sum_i \binom{2n}{2i+1} i^k = \sum_{d\geq 0} 2^{d+1}\binom{n+d}{d} S(k,n-1-d)/(2d+1)!!,$$

where $(2d+1)!! = \prod_{i\leq d}(2i+1)$.

Next we prove Proposition 1.3.

Proof of Proposition 1.3. Let $e=\lg(n)$. We have

$$P_n(x+2^t)-P_n(x) = \frac{1}{n!}\sum \binom{n}{2i+1}(2i+1)^x((2i+1)^{2^t}-1) = \sum_{k\geq 0}\sum_{j>0} T_{k,j},$$

where $T_{k,j} = \binom{x}{k}\binom{2^t}{j}\Phi_n(k+j)$, By Proposition 4.1, $\nu(T_{k,j}) \geq t-e+1$ if $j<2^e$, while if $j\geq 2^e$, by Proposition 3.14 we obtain

$$\nu(T_{k,j}) \geq t-\nu(j)+j-[n/2] \geq t+2^e-e-[n/2] \geq t-e+1. \qquad \square$$

Now we present the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $e\geq 2$, $1\leq \Delta\leq 4$, $0\leq p<2^{e-1}$, p_2 its mod 2 reduction, and

$$c = \begin{cases} 2 & \text{if } (\Delta,p_2)=(3,0) \text{ or } (4,1),\ e=2 \\ 1 & \text{if } (\Delta,p_2)=(3,0) \text{ or } (4,1),\ e>2 \\ 0 & \text{otherwise.} \end{cases}$$

By Corollary 3.2, the theorem will follow from $\nu(P_{2^e+\Delta}(p))\geq c$, and, for $d\geq 0$,

$$\nu(P_{2^e+\Delta}(2^{e-1}(x+2^d)+p)-P_{2^e+\Delta}(2^{e-1}x+p))-d = c. \tag{4.4}$$

Table 4.5: Conclusions about $\nu(T_{k,j})$ when $e\ge 3$

Δ	p	$\nu(T_{k,j})$	equality iff
$1,2$	any	≥ 0	$j=2^{e-1}$, $k=0$
3	odd	≥ 0	$j=2^{e-1}$, $k=1$
3	$0\ (4)$	≥ 1	$j=2^{e-1}$, $k=0$
3	$2\ (4)$	≥ 1	$j=2^{e-1}$, $k=0,1,2$
4	even	≥ 0	$j=2^{e-1}$, $k=0$
4	odd	≥ 0	$j=2^{e-1}$, $k=0,1$

Proposition 4.1 implies $\nu(P_{2^e+\Delta}(p))\ge 0$ and

$$P_{2^e+\Delta}(p)\equiv\sum\tbinom{p}{k}\tbinom{2^{e-1}-1-[\Delta/2]}{k-\Delta}\mod 2,$$

which is easily seen to be 0 mod 2 if $(\Delta,p_2)=(3,0)$ or $(4,1)$. Showing $\nu(P_{2^e+\Delta}(p))\ge 2$ when $e=2$ and $(\Delta,p_2)=(3,0)$ or $(4,1)$ is accomplished by using 3.14 to eliminate all but some very small values of k and checking these by direct computation.

Now we prove (4.4). The LHS equals $\sum_{k\ge 0,\ j>0}T_{k,j}$, where

$$T_{k,j}=\tbinom{2^{e-1}x+p}{k}\tfrac{1}{2^d}\tbinom{2^{d+e-1}}{j}\Phi_{2^e+\Delta}(j+k). \tag{4.6}$$

We first consider the case $e\ge 3$. We will prove the six statements in Table 4.5 together with Claim 4.7 and Claim 4.8, in which we assume $e\ge 3$.

Claim 4.7. *If $\Delta=4$ and p is odd, then $\nu(T_{k,j})=1$ iff $j=2^{e-1}$, $k=2$, and $p\equiv 3\ (4)$, or $j=2^{e-2}$, $k\equiv 0,1\ (4)$, and $\binom{p}{k}$ odd.*

Claim 4.8. *$\nu(\sum\binom{2^e+4}{2i+1}(2i)^{2^{e-1}})=\nu(\sum\binom{2^e+4}{2i+1}(2i)^{2^{e-1}+1})$ and their odd factors are both $\equiv 3\ (4)$.*

One easily checks that this implies the result. For example, if $\Delta=3$ and p is even, the table says that $\nu(\sum T_{k,j})$ is determined by exactly one or three terms having $\nu=1$. The only place where a little argument is required is the case $\Delta=4$, p odd. In this case, mod 4, we get $(p+1)$ from $j=2^{e-1}$, $k=0,1$, and also get 2 if $p\equiv 3\ (4)$. The sum of these is 2. Here we have used Claims 4.8 and 4.7 and the fact that if p is odd and $k\equiv 0$ mod 4, then $\binom{p}{k}\equiv\binom{p}{k+1}$ mod 2.

We have

$$V:=\nu(T_{j,k})=\nu\tbinom{p}{k}+e-1-\nu(j)+\nu(\Phi_{2^e+\Delta}(j+k)).$$

This is the quantity in the third column of Table 4.5. Now we verify the claims of Table 4.5 and Claims 4.7 and 4.8.

By Proposition 3.14, if $\nu(j) > e-1$ or $j = 3\cdot 2^{e-2}$, then $V \gg 0$, and so we need not consider these cases when looking for $V=0$ or $V=1$. By Proposition 4.1, we deduce that $V \ge 0$ with equality iff $j = 2^{e-1}$, $\binom{p}{k}$ odd, and $\binom{2^{e-1}-1-[\Delta/2]}{2^{e-1}+k-\Delta}$ odd. The latter condition is equivalent to $\binom{\Delta-1-k}{[\Delta/2]}$ odd, which, incorporating also $\binom{p}{k}$ odd, comprises the pairs $(\Delta,k) = (1,0)$, $(2,0)$, $(3,1)$, $(4,0)$, and $(4,1)$. This implies all of our claims regarding when $V=0$.

Now let $(\Delta,p_2) = (3,0)$. If $j = 2^{e-2}$, then $V=1$ iff $\binom{p}{k}$ is odd and (using Proposition 4.1) $\binom{2^{e-1}-2}{2^{e-2}+k-3}$ is odd, which is impossible. If $j = 2^{e-1}$, then $V=1$ if $\nu\binom{p}{k} = 1$ and $k=1$ or if $\binom{p}{k}$ is odd and $\Phi_{2^e+3}(2^{e-1}+k) \equiv 2\ (4)$. By Proposition 4.2, $\Phi_{2^e+3}(2^{e-1}) \equiv 2\ (4)$, and by Proposition 4.3

$$\Phi_{2^e+3}(2^{e-1}+k) \equiv \begin{cases} 2 & k=2 \\ 0 & k>2 \end{cases} \mod 4, \tag{4.9}$$

implying our claims in this case.

Finally let $(\Delta,p_2) = (4,1)$. Claim 4.8 is proved using Proposition 4.1, and 4.2 for the first sum and 4.3 for the second. If $j = 2^{e-2}$, then $V=1$ iff $\binom{p}{k}$ and $\binom{2^{e-1}-3}{2^{e-2}+k-4}$ are odd, and this happens in the asserted situations. If $j = 2^{e-1}$, then $V=1$ iff $\binom{p}{k} \equiv 2\ (4)$ and $k = 0,1$, which is impossible (since p is odd), or if $\binom{p}{k}$ is odd and $\Phi_{2^e+4}(2^{e-1}+k) \equiv 2\ (4)$. Since (4.9) is also true for Φ_{2^e+4} by 4.3, Claim 4.7 is clear.

The main difference when $e=2$ is that $j = 2^e$ can play a significant role. When $e>2$, Proposition 3.14 implies that $\nu(\Phi_{2^e+\Delta}(j+k))$ would be too large to have an effect.

The case $e=2$ involves P_5, P_6, P_7, and P_8. The result for them was part of [7, Theorem 2.1], although detailed proofs were not presented there for P_7 and P_8. As when $e>2$, the argument considers the terms $T_{k,j}$ of (4.6). We illustrate with the case $e=2$, $\Delta=3$, $p=0$, $d\ge 1$. We wish to prove that $\nu(\sum T_{k,j}) = 2$, where $T_{k,j} = \binom{2x}{k}\frac{1}{2^d}\binom{2^{d+1}}{j}\Phi_7(j+k)$, $k\ge 0$, $j>0$. We can use Proposition 3.14 to eliminate large values of $j+k$. If $x\equiv 0 \mod 4$, the only terms that are nonzero mod 8 occur when $k=0$ and $j=1$, 3, and 4. These three terms have 2-exponents 2, 1, and 1, respectively, and the sum of the last two is divisible by 8.

If $x \equiv 2 \mod 4$, or x odd, several additional terms are involved, but the same conclusion is obtained. □

Proof of Theorem 3.11. The inequality in (3.12) follows easily from Proposition 4.2 and [7, Lemma 2.40] (and Proposition 3.14 to handle $j = 2^e$).

These results also imply that, if $\varepsilon = 0 = \delta$, equality is obtained in (3.12) iff $k = p$, $\binom{n-1-j-k}{j+k}$ is odd, and $j = 2^e - 2^h$ with $2^h > k$. Thus the case $\varepsilon = 0 = \delta$ of the theorem follows from the following lemma.

Lemma 4.10. *If $\binom{n-1-p}{p}$ is odd, $e = \lg(n-1)$, $p < 2^h < 2^e$, and $p < [(n - 3 \cdot 2^{e-1})/2]$, then*

$$\binom{n-1-p-2^e+2^h}{p+2^e-2^h} \text{ is odd iff } h = \lg(2^{e+1} - n + p).$$

Proof. Let $\ell = \lg(2^{e+1} - n + p)$, $A = n-1-p-2^e+2^h$, and $B = p+2^e-2^h$. If $h < \ell$, then $2^{h+1} \le 2^{e+1} - n + p$, from which is follows that $A < B$, and hence $\binom{A}{B} = 0$.

If $h > \ell$, it follows that $2^e \le A < 3 \cdot 2^{e-1}$ and $2^{e-1} \le B < 2^e$, so $\binom{A}{B}$ is even, due to the 2^{e-1} position. If $h = \ell$, it is immediate that $n - 1 - p = 2(2^e - 2^h) + L$ with $0 \le L < 2^h$ and $\binom{L}{p}$ odd. This implies that $\binom{A}{B} = \binom{2^e-2^h+L}{2^e-2^h+p}$ is odd. □

If $\varepsilon = 1$ and $\delta = 0$, the above methods together with [7, Lemma 2.44] show that equality in (3.12) is obtained only for $(j, k) = (2^e - 2^\ell, p)$, as claimed.

Now let $\delta = 1$ and $\varepsilon = 0$. If $k \le p$, the $\delta = 0$ analysis applies to give the $(j, k) = (2^e - 2^\ell, p)$ solution. If $k = 2^{e-1} + \Delta$ with $\Delta \ge 0$, then analysis similar to that performed above implies that equality is obtained in (3.12) iff $\Delta = p$, $\phi(j, 2^{e-1} + \Delta) = e - 1$, where ϕ is as in [7, Lemma 2.40], and $\binom{n-1-j-2^{e-1}-\Delta}{j+2^{e-1}+\Delta}$ is odd with $2(j + 2^{e-1} + \Delta) < n$. Part 3 of [7, Lemma 2.44] (with its $e-1$ corresponding to our e) gives two possibilities for $\phi = e-1$. The first one does not satisfy $2(j + 2^{e-1} + \Delta) < n$. The second one reduces to $j = 2^{e-1} - 2^h$ with $\Delta < 2^h$. Since $\Delta = p$, the required oddness of the binomial coefficient becomes exactly the condition of Lemma 4.10, and so we obtain that h equals the ℓ of our Theorem 3.11.

If $p < k < 2^{e-1}$, the condition for equality in (3.12) becomes $\alpha(2^{e-1} + p - k) = 1$, $\binom{n-1-j-k}{j+k}$ odd with $2(j + k) < n$, and $\phi(j, k) = e$. The first of these says $k = 2^{e-1} + p - 2^h$ with $h < e - 1$. By part 2 of [7, Lemma 2.44], $j = 2^e - 2^t$ with $2^t > 2^{e-1} + p - 2^h$, which implies $t = e - 1$ since $h < e - 1$. Thus $j = 2^{e-1}$, and the odd binomial coefficient is again handled by Lemma 4.10, implying that h equals the ℓ of the theorem.

The case $\delta = 1 = \varepsilon$ is established using the same methods. □

Proof of last part of Proposition 4.3. We extend the proof in [7]. We have

$$\sum_i \binom{2n}{2i+1} i^k = \sum_\ell C_{2n,\ell,1}\ell!S(k,\ell),$$

where $C_{2n,\ell,1} = \sum\limits_i \binom{2n}{2i+1}\binom{i}{\ell}$. We will prove the possibly new result

$$\sum \binom{2n}{2i+1}\binom{i}{n-d-1} = 2^{2d+1}\binom{n+d}{2d+1}. \tag{4.11}$$

Then, using (4.11) at the second step,

$$\begin{aligned} \tfrac{1}{n!}\sum \binom{2n}{2i+1} i^k &= \tfrac{1}{n!}\sum_{d\geq 0} C_{2n,n-d-1,1}(n-d-1)!S(k,n-d-1) \\ &= \tfrac{1}{n!}\sum_d 2^{2d+1}\binom{n+d}{2d+1}(n-d-1)!S(k,n-d-1) \\ &= \sum_d 2^{2d+1}\tfrac{(n+d)!}{n!(2d+1)!}S(k,n-d-1), \end{aligned}$$

and the result follows since $\frac{d!}{(2d+1)!} = 1/(2^d(2d+1)!!)$.

We prove (4.11) with help from [14] and the associated software. Let d be fixed, and

$$F(n,i) = \frac{\binom{2n}{2i+1}\binom{i}{n-d-1}}{2^{2d+1}\binom{n+d}{2d+1}}.$$

We will show that

$$\sum_i F(n+1,i) = \sum_i F(n,i). \tag{4.12}$$

Since $\sum_i F(d+1,i) = 1$, this implies that $\sum_i F(n,i) = 1$ for all n, our desired result.

To prove (4.12), let

$$G(n,i) = \frac{(2n+1-i)(2i+1)(i+d+1-n)}{(d+n+1)(n-i)(-2n+2i-1)}F(n,i).$$

(This is what was discovered by the software.) Then one can verify

$$F(n+1,i) - F(n,i) = G(n,i+1) - G(n,i).$$

When summed over i, the RHS equals 0, implying (4.12). $\square$

Proof of second part of Proposition 4.1. We want to know when $\Phi_{2^e+\Delta}(k)$ is odd. We expand $(2i)^k$ as $\sum(-1)^j\binom{k}{j}(2i+1)^j$. Using Proposition 1.2, we are reduced to proving, when $\Delta = 2d+1$,

$$\binom{2^{e-1}-1-d}{k-2d-1} \equiv \sum \binom{k}{j}\binom{j-2^{e-1}-d-1}{2^{e-1}+d} \quad \text{mod } 2 \tag{4.13}$$

and a similar result when $\Delta = 2d$. If $k < 2^{e-1}+d+1$, the RHS equals

$$\sum [x^j](1+x)^k \cdot [x^{2^{e-1}+d-j}](1+x)^{-2^{e-1}-d-1} = \binom{-2^{e-1}-d-1+k}{2^{e-1}+d}.$$

Thus both sides of (4.13) are odd iff the binary expansions of d and $k-2d-1$ never have 1's in the same position. If $k \ge 2^{e-1}+d+1$, the LHS of (4.13) is 0. To evaluate the RHS, note that $\binom{j-2^{e-1}-d-1}{2^{e-1}+d} \equiv \binom{j-2^eA-2^{e-1}-d-1}{2^{e-1}+d}$ mod 2. Choose 2^eA so that $k-2^eA-2^{e-1}-d-1<0$. Then the RHS becomes $\binom{-2^{e-1}-d-1+k}{2^eA+2^{e-1}+d} = 0$. □

5. Proof of Theorem 2.6

The proof of Theorem 2.6 will be aided by two lemmas.

Lemma 5.1. *For* $0 < d < 2^e$,

$$\frac{2^{2^e+d-1-\alpha(d)}}{d!2^e!} \equiv \frac{2^{2^e+d-1-\alpha(d)}}{(2^e+d)!} \quad \text{mod } 2^{e-\lg(d)}.$$

Proof. First note that $\nu(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{d}) = -\lg(d)$, as is easily proved by induction on d. From this, we obtain,

$$\frac{(2^e+d)!/2^e!-d!}{d!} = \sum_{j\ge1} 2^{je}\sigma_j(\tfrac{1}{1},\ldots,\tfrac{1}{d}) \equiv 0 \quad \text{mod } 2^{e-\lg(d)},$$

where σ_j is the elementary symmetric polynomial. The terms with $j \ge 2$ are easily seen to have 2-exponent larger than that with $j=1$ by consideration of the largest 2-exponent in the denominator of any term of σ_j. Multiplying the above by the odd number $2^{2^e+d-1-\alpha(d)}/(2^e+d)!$ yields the claim of the lemma. □

Lemma 5.2. *If* $0 < r \le D$, *then* $\nu(2^{2^e-r-1}/(2^e-r)!) \ge e-1-\lg(D)$.

Proof. The indicated exponent equals $\alpha(2^e-r)-1 = e-\alpha(r-1)-1$, and $\alpha(r-1) \le \lg(D)$. □

Proof of Theorem 2.6. We have

$$\begin{aligned}
P_{2^e+\Delta}(x) &= \sum_{i\ge 0}\frac{1}{(2i+1)!(2^e+\Delta-2i-1)!}\sum_{k=0}^{x}S(x,k)k!\tbinom{2i+1}{k}\\
&= \sum_{k=0}^{x}S(x,k)\frac{1}{(2^e+\Delta-k)!}\sum_{i\ge 0}\binom{2^e+\Delta-k}{2i+1-k}\\
&= \sum_{k=0}^{x}S(x,k)\frac{1}{(2^e+\Delta-k)!}2^{2^e+\Delta-k-1}\\
&\equiv \sum_{k=0}^{\Delta}S(x,k)\frac{1}{(2^e+\Delta-k)!}2^{2^e+\Delta-k-1} \quad \bmod 2^{e-\lg(x-\Delta)-1}\\
&\equiv \sum_{k=0}^{\Delta}S(x,k)\frac{1}{2^e!(\Delta-k)!}2^{2^e+\Delta-k-1} \quad \bmod 2^{e-\lg(\Delta)+1}\\
&= \frac{1}{\mathrm{U}(2^e!)\Delta!}\sum_{k=0}^{\Delta}S(x,k)k!2^{\Delta-k}\tbinom{\Delta}{k}\\
&= \frac{1}{\mathrm{U}(2^e!)\Delta!}\sum_{k=0}^{\Delta}\sum_{j}(-1)^{k+j}\tbinom{k}{j}j^x2^{\Delta-k}\tbinom{\Delta}{k}\\
&= \frac{1}{\mathrm{U}(2^e!)\Delta!}\sum_{j}j^x\tbinom{\Delta}{j}.
\end{aligned}$$

Lemmas 5.2 and 5.1 are used to prove the two congruences. Equality occurs at the first congruence if $\Delta \ge x$ since $S(x,k)=0$ when $k>x$. To see the last step, let $\ell=\Delta-k$ and obtain

$$\sum_{k=0}^{\Delta}(-1)^{k+j}\tbinom{k}{j}2^{\Delta-k}\tbinom{\Delta}{k}=\tbinom{\Delta}{j}(-1)^{\Delta-j}\sum_{\ell=0}^{\Delta}(-2)^{\ell}\tbinom{\Delta-j}{\ell}=\tbinom{\Delta}{j}. \qquad \square$$

References

[1] M. Bendersky and D. M. Davis, *2-primary v_1-periodic homotopy groups of* SU(n), Amer J Math **114** (1991) 529–544. MR1165351

[2] L. Carlitz, *Congruences for generalized Bell and Stirling numbers*, Duke Math Jour **22** (1955) 193–205. MR0069206

[3] F. Clarke, *Hensel's Lemma and the divisibility of Stirling-like numbers*, Jour Number Theory **52** (1995) 69–84. MR1331766

[4] M. C. Crabb and K. Knapp, *The Hurewicz map on stunted complex projective spaces*, Amer Jour Math **110** (1988) 783–809. MR0961495

[5] D. M. Davis, *Divisibility by 2 of Stirling-like numbers*, Proc Amer Math Soc **110** (1990) 597–600. MR1036984

[6] D. M. Davis, *Divisibility by 2 and 3 of certain Stirling numbers*, Integers **8** (2008) A56, pp. 25. MR2472074

[7] D. M. Davis, *Divisibility by 2 of partial Stirling numbers*, Funct et Approx Comm Math **49** (2013) 29–56. MR3127897

[8] D. M. Davis, *Binomial coefficients involving infinite powers of primes*, to appear in American Math Monthly.

[9] D. M. Davis, v_1*-periodic 2-exponents of* $SU(2^e)$ *and* $SU(2^e+1)$, Jour Pure Appl Alg **216** (2012) 1268–1272. MR2890499

[10] D. M. Davis and Z. W. Sun, *A number-theoretic approach to homotopy exponents of* $SU(n)$, Jour Pure Appl Alg **209** (2007) 57–69. MR2288289

[11] Y. H. Kwong, *Minimum periods of* $S(n,k)$ *modulo* M, Fibonacci Quar **27** (1989) 217–221. MR1002064

[12] A. T. Lundell, *A divisibility property for Stirling numbers*, Jour Number Theory **10** (1978) 35–54. MR0460135

[13] A. T. Lundell, *Generalized e-invariants and the numbers of James*, Quar Jour Math Oxford **25** (1974) 427–440. MR0375312

[14] M. Petkovsek, H. S. Wilf, and D. Zeilberger, $A = B$, (1995) A.K. Peters. MR1379802

[15] P. T. Young, *Congruences for degenerate number sequences*, Discrete Math **270** (2003) 279–289. MR1997904

Donald M. Davis
Department of Mathematics
Lehigh University
Bethlehem, PA 18015
USA
E-mail address: dmd1@lehigh.edu

Received February 3, 2014