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Harmonic vectors and matrix tree theorems

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This paper describes an explicit combinatorial formula for a harmonic vector for the Laplacian of a directed graph with arbitrary edge weights. This result was motivated by questions from mathematical economics, and the formula plays a crucial role in recent work of the author on the emergence of prices and money in an exchange economy.

It turns out that the formula is closely related to well-studied problems in graph theory, in particular to the so-called weighted matrix tree theorem due to W. Tutte and independently to R. Bott and J. Mayberry. As a further application of our considerations, we obtain a short new proof of both the matrix tree theorem as well as its generalization due to S. Chaiken.

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1. Introduction

In this paper we prove a new result in graph theory that was motivated by considerations in mathematical economics, more precisely, by the problem of price formation in an exchange economy [4]. The aggregate demand/supply in the economy is described by an $n \times n$ matrix $A = (a_{ij})$ where a_{ij} is the amount of commodity j that is on offer for commodity i. In this context one defines a *market-clearing* price vector to be a vector p with strictly positive components p_i, which satisfies the equation

$$\sum_j a_{ij} p_j = \sum_j a_{ji} p_i \text{ for all } i. \tag{1}$$

The left side of (1) represents the total value of all commodities being offered for commodity i, while the right side represents the total value of commodity i in the market. It was shown in [4] that if the matrix A is irreducible, *i.e.* if it cannot be permuted to block upper-triangular form, then (1) admits a positive solution vector p, which is unique up to a positive multiple.

The primary purpose of the present paper is to describe an explicit combinatorial formula for p. The formula and its proof are completely elementary, but nonetheless the result seems to be new. This formula plays a crucial role in [5], which seeks to address a fundamental question in mathematical economics: *How do prices and money emerge in a barter economy?* We show in [5] that among a reasonable class of exchange mechanisms, trade via a commodity money, even in the absence of transactions costs, minimizes complexity in a very precise sense.

It turns out however that equation (1) is closely related to well-studied problems in graph theory, in particular to the so-called matrix tree theorems. Therefore as an additional application of our formula, we give an elementary proof of the matrix tree theorem of W. Tutte [6], which was independently discovered by R. Bott and J. Mayberry [2] coincidentally also in an economic context. With a little additional effort, we also obtain a short new proof of S. Chaiken's generalization of the matrix tree theorem [3].

2. Harmonic vectors

We first give a slight reformulation and reinterpretation of equation (1) in standard graph-theoretic language. Let G be a simple directed graph (digraph) on the vertices $1, 2, \ldots, n$, with weight a_{ij} attached to the edge ij from i to j. The weighted *adjacency* matrix of G is the $n \times n$ matrix $A = (a_{ij})$, where $a_{ij} = 0$ for missing edges. The weighted *degree* matrix D is the diagonal matrix with diagonal entries $(d_1, \ldots, d_n)$, where $d_i = \sum_j a_{ji}$ is the *weighted in-degree* of the vertex i. The *Laplacian* of G is the matrix $L = D - A$ and we say that a vector $\mathbf{x} = (x_i)$ is *harmonic* if $\mathbf{x}$ is a null vector of L, *i.e.* if it satisfies

$$L\mathbf{x} = \mathbf{0}. \tag{2}$$

It is easy to see that equation (1) is equivalent to equation (2), *i.e.* the market-clearing condition is the same as harmonicity of p.

To describe our construction of a harmonic vector, we introduce some terminology. A *directed tree*, also known as an *arborescence*, is a digraph with at most one incoming edge ij at each vertex j, and whose underlying undirected graph is acyclic and connected (*i.e.* a tree). Following the edges backwards from any vertex we eventually arrive at the same vertex called the *root.* Dropping the connectivity requirement leads to the notion of a *directed forest,* which is simply a vertex-disjoint union of directed trees. We define a *dangle* to be a digraph D that is an edge-disjoint union of a directed forest

F and a directed cycle C linking the roots of F; note that D determines C and F uniquely, the former as its unique simple cycle.

In the context of the digraph G, we will use the term i-tree to mean a directed spanning tree of G with root i, and i-dangle to mean a spanning dangle whose cycle contains i. We define the weight $wt(\Gamma)$ of a subgraph Γ of G to be the product of weights of all the edges of Γ, and we define the *weight vector* of G to be $\mathbf{w} = (w_i)$ where w_i is the weighted sum of all i-trees.

Theorem 1. *The weight vector of a digraph is harmonic.*

Proof. If Γ is an i-dangle in G with cycle C, and ij and ki are the unique outgoing and incoming edges at i in C, then deleting one of these edges from Γ gives rise to an j-tree and a i-tree, respectively. The dangle can be recovered uniquely from each of the two trees by reconnecting the respective edges; thus, writing $\mathcal{T}_i$ for the set of i-trees, we obtain bijections from the set of i-dangles to each of the following sets

$$\{(ij, t) : t \in \mathcal{T}_j\}, \quad \{(ki, t) : t \in \mathcal{T}_i\},$$

where ij and ki range over all outgoing and incoming edges at i in G.

Thus if v_i is the weighted sum of all i-dangles, we get

$$\sum_j a_{ij} w_j = v_i = \sum_k a_{ki} w_i.$$

Rewriting this we get $A\mathbf{w} = D\mathbf{w}$, and hence $(D - A)\mathbf{w} = \mathbf{0}$, as desired. $\square$

3. The matrix tree theorem

In this section we use Theorem 1 to derive the *weighted matrix tree theorem* due to [6] (see also [2]). This is the following formula for the cofactors of the Laplacian L, which generalizes a classical formula of Kirchoff for the number of spanning trees in an undirected graph.

Theorem 2. *The ij-th cofactor of the Laplacian L is given by*

$$c_{ij}(L) = \sum_{t \in \mathcal{T}_j} wt(t) \ \textit{for all } i, j.$$

We will prove this in a moment after some discussion on cofactors.

3.1. Interlude on cofactors

We recall that ij-th cofactor of an $n \times n$ matrix X is

$$c_{ij}(X) = (-1)^{i+j} \det X_{ij},$$

where X_{ij} is the matrix obtained from X by deleting row i and column j. The *adjoint* of X is the $n \times n$ matrix $\operatorname{adj}(X)$ whose ij-th entry is $c_{ji}(X)$.

Lemma 3. *If* $\det X = 0$ *then the columns of* $\operatorname{adj}(X)$ *are null vectors of* X*; moreover these are the* same *null vector if the columns of* X *sum to* 0.

Proof. By standard linear algebra we have $X \operatorname{adj}(X) = (\det X) I_n$. If $\det X = 0$ then $X \operatorname{adj}(X)$ is the zero matrix, which implies the first part. For the second part we note that if X has zero column sums then necessarily $\det X = 0$. In view of the first part it suffices to show that $c_{ij}(L) = c_{i+1,j}(L)$ for all i, j; or equivalently that

$$\det L_{ij} + \det L_{i+1,j} = 0.$$

The left side above equals $\det P$, where P is obtained from L by deleting column j and replacing rows i and $i+1$ by the single row consisting of their sum. But P too has zero column sums, and so $\det P = 0$. □

3.2. Proof of the matrix tree theorem

Proof. It suffices to prove Theorem 2 for the complete simple digraph G_n on n vertices, with edge weights $\{a_{ij} \mid i \neq j\}$ regarded as variables, and we work over the field of rational functions $\mathbb{C}(a_{ij})$. The Laplacian L has zero column sums by construction, and so by the previous lemma, $c_j := c_{ij}(L)$ is independent of i and the vector $\mathbf{c} = (c_1, \ldots, c_n)^t$ is a null vector for L. To complete the proof it suffices to show that the null vectors $\mathbf{c}$ and $\mathbf{w}$ are equal. Now the null space of L is 1-dimensional since $c_{ij}(L) \neq 0$, and hence

$$c_i w_j = c_j w_i \text{ for all } i, j. \tag{3}$$

Note that c_j and w_j belong to the polynomial ring $\mathbb{C}[a_{ij}]$. We claim that the polynomials c_j are distinct and irreducible. Consider first $c_n = \det B$ where $B = L_{nn}$ has entries

$$b_{ij} = \begin{cases} -a_{ij} & \text{if } i \neq j \\ a_{nj} + \sum_{k=1}^{n-1} a_{kj} & \text{if } i = j \end{cases}; \quad \text{for } 1 \leq i, j \leq n-1.$$

This is an *invertible* $\mathbb{C}$-*linear* map relating $\{b_{ij}\}$ to the $(n-1)^2$ variables

$$\{a_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq n-1, i \neq j\},$$

which occur in c_n. Thus the irreducibility of c_n follows from the irreducibility of the determinant as a polynomial in the matrix entries [1]. The argument for the other c_i is similar, and their distinctness is obvious.

Since c_i and c_j are distinct and irreducible, we conclude from formula (3) that c_i divides w_i. Since c_i and w_i both have total degree $n-1$, we conclude that $w_i = \alpha c_i$ for some $\alpha \in \mathbb{C}$. To prove that $\alpha = 1$, it suffices to note that the monomial $m_i = \prod_{j \neq i} a_{ij}$ occurs in both c_i and w_i with coefficient 1. □

4. The all minors theorem

The *all minors* theorem [3] is a formula for $\det L_{IJ}$, where L_{IJ} is the submatrix of L obtained by deleting rows I and columns J. It turns out this follows from Theorem 2 by a specialization of variables. We will state and prove this below after a brief discussion on signs of permutations and bijections.

4.1. Interlude on signs

Let I, J be equal-sized subsets of $\{1, \ldots, n\}$ and let Σ_I, Σ_J denote the sums of their elements. If $\beta : J \to I$ is a bijection, we write $inv(\beta)$ for the number of inversions in β, *i.e.* pairs $j < j'$ in J such that $\beta(j) > \beta(j')$ and we define

$$\varepsilon(\beta) = (-1)^{inv(\beta)+\Sigma_I+\Sigma_J}.$$

Note that if $J = I$ then $\varepsilon(\sigma) = (-1)^{inv(\sigma)}$ is the sign of σ as a permutation.

Lemma 4. *If $\beta : J \to I$, $\alpha : I \to H$ are bijections then $\varepsilon(\alpha\beta) = \varepsilon(\alpha)\varepsilon(\beta)$.*

Proof. This follows by combining the following mod 2 congruences

$$\Sigma_H + \Sigma_I + \Sigma_I + \Sigma_J \equiv \Sigma_H + \Sigma_J$$
$$inv(\alpha) + inv(\beta) \equiv inv(\alpha\beta)$$

the first of which is obvious. To establish the second congruence we replace α, β by the permutations $\lambda\alpha, \beta\mu$ of I, where $\lambda : H \to I, \mu : I \to J$ are the unique order-preserving bijections; this does not affect $inv(\alpha)$ *etc.,* and reduces the second congruence to a standard fact about permutations. □

The meaning of $\varepsilon(\beta)$ is clarified by the following result. For a bijection $\beta : J \to I$ and any $n \times n$ matrix X, let X_β be the matrix obtained from X by replacing, for each $j \in J$, the jth column of X by the unit vector $\mathbf{e}_{\beta(j)}$.

Lemma 5. *We have* $\det X_\beta = \varepsilon(\beta) \det X_{IJ}$.

Proof. If σ is a permutation of I then by the previous lemma, and standard properties of the determinant, we have

$$\varepsilon(\sigma\beta) = \varepsilon(\sigma)\,\varepsilon(\beta)\,, \;\det X_{\sigma\beta} = \varepsilon(\sigma) \det X_\beta.$$

Thus replacing β by a suitable $\sigma\beta$, we may assume $inv(\beta) = 0$ and write

$$I = \{i_1 < \cdots < i_p\}\,, \;J = \{j_1 < \cdots < j_p\} \text{ with } \beta(j_k) = i_k \text{ for all } k.$$

The lemma now follows from the identity

$$\det X_\beta = (-1)^{i_p + j_p} \cdots (-1)^{i_1 + j_1} \det X_{IJ} = (-1)^{\Sigma_I + \Sigma_J} \det X_{IJ}$$

obtained by iteratively expanding $\det X_\beta$ along columns $j_p, \ldots, j_1$. □

4.2. Directed forests

Let $\mathcal{F}(J)$ be the set of all directed spanning forests f of G with root set J. Let $\mathcal{F} \subset \mathcal{F}(J)$ be the subset consisting of those forests f such that each tree of f contains a unique vertex of I. Note that the trees of $f \in \mathcal{F}$ give a bijection β_f: $J \to I$. The all minors theorem is the following formula [3].

Theorem 6. *We have* $\det L_{IJ} = \sum_{f \in \mathcal{F}} \varepsilon(\beta_f) \operatorname{wt}(f)$.

We fix a bijection $\beta : J \to I$ and define $\sigma_f = \beta^{-1}\beta_f : J \to J$. In view of Lemmas 4 and 5, it suffices to prove the following reformulation of the previous theorem.

Theorem 7. *We have* $\det L_\beta = \sum_{f \in \mathcal{F}} \varepsilon(\sigma_f) \operatorname{wt}(f)$.

Proof. As usual it is enough to treat the complete digraph G_n with arbitrary edge weights a_{ij}. We fix an index $j_0 \in J$ and put $i_0 = \beta(j_0)$, $J_0 = J \setminus \{j_0\}$. We now consider a particular specialization $\bar{a}_{ij}$ of a_{ij}, and the entries $\bar{l}_{ij}$ of the specialized Laplacian $\bar{L}$. For $j \notin J_0$ we set $\bar{a}_{ij} = a_{ij}$ and hence $\bar{l}_{ij} = a_{ij}$; while for $j \in J_0$ we set

$$\bar{a}_{ij} = \begin{cases} 1 & \text{if } i = i_0 \\ -1 & \text{if } i = \beta(j) \\ 0 & \text{otherwise} \end{cases} \implies \bar{l}_{ij} = \begin{cases} -1 & \text{if } i = i_0 \\ 1 & \text{if } i = \beta(j) \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Note that $\bar{L}$ and L_β have the same entries outside of row i_0 and column j_0; hence we get $\det L_\beta = c_{i_0 j_0}(L_\beta) = c_{i_0 j_0}(\bar{L})$ and it remains to show that

$$c_{i_0 j_0}\left(\bar{L}\right) \overset{?}{=} \sum_{f \in \mathcal{F}} \varepsilon\left(\sigma_f\right) \mathrm{wt}\left(f\right). \tag{5}$$

Specializing Theorem 2 we get

$$c_{i_0 j_0}\left(\bar{L}\right) = \sum_{f \in \mathcal{F}(J)} \psi\left(f\right) \mathrm{wt}\left(f\right), \ \psi\left(f\right) := \sum_{t \in \mathcal{A}_f} (-1)^{p(t)},$$

where $\mathcal{A}_f$ is the set of j_0-trees t such that for each $j \in J_0$ the unique edge ij in t satisfies $i = i_0$ or $i = \beta(j)$, and for which deleting all such edges from t yields the forest f; and where $p(t)$ is the number of edges in t of type $i_0 j$, $j \in J_0$. Therefore to prove equality in formula (5) it suffices to show

$$\psi\left(f\right) \overset{?}{=} \begin{cases} 0 & \text{if } f \notin \mathcal{F} \\ \varepsilon\left(\sigma_f\right) & \text{if } f \in \mathcal{F} \end{cases}.$$

First suppose $f \notin \mathcal{F}$. In this case if $t \in \mathcal{A}_f$ there is some $j \in J_0$ such that the j-subtree contains no I vertex. Choose the largest such j and change the edge ij, from $i = i_0$ to $i = \beta(j)$ or vice versa. This is a sign-reversing involution on $\mathcal{A}_f$ and hence we get $\psi(f) = 0$.

Now let $f \in \mathcal{F}$, and for each subset $S \subset J_0$ consider the graph obtained from f by adding the edges $i_0 j$ for $j \in S$, and $\beta(j)\, j$ for $j \in J_0 \setminus S$. This graph is a tree in $\mathcal{A}_f$ iff S meets every cycle c of the permutation σ_f of J, and is disconnected otherwise. Thus a tree $t \in \mathcal{A}_f$ is prescribed uniquely by choosing, for each cycle c of σ_f, a nonempty subset S_c of its vertex set J_c. By definition we have $(-1)^{p(t)} = \prod_c (-1)^{|J_c| - |S_c|}$, and so $\psi(f)$ factors as

$$\psi\left(f\right) = \prod_c \psi\left(c\right), \ \psi\left(c\right) := \sum_{J_c \supseteq S_c \neq \emptyset} (-1)^{|J_c| - |S_c|}.$$

Now we get $\psi(c) = (-1)^{|J_c| - 1}$ using the elementary identity

$$\sum_{k=1}^{m} \binom{m}{k} (-1)^{m-k} = (1-1)^m - (-1)^m = (-1)^{m-1}.$$

Thus $\psi(f)$ agrees with the standard formula $\prod_c (-1)^{|J_c| - 1}$ for $\varepsilon(\sigma_f)$. □

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