The $q = -1$ phenomenon via homology concentration

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We introduce a homological approach to exhibiting instances of Stembridge's $q = -1$ phenomenon. This approach is shown to explain two important instances of the phenomenon, namely that of partitions whose Ferrers diagrams fit in a rectangle of fixed size and that of plane partitions fitting in a box of fixed size. A more general framework of invariant and coinvariant complexes with coefficients taken mod 2 is developed, and as a part of this story an analogous homological result for necklaces is conjectured.

1. Introduction

There is a rich history surrounding the enumeration of partitions in a rectangle or higher dimensional box, as well as the enumeration of classes of partitions possessing various symmetries (see e.g. [19], [4]). One reason for so much interest comes from connections to physics, while another is the important role they play in representation theory, specifically in the theory of canonical bases (see e.g. [22] and [23]). Richard Stanley used the Littlewood-Richardson rule in [19] to prove a recursive formula for the number of self-complementary plane partitions of bounded value. John Stembridge proved that semistandard domino tableaux are counted by this same formula by showing that their enumeration formula satisfies the same recurrence. This proved that the set of fixed points in a fundamental involution of Lusztig on a type A canonical basis also has this same cardinality, by virtue of a bijection due to Berenstein and Zelevinsky between the elements of a canonical basis and semistandard Young tableaux (actually Gelfand-Tsetlin patterns, objects also in bijection with semistandard Young tableaux) such that this bijection sends Lusztig's involution to evacuation. Stembridge examined this connection between self-complementary partitions and canonical bases more

closely, unveiling in the process a phenomenon he dubbed the "$q = -1$ phenomenon".

In [23], Stembridge defines the $q = -1$ phenomenon as follows. Given a finite set B and a weight function $w : B \to \mathbb{N}_0$, one defines the generating function

$$X(q) := \sum_{b \in B} q^{w(b)}.$$

The $q = -1$ phenomenon occurs when there is some "natural" involution t on B such that $X(-1)$ is the number of fixed points of t. Stembridge exhibited various instances of the $q = -1$ phenomenon in which B is an interesting collection of combinatorial objects with a natural weight function w, by showing in each case that $X(-1)$ is the trace of a matrix that is conjugate to the permutation matrix for t.

Here we introduce another method of proving that the $q = -1$ phenomenon occurs. It is natural to ask if the phenomenon may be explained in interesting cases by an Euler characteristic computation. That is, given B, t as above, we seek a chain complex $\mathcal{C}$ such that, for each i, the rank of the ith chain group C_i is the coefficient of q^i in $X(q)$. Under these conditions, the Euler characteristic of $\mathcal{C}$ is $X(-1)$. We aim to use the fact that the Euler characteristic is also the alternating sum of the ranks of the homology groups and to choose such a complex $\mathcal{C}$ so that its homology $H_*(\mathcal{C})$ is concentrated in even dimensions and has a basis indexed by the set of fixed points of t, or at least by a set admitting a nice bijection to the fixed point set. Thus the $q = -1$ phenomenon may be proven using the Euler-Poincaré formula. This approach amounts to a categorification of the $q = -1$ phenomenon.

In order to apply our method, we need some way of computing the homology of $\mathcal{C}$. In Section 3, we describe a rudimentary algebraic version of discrete Morse theory that suffices in the cases we examine herein. In these cases, the chain groups C_i have basis indexed by elements of B. To compute $X(-1)$, one might find a fixed-point-free involution m on some large subset S of B such that $|w(m(s)) - w(s)| = 1$ for all $s \in S$. Then $X(-1) = |B \setminus S|$. The idea behind algebraic discrete Morse theory is that, under the right conditions, m can be used to reduce $\mathcal{C}$ to a smaller complex $\mathcal{C}'$ such that $\mathcal{C}$ and $\mathcal{C}'$ have the same homology and $H_*(\mathcal{C}')$ is very easy to compute.

We will carry out our homological approach in two quite central cases, namely when B is the set of partitions whose Ferrers diagram fits in a rectangle of fixed size and when B is the set of 3-dimensional partitions that fit in a box of fixed size. In each case, the involution t sends a partition to its complement in the given rectangle or box. Thus, the t-fixed partitions are exactly those that are self-complementary.

In the first case, our complex arises as a special case of the following general construction, which is developed in Section 2. For a positive integer n, let t be the involution on the power set $2^{[n]}$ mapping Y to its complement $[n] \setminus Y$. For any subgroup G of the symmetric group $\mathfrak{S}_n$, the action of G on $[n]$ determines an action on $2^{[n]}$, and t commutes with each element of G in this action. Thus t acts on the set $\mathcal{O}(G)$ of orbits of G. We describe in Section 2 four chain complexes, each of which has Euler characteristic equal to the number of fixed points of t on $\mathcal{O}(G)$. Moreover, in each such complex, the rank of the ith chain space is the number of orbits of G on the set $\binom{[n]}{i}$ of i-subsets of $[n]$. Thus if we define the weight function w on $\mathcal{O}(G)$ by mapping an orbit to the size of any representative, the resulting generating function $X(q)$ will give the number of orbits fixed by t upon substituting -1 for q. Thus we hope that the homology of one of these complexes is concentrated in even dimensions and has a basis indexed by the set of self-complementary G-orbits. Our hope is realized when some G-orbit on $[n]$ has odd size, in which case the complexes are acyclic. In particular, the complexes are acyclic when $|G|$ is odd. We know also that, for arbitrary $G \leq S_n$, each of these complexes has trivial first homology group.

In Section 4, we indeed use one of the complexes from Section 2 with G a wreath product of symmetric groups and the set B being the Ferrers diagrams fitting in a rectangle of fixed size. To this end, we make use of the following idea. We factor n as $n = k\ell$ and identify $[n]$ with the set of boxes in a $k \times \ell$ rectangle. We consider the partition of this set of boxes into k subsets of size l, each subset consisting of all the boxes in a fixed row of the rectangle. The stabilizer G of this partition in $\mathfrak{S}_n$ is isomorphic with the wreath product $\mathfrak{S}_\ell \wr \mathfrak{S}_k$. Each G-orbit on $2^{[n]}$ contains the Ferrers diagram of a unique partition. We show that indeed the homology of one of the associated complexes is concentrated in even dimensions and has a basis indexed by objects that are easily seen to be in bijection with the set of self-complementary partitions in our rectangle.

In Section 5 we turn to the case of partitions in a box. We do not use the general construction from Section 2. Rather, we use the fact that the set of plane partitions fitting in an $r \times c \times t$ box is equinumerous with the set of semi-standard Young tableaux of shape (c^r) with entries from $\{0, 1, \ldots, r + t - 1\}$. Moreover, there is a bijection between these two sets that intertwines the complementation map on the set of plane partitions and the evacuation map on the set of semi-standard Young tableaux. We define a chain complex whose i^{th} chain space has a basis indexed by the set of semi-standard Young tableaux under consideration whose entries sum to $i + c\binom{r}{2}$. We use the results of Section 3 to show that the homology of

this complex is concentrated in even dimensions and has a basis indexed by semi-standard domino tableaux of suitable bounded value. Thus, the objects which previously arose in Stembridge's work also emerge in a completely natural way from our complexes and our Morse theory. Section 5 concludes with a poset-theoretic result regarding posets playing an important role in our application of algebraic Morse theory to the chain complexes studied therein. We give a decomposition of each of these posets into a disjoint union of Boolean algebras, which yields a formula for a principal specialization of the Schur function associated to the partition (c^r).

In Section 6, we examine further the orbit complexes introduced in Section 2. We describe an infinite family of permutation groups G for which the homology of the associated complexes is not concentrated in even dimensions. In fact, such concentration fails to occur whenever n is even and a Sylow 2-subgroup of G contains no derangement. Finally, we conjecture that this homology is concentrated in even dimensions when $G \leq \mathfrak{S}_n$ is generated by an n-cycle. In fact, we give a conjectured generating function for the the ranks of the homology groups in this case.

2. Chain complexes resulting from a group action

In this section, we define an associated chain complex for each permutation group on a finite set. Certain complexes of this type will be used later to give a homological proof of the $q = -1$ phenomenon in the case of partitions in a rectangle.

Let G be a subgroup of $\mathfrak{S}_n$. So, G acts on $[n] := \{1, 2, \ldots, n\}$ and this action determines an action of G on the set $\binom{[n]}{i}$ of subsets of cardinality i from $[n]$ for each i. Thus we have an action of G on the set $2^{[n]}$ of all subsets of $[n]$. For $S \subseteq [n]$, let $\overline{S}$ be the G-orbit containing S in this action, and let $2^{[n]}/G$ be the set of all G-orbits on $2^{[n]}$. We consider the generating function

$$
\begin{aligned}
X(G, q) \quad &:= \sum_{\overline{S} \in 2^{[n]}/G} q^{|S|} \\
&= \sum_{i=0}^{n} \left| \binom{[n]}{i} \Big/ G \right| q^i,
\end{aligned}
$$

counting G-orbits on $2^{[n]}$ according to the cardinalities of their representatives. (Here $\binom{[n]}{i}/G$ is the set of G-orbits on $\binom{[n]}{i}$.) The polynomial $X(G, q)$ has been well-studied via algebraic means, perhaps first in the work of Redfield and Pólya. Key in this study is the introduction of a vector space V

over an arbitrary field $\mathbb{F}$ with a basis $\{e_0, e_1\}$. Having fixed $G \leq \mathfrak{S}_n$, we define the tensor product $\mathcal{C}(\mathbb{F}) = V^{\otimes n}$. For $S \subseteq [n]$, we define

$$e_S := e_{i_1} \otimes \cdots \otimes e_{i_n} \in \mathcal{C}(\mathbb{F}),$$

where

$$i_j := \begin{cases} 1 & \text{if } j \in S, \\ 0 & \text{if } j \notin S. \end{cases}$$

Then $\{e_S : S \subseteq [n]\}$ is a basis for $\mathcal{C}(\mathbb{F})$ that is permuted by G according the the action of G on $2^{[n]}$ and thus we obtain a linear action of G on $\mathcal{C}(\mathbb{F})$. The vector space $\mathcal{C}(\mathbb{F})$ admits a natural grading

$$\mathcal{C}(\mathbb{F}) = \bigoplus_{i=0}^{n} \mathcal{C}(\mathbb{F})_i,$$

where

$$\mathcal{C}(\mathbb{F})_i := \mathbb{F}\{e_S : S \in \binom{[n]}{i}\}.$$

Now G acts linearly on each $\mathcal{C}(\mathbb{F})_i$, and we may define the spaces of G-*invariants* as

$$\mathcal{C}(\mathbb{F})_i^G := \{v \in \mathcal{C}(\mathbb{F})_i : g(v) = v \text{ for all } g \in G\}$$

and the spaces of G-*coinvariants* as

$$(\mathcal{C}(\mathbb{F})_i)_G := \mathcal{C}(\mathbb{F})_i / \mathbb{F}\{u - g(u) : u \in \mathcal{C}(\mathbb{F})_i, g \in G\}.$$

We define also

$$\mathcal{C}(\mathbb{F})^G := \bigoplus_{i=0}^{n} \mathcal{C}(\mathbb{F})_i^G$$

and

$$\mathcal{C}(\mathbb{F})_G := \bigoplus_{i=0}^{n} (\mathcal{C}(\mathbb{F})_i)_G.$$

Note that, for $0 \leq i \leq n$,

$$(2.1) \qquad \dim \mathcal{C}(\mathbb{F})_i^G = \left| \binom{[n]}{i} / G \right| = \dim(\mathcal{C}(\mathbb{F})_i)_G.$$

Indeed, for each $\Omega \in \binom{[n]}{i}/G$, set

$$e_\Omega := \sum_{S \in \Omega} e_S.$$

Then $\{e_\Omega : \Omega \in \binom{n}{i}/G\}$ is a basis for $\mathcal{C}(\mathbb{F})_i^G$. Similarly, if we define $\overline{e_S}$ to be the coset represented by e_S in $(\mathcal{C}(\mathbb{F})_i)_G$, then we get a basis $\{\overline{e_S}\}$ for $(\mathcal{C}(\mathbb{F})_i)_G$ by selecting one representative S from each G-orbit on $\binom{[n]}{i}$.

See [15, Corollary 6.2] for the next result, which is due to N. G. de Bruijn.

Theorem 2.1. *(de Bruijn) $X(G, -1)$ is the number of G-orbits $\bar{S}$ of subsets of $[n]$ which are self-complementary in the sense that $\bar{S} = \overline{[n]\backslash S}$.*

For example, if $G = \langle (1234) \rangle$, then $\overline{S} = \{\{1, 3\}, \{2, 4\}\}$ is a self-complementary orbit.

We include a proof of Theorem 2.1, both because it is quite elegant and compact, and also so that it may be compared with our homological approach.

Proof. The action of $GL_2(\mathbb{Q})$ on V induces an action on $\mathcal{C}(\mathbb{Q})$, defined by

$$h(v_1 \otimes \cdots \otimes v_n) = h(v_1) \otimes \cdots \otimes h(v_n).$$

This action commutes with the action of G, and it follows that $\mathcal{C}(\mathbb{Q})^G$ is stabilized by $GL_2(\mathbb{Q})$.

Consider the two elements of $GL_2(\mathbb{Q})$,

$$g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad g' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Note that g, g' are conjugate in $GL_2(\mathbb{Q})$. So, they will act by conjugates on $\mathcal{C}(\mathbb{Q})^G$, and hence have the same trace on $\mathcal{C}(\mathbb{Q})^G$.

Since g swaps e_0 and e_1 in each copy of V, it swaps the basis element $e_{\bar{S}}$ in $\mathcal{C}(\mathbb{Q})^G$ with the basis element $e_{\overline{[n]\backslash S}}$; thus it acts with trace equalling the number of self-complementary G-orbits of subsets.

On the other hand, g' is the specialization to $q = -1$ of the diagonal matrix

$$g(q) = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}.$$

Since $g(q)$ acts on V by sending e_0 to itself and e_1 to qe_1, the induced action of $g(q)$ on $\mathcal{C}(\mathbb{Q})^G$ has $e_{\bar{S}}$ as an eigenvector with eigenvalue $q^{|S|}$. Hence $g(q)$

acts on $\mathcal{C}(\mathbb{Q})^G$ with trace $X(G, q)$. In particular, $g' = g(-1)$ acts with trace $X(G, -1)$.   □

Now define the *up* and *down* maps $U, D : \mathcal{C}(\mathbb{F}) \to \mathcal{C}(\mathbb{F})$ by

$$U(e_S) = \sum_{i \in [n] \setminus S} e_{S \cup \{i\}}$$

$$D(e_S) = \sum_{j \in S} e_{S \setminus \{j\}}.$$

Note that U, D both commute with the G-action. As a consequence, they give well-defined maps on the graded vector spaces $\mathcal{C}(\mathbb{F})^G$ and $\mathcal{C}(\mathbb{F})_G$. Indeed, D and U map $\mathcal{C}(\mathbb{F})_i^G$ to $\mathcal{C}(\mathbb{F})_{i-1}^G$ and $\mathcal{C}(\mathbb{F})_{i+1}^G$, respectively, and a similar statement holds for coinvariants. In addition to Theorem 2.1, the algebraic approach yields the following result.

Theorem 2.2. *([18, 14]) The polynomial $X(G, q)$ has symmetric, unimodal coefficients.*

Indeed, one can verify that the Lie subalgebra of $\mathfrak{gl}(\mathcal{C}(\mathbb{C}))$ generated by U and D is isomorphic with $\mathfrak{sl}_2(\mathbb{C})$ and obtain Theorem 2.2 from classical results in representation theory.

We use D and U for a different purpose, namely, to define the desired chain complexes, which have Euler characteristic $X(G, -1)$. As mentioned in the introduction, this will allow us to prove that the $q = -1$ phenomenon occurs in the case of partitions in a rectangle without the use of Theorem 2.1.

Proposition 2.3.   *1. The maps induced by D on the graded vector spaces $\mathcal{C}(\mathbb{F}_2)^G$ and $\mathcal{C}(\mathbb{F}_2)_G$ are both chain maps, i.e., $D^2 = 0$ in each case. Likewise, the maps induced by U on $\mathcal{C}(\mathbb{F}_2)^G$ and $\mathcal{C}(\mathbb{F}_2)_G$ are both cochain maps, i.e., $U^2 = 0$.*
   2. Each of the four complexes $(\mathcal{C}(\mathbb{F}_2)^G, D)$, $(\mathcal{C}(\mathbb{F}_2)_G, D)$, $(\mathcal{C}(\mathbb{F}_2)^G, U)$, $(\mathcal{C}(\mathbb{F}_2)_G, U)$ has Euler characteristic $X(G, -1)$.

Proof. The maps D, U on $\mathcal{C}(\mathbb{F}_2)$ coincide with the boundary and coboundary maps ∂_i and ∂^i in the usual simplicial chain complex for a simplex having vertex set $[n]$. Hence $D^2 = U^2 = 0$, and the same holds after taking G-invariants or G-coinvariants. This gives (1). Assertion (2) follows from (2.1).   □

As above, given a G-orbit $\bar{S}$, we set $e_{\bar{S}} := \sum_{S' \in \bar{S}} e_{S'}$ and write $\overline{e_S}$ for the basis element of $\mathcal{C}(\mathbb{F})_G$ corresponding to $\bar{S}$. For a finite cochain complex $\mathcal{C}$,

$\mathcal{C}^{\mathrm{op}}$ will denote the opposite chain complex that one obtains by reindexing in the opposite order and reversing all the arrows.

Proposition 2.4. *The map sending $e_{\bar{S}} \mapsto \overline{e_{\bar{S}}}$ induces isomorphisms of complexes*

$$(\mathcal{C}(\mathbb{F}_2)^G, U) \cong \mathrm{Hom}_{\mathbb{F}_2}((\mathcal{C}(\mathbb{F}_2)_G, D), \mathbb{F}_2)$$
$$(\mathcal{C}(\mathbb{F}_2)^G, D) \cong \mathrm{Hom}_{\mathbb{F}_2}((\mathcal{C}(\mathbb{F}_2)_G, U), \mathbb{F}_2).$$

The map sending $e_{\bar{S}} \mapsto \overline{e_{[n] \setminus S}}$ induces isomorphisms of complexes

$$(\mathcal{C}(\mathbb{F}_2)^G, D) \cong (\mathcal{C}(\mathbb{F}_2)^G, U)^{\mathrm{op}}$$
$$(\mathcal{C}(\mathbb{F}_2)_G, D) \cong (\mathcal{C}(\mathbb{F}_2)_G, U)^{\mathrm{op}}.$$

Proof. Let $\bar{S}, \bar{T}$ be G-orbits of subsets with $|T| = |S| + 1$. Then the boundary map coefficient $D_{\bar{S}, \bar{T}}$ in $\mathcal{C}(\mathbb{F}_2)^G$ is the number of elements T' in the orbit $\bar{T}$ that *contain* the fixed set S in $\bar{S}$. Meanwhile the boundary coefficient $D_{\bar{S}, \bar{T}}$ in $\mathcal{C}(\mathbb{F}_2)_G$ is the number of elements S' in the orbit $\bar{S}$ that are *contained in* the fixed set T in $\bar{T}$. There are similar formulae for the coefficients $U_{\bar{S}, \bar{T}}$ in the two complexes. The isomorphisms are not hard to verify, using the fact that set-complementation is an inclusion-reversing bijection. $\square$

In light of the previous proposition, one may consider any one of the four complexes, as its homology determines the homology of the others (either by turning it around in homological degree, or by taking dual $\mathbb{F}_2$-vector spaces, or both).

Theorem 2.5. *The complex $(\mathcal{C}(\mathbb{F}_2)^G, D)$ is acyclic when n is odd. More generally, it is acyclic whenever G has at least one orbit of odd cardinality in its action on $[n]$. If $(\mathcal{C}(\mathbb{F}_2)^G, D)$ is not acyclic, then $H_0(\mathcal{C}(\mathbb{F}_2)^G, D) = \mathbb{F}_2$ and $H_1(\mathcal{C}(\mathbb{F}_2)^G, D) = 0$.*

Proof. Let S be the set of elements of $[n]$ belonging to a G-orbit of odd size, which in particular ensures that S is G-stable (although not necessarily pointwise fixed by G). Then one forms S-*masked* version of the up map in $\mathcal{C}(\mathbb{F}_2)$ as follows:

$$U^{(S)}(e_T) := \sum_{i \in S \setminus T} e_{T \cup \{i\}}$$

The G-stability of S implies that $U^{(S)}$ commutes with the G-action. Indeed, the crucial point in all the requisite calculations is that $g(S) = S$, so that, for example,

$$S \setminus g(T) = g(S \setminus T),$$
$$S \cap g(T) = g(S \cap T).$$

Hence one gets an induced S-masked up map $U^{(S)}$ on $\mathcal{C}(\mathbb{F}_2)^G$.

A straightforward calculation shows that

$$(D \cdot U^{(S)} - U^{(S)} \cdot D)(e_T) = (|S \backslash T| - |S \cap T|) \cdot e_T$$
$$= (|S| - 2|S \cap T|) \cdot e_T.$$

As we are working in characteristic two, this gives

$$D \cdot U^{(S)} + U^{(S)} \cdot D = |S| \cdot I.$$

Thus when $|S|$ is odd, the S-masked up map $U^{(S)}$ gives an algebraic chain-contraction, showing that $(\mathcal{C}(\mathbb{F}_2)^G, D)$ is acyclic.

Next we verify the assertions about H_0 and H_1 by closely examining the first few boundary maps in $(\mathcal{C}(\mathbb{F}_2)^G, D)$.

The boundary map D out of $\mathcal{C}(\mathbb{F}_2)_0^G = \mathbb{F}_2$ is *always* the zero map, regardless of G. If all G-orbits have even cardinality, the boundary map D out of $\mathcal{C}(\mathbb{F}_2)_1^G$ will also be the zero map, so the assertion about H_0 follows.

It remains to show that when all G-orbits have even cardinality, the map D out of $\mathcal{C}(\mathbb{F}_2)_2^G$ is surjective. But for this we can work within each G-orbit X on $[n]$. That is, it suffices to show that there is some G-orbit $Y = \overline{\{i,j\}}$ of pairs with $i, j \in X$ for which $D(e_Y)$ has coefficient 1 on e_X, not zero. However, fixing X, one can see that the sum of all of such boundary map coefficients incident to X and *coming from G-orbits of pairs contained in X* will be $|X| - 1$, an odd number. Thus one of them must be non-zero in $\mathbb{F}_2$, as desired. $\square$

It follows from Theorem 2.1 and Proposition 2.3(2) that the Euler characteristic of all of the complexes under consideration is non-negative. Given this fact and the calculations of H_0, H_1 in Theorem 2.5, one might be tempted to make the conjecture (true up through $n = 5$) that the homology of $\mathcal{C}(\mathbb{F}_2)^G$ is always concentrated in even dimensions. However, we shall see in Section 6 that this is not the case. In Section 4 we will nonetheless confirm this behavior for wreath products of symmetric groups in their natural imprimitive actions, and in Section 6 we provide evidence that this holds for cyclic groups as well.

3. A Morse matching lemma

In this section we prove Lemma 3.2, which is an algebraic version of the main result in R. Forman's discrete Morse theory. Similar results appear in the work of Jöllenbeck-Welker and that of Sköldberg (see [11] and [17]), in

which much more extensive versions of algebraic discrete Morse theory are developed. Lemma 3.2 will be applied in Sections 4 and 5 to obtain homology bases for chain complexes under consideration, and to thereby compute their Euler characteristics.

Definition 3.1. *Let P be a graded poset and, for $i \geq 0$, let P_i be the set of elements of rank i in P. Let $(\mathcal{C}, d)$ be a chain complex of $\mathbb{F}$-vector spaces $\mathcal{C}_i$ such that $\mathcal{C}_i \neq 0$ if and only if $P_i \neq \emptyset$. We say that P supports $(\mathcal{C}, d)$ if each $\mathcal{C}_i$ has a basis (indexed by) P_i and, for $p \in P_i$ and $q \in P_{i-1}$, the coefficient $d_{p,q}$ of q in $d_i(p)$ is zero unless $q <_P p$.*

We consider the Hasse diagram of P as a graph. A (partial) matching M on this Hasse diagram H is said to be an acyclic matching, or a Morse matching, if the digraph $D(P, M)$, obtained by first directing all edges in H downward and then reversing all the directions of the edges in M, is acyclic (see e.g. [3], [2], [8]).

Given an acyclic matching M on P, let the subsets

$$P^{\mathrm{M}}, P^{\mathrm{unM}}, P_i^{\mathrm{M}}, P_i^{\mathrm{unM}}$$

respectively denote the M-matched and M-unmatched elements in P, and the same sets restricted to rank i. The elements of P^{unM} are called critical elements. We write $q = M(p)$ if the edge $\{p, q\}$ of H lies in M. We write $<_{D(P,M)}$ for the partial order on P that results from taking the transitive closure of $D(P, M)$.

Lemma 3.2. *Let P be a graded poset supporting an algebraic complex $(\mathcal{C}, d)$ and assume P has a Morse matching M such that for all $q = M(p)$ with $q < p$, one has $d_{p,q} \in \mathbb{F}^{\times}$. Let Q_i be the set of all $p \in P_i^M$ such that $M(p) \in P_{i-1}$. Then*

(i) $\dim H_i(\mathcal{C}, d) \leq |P_i^{\mathrm{unM}}|$.
(ii) *If $|Q_i| = rank(d_i)$ for every i, then $\dim H_i = |P_i^{\mathrm{unM}}|$. (For example, this condition is met whenever all unmatched poset elements live in ranks of the same parity.)*
(iii) *If $d_{q,p} = d_{p,r} = 0$ for all $p \in P^{\mathrm{unM}}$ and all $q, r \in P$, then the homology $H(\mathcal{C}, d)$ has $\mathbb{F}$-basis $\{e_p : p \in P^{\mathrm{unM}}\}$.*

Proof. To prove (i), we will show that the matrices of the boundary maps d_i have sufficiently large ranks by showing that they have large, nonsingular square submatrices. Order Q_i (as defined in the statement of this lemma) as $q_1, \ldots, q_r$ by any linear extension of the partial order $<_{D(P,M)}$. We claim that the square submatrix of d_i having columns indexed $q_1, \ldots, q_r$ and rows indexed by $M(q_1), \ldots, M(q_r)$ is invertible upper-triangular.

To prove the claim, note that the hypothesis $d_{p,q} \in \mathbb{F}^{\times}$ for $q < p$ implies that the diagonal entries of this square matrix are all in $\mathbb{F}^{\times}$, since $q = M(p)$ implies $q < p$ or $p < q$. Hence one only needs to verify upper-triangularity. So assume that the boundary map d_i has $(d_i)_{q_j, M(q_k)} \neq 0$ for some $k \neq j$. Since the complex $\mathcal{C}$ is supported on P, we see that $q_j >_P M(q_k)$ and hence $D(P, M)$ has a directed edge $q_j \to M(q_k)$. There is also the matching edge in $D(P, M)$, directed upward as $M(q_k) \to q_k$, and thus by transitivity, $q_j <_{D(P,M)} q_k$. Hence $j < k$, yielding the claim.

The claim implies that $\mathrm{rank}(d_i) \geq |Q_i|$ for all i. As usual letting $Z_i = \ker d_i$ and $B_i = \mathrm{im}\, d_{i+1}$, note that

$$
\begin{aligned}
\dim H_i &= \dim Z_i - \dim B_i \\
&= |P_i| - (\mathrm{rank}(d_i) + \mathrm{rank}(d_{i+1})) \\
&\leq |P_i| - (|Q_i| + |Q_{i+1}|) \\
&= |P_i| - |P_i^{\mathrm{M}}| \\
&= |P_i^{\mathrm{unM}}|
\end{aligned}
$$

as desired.

Now the hypothesis in (ii) makes the weak inequality into an equality in the above string of equalities and weak inequalities, implying the desired equality in (ii). To prove (iii), note that the hypothesis $d_{p,r} = 0$ for all $r < p$ ensures that $\{e_p | p \in P_i^{\mathrm{unM}}\}$ is contained in the kernel of d_i for each i, while $d_{q,p} = 0$ for all $q > p$ implies linear independence of this set in $H_i(\mathcal{C}, d)$. Since $|P_i^{\mathrm{unM}}| \geq \dim H_i(\mathcal{C}, d)$, this set also must span $H_i(\mathcal{C}, d)$, hence is a homology basis. $\qquad\square$

4. Application to partitions in a rectangle

Let $P(k, \ell)$ be the set of integer partitions (see, for example, [1]) whose Ferrers diagrams lie inside a $k \times \ell$ rectangle. Any $\lambda \in P(k, \ell)$ may be written as $\lambda = (\lambda_1, \ldots, \lambda_k)$ with

$$
\ell \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0.
$$

$P(k, \ell)$ is a ranked poset under inclusion of Ferrers diagrams. The rank generating function of $P(k, \ell)$ is given by the q-binomial coefficient

$$
X(k, \ell, q) = \begin{bmatrix} k + \ell \\ k \end{bmatrix}_q.
$$

It is well-known that the number of self-complementary partitions in $P(k, \ell)$ is $X(k, \ell, -1)$. Here we will prove this fact by applying Proposition 2.3(2) and Lemma 3.2, thus avoiding the use of Theorem 2.1.

First let us check that Proposition 2.3 may be applied. Consider the partition of the set C of all $k\ell$ boxes in the $k \times \ell$ rectangle into k subsets of size ℓ, each subset consisting of the boxes in a given row of the rectangle. The wreath product $G = \mathfrak{S}_\ell \wr \mathfrak{S}_k$ is the full stabilizer of this partition in $\mathfrak{S}_C$. An element of G first permutes rows wholesale and then permutes boxes within each row independently. It is not hard to see that each G-orbit on the set of subsets of C contains (the Ferrers diagram of) a unique partition and that $(\mathcal{C}(\mathbb{F}_2)^G, D)$ is supported on $P(k, \ell)$.

We will use Lemma 3.2 to show that the homology of $(\mathcal{C}(\mathbb{F}_2), D)$ is concentrated in even dimensions and to exhibit a basis for the homology. Then we will describe a bijection between this basis and the set of self-complementary partitions in $P(k, \ell)$.

One may check directly that the entries in the boundary maps D take the following form.

Proposition 4.1. *If μ is obtained from λ by reducing the part λ_i to $\lambda_i - 1$, then*

$$(4.1) \qquad D_{\lambda,\mu} = (\ell - \lambda_i + 1)(\mathrm{mult}_\lambda(\lambda_i - 1) + 1)$$

where $\mathrm{mult}_\lambda(s)$ is the multiplicity of the part s in λ.

In order to define our Morse matching M on $P(k, \ell)$, we need to specify two parts of each partition therein. Given $\lambda \in P(k, \ell)$,

(1) $A(\lambda)$ is the last nonzero part of λ having the same parity as ℓ, if such a part exists. If no such part exists, then $A(\lambda) = \infty$. Also,
(2) $B(\lambda)$ is the first occurrence of the smallest part of λ having odd multiplicity and opposite parity to ℓ, if such a part exists. If no such part exists, $B(\lambda) = \infty$.

Note that while $A(\lambda) > 0$ always holds, it is possible that $B(\lambda) = 0$. Now we define $M(\lambda)$ for each $\lambda \in P(k, \ell)$ satisfying

$$min\{A(\lambda), B(\lambda)\} < \infty.$$

We will see that M determines a partial matching on $P(k, \ell)$.

Definition 4.2. *Suppose that $\lambda \in P(k, \ell)$ and $min\{A(\lambda), B(\lambda)\} < \infty$. The matching $M(\lambda) = \mu$ is defined as follows.*

1. *If $A(\lambda) < B(\lambda)$, then μ is obtained from λ by replacing $A(\lambda)$ with $A(\lambda) - 1$.*
2. *If $A(\lambda) > B(\lambda)$, then μ is obtained from λ by replacing $B(\lambda)$ with $B(\lambda) + 1$.*

Note that $A(\lambda)$ and $B(\lambda)$ have opposite parities if they are both finite and are therefore distinct. Since $A(\lambda) > 0$ and $B(\lambda)$ has the opposite parity to the largest allowed part ℓ, we see that $M(\lambda) \in P(k, \ell)$ whenever $\min\{A(\lambda), B(\lambda)\} < \infty$. So the function M is a well-defined on its domain, and $M(\lambda)$ either covers or is covered by λ in $P(k, \ell)$.

Next we check that M is an involution on $P(k, \ell)$. If $A(\lambda) < B(\lambda)$, then $A(M(\lambda)) \geq A(\lambda)$ and $B(M(\lambda)) = A(\lambda) - 1$, because the parts of size $A(\lambda) - 1$ have odd multiplicity in $M(\lambda)$. Thus applying M again we see that $M(M(\lambda)) = \lambda$. If $A(\lambda) > B(\lambda)$, then $B(M(\lambda)) \geq B(\lambda) + 2$, and $A(M(\lambda)) = B(\lambda) + 1$, so $M(M(\lambda)) = \lambda$.

The M-unmatched partitions are those such that $A(\lambda) = B(\lambda) = \infty$.

Proposition 4.3. *The M-unmatched partitions $\lambda \in P(k, \ell)$ are exactly those satisfying*

1. *every nonzero part of λ has even multiplicity and opposite parity to ℓ, and*
2. *if ℓ is odd then 0 has even multiplicity in λ.*

Note that if k and ℓ are both odd, $P^{\mathrm{unM}} = \varnothing$. Indeed, if ℓ is odd and a critical cell μ of M exists, then μ has k parts (including 0), all of which are even, with even multiplicity. So k must be even.

Theorem 4.4. *The matching M on the poset $P = P(k, \ell)$ is acyclic, with the partitions λ in P^{unM} being those described in Proposition 4.3. Moreover, the homology of $\mathcal{C}((\mathbb{F}_2)^G, D)$ is concentrated in even dimensions.*

Proof. We need to verify the non-vanishing condition $D_{p,q} \in \mathbb{F}_2^\times$, the hypotheses in (ii) and (iii) of Lemma 3.2, and the acyclicity of M.

Let p be M-matched with $M(p) = q < p$. We must show that $D_{p,q} = 1 \neq 0 \in \mathbb{F}_2$. The part $A(p)$ which is reduced by one by the matching M has the same parity as ℓ, so $\ell - A(p) + 1 = 1 \in \mathbb{F}_2$. Since all parts of p smaller than $A(p)$ have parity distinct from that of ℓ, and $B(p) > A(p)$, the multiplicity of $A(p) - 1$ is even, so $mult_p(A(p) - 1) + 1 = 1 \in \mathbb{F}_2$. Thus by Proposition 4.1, $D_{p,q} = 1 \neq 0 \in \mathbb{F}_2$.

From Proposition 4.3, each unmatched λ is a partition of an even integer, so (ii) is clear.

Let p be M-unmatched. By Proposition 4.3 all parts of p have opposite parity from ℓ so Proposition 4.1 implies $D_{p,r} = 0 \in \mathbb{F}_2$. Similarly $D_{q,p} = 0 \in \mathbb{F}_2$, because, with q_i as defined in Proposition 4.1, the multiplicity of $q_i - 1$ in q must be odd.

Let us now check acyclicity of M. Suppose there is a directed cycle C. Then C alternates M-edges traversed upward and non-M Hasse edges traversed downward. Let E be the smallest part incremented or decremented on C. E cannot be decremented, since all increments would then be of larger integers, and C is a cycle. So E is incremented, has the same parity as ℓ. All parts less than E are fixed in C. When $E = B(\lambda)$ was incremented, λ has no other parts less than E of the same parity as ℓ. So there are no parts less than E of the same parity as ℓ anywhere on C. Moreover all parts less than E of the opposite parity from ℓ always have even multiplicity. The multiplicity of E in λ is odd. Find the first edge f in C where $E + 1$ is decremented to E in μ along a non-M Hasse edge. At this time $A(\mu) = E + 1$ because μ has no smaller parts with the same parity as ℓ. Also $B(\mu) \geq E + 2$, because the multiplicity of E in μ is even. Hence the edge f is an M-edge, which is a contradiction. $\qquad\square$

Theorem 4.5. *There is a bijection ϕ between the set of critical cells of M and the set of self-complementary partitions in a $k \times \ell$ rectangle.*

Proof. Let $\lambda \in P(k, \ell)$. Walking from the southwest corner of the $k \times \ell$ rectangle to the northeast corner along the southeast boundary of the Ferrers diagram of λ (which includes the parts of the western and northern borders of the rectangle that are not contained in the Ferrers diagram), we take k steps of length one to the north and ℓ steps of length one to the east, in some order. Recording these steps in the order they are taken, we get a word $W(\lambda)$ of length $k + \ell$ in the alphabet $\{N, E\}$. Collecting like terms, we write

$$(4.2) \qquad\qquad W(\lambda) = N^{k_1} E^{\ell_1} \ldots N^{k_t} E^{\ell_t},$$

where $k_1, \ell_t \geq 0$ and all other k_i, ℓ_j are positive, $\sum_{i=1}^{t} k_i = k$ and $\sum_{j=1}^{t} \ell_j = \ell$.

Note that if μ is complementary to λ in our rectangle then $W(\mu)$ is the word $W(\lambda)^{\mathsf{rev}}$ obtained by writing $W(\lambda)$ backwards. Therefore, λ is self-complementary if and only if $W(\lambda) = W(\lambda)^{\mathsf{rev}}$.

We define ϕ depending on the parities of k, ℓ as follows. If k, ℓ are both odd then both sets are empty and there is nothing to do.

Now fix k, ℓ such that $k\ell$ is even, and let λ be a critical cell of M.

Say k is even and ℓ is odd. It follows from Proposition 4.3 that, with $W(\lambda)$ written as in (4.2), each k_i is even, ℓ_j is even for $1 \leq j \leq t-1$ and ℓ_t is odd. We set

$$W'(\lambda) := N^{k_1/2} E^{\ell_1/2} \ldots N^{k_t/2} E^{(\ell_t-1)/2},$$

and define $\phi(\lambda)$ to be the partition satisfying

$$W(\phi(\lambda)) = W'(\lambda) E (W'(\lambda))^{\mathsf{rev}}.$$

Now say k and l are both even. Then, in $W(\lambda)$, each k_i is even. If λ is the empty partition then $W(\lambda) = N^k E^\ell$ and we define $\phi(\lambda)$ to be the partition satisfying

$$W(\phi(\lambda)) = N^{k/2} E^\ell N^{k/2}.$$

Otherwise, in $W(\lambda)$, ℓ_1 and ℓ_t are odd and all other ℓ_j are even. We set

$$W'(\lambda) := N^{k_1/2} E^{(\ell_1+1)/2} N^{k_2/2} E^{\ell_2/2} \ldots N^{k_t/2} E^{(\ell_t-1)/2},$$

and define $\phi(\lambda)$ to be the partition satisfying

$$W(\phi(\lambda)) = W'(\lambda) W'(\lambda)^{\mathsf{rev}}.$$

Say k is odd and ℓ is even. Then k_1 is odd and k_i is even for $2 \leq i \leq t$. If λ is the empty partition, so $W(\lambda) = N^k E^\ell$, we define $\phi(\lambda)$ to be the partition satisfying

$$W(\phi(\lambda)) = N^{(k-1)/2} E^{\ell/2} N E^{\ell/2} N^{(k-1)/2}.$$

Otherwise, ℓ_1 and ℓ_t are odd, while ℓ_j is even for $2 \leq j \leq t-1$. We set

$$W'(\lambda) := N^{(k_1-1)/2} E^{(\ell_1+1)/2} \ldots N^{k_t/2} E^{(\ell_t-1)/2},$$

and define $\phi(\lambda)$ by

$$W(\phi(\lambda)) = W'(\lambda) N (W'(\lambda))^{\mathsf{rev}}.$$

We leave it to the reader to find ϕ^{-1} in each case, thus confirming that ϕ is a bijection. $\qquad\square$

5. Application to plane partitions of bounded value in a rectangle

Plane partitions of bounded value inside a rectangle (see e.g. [1]), or equivalently 3-dimensional partitions which lie inside an $r \times c \times t$ box, generalize the objects considered in the previous section, namely the integer partitions which lie inside an $r \times c$ rectangle. Specifically, these integer partitions comprise the $t = 1$ case. The generating function for all plane partitions inside an $r \times c \times t$ box, counted by volume, is given by the Macmahon box formula [1]

$$X(r, c, t, q) = \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{1 - q^{r+t+j-i}}{1 - q^{r+c-j-i+1}}.$$

Stembridge [21] proved that the number of self-complementary plane partitions inside the $r \times c \times t$ box is $X(r, c, t, -1)$.

In this section, we construct an $\mathbb{F}_2$-complex $(\mathcal{C}(r, c, t), d)$ whose i-dimensional cells are indexed by plane partitions of volume i in a $r \times c \times t$ box. We prove homology concentration in Theorem 5.8 and also give a homology basis comprised of objects that are known to be equinumerous with the set of self-complementary plane partitions.

Rather than using the set of plane partitions as our indexing set, it will be more convenient to index bases for our chain spaces by another set of equal cardinality, namely the set of semistandard Young tableaux (SSYT) of shape $\lambda = (c)^r$ in which all entries lie weakly between 0 and $r + t - 1$. We think of a plane partition inside an $r \times c$ rectangle as a Ferrers diagram of shape λ with weakly increasing rows and columns with integers entries lying weakly between 0 and t. A bijective map sending such a plane partition to a SSYT is obtained by adding $i - 1$ to each entry in the i-th row for each $i \in [r]$.

We now define the complex $(\mathcal{C}(r, c, t), d)$, with coefficients in $\mathbb{F}_2$, by letting the chain group generators be indexed by the SSYT of $r \times c$ rectangular shape with entries between 0 and $t + r - 1$, denoted $SSYT(r, c, t)$. We partially order this set $SSYT(r, c, t)$ by setting $T_1 \leq T_2$ if each entry of T_1 is no larger than the corresponding entry of T_2. The poset $SSYT(r, c, t)$ is ranked with rank function

$$(5.1) \qquad rank(T) = (\text{sum of entries of } T) - c\binom{r}{2} = ||T|| - c\binom{r}{2}.$$

Define the boundary map d as follows. For $T \in SSYT(r, c, t)$, dT is a sum over all possible S in $SSYT(r, c, t)$ obtained by subtracting one from

the leftmost copy in some row R of some odd value $2k + 1$ such that $2k + 1$ has the following property: there are an odd number of copies of $2k + 1$ in row R that do not have $2k$ immediately above them in row $R - 1$. In other words,

$$dT = \sum_{T' \in B(T)} T'$$

for $B(T)$ the set of $SSYT(r, c, t)$ obtained by subtracting one from an odd entry $\lambda_{i,j}$ of T located at position (i, j), with the further requirement that there are an odd number of occurrences of the value $\lambda_{i,j}$ in row i that do not have the value $\lambda_{i,j} - 1$ in the position just above them in row $i - 1$.

Proposition 5.1. *$(\mathcal{C}(r, c, t), d)$ is a chain complex, i.e., $d^2 = 0$.*

Proof. Any pair of distinct values that may both be decremented when we apply d twice to T may be decremented in either order, so that the resulting SSYT with the two values decremented occurs with coefficient of 0 in $d^2 T$, because we are working mod 2. One fact this relies upon is that decrementing an odd value will not impact the parity calculation involved in the description of d for other odd values in the same row or for odd values in other rows.

The parity requirements preclude d^2 from decrementing a single value by two or decrementing the same value at two different positions in the same row. $\square$

Definition 5.2. *Let T be in $SSYT(r, c, t)$. Consider the rows in T having at least one of the following properties:*

1. *There is an odd value i such that there are an odd number of occurrences of i in the row which do not have $i - 1$ immediately above them in the prior row,*
2. *There is an even value $i < r + t - 1$ not having the value $i + 1$ immediately below it such that the value $i + 1$ occurs an even (possibly zero) number of times in i's row in positions that do not have i immediately above them.*

In (1) and (2), any condition referring to row 0 or row $r + 1$ is vacuously fulfilled.

If there is some row of T satisfying (1) or (2), define $M(T) \in SSYT(r, c, t)$ to be the tableau obtained from T by

1. *choosing the smallest row R of T containing some i that satisfies (1) or (2)*

2. *choosing the smallest i in row R such that i satisfies (1) or (2)*
3. *subtracting one from the leftmost copy of i in row R if i is odd, and adding one to the rightmost copy of i in R if i is even.*

Next we verify that M is well-defined. In case (1), decrementing the odd entry i is allowed, and $M(T) \in SSYT(r, c, t)$. In case (2), incrementing the even entry i is allowed, and $M(T) \in SSYT(r, c, t)$ because $i + 1 \leq r + t - 1$.

Now let us show that M is a matching, namely that $M(M(T)) = T$ whenever some row of T satisfies one of the conditions (1), (2) of Definition 5.2. Suppose that $M(T)$ is obtained from T by decrementing an odd entry i in row R. Then in $M(T)$, row R satisfies the condition (2) with respect to $i - 1$. Indeed, the copy of $i - 1$ in row R of $M(T)$ which was an i in T cannot lie directly above an i in $M(T)$, as T has strictly increasing columns. Moreover, there are oddly many copies of i in row R of T that do not have an $i - 1$ immediately above them. One of these must be the copy that was decremented to obtain $M(T)$, as this is the leftmost copy. Thus, there are evenly many such copies of i in row R in $M(T)$. Given that we obtain $M(T)$ without making any changes to the first $R - 1$ rows of T or to any entry smaller than i in row R of T, it follows that $M(M(T))$ is obtained from $M(T)$ by incrementing the same entry of $M(T)$ that was decremented in T to obtain $M(T)$. Thus, $M(M(T)) = T$ in this case. A similar argument handles the case where $M(T)$ is obtained from T by instead incrementing an entry.

Example 5.3. *If $r = c = t = 2$, the matching M is*

$$
\begin{matrix} 0 & 2 \\ 3 & 3 \end{matrix} \leftrightarrow \begin{matrix} 1 & 2 \\ 3 & 3 \end{matrix}' \quad
\begin{matrix} 0 & 2 \\ 2 & 3 \end{matrix} \leftrightarrow \begin{matrix} 1 & 2 \\ 2 & 3 \end{matrix}' \quad
\begin{matrix} 1 & 1 \\ 2 & 2 \end{matrix} \leftrightarrow \begin{matrix} 1 & 1 \\ 2 & 3 \end{matrix}' \quad
\begin{matrix} 0 & 0 \\ 3 & 3 \end{matrix} \leftrightarrow \begin{matrix} 0 & 1 \\ 3 & 3 \end{matrix}
$$

$$
\begin{matrix} 0 & 0 \\ 2 & 3 \end{matrix} \leftrightarrow \begin{matrix} 0 & 1 \\ 2 & 3 \end{matrix}' \quad
\begin{matrix} 0 & 0 \\ 1 & 3 \end{matrix} \leftrightarrow \begin{matrix} 0 & 1 \\ 1 & 3 \end{matrix}' \quad
\begin{matrix} 0 & 0 \\ 2 & 2 \end{matrix} \leftrightarrow \begin{matrix} 0 & 1 \\ 2 & 2 \end{matrix}' \quad
\begin{matrix} 0 & 0 \\ 1 & 2 \end{matrix} \leftrightarrow \begin{matrix} 0 & 1 \\ 1 & 2 \end{matrix}
$$

with unmatched points

$$
\begin{matrix} 2 & 2 \\ 3 & 3 \end{matrix}' \quad
\begin{matrix} 1 & 1 \\ 3 & 3 \end{matrix}' \quad
\begin{matrix} 0 & 2 \\ 1 & 3 \end{matrix}' \quad
\begin{matrix} 0 & 0 \\ 1 & 1 \end{matrix}.
$$

Proposition 5.4. *M is a Morse matching.*

Proof. Our proof that M is acyclic is similar to the one given in the previous section. Assume for sake of contradiction that C is a directed cycle in the directed graph $D(P, M)$ obtained from M. Find T and $M(T)$ in C such that $T \leq M(T)$, chosen so that the row R in which T and $M(T)$ differ is as

small as possible within the cycle C, and among such semistandard Young tableaux S in C such that $M(S)$ is obtained from S by incrementing an entry in row R, the entry i of R which is incremented to obtain $M(T)$ from T is as small as possible. Since C is a cycle, at some point in C the resulting entry $i + 1$ that is present in row R of $M(T)$ but instead has value i in T must be decremented. Say the first time (after obtaining $M(T)$ from T in C) that such decrementation occurs, we obtain tableau U from tableau V. Then it follows that $V = M(U)$, which contradicts the cycle C having a directed edge downward from V to U. □

Now we describe P^{unM}.

Lemma 5.5. *In any element of P^{unM}, every even value $2i$ with $2i < t+r-1$ has the odd value $2i+1$ just below it. For each odd value $2i+1$, the number of copies of $2i+1$ in a given row not having $2i$ immediately above them is even.*

Proof. Start with the top row, and proceed downward from row R to row $R+1$ by induction as follows. In row 1, notice that each odd value must occur with even multiplicity, since otherwise we could match by decrementing the leftmost copy of the odd value. Thus, each even value $2i$ in row 1 will have an even number of copies (possibly 0) of $2i+1$ just to its right; therefore we could match by increasing the rightmost copy of $2i$ to $2i+1$ unless there is a $2i+1$ just below it. Now, since our fillings are semistandard, we also must have $2i+1$ just below all the other copies of $2i$ in that row. This handles row 1.

The same argument works at row $R+1$ once the claim has been proven through row R. That is, we may have some odd values in row $R+1$ which were forced to be there by virtue of even values in row R, but all remaining spots to be filled in row $R+1$ will have odd values just above them, which ensures that each odd value $2i+1$ that is put in any of these remaining positions must occur with even multiplicity (not counting the occurrences of the value already forced by row R) to avoid matching by decrementing $2i+1$. And again any even value $2i$ in row $R+1$ will have an even number of decrementable copies of $2i+1$ to its right, allowing matching by incrementing the rightmost $2i$ unless either it has a copy of $2i+1$ just below it or it is the absolute largest allowable value. □

One may check that M is a Morse matching meeting the requirements of Lemma 3.2.

Corollary 5.6. *If $T \in P^{\text{unM}}$ is a critical cell, then $rank(T)$ in the poset $SSYT(r, c, t)$ is even.*

Proof. We will use (5.1) to find the parity of $rank(T)$. For any $T \in P^{\mathrm{unM}}$, the first row sum of T is even. If the row length c is odd, then the row sums alternate in parity. Otherwise, all row sums are even. So the sum of all entries in T is even if c is even, and is odd only when c is odd, and $r \equiv 2, 3$ mod 4. However, $c\binom{r}{2}$ is also odd only when c is odd, and $r \equiv 2, 3 \mod 4$, so $rank(T)$ must be even. □

From Corollary 5.6 the elements of P^{unM} are equinumerous with self-complementary plane partitions. The next Proposition describes the elements of P^{unM} as domino tableaux.

Proposition 5.7. *The elements of P^{unM} are in bijection with*

1. *the semistandard domino tableaux of rectangular shape $r \times c$ filled with odd values between 0 and $r + t - 1$, when $r + t - 1$ is odd,*
2. *the semistandard domino tableaux of rectangular shape $r \times c$ filled with odd values between 0 and $r + t - 2$, allowing also monominos in the last row with value $r + t - 1$, when $r + t - 1$ is even.*

Proof. We use Lemma 5.5 to show that our description of P^{unM} amounts to saying that $r \times c$ is tiled by two types of dominos:

1. horizontal dominos in which both entries of the domino have odd value $2i + 1$, and
2. vertical dominos in which the top entry is $2i$ and the bottom entry is $2i + 1$, along with perhaps some monominos in the bottom row of maximal allowed value $r + t - 1$, in the case that $r + t - 1$ is even.

These tilings are transformed into the desired set of semistandard domino tableaux by simply replacing each $2i$ in the vertical dominos with $2i + 1$. To see that we indeed get such a tiling, first notice that each non-maximal even value must belong to one of our vertical dominos, and then notice that the remaining odd values of non-maximal value occur in horizontal pairs, i.e. horizontal dominos, by our even parity requirement above.

It remains to consider the maximal value $r + t - 1$, which must be in the last row. If $r + t - 1$ is odd, the number of $r + t - 1$ in the last row which do not have $r + t - 2$ above them must be even, otherwise $r + t - 1$ is available to be matched by case (1). Thus we tile this portion of the last row with horizontal dominos labelled with the odd number $r + t - 1$. If $r + t - 1$ is even, there is no restriction on the even entries $r + t - 1$ in the last row, place monominos there. □

Note that if r, c, and t are all odd, $P^{\mathrm{unM}} = \varnothing$, because $r + t - 1$ is odd, rc is odd, and no monominos are allowed. This is expected, because there are no self-complementary plane partitions in this case.

Theorem 5.8. *The complex $(\mathcal{C}(r, c, t), d)$ has homology concentrated in ranks of even parity. Moreover, a basis for the homology is given by Lemma 3.2(iii) and Lemma 5.5.*

It now follows from [23, Theorem 3.1(d)] and the comments thereafter that the number of self-complementary plane partitions in an $r \times c \times t$ box is indeed $X(r, c, t, -1)$, since these are equinumerous with semistandard domino tableaux of rectangular shape $r \times c$ with appropriate corresponding bound on value.

We next use the Morse matching M to decompose $SSYT(r, c, t)$ into a disjoint union of Boolean algebras, generalizing [6, Proposition 11] for $P(k, \ell)$.

Let $Bottom(r, c, t)$ be the set of all $T \in SSYT(r, c, t)$ for which no odd entry satisfies (1) in Definition 5.2. For each $T_{bottom} \in Bottom(r, c, t)$, let T_{top} be the SSYT obtained by incrementing all possible even values which occur in case (2) in any row. Since these incrementations may be done independently, the interval $[T_{bottom}, T_{top}]$ is a Boolean algebra in $SSYT(r, c, t)$.

Theorem 5.9. *As a set, $SSYT(r, c, t)$ is a disjoint union of Boolean algebras*

$$SSYT(r, c, t) = \cup_{T_{bottom} \in Bottom(r, c, t)} [T_{bottom}, T_{top}].$$

Proof. For any $T \in SSYT(r, c, t)$, consider the set

$$D(T) = \{(R, i) : \text{an odd } i \text{ may be decremented to } i - 1$$
$$\text{in row } R \text{ of } T \text{ by case (1) of } 5.2\}$$

of possible decrementations of odd values, and the analogous set of incrementations

$$E(T) = \{(R, i) : \text{an even } i \text{ may be incremented to } i + 1$$
$$\text{in row } R \text{ of } T \text{ by case (2) of } 5.2\}.$$

First we check that $(R, i) \in D(T)$ implies $(R, i - 1) \notin E(T)$: in row R, T has an odd number of $(i - 1) + 1 = i$ to the right of any $i - 1$ which do not have $i - 1$ immediately above them. Similarly one may check that $(R, i) \in E(T)$ implies $(R, i + 1) \notin D(T)$.

If $i \in T$ has been decremented in row R to obtain T', one may check that

$$D(T') = D(T) - \{(R, i)\}, \quad E(T') = E(T) \cup \{(R, i - 1)\},$$

and also if $i \in T$ has been incremented in row R to obtain T', one may check that

$$E(T') = E(T) - \{(R, i)\}, \quad D(T') = D(T) \cup \{(R, i + 1)\}.$$

Thus for a given $T \in SSYT(r, c, t)$, T_{top} is found by incrementing T until $E(T') = \varnothing$, and T_{bottom} is found by decrementing T until $D(T') = \varnothing$.

The interval $[T_{bottom}, T_{top}]$ is a Boolean algebra, because T_{top} agrees with T_{bottom}, except at $E(T_{bottom})$, where they differ by one. $\square$

Example 5.10. *The Boolean algebras in Theorem 5.9 for* $r = c = t = 2$ *consist of the four unmatched points in Example 5.3, and the intervals*

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 2' & 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 1 & 2 \\ 2 & 3' & 3 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2' & 2 & 3 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 3 & 3' & 3 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 2' & 2 & 3 \end{bmatrix}.$$

Theorem 5.9 immediately yields the following enumerative result regarding rank-generating functions of posets by virtue of our decomposition into Boolean algebras and the combinatorial definition of a Schur function as a weighted sum over semistandard Young tableaux.

Corollary 5.11. *Let* θ *be the rectangular partition* c^r. *Then the principally specialized Schur function has the expansion*

$$s_\theta(1, q, \ldots, q^{r+t-1}) = q^{c\binom{r}{2}} X(r, c, t, q) = \sum_{T \in Bottom(r,c,t)} q^{||T||} (1 + q)^{|E(T)|}.$$

Again the Boolean algebras of rank 0 consist of our M-unmatched elements, $T = T_{top} = T_{bottom}$.

6. A counterexample and a conjecture

Here we show that, despite the positive indications obtained from Theorem 2.5 and Theorem 4.4, there are infinitely many permutation groups G such that the homology of $(\mathcal{C}(\mathbb{F}_2)^G, D)$ is not concentrated in even dimensions. On the other hand, we conjecture that if $G \leq \mathfrak{S}_n$ is generated by an n-cycle then

such homology concentration in even dimensions exists. We provide evidence for this conjecture, along with a stronger conjecture giving a formula for the ranks of all homology groups under consideration.

Let $G \leq \mathfrak{S}_n$. By Theorem 2.1 and Proposition 2.3(2), if G has no self-complementary orbit on $\binom{[n]}{n/2}$ then the Euler characteristic of $(\mathcal{C}(\mathbb{F}_2)^G, D)$ is zero. On the other hand, if n is even and G is transitive, then $H_0((\mathcal{C}(\mathbb{F}_2)^G, D))$ is nontrivial, by Theorem 2.5. The next result follows.

Proposition 6.1. *Let n be an even integer. If $G \leq \mathfrak{S}_n$ is transitive but has no self-complementary orbit on $\binom{[n]}{n/2}$ then the homology of $(\mathcal{C}(\mathbb{F}_2)^G, D)$ is not concentrated in even dimensions.*

As shown by J. R. Isbell in [10], $G \leq \mathfrak{S}_n$ has a self-complementary orbit on $\binom{[n]}{n/2}$ if and only if G contains a derangement whose order is a power of two. (This is not hard to prove.) Isbell also shows in [10] that if $n = 2b$ with $b > 1$ odd, then there exists a transitive subgroup of $\mathfrak{S}_n$ containing no derangement whose order is a power of two. As we found copies of [10] to be not easily available, we mention here a construction of such subgroups. Given an odd integer $b > 1$, find the smallest integer d such that b divides $2^d - 1$. Let $V = \mathbb{F}_2^d$. There exists a cyclic subgroup $C \leq GL(V)$ such that C has order b and acts irreducibly on V. Form the semidirect product $G := C \ltimes V$. Choose an ordered basis $e_1, \ldots, e_{d-1}, e_d$ for V such that the matrix of a generator c for C is in rational canonical form with respect to this basis. So, $e_i c = e_{i+1}$ for $1 \leq i \leq d - 1$ and $e_d c = \sum_{i=1}^{d} \alpha_i e_i$, where the characteristic polynomial of c is $x^d + \sum_{i=1}^{d} \alpha_i x^{i-1}$. As c acts irreducibly on V, 1 is not a root of this characteristic polynomial. It follows that $\sum_{i=1}^{d} \alpha_i \equiv 0$. Set

$$H := \left\{ \sum_{i=1}^{d} a_i e_i \in V : \sum_{i=1}^{d} a_i \equiv 0 \right\}.$$

Now H is a hyperplane in V and therefore a subgroup of G. Note that $[G : H] = 2b$. As C acts irreducibly on V, H contains no nontrivial normal subgroup of G. Thus the action of G on the cosets of H by translation gives an embedding of G in $\mathfrak{S}_{2b}$. The conjugacy classes of nontrivial elements of 2-power order in G are exactly the orbits of C on V. Therefore, to show that no 2-element of G is a derangement, it suffices to show that every such orbit intersects H nontrivially. Pick some nonempty $I \subseteq [d]$ and let $v = \sum_{i \in I} e_i \in V$. If $|I|$ is even then $v \in H$. If $|I|$ is odd, let m be the largest

element of I. Then

$$vc^{d-m+1} = e_d c + \sum_{j \in I \setminus \{m\}} e_{j+d-m} \in H.$$

We have the following conclusion.

Proposition 6.2. *If $b > 1$ is odd, there exists some $G \leq \mathfrak{S}_{2b}$ such that the homology of $(\mathcal{C}(\mathbb{F}_2)^G, D)$ is not concentrated in even dimensions.*

Before turning to n-cycles, we remark that Isbell conjectured that if we fix an odd number b, then for all large enough a, every transitive subgroup of $\mathfrak{S}_{2^a b}$ contains a derangement whose order is a power of two.

Although the homology of the complexes introduced in Section 2 is not always concentrated in even dimensions, we are still interested in families of groups G for which this concentration does occur. In this situation, the Poincaré polynomial for the homology of, say, $(\mathcal{C}(\mathbb{F}_2)^G, D)$, can be interpreted as giving a grading on the set of self-complementary G-orbits.

We conclude with evidence that this concentration holds for G a cyclic group C_n generated by an n-cycle in $\mathfrak{S}_n$. In this case the orbits are necklaces, as in [16]. Here are homology calculations from `Mathematica` on C_n for n even; the case where n is odd is settled by Theorem 2.5.

n	homology ranks
2	$1, 0, 0$
4	$1, 0, 1, 0, 0$
6	$1, 0, 0, 0, 1, 0, 0$
8	$1, 0, 1, 0, 1, 0, 1, 0, 0$
10	$1, 0, 0, 0, 2, 0, 0, 0, 1, 0, 0$
12	$1, 0, 1, 0, 2, 0, 2, 0, 1, 0, 1, 0, 0$
14	$1, 0, 0, 0, 3, 0, 2, 0, 3, 0, 0, 0, 1, 0, 0$
16	$1, 0, 1, 0, 3, 0, 5, 0, 5, 0, 3, 0, 1, 0, 1, 0, 0$
18	$1, 0, 0, 0, 4, 0, 6, 0, 8, 0, 6, 0, 4, 0, 0, 0, 1, 0, 0$

The following calculation of $X(C_n, q)$ and $X(C_n, -1)$ may be done using Polya-Redfield theory or Burnside's lemma.

Proposition 6.3. *For $n > 2$,*

$$X(C_n, q) = \frac{1}{n} \sum_{d: d|n} \varphi(d)(1 + q^d)^{\frac{n}{d}}, \ \ and$$

$$X(C_n, -1) = \frac{1}{n} \sum_{\substack{d:d|n, \\ d \text{ even}}} \varphi(d) 2^{\frac{n}{d}},$$

where $\phi(d)$ denotes Euler's totient function.

Note that the above homology data suggest that

- $H_i(\mathcal{C}^{C_{2n}}) = 0$ for i odd, and for $i = 2n - 1, 2n$, and
- $H_{2j}(\mathcal{C}^{C_{2n}}) = H_{2n-2-2j}(\mathcal{C}^{C_{2n}})$.

But one can be much more precise. Note that using the formula in Proposition 6.3 for $X(C_n, -1)$, one can easily check that it satisfies the recursion

$$X(C_{2n}, -1) = \frac{X(C_n, -1) + X(C_n, 1)}{2}.$$

This recursion may be modified to a recursion predicting the homology Poincaré series. For notational convenience, define

$$A_n(q) := \sum_{i=0}^{n} \dim H_i((\mathcal{C}(\mathbb{F}_2)^{C_n}, D))q^i$$

$$X_n(q) := X(C_n, q).$$

Conjecture 6.4. *For any positive integer n, one has $H_i((\mathcal{C}^{C_{2n}}, D)) = 0$ if i is odd, so that $A_{2n}(q)$ is a polynomial in q^2, and one has the recursion*

$$A_{2n}(q^{1/2}) = \frac{qA_n(q) + X_n(q)}{1 + q}.$$

We have some strong evidence for Conjecture 6.4. It is correct at $q = 1$. It holds through $n = 18$, as shown earlier. We have proven it for n odd, although we do not include a proof here. In addition, it is correct with regard to its prediction about $H_2((\mathcal{C}^{C_{2n}}, D))$.

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